The universal Gröbner basis of a binomial edge ideal

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Abstract

We show that the universal Gröbner basis and the Graver basis of a binomial edge ideal coincide. We provide a description for this basis set in terms of certain paths in the underlying graph. We conjecture a similar result for a parity binomial edge ideal and prove this conjecture for the case when the underlying graph is the complete graph.

Keywords: Binomial edge ideals; Parity binomial edge ideals; universal Gröbner basis; Graver basis

1 Introduction

For $n \in \mathbb{N}_{>0}$, $[n] := \{1, \ldots, n\}$. Let G be a simple graph on the vertex set $[n]$, that is, G has no loops and no multiple edges. Let $E(G)$ denote the edge set of G. Let F be a field and let $S = F[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables. The binomial edge ideal of G was introduced and studied independently by Herzog, Hibi, Hreinsdóttir, Kahle and Rauh [2] and Ohtani [6].

Definition 1.1. The *binomial edge ideal* of G is

$$\mathcal{J}_G := \langle x_i y_j - x_j y_i : \{i, j\} \in E(G) \rangle \subseteq S. \tag{1.1}$$

The parity binomial edge ideal of G was introduced and studied by Kahle, Sarmiento and Windisch [5] but had previously been examined by Herzog, Macchia, Madani and Welker [3].

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Definition 1.2. The *parity binomial edge ideal* of G is

$$\mathcal{I}_G := \langle x_i x_j - y_i y_j : \{i, j\} \in E(G)\rangle \subseteq S. \tag{1.2}$$

These ideals appear in various settings and applications in mathematics and statistics and belong to an important class of binomial ideals which may be defined as follows. If we let $R = F[x_1, \ldots, x_n]$ then an ideal I of R is a *pure difference ideal* (also known in the literature as a *pure binomial ideal*) if I is generated by differences of monic monomials i.e. binomials of the form $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$. There are several well-known distinguished subsets of binomials in such an ideal I, two of which we now mention. A binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I$ is called *primitive* if there exists no other binomial $\mathbf{x}^{\mathbf{u}'} - \mathbf{x}^{\mathbf{v}'} \in I$ such that $\mathbf{x}^{\mathbf{u}'}$ divides $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}'}$ divides $\mathbf{x}^{\mathbf{v}}$. The set of primitive binomials in I is called the *Graver basis* of I and denoted $\mathrm{Gr}(I)$. The union of all of the reduced Gröbner bases of I is called the *universal Gröbner basis* of I and denoted $\mathcal{U}(I)$. Graver bases were originally defined for toric ideals by Sturmfels [8]. Charalambous, Thoma and Vladoiu [1] recently generalised the concept to an arbitrary pure difference ideal I. They show that $\mathrm{Gr}(I)$ is finite and extend an argument of Sturmfels to prove that $\mathcal{U}(I) \subseteq \mathrm{Gr}(I)$.

One open problem that arises in the literature is providing a combinatorial characterisation of toric ideals for which the universal Gröbner basis and the Graver basis are equal (many examples have been discovered, see Petrović, Thoma and Vladoiu [7] and references therein). We consider this problem for certain classes of pure difference ideals that are not lattice ideals. In particular, we show that $\mathcal{U}(\mathcal{J}_G) = \mathrm{Gr}(\mathcal{J}_G)$ and provide a description for this basis set in terms of certain paths in G. We conjecture a similar result for $\mathcal{I}_G$ and prove this conjecture for the case when G is the complete graph.

1.1 Preliminaries

Throughout the paper we assume that G is finite, undirected and connected. For any $W \subseteq [n]$, let $G[W]$ denote the induced subgraph on W, and for a sequence of vertices $\pi = (i_0, \ldots, i_r) \in [n]^{r+1}$, $G[\pi] := G[\{i_0, \ldots, i_r\}]$. A (v, w)-*path* of *length* r is a sequence of vertices $v = i_0, i_1, \ldots, i_r = w$ such that $\{i_k, i_{k+1}\} \in E(G)$ for all $k = 0, \ldots, r-1$. The path is *odd (even)* if its length is odd (even). The *interior* of a (v, w)-path $\pi = (i_0, \ldots, i_r)$ is the set $\mathrm{int}(\pi) = \{i_0, \ldots, i_r\} \setminus \{v, w\}$. The *inverse* π^{-1} of a (v, w)-path $\pi = (i_0, \ldots, i_r)$ is the (w, v)-path $(i_r, i_{r-1}, \ldots, i_0)$. For the vertex set of a graph H we sometimes use the notation $V(H)$. For a monomial $\mathbf{x}^{\mathbf{u}} = x_1^{d_1} y_1^{e_1} \cdots x_n^{d_n} y_n^{e_n}$ in S the set $\{i : d_i \neq 0 \text{ or } e_i \neq 0\} \subseteq [n]$ is denoted by $V(\mathbf{x}^{\mathbf{u}})$.

2 Binomial Edge Ideals

In this section we will use two different gradings on S, the first is the $\mathbb{N}^2$-grading by considering the letter of a variable, so we let $\mathrm{ldeg}(x_i) = (1, 0)$ and $\mathrm{ldeg}(y_i) = (0, 1)$ for all $i \in [n]$. The second is the $\mathbb{N}^n$-grading which considers the vertex of a variable and we set $\mathrm{gdeg}(x_i) = \mathrm{gdeg}(y_i) = \mathbf{e}_i$ for all $i \in [n]$, where $\mathbf{e}_i$ is the ith standard basis vector in $\mathbb{N}^n$. The ideal $\mathcal{J}_G$ is homogeneous with respect to both of these gradings and we combine them into what we call the *multidegree* of a monomial $\mathrm{mdeg}(\mathbf{x}^{\mathbf{u}}) := (\mathrm{ldeg}(\mathbf{x}^{\mathbf{u}}), \mathrm{gdeg}(\mathbf{x}^{\mathbf{u}})) \in \mathbb{N}^2 \times \mathbb{N}^n$.

We now recall the definition of admissible paths and the description of the Gröbner basis of $\mathcal{J}_G$ with respect to the lexicographic order which was independently obtained by Herzog et al. [2] and Ohtani [6].

Definition 2.1. Fix a permutation $\sigma \in S_n$ of $[n]$ and let $i, j \in [n]$ satisfy $\sigma^{-1}(i) < \sigma^{-1}(j)$. An (i, j)-path $\pi = (i_0, \dots, i_r)$ in G is called σ-*admissible*, if

(i) $i_k \neq i_l$ if $k \neq l$;

(ii) $j_0, \dots, j_s$ is not a path from i to j for any proper subset $\{j_0, \dots, j_s\}$ of $\{i_0, \dots, i_r\}$;

(iii) for each $k = 1, \dots, r-1$, either $\sigma^{-1}(i_k) < \sigma^{-1}(i)$ or $\sigma^{-1}(i_k) > \sigma^{-1}(j)$.

Given a σ-admissible (i, j)-path $\pi = (i_0, \dots, i_r)$ in G, where $\sigma^{-1}(i) < \sigma^{-1}(j)$,

$$u_\pi := \prod_{\sigma^{-1}(i_k) < \sigma^{-1}(i)} y_{i_k} \prod_{\sigma^{-1}(i_k) > \sigma^{-1}(j)} x_{i_k}.$$

Theorem 2.2 ([2], [6]). *The set of binomials*

$$\mathcal{G}_{G,\sigma} := \bigcup_{\sigma^{-1}(i) < \sigma^{-1}(j)} \{u_\pi(x_i y_j - x_j y_i) : \pi \text{ is a } \sigma\text{-admissible } (i,j)\text{-path in } G\}$$

is the reduced Gröbner basis of $\mathcal{J}_G$ w.r.t. the lexicographic monomial order on S induced by $x_{\sigma(1)} \succ \dots \succ x_{\sigma(n)} \succ y_{\sigma(1)} \succ \dots \succ y_{\sigma(n)}$.

Our first result is a characterisation of the binomials in $\mathcal{J}_G$. For this we need to introduce the following notations. We denote by $d_G(v, w)$ the length of a shortest (v, w)-path in G. For a monomial $\mathbf{x}^{\mathbf{u}} = x_1^{d_1} y_1^{e_1} \cdots x_n^{d_n} y_n^{e_n} \in S$, we sometimes use the notation $\deg_{x_i}(\mathbf{x}^{\mathbf{u}})$ for d_i and $\deg_{y_i}(\mathbf{x}^{\mathbf{u}})$ for e_i. For an induced subgraph H of G, we define the *restriction* of $\mathbf{x}^{\mathbf{u}}$ to H to be $\mathrm{res}_H(\mathbf{x}^{\mathbf{u}}) = \Pi_{i \in V(H)} x_i^{d_i} y_i^{e_i}$.

Lemma 2.3. *Let $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ be a multi-homogeneous binomial such that $G[V(\mathbf{x}^{\mathbf{u}})]$ is a connected graph. Then $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{J}_G$.*

Proof. Suppose that $m = \mathbf{x}^{\mathbf{u}}$ is a monomial such that there is a pair of indices $i < j$ with $\deg_{x_i}(m) \geqslant 1$ and $\deg_{y_j}(m) \geqslant 1$. Let $G' = G[V(m)]$. We can assume that i and j are chosen such that $d_{G'}(i, j)$ is minimal. Now we consider an (i, j)-path $\pi = (i_0, \dots, i_r)$ of minimal length in G'. Since π is of minimal length we can conclude that $i_k \neq i_l$ for $k \neq l$ and that no proper subset $\{j_0, \dots, j_s\}$ of $\{i_0, \dots, i_r\}$ is a path from i to j. Suppose that there is a k such that $i < i_k < j$, then either $\deg_{x_{i_k}}(m) \geqslant 1$, in which case $d_{G'}(i_k, j) < d_{G'}(i, j)$ which contradicts the minimality of $d_{G'}(i, j)$, or $\deg_{y_{i_k}}(m) \geqslant 1$, in which case $d_{G'}(i, i_k) < d_{G'}(i, j)$ which again contradicts the minimality of $d_{G'}(i, j)$. We may thus conclude that π is a σ-admissible (i, j)-path in G', where $\sigma = \mathrm{id}$, the identity permutation in S_n.

Now we consider the vertex i_k on π. By the minimality of $d_{G'}(i,j)$, if $i_k < i$ then $\deg_{x_{i_k}}(m) = 0$ and if $i_k > j$ then $\deg_{y_{i_k}}(m) = 0$. We may thus conclude that $u_\pi x_i y_j$ divides m, and therefore m is reducible with respect to $\mathcal{G}_{G,\mathrm{id}}$. This shows that an irreducible monomial of the same multidegree as $\mathbf{x}^{\mathbf{u}}$ has the form $\mathbf{x}^{\mathbf{w}} = y_{i_1}^{e_1} \cdots y_{i_k}^{e_k} x_{i_k}^{d_k} \cdots x_{i_l}^{d_l}$ where $i_1 < i_2 < \cdots < i_l$. Since there is only one such monomial in a given multidegree, we can conclude that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ reduces to zero with respect to $\mathcal{G}_{G,\mathrm{id}}$ and thus $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{J}_G$. □

Lemma 2.4. *A multi-homogeneous binomial* $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ *lies in* $\mathcal{J}_G$ *if and only if*

$$\mathrm{mdeg}(\mathrm{res}_C(\mathbf{x}^{\mathbf{u}})) = \mathrm{mdeg}(\mathrm{res}_C(\mathbf{x}^{\mathbf{v}}))$$

for all connected components C *of* $G[V(\mathbf{x}^{\mathbf{u}})]$.

Proof. Let $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{J}_G$, then we can write

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_k \mathbf{x}^{\mathbf{w}_k}(x_{i_k} y_{j_k} - x_{j_k} y_{i_k})$$

where $\{i_k, j_k\} \in E(G)$ for all k. Let C be a component in $G[V(\mathbf{x}^{\mathbf{u}})]$, then by restricting to C we get

$$\mathrm{res}_C(\mathbf{x}^{\mathbf{u}}) - \mathrm{res}_C(\mathbf{x}^{\mathbf{v}}) = \sum_{k, i_k \in V(C)} \mathrm{res}_C(\mathbf{x}^{\mathbf{w}_k})(x_{i_k} y_{j_k} - x_{j_k} y_{i_k})$$

since $j_k \in V(C)$ if and only if $i_k \in V(C)$. Thus we see that $\mathrm{res}_C(\mathbf{x}^{\mathbf{u}}) - \mathrm{res}_C(\mathbf{x}^{\mathbf{v}}) \in \mathcal{J}_G$, and therefore $\mathrm{mdeg}(\mathrm{res}_C(\mathbf{x}^{\mathbf{u}})) = \mathrm{mdeg}(\mathrm{res}_C(\mathbf{x}^{\mathbf{v}}))$.

For the converse, suppose $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ satisfies that $\mathrm{mdeg}(\mathrm{res}_C(\mathbf{x}^{\mathbf{u}})) = \mathrm{mdeg}(\mathrm{res}_C(\mathbf{x}^{\mathbf{v}}))$ for all connected components $C_1, \ldots C_r$ of $G[V(\mathbf{x}^{\mathbf{u}})]$, then, by Lemma 2.3, $\mathrm{res}_{C_i}(\mathbf{x}^{\mathbf{u}}) - \mathrm{res}_{C_i}(\mathbf{x}^{\mathbf{v}}) \in \mathcal{J}_G$ for all r, and we can write

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_{i=1}^{r} \frac{\mathbf{x}^{\mathbf{u}} \prod_{j=1}^{i-1} \mathrm{res}_{C_i}(\mathbf{x}^{\mathbf{v}})}{\prod_{j=1}^{i} \mathrm{res}_{C_i}(\mathbf{x}^{\mathbf{u}})} (\mathrm{res}_{C_i}(\mathbf{x}^{\mathbf{u}}) - \mathrm{res}_{C_i}(\mathbf{x}^{\mathbf{v}})),$$

which shows that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{J}_G$. □

Definition 2.5. An (i,j)-path $\pi = (i_0, \ldots, i_r)$ in G is called *weakly admissible* if it satisfies conditions (i) and (ii) of the definition of a σ-admissible path.

Given a weakly admissible (i,j)-path $\pi = (i_0, \ldots, i_r)$ in G,

$$\mathcal{S}_\pi := \{t_{i_1} t_{i_2} \cdots t_{i_{r-1}}(x_i y_j - x_j y_i) : t_k \in \{x_k, y_k\}\}$$

and $\mathcal{S}(\mathcal{J}_G) := \bigcup_\pi \mathcal{S}_\pi \setminus \{0\}$ where π runs over all weakly admissible paths in G. Notice that if π is an (i,i)-path in G, then π is weakly admissible if and only if π is the path (i) of length 0, in which case $\mathcal{S}_\pi = \{0\}$.

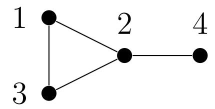

Figure 1: See Examples 1 and 2.

Example 1. Let G be the graph in Figure 1. The weakly admissible paths in G are the paths (1), (2), (3), (4), $(1,2)$, $(1,3)$, $(2,3)$, $(2,4)$, $(1,2,4)$ and $(3,2,4)$, together with their inverses. Hence $|\mathcal{S}(\mathcal{J}_G)| = 16$.

Theorem 2.6. *The sets $\mathcal{S}(\mathcal{J}_G)$, $\mathcal{U}(\mathcal{J}_G)$ and $\mathrm{Gr}(\mathcal{J}_G)$ coincide.*

Proof. We prove the theorem in three steps; the containments $\mathcal{S}(\mathcal{J}_G) \subseteq \mathcal{U}(\mathcal{J}_G)$, $\mathcal{U}(\mathcal{J}_G) \subseteq \mathrm{Gr}(\mathcal{J}_G)$ and $\mathrm{Gr}(\mathcal{J}_G) \subseteq \mathcal{S}(\mathcal{J}_G)$.

Step 1. $\mathcal{S}(\mathcal{J}_G) \subseteq \mathcal{U}(\mathcal{J}_G)$: Let $\pi = (i_0, \ldots, i_r)$ be a weakly admissible (i,j)-path in G and let $f = t_{i_1} t_{i_2} \cdots t_{i_{r-1}}(x_i y_j - x_j y_i)$ be a corresponding binomial in $\mathcal{S}(\mathcal{J}_G)$. Now let $\sigma \in S_n$ be a permutation such that $\sigma^{-1}(i) < \sigma^{-1}(j)$, $\sigma^{-1}(i_k) < \sigma^{-1}(i)$ for all k such that $t_{i_k} = y_{i_k}$ and $\sigma^{-1}(i_k) > \sigma^{-1}(j)$ for all k such that $t_{i_k} = x_{i_k}$. Then $f \in \mathcal{G}_{G,\sigma}$ and thus $f \in \mathcal{U}(\mathcal{J}_G)$.

Step 2. $\mathcal{U}(\mathcal{J}_G) \subseteq \mathrm{Gr}(\mathcal{J}_G)$: [1, Proposition 4.2].

Step 3. $\mathrm{Gr}(\mathcal{J}_G) \subseteq \mathcal{S}(\mathcal{J}_G)$: Let $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ be a primitive binomial in $\mathcal{J}_G$ and let C be a component of $G[V(\mathbf{x}^{\mathbf{u}})]$. By Lemma 2.4 we have that $\mathrm{res}_C(\mathbf{x}^{\mathbf{u}}) - \mathrm{res}_C(\mathbf{x}^{\mathbf{v}}) \in \mathcal{J}_G$, so by primitivity we can conclude that $C := G[V(\mathbf{x}^{\mathbf{u}})]$ is connected.

Since $\mathbf{u} \neq \mathbf{v}$ we can choose $i, j \in V(\mathbf{x}^{\mathbf{u}})$ such that $d_C(i,j)$ is minimal among all pairs i, j with $\deg_{x_i}(\mathbf{x}^{\mathbf{u}}) > \deg_{x_i}(\mathbf{x}^{\mathbf{v}})$ and $\deg_{y_j}(\mathbf{x}^{\mathbf{u}}) > \deg_{y_j}(\mathbf{x}^{\mathbf{v}})$. Now let $\pi = (i_0, \ldots, i_r)$ be an (i,j)-path in C of minimal length. Suppose that there is a $k \in \{1, \ldots, r-1\}$ such that $\deg_{x_{i_k}}(\mathbf{x}^{\mathbf{u}}) \neq \deg_{x_{i_k}}(\mathbf{x}^{\mathbf{v}})$. Then either $\deg_{x_{i_k}}(\mathbf{x}^{\mathbf{u}}) > \deg_{x_{i_k}}(\mathbf{x}^{\mathbf{v}})$, in which case $d_C(i_k, j)$ would contradict the minimality of $d_C(i,j)$, or $\deg_{x_{i_k}}(\mathbf{x}^{\mathbf{u}}) < \deg_{x_{i_k}}(\mathbf{x}^{\mathbf{v}})$, in which case by homogeneity we would have $\deg_{y_{i_k}}(\mathbf{x}^{\mathbf{u}}) > \deg_{y_{i_k}}(\mathbf{x}^{\mathbf{v}})$ and thus $d_C(i, i_k)$ would contradict the minimality of $d_C(i,j)$. So for $k \in \{1, \ldots, r-1\}$ we have $\deg_{x_{i_k}}(\mathbf{x}^{\mathbf{u}}) = \deg_{x_{i_k}}(\mathbf{x}^{\mathbf{v}})$ and hence by homogeneity $\deg_{y_{i_k}}(\mathbf{x}^{\mathbf{u}}) = \deg_{y_{i_k}}(\mathbf{x}^{\mathbf{v}})$. We can then, for $k = 1, \ldots, r-1$, let $z_{i_k} \in \{x_{i_k}, y_{i_k}\}$ such that z_{i_k} divides $\mathbf{x}^{\mathbf{u}}$ and thus also $\mathbf{x}^{\mathbf{v}}$. Then $z_{i_1} \cdots z_{i_{r-1}}(x_i y_j - x_j y_i) \in \mathcal{J}_G$ with $z_{i_1} \cdots z_{i_{r-1}} x_i y_j | \mathbf{x}^{\mathbf{u}}$ and $z_{i_1} \cdots z_{i_{r-1}} x_j y_i | \mathbf{x}^{\mathbf{v}}$, which implies that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = z_{i_1} \cdots z_{i_{r-1}}(x_i y_j - x_j y_i) \in \mathcal{S}(\mathcal{J}_G)$. □

3 Parity Binomial Edge Ideals

In this section we will use two different gradings on S, but not exactly as in the previous section. The first grading is the $\mathbb{Z}_2^2$-grading by considering the letter of a variable, so we let $\mathrm{ldeg}(x_i) = (1,0) \in \mathbb{Z}_2^2$ and $\mathrm{ldeg}(y_i) = (0,1) \in \mathbb{Z}_2^2$ for all $i \in [n]$. The second is the $\mathbb{N}^n$-grading as in the previous section. The ideal $\mathcal{I}_G$ is homogeneous with respect to both of these gradings and we combine them into what we call the *multidegree* of a monomial $\mathrm{mdeg}(\mathbf{x}^{\mathbf{u}}) := (\mathrm{ldeg}(\mathbf{x}^{\mathbf{u}}), \mathrm{gdeg}(\mathbf{x}^{\mathbf{u}})) \in \mathbb{Z}_2^2 \times \mathbb{N}^n$.

Lemma 3.1 ([5])**.** *Let $\pi = (i_0, \ldots, i_r)$ be an (i,j)-path in G and for $k \in \mathrm{int}(\pi)$, let $t_k \in \{x_k, y_k\}$ be arbitrary. If π is odd, then*

$$(x_i x_j - y_i y_j) \prod_{k \in \mathrm{int}(\pi)} t_k \in \mathcal{I}_G.$$

If π is even, then

$$(x_i y_j - y_i x_j) \prod_{k \in \mathrm{int}(\pi)} t_k \in \mathcal{I}_G.$$

Definition 3.2. An (i,j)-path π in G is called *minimal*, if

(i) for no $k \in \mathrm{int}(\pi)$ there is an (i,j)-path with the same parity as π in $G[\pi \setminus \{k\}]$;

(ii) there is no shorter (i,j)-path π' in G satisfying $\mathrm{parity}(\pi') = \mathrm{parity}(\pi)$ and $\mathrm{int}(\pi') = \mathrm{int}(\pi)$.

For a minimal (i,j)-path π in G, we define a set of binomials $\mathcal{S}_\pi$ as follows. If π is odd, then $\mathcal{S}_\pi := \mathcal{S}^+_{\pi,o} \bigcup \mathcal{S}^-_{\pi,o}$ where

$$\mathcal{S}^+_{\pi,o} := \{(x_i x_j - y_i y_j) \prod_{k \in \mathrm{int}(\pi)} t_k : t_k \in \{x_k, y_k\}\},$$
$$\mathcal{S}^-_{\pi,o} := \{(y_i y_j - x_i x_j) \prod_{k \in \mathrm{int}(\pi)} t_k : t_k \in \{x_k, y_k\}\}.$$

If π is even, then $\mathcal{S}_\pi := \mathcal{S}_{\pi,e}$ where

$$\mathcal{S}_{\pi,e} := \{(x_i y_j - y_i x_j) \prod_{k \in \mathrm{int}(\pi)} t_k : t_k \in \{x_k, y_k\}\}.$$

$\mathcal{S}(\mathcal{I}_G) := \bigcup_\pi \mathcal{S}_\pi \setminus \{0\}$ where π runs over all minimal paths in G. Notice that if π is an even (i,i)-path in G, then π is minimal if and only if π is the path (i) of length 0, in which case $\mathcal{S}_\pi = \{0\}$.

Example 2. Let G be the graph in Figure 1. The minimal paths in G are the paths

$(1),\ (2),\ (3),\ (4),$
$(1,2),\ (1,3),\ (2,3),\ (2,4),$
$(1,2,3),\ (1,2,4),\ (1,3,2),\ (2,1,3),\ (3,2,4),$
$(1,2,3,1),\ (1,3,2,4),\ (2,1,3,2),\ (3,1,2,3),\ (3,1,2,4),$
$(2,1,3,2,4),\ (2,3,1,2,4)$
and $(4,2,1,3,2,4),$

together with their inverses. Hence $|\mathcal{S}(\mathcal{I}_G)| = 92$.

Given a graph G, it is clear that the set of its weakly admissible paths is a subset of the set of its minimal paths. If π is a minimal (i,j)-path in G which is not a weakly admissible path, π contains repeated vertices or $G[\pi]$ contains an (i,j)-path π' of parity opposite to that of π such that $\mathrm{int}(\pi') \subsetneq \mathrm{int}(\pi)$.

Conjecture 3.3. *The sets $\mathcal{S}(\mathcal{I}_G)$, $\mathcal{U}(\mathcal{I}_G)$ and $\mathrm{Gr}(\mathcal{I}_G)$ coincide.*

We have partially tested Conjecture 3.3 for small graphs using the software GFAN [4] and have found no counterexamples so far. It must be said, however, that we are not currently aware of any algorithm for computing the Graver basis of an arbitrary pure difference ideal. The main result of this section is a proof that Conjecture 3.3 holds when G is the complete graph K_n on the vertex set $[n]$. The rest of the section is arranged as follows. In Lemmas 3.4 through 3.7 we describe a reduced Gröbner basis of $\mathcal{I}_{K_n}$. In Lemmas 3.8 through 3.12 we characterise the binomials in $\mathcal{I}_{K_n}$. In Theorem 3.13 the main result is proved.

Lemma 3.4. *The minimal paths in K_n are all*

$$(i),\ (i,j),\ (i,k,j) \text{ and } (i,k,l,i)$$

where i, j, k and l are distinct elements of $[n]$.

Proof. Let $\pi = (i_0, \dots, i_r)$ be an (i,j)-path in K_n. Suppose that π is odd. If $\mathrm{int}(\pi) = \varnothing$ then π is necessarily of the form $(i,j,i,j,\dots,i,j)$ and thus minimal if and only if $\pi = (i,j)$. If $\mathrm{int}(\pi) \neq \varnothing$ then there are two cases: $i = j$ and $i \neq j$. If $i = j$ then $|\mathrm{int}(\pi)| \geqslant 2$ so let $i_{s_1} \neq i_{s_2} \in \mathrm{int}(\pi)$ and notice that K_n contains the odd path $\pi' = (i_0, i_{s_1}, i_{s_2}, i_0)$. Now $K_n[\pi' \setminus \{i_{s_1}\}] \cong K_n[\pi' \setminus \{i_{s_2}\}] \cong K_2$ which does not contain an odd cycle, hence π' is minimal. It follows that π is minimal if and only if $\pi = (i,k,l,i)$, where $\{k,l\} = \mathrm{int}(\pi)$. If $i \neq j$ then for all $k \in \mathrm{int}(\pi) \neq \varnothing$ the graph $K_n[\pi \setminus \{k\}]$ contains the odd (i,j)-path $\pi' = (i,j)$, hence π is not minimal in this case. The proof for an even path is similar and omitted. ☐

Given a permutation $\sigma \in S_n$ of $[n]$ and a set $L \subseteq [n]$ let $\succ$ denote the lexicographic monomial order on S induced by

$$t_{\sigma(1)} \succ \cdots \succ t_{\sigma(n)} \succ t'_{\sigma(1)} \succ \cdots \succ t'_{\sigma(n)}$$

where $t_{\sigma(i)} = x_{\sigma(i)}$, $t'_{\sigma(i)} = y_{\sigma(i)}$ for all $i \in [n] \setminus L$ and $t_{\sigma(i)} = y_{\sigma(i)}$, $t'_{\sigma(i)} = x_{\sigma(i)}$ for all $i \in L$. For $i, j \in [n]$ write $i \succ j$ if $\sigma^{-1}(i) < \sigma^{-1}(j)$. Let $\mathcal{G}_\succ(G)$ denote the reduced Gröbner basis of $\mathcal{I}_G$ with respect to $\succ$. For a nonzero $f \in S$ let $N_{\mathcal{G}_\succ(G)}(f)$ denote the normal form of f with respect to $\mathcal{G}_\succ(G)$ and let $in_\succ(f)$ denote the initial monomial of f with respect to $\succ$. For the next three lemmas (3.5 to 3.7) fix a permutation $\sigma \in S_n$ and a set $L \subseteq [n]$. For $v \in [n]$

$$c_v := \begin{cases} +1 & \text{if } \sigma^{-1}(v) \notin L \\ -1 & \text{if } \sigma^{-1}(v) \in L; \end{cases} \qquad r_v := \begin{cases} y_v & \text{if } \sigma^{-1}(v) \notin L \\ x_v & \text{if } \sigma^{-1}(v) \in L. \end{cases}$$

For $i, j, k, l \in [n]$ let $B_{(i,j)} = \{c_i(x_ix_j - y_iy_j) : i \succ j\}$, $B_{(i,k,j)} = \{c_i(x_iy_j - y_ix_j)r_k : i, k \succ j\}$ and $B_{(i,k,l,i)} = \{c_i(x_i^2 - y_i^2)r_kr_l : k, l \succ i\}$. Notice that for an element $f \in \big(B_{(i,j)} \cup B_{(i,k,j)} \cup B_{(i,k,l,i)}\big)$, the value of c_i ensures that the coefficient of the initial monomial $in_\succ(f)$ is 1. Finally let the set $\Gamma \subseteq B_{(i,k,j)}$ consist of all binomials $f = c_i(x_iy_j - y_ix_j)r_k \in B_{(i,k,j)}$ satisfying $i \succ k \succ j$ and $|\{\sigma^{-1}(i), \sigma^{-1}(k)\} \cap L| = 1$.

Lemma 3.5. *Let $f \in S$ be a nonzero binomial corresponding to a minimal path π in K_n (in the sense of Lemma 3.1). Then f is reduced with respect to $\succ$ if and only if $f \in \Lambda := \big(B_{(i,j)} \cup B_{(i,k,j)} \cup B_{(i,k,l,i)}\big) \setminus \Gamma$.*

Proof. By Lemma 3.4 it suffices to consider only binomials corresponding to the paths (i, j), (i, k, j) and (i, k, l, i) in K_n, where i, j, k and l are distinct elements of $[n]$. Without loss of generality we may assume that $i \succ j$. If f is the binomial corresponding to the path (i, j) then clearly f is reduced if and only if $f \in B_{(i,j)} \subseteq \Lambda$.

If f is a binomial corresponding to the path (i, k, j) then there are three conceivable cases: $i \succ j \succ k$, $i \succ k \succ j$ and $k \succ i \succ j$. For ease of notation $f^t_{(i,j,k)} := c_i(x_iy_k - y_ix_k)t_j$ where the superscript $t \in \{x, y\}$ indicates whether $t_j = x_j$ or $t_j = y_j$.

Case 1 ($i \succ j \succ k$). If $\sigma^{-1}(i) \notin L$ then $f^x_{(i,k,j)}$ is reduced by $x_ix_k - y_iy_k$; also $f^y_{(i,k,j)}$ is reduced by $f^y_{(i,j,k)}$ if $\sigma^{-1}(j) \notin L$, or $y_jy_k - x_jx_k$ if $\sigma^{-1}(j) \in L$. If $\sigma^{-1}(i) \in L$ then $f^x_{(i,k,j)}$ is reduced by $x_jx_k - y_jy_k$ if $\sigma^{-1}(j) \notin L$, or $f^x_{(i,j,k)}$ if $\sigma^{-1}(j) \in L$; also $f^y_{(i,k,j)}$ is reduced by $y_iy_k - x_ix_k$.

Case 2 ($i \succ k \succ j$). If $\sigma^{-1}(i) \notin L$ then $f^x_{(i,k,j)}$ is reduced by $x_ix_k - y_iy_k$; also $f^y_{(i,k,j)}$ is reduced by $y_ky_j - x_kx_j$ if $\sigma^{-1}(k) \in L$ but is irreducible if $\sigma^{-1}(k) \notin L$. If $\sigma^{-1}(i) \in L$ then $f^x_{(i,k,j)}$ is reduced by $x_kx_j - y_ky_j$ if $\sigma^{-1}(k) \notin L$ but is irreducible if $\sigma^{-1}(k) \in L$; also $f^y_{(i,k,j)}$ is reduced by $y_iy_k - x_ix_k$.

Case 3 ($k \succ i \succ j$). If $\sigma^{-1}(i) \notin L$ then $f^x_{(i,k,j)}$ is reduced by $x_kx_i - y_ky_i$ if $\sigma^{-1}(k) \notin L$ but is irreducible if $\sigma^{-1}(k) \in L$; also $f^y_{(i,k,j)}$ is reduced by $y_ky_j - x_kx_j$ if $\sigma^{-1}(k) \in L$ but is irreducible if $\sigma^{-1}(k) \notin L$. If $\sigma^{-1}(i) \in L$ then $f^x_{(i,k,j)}$ is reduced by $x_kx_j - y_ky_j$ if $\sigma^{-1}(k) \notin L$ but is irreducible if $\sigma^{-1}(k) \in L$; also $f^y_{(i,k,j)}$ is reduced by $y_ky_i - x_kx_i$ if $\sigma^{-1}(k) \in L$ but is irreducible if $\sigma^{-1}(k) \notin L$.

Finally let $f = c_i(x_i^2 - y_i^2)t_kt_l$ be a binomial corresponding to the path (i, k, l, i). If $i \succ k$ then f is reduced by one of $f^x_{(i,l,k)}, f^y_{(i,l,k)}$ or $c_i(x_ix_k - y_iy_k)$. The case $i \succ l$ is similar. If $k, l \succ i$ then by arguing as before one finds that f is irreducible if and only if $t_s = y_s$ whenever $\sigma^{-1}(s) \notin L$ and $t_s = x_s$ whenever $\sigma^{-1}(s) \in L$. □

Lemma 3.6. *Let π be an (i, j)-path in K_n and for $k \in \mathrm{int}(\pi)$, let $t_k \in \{x_k, y_k\}$ be arbitrary. Then $(x_ix_j - y_iy_j)\Pi_{k \in \mathrm{int}(\pi)}t_k$ if π is odd and $(x_iy_j - y_ix_j)\Pi_{k \in \mathrm{int}(\pi)}t_k$ if π is even, reduce to zero modulo Λ.*

Proof. It suffices to restrict to a minimal path π (if π is not minimal, then its binomial is a multiple of the binomial for a shorter path). If π is minimal, then Lemma 3.5 gives the result. □

Lemma 3.7. $\mathcal{G}_\succ(K_n) = \Lambda$.

Proof. The proof is by Buchberger's criterion and is similar to the proof of Theorem 3.6 in Kahle et al. [5]. Let $g, g' \in \Lambda$ be reduced binomials corresponding, respectively, to the odd path $\pi = (i,k,l,i)$ in K_n and the even path $\pi' = (u,v,w)$ in K_n. Consider the case that $|\{i,k,l\} \cup \{u,v,w\}| = 3$. Since $k,l \succ i$ and $u,v \succ w$ we may assume without loss of generality that $i = w$, $k = u$ and $l = v$. Thus

$$\begin{aligned} g &= c_i(x_i^2 - y_i^2)r_k r_l, \\ g' &= c_k(x_k y_i - y_k x_i)r_l. \end{aligned}$$

There are four possibilities for the constants c_i, c_k. By duality we need only consider $(c_i, c_k) = (1,1)$ and $(c_i, c_k) = (-1,1)$. If $(c_i, c_k) = (1,1)$ then $r_k = y_k$ and

$$\begin{aligned} \text{spol}(g, g') &= -(x_k y_i^3 - y_k x_i^3) y_k r_l \\ &= -(y_k y_i^2) \cdot g' + (y_k x_i) \cdot g + 0 \end{aligned}$$

and $in_\succ(\text{spol}(g,g')) \succeq in_\succ(-y_k y_i^2 \cdot g'),\ in_\succ(y_k x_i \cdot g)$. If $(c_i, c_k) = (-1,1)$ then $\text{spol}(g,g') = -(x_k x_i - y_k y_i) x_i y_k r_l$ which is a monomial multiple of the binomial corresponding to the path (k,i) in K_n and thus reduces to zero by Lemma 3.6. In a similar fashion to the above all $\text{spol}(g,g')$ (where $g, g' \in \Lambda$) reduce to zero with respect to Λ. Thus the set Λ fulfills Buchberger's criterion and hence is a Gröbner basis of $\mathcal{I}_{K_n}$. By Lemma 3.5 it follows that the elements of Λ are reduced with respect to $\succ$. □

Lemma 3.8. *Let $f = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in S$ be multi-homogeneous with $V(\mathbf{x}^\mathbf{u}) = \{i,j\}$. If* $\gcd(\mathbf{x}^\mathbf{u}, \mathbf{x}^\mathbf{v}) = 1$ *then* $f \in \{\pm(x_i^p x_j^q - y_i^p y_j^q), \pm(x_i^p y_j^q - y_i^p x_j^q) : p, q \geqslant 1,\ \text{parity}(p) = \text{parity}(q)\}$.

Proof. By homogeneity f is necessarily of the form

$$\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} = x_i^{d_i} y_i^{e_i} x_j^{d_j} y_j^{e_j} - x_i^{d'_i} y_i^{e'_i} x_j^{d'_j} y_j^{e'_j}$$

satisfying $d_i + e_i = d'_i + e'_i$ and $d_j + e_j = d'_j + e'_j$. By $\gcd(\mathbf{x}^\mathbf{u}, \mathbf{x}^\mathbf{v}) = 1$ it follows that if $d_i > 0$ then $d'_i = 0$ and similarly for all exponents. But if $d_i > 0$ then $d'_i > 0$ or $e'_i > 0$ i.e. $e'_i > 0$ (since $d'_i = 0$) which in turn implies $e_i = 0$. If $d_j > 0$ we get a similar result. If $d_j = 0$ then by $V(\mathbf{x}^\mathbf{u}) = \{i,j\}$ we have $e_j > 0$. By inverting the argument we obtain

$$\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in \{\pm(x_i^{d_i} x_j^{d_j} - y_i^{e'_i} y_j^{e'_j}), \pm(x_i^{d_i} y_j^{e_j} - y_i^{e'_i} x_j^{d'_j})\}.$$

The result follows from the implications of homogeneity. □

Lemma 3.9. *Let $\succ$ be the lexicographic monomial order on S corresponding to $\sigma = \text{id}$ and $L = \varnothing$. Let $\mathbf{x}^\mathbf{u} = x_1^{d_1} y_1^{e_1} \cdots x_n^{d_n} y_n^{e_n} \in S$ where $|V(\mathbf{x}^\mathbf{u})| > 2$. Let $k = \max\{i : i \in V(\mathbf{x}^\mathbf{u})\}$ and let $\gamma = \sum_{i=1}^k d_i$.*

$$N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{u}) = \begin{cases} y_1^{d_1+e_1} \cdots y_k^{d_k+e_k}, & \text{if } \gamma \text{ is even} \\ x_k y_1^{d_1+e_1} \cdots y_{k-1}^{d_{k-1}+e_{k-1}} y_k^{d_k+e_k-1}, & \text{if } \gamma \text{ is odd.} \end{cases} \tag{3.1}$$

Proof. The forms in (3.1) are clearly irreducible. By Lemma 3.7 we have $x_i x_j - y_i y_j \in \mathcal{G}_\succ(K_n)$ for all $i \succ j \in [n]$, so that $\mathbf{x}^\mathbf{u}$ can be reduced to

$$\mathbf{x}^{\mathbf{u}'} = x_s^{d_s - l} y_1^{d_1+e_1} \cdots y_{s-1}^{d_{s-1}+e_{s-1}} y_s^{e_s+l} y_{s+1}^{d_{s+1}+e_{s+1}} \cdots y_k^{d_k+e_k}$$

for some $1 \leqslant s \leqslant k$ where $l \in \mathbb{Z}_{\geqslant 0}$, $l \leqslant d_s$. By the homogeneity of $x_i x_j - y_i y_j$ parity$(\sum_{i=1}^k d_i)$ = parity$(d_s - l)$. If $d_s = l$ then we are done. Otherwise we consider the following two cases. If $1 \leqslant s \leqslant k-1$ then by $|V(\mathbf{x}^\mathbf{u})| > 2$ there exists $v \in V(\mathbf{x}^\mathbf{u}) \setminus \{s\}$, $v \neq k$. Since $v, s \succ k$, by Lemma 3.7 $f = y_v(x_s y_k - y_s x_k) \in \mathcal{G}_\succ(K_n)$. Using f and $x_s x_k - y_s y_k \in \mathcal{G}_\succ(K_n)$ in that order $\mathbf{x}^{\mathbf{u}'}$ can be reduced to

$$\mathbf{x}^{\mathbf{u}''} = x_s^{d_s - l - 2} y_1^{d_1+e_1} \cdots y_{s-1}^{d_{s-1}+e_{s-1}} y_s^{e_s+l+2} y_{s+1}^{d_{s+1}+e_{s+1}} \cdots y_k^{d_k+e_k}.$$

Repeated iteration of this step gives one of the forms in (3.1), depending on the parity of $d_s - l$. If $s = k$ then by $|V(\mathbf{x}^\mathbf{u})| > 2$ there exist $v_1, v_2 \in V(\mathbf{x}^\mathbf{u}) \setminus \{k\}$, $v_1 \neq v_2$. Since $v_1, v_2 \succ k$, by Lemma 3.7 $f = y_{v_1} y_{v_2}(x_k^2 - y_k^2) \in \mathcal{G}_\succ(K_n)$. Using f we can reduce $\mathbf{x}^{\mathbf{u}'}$ to one of the forms in (3.1), depending on the parity of $d_s - l$. ☐

Lemma 3.10. *Let $\succ$ be the lexicographic monomial order on S corresponding to $\sigma = \mathrm{id}$ and $L = \varnothing$. Let $\mathbf{x}^\mathbf{u} = x_i^{d_i} y_i^{e_i} x_j^{d_j} y_j^{e_j} \in S$ i.e. $|V(\mathbf{x}^\mathbf{u})| \leqslant 2$. Let $q = \min\{d_i, d_j\}$.*

$$N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{u}) = x_i^{d_i - q} y_i^{e_i+q} x_j^{d_j-q} y_j^{e_j+q}. \tag{3.2}$$

Proof. The monomial $\mathbf{x}^\mathbf{u}$ can be reduced to (3.2) by the binomial $x_i x_j - y_i y_j \in \mathcal{G}_\succ(K_n)$ (Lemma 3.7). No element of the set $\{in_\succ(g) : g \in \mathcal{G}_\succ(K_n)\}$ divides (3.2) hence (3.2) is irreducible. ☐

Lemma 3.11. *Let $f = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in S$ be such that $|V(\mathbf{x}^\mathbf{u})| > 2$. Then $f \in \mathcal{I}_{K_n}$ if and only if f is multi-homogeneous.*

Proof. "⇒": This is clear. "⇐": Suppose f is multi-homogeneous in degree $((\alpha_1, \alpha_2), \beta) \in \mathbb{Z}_2^2 \times \mathbb{N}^n$. Let $\succ$ be the lexicographic monomial order on S corresponding to $\sigma = \mathrm{id}$ and $L = \varnothing$. Let $k = \max\{i : i \in V(\mathbf{x}^\mathbf{u})\}$. If $\alpha_1 = 0$ then $N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{u}) = N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{v})$ is the first form in (3.1) and if $\alpha_1 = 1$ then $N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{u}) = N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{v})$ is the second form in (3.1). ☐

Lemma 3.12. *Let $f = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in S$ be such that $|V(\mathbf{x}^\mathbf{u})| \leqslant 2$. Then $f \in \mathcal{I}_{K_n}$ if and only if f is of the form $x_i^{d_i} y_i^{e_i} x_j^{d_j} y_j^{e_j} - x_i^{d_i-q} y_i^{e_i+q} x_j^{d_j-q} y_j^{e_j+q}$ where $q \in \mathbb{Z}$.*

Proof. "⇒": $f = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in \mathcal{I} \Rightarrow N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{u}) = N_{\mathcal{G}_\succ(K_n)}(\mathbf{x}^\mathbf{v})$ where $\succ$ is the lexicographic monomial order on S corresponding to $\sigma = \mathrm{id}$ and $L = \varnothing$. Apply Lemma 3.10. "⇐": Apply Lemma 3.10. ☐

Theorem 3.13. *Conjecture 3.3 holds for $G = K_n$.*

Proof. Let $G = K_n$ throughout. We prove the theorem in three steps; the containments $\mathcal{S}(\mathcal{I}_G) \subseteq \mathcal{U}(\mathcal{I}_G)$, $\mathcal{U}(\mathcal{I}_G) \subseteq \mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\mathcal{I}_G) \subseteq \mathcal{S}(\mathcal{I}_G)$.

Step 1. $\mathcal{S}(\mathcal{I}_G) \subseteq \mathcal{U}(\mathcal{I}_G)$: Here we invoke Lemmas 3.4 and 3.7. If π is a path of the form (i,k,l,i) in G then let $\sigma \in S_n$ be a permutation such that $k, l \succ i$. A suitable choice of $L \subseteq \{\sigma^{-1}(i), \sigma^{-1}(k), \sigma^{-1}(l)\}$ then provides that $(x_i^2 - y_i^2)r_k r_l \in \mathcal{G}_\succ(G)$ or $(y_i^2 - x_i^2)r_k r_l \in \mathcal{G}_\succ(G)$. The cases $\pi = (i,k,j)$ and $\pi = (i,j)$ are similar and omitted.

Step 2. $\mathcal{U}(\mathcal{I}_G) \subseteq \mathrm{Gr}(\mathcal{I}_G)$: [1, Proposition 4.2].

Step 3. $\mathrm{Gr}(\mathcal{I}_G) \subseteq \mathcal{S}(\mathcal{I}_G)$: Let $f = \mathbf{x^u} - \mathbf{x^v} \in \mathrm{Gr}(\mathcal{I}_G)$. If $|V(\mathbf{x^u})| \leqslant 2$ then by Lemma 3.12 we have $f = x_i^{d_i} y_i^{e_i} x_j^{d_j} y_j^{e_j} - x_i^{d_i - q} y_i^{e_i + q} x_j^{d_j - q} y_j^{e_j + q}$ where $q \in \mathbb{Z}$. If $q > 0$ then necessarily $f = x_i x_j - y_i y_j \in \mathcal{S}(\mathcal{I}_G)$. If $q < 0$ then necessarily $f = y_i y_j - x_i x_j \in \mathcal{S}(\mathcal{I}_G)$.

Now consider the case that $|V(\mathbf{x^u})| > 2$. First suppose that $f = t_k(\mathbf{x^{u'}} - \mathbf{x^{v'}})$ where $k \in V(\mathbf{x^u})$ and $t_k \in \{x_k, y_k\}$. Now $\mathbf{x^{u'}} - \mathbf{x^{v'}}$ is multi-homogeneous and it must be that $|V(\mathbf{x^{u'}})| = 2$ since otherwise by Lemma 3.11 $\mathbf{x^{u'}} - \mathbf{x^{v'}} \in \mathcal{I}_G$, contradicting the primitivity of f. Write

$$f = t_k(\mathbf{x^{u'}} - \mathbf{x^{v'}}) = t_k(x_i^{d_i} y_i^{e_i} x_j^{d_j} y_j^{e_j} - x_i^{d'_i} y_i^{e'_i} x_j^{d'_j} y_j^{e'_j}). \tag{3.3}$$

If $\gcd(\mathbf{x^{u'}}, \mathbf{x^{v'}}) \neq 1$ then it follows from the previous argument that $f = t_k t_l (x_s^{d_s} y_s^{e_s} - x_s^{d'_s} y_s^{e'_s})$ where $\{s, l\} \subseteq \{i, j\}$ and $t_l \in \{x_l, y_l\}$. Now $x_s^{d_s} y_s^{e_s} - x_s^{d'_s} y_s^{e'_s}$ is multi-homogeneous and nonzero. These criteria are minimally satisfied by $(d_s, e_s) \in \{(2,0),(0,2)\}$ i.e. $f = t_k t_l (x_s^2 - y_s^2) \in \mathcal{S}(\mathcal{I}_G)$ or $f = t_k t_l (y_s^2 - x_s^2) \in \mathcal{S}(\mathcal{I}_G)$. For $d_s + e_s > 2$ the primitivity of f is contradicted either by one of these binomials or by an element of the form $\pm(x_i x_j - y_i y_j) \in \mathcal{S}(\mathcal{I}_G)$. If instead in (3.3) $\gcd(\mathbf{x^{u'}}, \mathbf{x^{v'}}) = 1$ then by Lemma 3.8 and the primitivity of f we have $f = \pm t_k(x_i y_j - y_i x_j) \in \mathcal{S}(\mathcal{I}_G)$.

Suppose now that $f = \mathbf{x^u} - \mathbf{x^v}$ cannot be written as $t_k(\mathbf{x^{u'}} - \mathbf{x^{v'}})$ where $k \in V(\mathbf{x^u})$ and $t_k \in \{x_k, y_k\}$. Since f is multi-homogeneous and $|V(\mathbf{x^u})| > 2$ we can assume that for some $i, j \in V(\mathbf{x^u})$ either $x_i x_j | \mathbf{x^u}$ and $y_i y_j | \mathbf{x^v}$ or $y_i y_j | \mathbf{x^u}$ and $x_i x_j | \mathbf{x^v}$ i.e. in this case the primitivity of f is contradicted by an element of the form $\pm(x_i x_j - y_i y_j) \in \mathcal{S}(\mathcal{I}_G)$. □

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