On stacked triangulated manifolds

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Abstract

We prove two results on stacked triangulated manifolds in this paper: (a) every stacked triangulation of a connected manifold with or without boundary is obtained from a simplex or the boundary of a simplex by certain combinatorial operations; (b) in dimension $d \geqslant 4$, if Δ is a tight connected closed homology d-manifold whose ith homology vanishes for $1 < i < d-1$, then Δ is a stacked triangulation of a manifold. These results give affirmative answers to questions posed by Novik and Swartz and by Effenberger.

Keywords: Stacked manifolds; Triangulations of 3-manifolds; Tight triangulations

1 Introduction

Stacked triangulations of spheres are of fundamental interest, in particular in the study of convex polytopes and triangulations of spheres. Recently, the notion of stackedness was extended to triangulations of manifolds in [15]. In this paper, we prove two results on stacked triangulations of manifolds.

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A triangulation of a connected closed d-manifold is said to be *stacked* if it is the boundary of a triangulation of a $(d+1)$-manifold, and all of its interior faces have dimension $\geqslant d$. A triangulation of a closed d-manifold is said to be *locally stacked* if each vertex link is a stacked triangulation of a sphere. Kalai [10] proved that, for $d \geqslant 4$, every locally stacked triangulation of a connected closed d-manifold can be obtained from the boundary of a $(d+1)$-simplex by certain combinatorial operations. This result does not hold for 3-manifolds since there are triangulations of 3-manifolds which are locally stacked but cannot be obtained by these operations (see e.g. [5, Example 6.2]). On the other hand, since the notions of stackedness and local stackedness are equivalent in dimension $\geqslant 4$ [3, 15], Kalai's result also characterizes stacked triangulations of connected closed manifolds of dimension $\geqslant 4$. We give a similar characterization for stacked triangulations of manifolds with boundary (Theorem 4.5). As a consequence, we generalize the result of Kalai to stacked triangulations of closed manifolds of dimension $\geqslant 2$ (Corollary 4.6). This result and a recent result of Bagchi [1] solve a question posed by Novik and Swartz [16, Problem 5.3].

Our second result is about an equivalence of tightness and tight-neighborliness. For a field $\mathbb{F}$, a simplicial complex Δ is said to be $\mathbb{F}$-*tight* if it is connected and, for any induced subcomplex Γ of Δ, the natural map on reduced homology groups $\widetilde{H}_i(\Gamma;\mathbb{F}) \to \widetilde{H}_i(\Delta;\mathbb{F})$ induced by the inclusion map is injective for all $i \geqslant 0$. We simply say that a simplicial complex is *tight* if it is $\mathbb{F}$-tight for some field $\mathbb{F}$. See [11, 12] for background and motivations of tightness. On the other hand, an n-vertex triangulation Δ of a closed manifold of dimension $d \geqslant 3$ is said to be *tight-neighborly* if $\binom{n-d-1}{2} = \binom{d+2}{2}\beta_1(\Delta;\mathbb{F})$, where $\beta_i(\Delta;\mathbb{F})$ is the ith Betti number computed over a field $\mathbb{F}$. This condition is known to be equivalent to saying that Δ is stacked and neighborly, in particular, is independent of the choice of $\mathbb{F}$ (cf. Section 5). Tight-neighborliness was introduced by Lutz, Sulanke and Swartz. They conjectured that tight-neighborly triangulations are $(\mathbb{Z}/2\mathbb{Z})$-tight [13, Conjecture 13]. The conjecture was solved by Effenberger [9, Corollary 4.4] in dimension $\geqslant 4$ and by Burton, Datta, Singh and Spreer [7, Corollary 1.3] in dimension 3. Moreover, it was recently proved in [6] that, for triangulations of 3-manifolds, tightness is equivalent to tight-neighborliness. On the other hand, Effenberger [9, Question 4.5] asked if $(\mathbb{Z}/2\mathbb{Z})$-tight triangulations of connected sums of S^{d-1}-bundles over S^1 are tight-neighborly when $d \geqslant 4$.

We answer Effenberger's question affirmatively. More generally, we prove that, in dimension $d \geqslant 4$, every tight, closed, orientable, $\mathbb{F}$-homology d-manifold with $\beta_i(\Delta;\mathbb{F}) = 0$ for $1 < i < d-1$, is tight-neighborly (Corollary 5.4). This result and the results mentioned above say that, for triangulations of connected sums of S^{d-1}-bundles over S^1 with $d \geqslant 3$, tightness is equivalent to tight-neighborliness. Also, since tight-neighborly triangulations are vertex minimal triangulations, the result solves a special case of a conjecture of Kühnel and Lutz [12, Conjecture 1.3] which states that every tight combinatorial triangulation is vertex minimal.

This paper is organized as follows. In Section 2, we give a few basic definitions. In Section 3, we define an analogue of a combinatorial handle addition for homology manifolds with boundary and study its basic properties. In Section 4, we present a combinatorial characterization of stacked triangulated manifolds with and without boundary. In Section 5, we study the stackedness of tight triangulations.

2 Preliminaries

Recall that a *simplicial complex* is a collection of finite sets (sets of *vertices*) such that every subset of an element is also an element. All simplicial complexes here are finite and non-empty. For $i \geqslant 0$, the elements of size $i+1$ are called the *i-faces* (or *faces of dimension i*) of the complex. The empty set $\emptyset$ is a face (of dimension -1) of every simplicial complex. For a simplicial complex Δ, let $f_i(\Delta)$ be the number of i-faces of Δ. The maximum k such that Δ has a k-simplex is called the *dimension* of Δ and is denoted by $\dim(\Delta)$. A maximal face (under inclusion) in Δ is called a *facet* of Δ. If σ is a face of Δ then the *link* of σ in Δ is the subcomplex

$$\mathrm{lk}_{\Delta}(\sigma) = \{\tau \setminus \sigma : \sigma \subseteq \tau \in \Delta\}.$$

For any subset W of the vertex set of Δ, we write $\Delta[W] = \{\alpha \in \Delta : \alpha \subseteq W\}$ for the subcomplex of Δ *induced* by W. A simplicial complex is said to be *neighborly* if each pair of vertices forms an edge.

We say that a simplicial complex Δ is a *triangulation* of a manifold M if its geometric carrier $|\Delta|$ is homeomorphic to M. We are mainly interested in triangulations of manifolds, but we actually consider slightly more general objects called homology manifolds. Let $\widetilde{H}_i(\Delta; \mathbb{F})$ be the ith reduced homology group of a topological space (or a simplicial complex) Δ with coefficients in a field $\mathbb{F}$. The number $\beta_i(\Delta; \mathbb{F}) := \dim_{\mathbb{F}} \widetilde{H}_i(\Delta; \mathbb{F})$ is called the *ith Betti number* of Δ with respect to $\mathbb{F}$. For a field $\mathbb{F}$, a simplicial complex S of dimension d is said to be an *$\mathbb{F}$-homology d-sphere* if, for each face σ of dimension $i \geqslant -1$, $\mathrm{lk}_S(\sigma)$ has the same $\mathbb{F}$-homology groups as the $(d-i-1)$-sphere. A simplicial complex B of dimension d is said to be an *$\mathbb{F}$-homology d-ball* if (i) B has trivial reduced $\mathbb{F}$-homology groups, (ii) for each face σ of dimension $i \leqslant d-1$, the reduced $\mathbb{F}$-homology groups of $\mathrm{lk}_B(\sigma)$ are trivial or the same as those of the $(d-i-1)$-sphere and (iii) the boundary

$$\partial B = \{\sigma \in B : -1 < \dim(\sigma) < d \text{ and } \widetilde{H}_{d-\dim(\sigma)-1}(\mathrm{lk}_B(\sigma); \mathbb{F}) = 0\} \cup \{\emptyset\} \tag{1}$$

is an $\mathbb{F}$-homology $(d-1)$-sphere. A simplicial complex is said to be an *$\mathbb{F}$-homology d-manifold* if each vertex link is either an $\mathbb{F}$-homology $(d-1)$-sphere or an $\mathbb{F}$-homology $(d-1)$-ball. Note that a triangulation of a d-manifold is an $\mathbb{F}$-homology d-manifold for every field $\mathbb{F}$. In the rest of the paper, we fix a field $\mathbb{F}$ and by a *homology manifold/ball/sphere* we shall mean an $\mathbb{F}$-homology manifold/ball/sphere.

We define the *boundary* $\partial\Delta$ of a homology d-manifold Δ in the same way as in (1). If $\partial\Delta = \{\emptyset\}$, then Δ is called a *closed homology d-manifold* (or a *homology d-manifold without boundary*), otherwise Δ is called a *homology d-manifold with boundary.* If Δ is a homology d-manifold with boundary, then $\partial\Delta$ becomes a closed homology $(d-1)$-manifold. We say that a connected, closed, $\mathbb{F}$-homology d-manifold Δ is *$\mathbb{F}$-orientable* (or simply, *orientable*) if $\widetilde{H}_d(\Delta; \mathbb{F}) \cong \mathbb{F}$. Similarly, a connected homology d-manifold Δ with boundary is *orientable* if $H_d(\Delta, \partial\Delta; \mathbb{F}) \cong \mathbb{F}$. We note the following easy fact.

Lemma 2.1. *Let Δ be an orientable homology d-manifold with boundary. If Δ has trivial reduced homology groups then Δ is a homology d-ball.*

Proof. It is clear that Δ satisfies conditions (i) and (ii) of homology balls. The fact that $\partial\Delta$ is a homology $(d-1)$-sphere follows from the long exact sequence of the pair $(\Delta, \partial\Delta)$ and the Poincaré–Lefschetz duality [17, Theorem 6.2.19]. $\square$

We also recall combinatorial manifolds. A *combinatorial d-ball* (resp. *d-sphere*) is a simplicial complex which is PL homeomorphic to the d-simplex (resp. the boundary of the $(d+1)$-simplex). A simplicial complex is said to be a *combinatorial d-manifold* if each vertex link is either a combinatorial $(d-1)$-ball or a combinatorial $(d-1)$-sphere. Any combinatorial manifold is a triangulation of an actual PL manifold.

An $\mathbb{F}$-homology d-manifold with non-empty boundary is said to be *stacked* if all its interior faces have dimension $\geqslant d-1$. An $\mathbb{F}$-homology d-manifold is said to be *stacked* if it is the boundary of a stacked triangulation of an $\mathbb{F}$-homology $(d+1)$-manifold. An $\mathbb{F}$-homology d-manifold is said to be *locally stacked* if each vertex link is stacked. Clearly, a stacked homology manifold is locally stacked. Since any stacked homology ball (resp., sphere) is a combinatorial ball (resp., sphere), it follows that every (locally) stacked homology manifold is a combinatorial manifold. Thus we simply call them (*locally*) *stacked manifolds* (*with or without boundary*).

Next, we recall Walkup's class $\mathcal{H}^d$. Let Δ be a connected, closed, homology manifold and let σ and τ be facets of Δ. We say that a bijection $\psi : \sigma \to \tau$ is *admissible* if $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Delta(\psi(v)) = \{\emptyset\}$ for each vertex $v \in \sigma$. Note that, for the existence of an admissible bijection $\psi : \sigma \to \tau$, σ and τ must be disjoint. For an admissible bijection $\psi : \sigma \to \tau$, let Δ^ψ be the simplicial complex obtained from $\Delta \setminus \{\sigma, \tau\}$ by identifying v and $\psi(v)$ for all $v \in \sigma$. The simplicial complex Δ^ψ is said to be obtained from Δ by a *combinatorial handle addition*.

Definition 2.2 (Walkup's class $\boldsymbol{\mathcal{H}^d}$)**.** Let $d \geqslant 3$ be an integer. We recursively define the class $\mathcal{H}^d(k)$ as follows.

(a) $\mathcal{H}^d(0)$ is the set of stacked triangulations of the $(d-1)$-sphere.

(b) A simplicial complex Δ is in $\mathcal{H}^d(k+1)$ if it is obtained from a member of $\mathcal{H}^d(k)$ by a combinatorial handle addition.

Walkup's class $\mathcal{H}^d$ is the union $\mathcal{H}^d = \bigcup_{k\geqslant 0} \mathcal{H}^d(k)$.

Kalai [10, Corollary 8.4] proved that every connected, locally stacked, closed, d-manifold is a member of Walkup's class $\mathcal{H}^{d+1}$ when $d \geqslant 4$, and as a consequence it follows that $\mathcal{H}^{d+1}$ is the set of all (locally) stacked, connected, closed, d-manifolds for $d \geqslant 4$. Although Kalai's result is not true for $d = 3$ (see e.g. [5, Example 6.2]), we prove that $\mathcal{H}^{d+1}$ is the set of all connected, stacked, closed, d-manifolds for $d \geqslant 2$.

3 Simplicial handle addition

In this section, we define an analogue of combinatorial handle additions for homology manifolds with boundary.

All homology groups are with coefficients in an arbitrary field $\mathbb{F}$, which is fixed throughout, and $\widetilde{H}_i(\Delta;\mathbb{F})$ and $\beta_i(\Delta;\mathbb{F})$ will be denoted by $\widetilde{H}_i(\Delta)$ and $\beta_i(\Delta)$, respectively. Let Δ be a homology d-manifold with boundary on the vertex set V and let σ and τ be facets of $\partial\Delta$. We say that a bijection $\psi : \sigma \to \tau$ is *admissible* if, for every vertex $v \in \sigma$, $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Delta(\psi(v)) = \{\emptyset\}$. For an admissible bijection $\psi : \sigma \to \tau$, let $\Delta^{\overline{\psi}}$ be the simplicial complex obtained from Δ by identifying v and $\psi(v)$ for all $v \in \sigma$. (The main difference between Δ^ψ and $\Delta^{\overline{\psi}}$ is that we do not remove σ and τ for the definition of $\Delta^{\overline{\psi}}$.) Thus, if we define ψ_+ to be the map from V to $V \setminus \sigma$ by $\psi_+(v) = \psi(v)$ if $v \in \sigma$ and $\psi_+(v) = v$ otherwise, then we can consider $\Delta^{\overline{\psi}}$ on the vertex set $V \setminus \sigma$ as

$$\Delta^{\overline{\psi}} = \{\psi_+(\alpha) : \alpha \in \Delta\}.$$

If Δ is connected, then we say that $\Delta^{\overline{\psi}}$ is obtained from Δ by a *simplicial handle addition*. If Δ has two connected components Δ_1 and Δ_2 and if $\sigma \in \Delta_1$ and $\tau \in \Delta_2$, then we write $\Delta^{\overline{\psi}} = \Delta_1 \cup_\psi \Delta_2$ and call it a *simplicial connected union* of Δ_1 and Δ_2. Below we give some basic properties of $\Delta^{\overline{\psi}}$. For $\sigma \in \Delta$, we write $\overline{\sigma}$ for the simplicial complex having a single facet σ.

Lemma 3.1. *Let Δ and Γ be two homology d-balls. If $\Delta \cap \Gamma = \partial\Delta \cap \partial\Gamma = \overline{\alpha}$, where α is a $(d-1)$-simplex, then $\Delta \cup \Gamma$ is a homology d-ball.*

Proof. We use induction on d. The statement is obvious when $d = 1$. Suppose $d > 1$. Since $\Delta \cap \Gamma = \overline{\alpha}$, the exactness of the Mayer–Vietoris sequence implies that $\Delta \cup \Gamma$ has trivial reduced homology groups. Let v be a vertex of $\Delta \cup \Gamma$. If $v \notin \alpha$ then $\mathrm{lk}_{\Delta\cup\Gamma}(v)$ is equal to either $\mathrm{lk}_\Delta(v)$ or $\mathrm{lk}_\Gamma(v)$ and hence a homology $(d-1)$-sphere or $(d-1)$-ball. If $v \in \alpha$ then $v \in \partial\Delta \cap \partial\Gamma$ and hence $\mathrm{lk}_\Delta(v)$ and $\mathrm{lk}_\Gamma(v)$ are homology $(d-1)$-balls and $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Gamma(v) = \overline{\alpha \setminus \{v\}}$. Since $\mathrm{lk}_{\Delta\cup\Gamma}(v) = \mathrm{lk}_\Delta(v) \cup \mathrm{lk}_\Gamma(v)$, $\mathrm{lk}_{\Delta\cup\Gamma}(v)$ is a homology $(d-1)$-ball by the induction hypothesis. The lemma now follows from Lemma 2.1. $\square$

It follows from Lemma 3.1 that the simplicial connected union of two homology d-balls is a homology d-ball. Note that the same property holds for combinatorial manifolds.

Lemma 3.2. *For $d \geqslant 2$, let Δ be a (not necessary connected) homology d-manifold with boundary. Let σ and τ be two facets of $\partial\Delta$. If $\psi : \sigma \to \tau$ is an admissible bijection then*

(i) *$\Delta^{\overline{\psi}}$ is a homology d-manifold with boundary,*

(ii) *$(\beta_0(\Delta^{\overline{\psi}}), \beta_1(\Delta^{\overline{\psi}})) = (\beta_0(\Delta), \beta_1(\Delta) + 1)$ or $(\beta_0(\Delta) - 1, \beta_1(\Delta))$ and*

(iii) *$\Delta^{\overline{\psi}}$ is stacked if and only if Δ is stacked.*

Proof. (i) For every $\alpha \in \Delta^{\overline{\psi}}$ with $\alpha \not\subseteq \tau$, there is a unique face $\gamma \in \Delta$ such that $\alpha = \psi_+(\gamma)$ and $\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha)$ is combinatorially isomorphic to $\mathrm{lk}_\Delta(\gamma)$. Thus, to prove the statement, it is enough to show that, for every $\alpha \subseteq \tau$, $\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha)$ is either a homology $(d - \dim(\alpha) - 1)$-sphere or $(d - \dim(\alpha) - 1)$-ball. It is clear that $|\mathrm{lk}_{\Delta^{\overline{\psi}}}(\tau)| \cong S^0$. For a proper face α of τ, a straightforward computation implies

$$\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha) = \mathrm{lk}_\Delta(\alpha) \cup_{\psi'} \mathrm{lk}_\Delta(\psi^{-1}(\alpha)),$$

where $\psi' : \psi^{-1}(\tau \setminus \alpha) \to \tau \setminus \alpha$ is the restriction of ψ to $\psi^{-1}(\tau \setminus \alpha)$. By Lemma 3.1, $\mathrm{lk}_{\Delta^{\overline{\psi}}}(\alpha)$ is a homology $(d - \dim(\alpha) - 1)$-ball.

(ii) It is clear that $\beta_0(\Delta^{\overline{\psi}}) = \beta_0(\Delta) - 1$ if σ and τ belong to different connected components and $\beta_0(\Delta^{\overline{\psi}}) = \beta_0(\Delta)$ if σ and τ are in the same connected component. Observe that $\widetilde{H}_i(|\Delta^{\overline{\psi}}|) \cong \widetilde{H}_i(|\Delta^{\overline{\psi}}|, |\tau|) \cong \widetilde{H}_i(|\Delta|, |\sigma| \cup |\tau|)$ for all i. Then the desired statement follows from the following exact sequence of pairs

$$\begin{array}{lllll} 0 = \widetilde{H}_1(|\sigma| \cup |\tau|) & \longrightarrow & \widetilde{H}_1(|\Delta|) & \longrightarrow & \widetilde{H}_1(|\Delta|, |\sigma| \cup |\tau|) \\ \longrightarrow \widetilde{H}_0(|\sigma| \cup |\tau|) & \longrightarrow & \widetilde{H}_0(|\Delta|) & \longrightarrow & \widetilde{H}_0(|\Delta|, |\sigma| \cup |\tau|) \quad \longrightarrow \quad 0. \end{array}$$

(iii) This statement follows from the proof of (i) since it says that the interior faces of $\Delta^{\overline{\psi}}$ are τ and $\psi_+(\alpha)$ for all interior faces α of Δ. □

The proof of Lemma 3.2 (i) also says that if Δ is connected then $\partial(\Delta^{\overline{\psi}}) = (\partial\Delta)^{\psi}$. Also $|\partial(\Delta_1 \cup_\psi \Delta_2)|$ is a connected sum of $|\partial\Delta_1|$ and $|\partial\Delta_2|$.

We can consider the inverse of the construction of $\Delta^{\overline{\psi}}$, which we call a simplicial handle deletion. Indeed, we prove the following statement.

Theorem 3.3. *Let Δ be a connected homology d-manifold with boundary. If Δ has an interior $(d-1)$-face σ with $\partial\overline{\sigma} \subseteq \partial\Delta$, then there exists a homology d-manifold with boundary $\widetilde{\Delta}$ such that Δ is obtained from $\widetilde{\Delta}$ by a simplicial handle addition or by taking a simplicial connected union.*

We say that $\widetilde{\Delta}$ in the above theorem can be obtained from Δ by a *simplicial handle deletion* over σ. Intuitively, the simplicial complex $\widetilde{\Delta}$ can be considered as the simplicial complex obtained from Δ by cutting it along an interior face σ of codimension one. On the other hand, one needs to be careful with this intuitive understanding since it is not clear for us if the above theorem holds in the class of triangulated (or combinatorial) manifolds. For example, we are not sure if the converse of Lemma 3.1 holds for combinatorial balls, that is, if Δ and Γ are combinatorial balls when $\Delta \cup \Gamma$ is a combinatorial d-ball and $\Delta \cap \Gamma$ is a $(d-1)$-simplex.

In the rest of the section, we will prove Theorem 3.3 by giving an explicit construction of $\widetilde{\Delta}$. To do this, we need the following two technical lemmas. Recall that, for any homology d-manifold Δ and its interior $(d-1)$-face σ, $\mathrm{lk}_\Delta(\sigma)$ consist of two vertices since it has the same homology groups as S^0.

Lemma 3.4. *Let B be a homology d-ball with vertex set V, σ an interior $(d-1)$-face of B with $\partial\overline{\sigma} \subseteq \partial B$. Then, $B[V \setminus \sigma]$ contains exactly two connected components, say C_1 and C_2. Moreover, If $\mathrm{lk}_B(\sigma) = \{x, y\}$, then one of x and y is in C_1 and the other is in C_2.*

Proof. Let $v * \partial B = \partial B \cup \{\{v\} \cup \alpha : \alpha \in \partial B\}$ be the cone over ∂B, where v is a new vertex. It is easy to see that $S = B \cup (v * \partial B)$ is a homology d-sphere. Then

$$\widetilde{H}_0(S[V \setminus \sigma]) \cong \widetilde{H}_{d-1}(S[\sigma \cup \{v\}]) \cong \widetilde{H}_{d-1}(S[\sigma \cup \{v\}], (v * \partial B)[\sigma \cup \{v\}])),$$

where the first isomorphism follows from the Alexander duality [17, Theorem 6.2.17] and the second isomorphism follows from the long exact sequence of pairs since $\widetilde{H}_i((v * \partial B)[\sigma \cup \{v\}]) = 0$ for all i. Since $B[V \setminus \sigma] = S[V \setminus \sigma]$ and since

$$\widetilde{H}_{d-1}(S[\sigma \cup \{v\}], (v * \partial B)[\sigma \cup \{v\}])) \cong \widetilde{H}_{d-1}(B[\sigma], (\partial B)[\sigma])) = \widetilde{H}_{d-1}(\overline{\sigma}, \partial\overline{\sigma}) \cong \mathbb{F},$$

$B[V \setminus \sigma]$ has exactly two connected components. This proves the first statement.

Let $B_i = B[V(C_i) \cup \sigma]$ for $i = 1, 2$, where $V(C_i)$ is the vertex set of C_i. Clearly $B_1 \cap B_2 = \overline{\sigma}$. Also, any facet of B belongs either B_1 or B_2. Indeed, if α is a facet of B, then $\alpha \setminus \sigma \in B[V \setminus \sigma]$ is contained either in C_1 or in C_2, which implies that α is either in B_1 or in B_2. Let $u \in V(C_1)$ and $w \in V(C_2)$, and let α and γ be facets of B with $v \in \alpha$ and $w \in \gamma$. Then $\alpha \in B_1$ and $\gamma \in B_2$. Since B is a homology d-ball, it is a d-dimensional pseudomanifold and hence there is a sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \gamma$ of facets such that $\alpha_{i-1} \cap \alpha_i$ has dimension $d-1$ for $1 \leqslant i \leqslant k$ (see [17, p. 150 and 278]). Let j be a number such that $\alpha_{j-1} \in B_1$ and $\alpha_j \in B_2$. Then $\alpha_{j-1} \cap \alpha_j$ must be σ. Since $\{x\} \cup \sigma$ and $\{y\} \cup \sigma$ are the only facets containing σ, they must be α_{j-1} and α_j. This proves the second statement. $\square$

We say that $B_i = B[V(C_i) \cup \sigma]$ in Lemma 3.4 is the *x-component* (resp. *y-component*) of B with respect to σ if it contains x (resp. y). We later see in the proof of Theorem 3.3 that B_1 and B_2 are homology balls.

Let Δ be a homology d-manifold with boundary. Suppose that Δ has an interior $(d-1)$-face $\sigma = \{z_1, \ldots, z_d\}$ with $\partial\overline{\sigma} \subseteq \partial\Delta$. Let $\{x\} \cup \sigma$ and $\{y\} \cup \sigma$ be the facets of Δ containing σ. Consider

$$R = \{\alpha \in \Delta : \alpha \cap \sigma \neq \emptyset,\ \alpha \not\subseteq \sigma\}.$$

Observe that, for each $\tau \in \partial\overline{\sigma}$, $\mathrm{lk}_\Delta(\tau)$ is a homology ball satisfying the assumption of Lemma 3.4 in the sense that $\sigma \setminus \tau$ is an interior face of $\mathrm{lk}_\Delta(\tau)$ with $\partial(\overline{\sigma \setminus \tau}) \subseteq \partial(\mathrm{lk}_\Delta(\tau))$. Consider the collection of simplices (which is not a subcomplex of Δ)

$$R_x(k) = \{\alpha \cup \{z_k\} : \alpha \in R \text{ and } \alpha \text{ is in the } x\text{-component of } \mathrm{lk}_\Delta(z_k) \text{ w.r.t. } \sigma \setminus \{z_k\}\}.$$

Also, define $R_y(k)$ similarly. Let

$$X = \bigcup_{k=1}^{d} R_x(k) \text{ and } Y = \bigcup_{k=1}^{d} R_y(k).$$

Note that $R = X \cup Y$.

Lemma 3.5. *If $\sigma = \{z_1, \ldots, z_d\}$, $R_x(k)$, $R_y(k)$, X and Y are as above, then $X \cap Y = \emptyset$. Also, $\overline{\sigma \setminus \{z_k\}} \cup \{\alpha \setminus \{z_k\} : z_k \in \alpha \in X\}$ is the x-component of $\mathrm{lk}_\Delta(z_k)$ and $\overline{\sigma \setminus \{z_k\}} \cup \{\alpha \setminus \{z_k\} : z_k \in \alpha \in Y\}$ is the y-component of $\mathrm{lk}_\Delta(z_k)$ for $1 \leqslant k \leqslant d$.*

Proof. To prove the first statement, we must prove that $R_x(k) \cap R_y(\ell) = \emptyset$ for all $k \neq \ell$. Suppose to the contrary that $\alpha \in R_s(k) \cap R_y(\ell)$ for some $k \neq \ell$. Then $\alpha \setminus \{z_k, z_\ell\}$ is in the x-component and the y-component of $\mathrm{lk}_\Delta(\{z_k, z_\ell\})$ with respect to $\sigma \setminus \{z_k, z_\ell\}$ and hence $\alpha \subseteq \sigma$, a contradiction since $\alpha \in R$.

We prove the second statement only for the x-components (the proof for y-components is similar). Since $\overline{\sigma \setminus \{z_k\}} \cup \{\alpha \setminus \{z_k\} : \alpha \in R_x(k)\}$ is the x-component of $\mathrm{lk}_\Delta(z_k)$, to prove the statement, it is enough to show $\{\alpha \in X : z_k \in \alpha\} = R_x(k)$. Let $\alpha \in X$ with $z_k \in \alpha$. Then $\alpha \in R_x(\ell)$ for some ℓ. If $\ell = k$ then $\alpha \in R_x(k)$. Otherwise, $\alpha \setminus \{z_k, z_\ell\}$ and x are in the same component of $\mathrm{lk}_\Delta(\{z_k, z_\ell\})$. Since $\mathrm{lk}_\Delta(z_k) \supseteq \mathrm{lk}_\Delta(\{z_k, z_\ell\})$, we have $\alpha \in R_x(k)$. These prove that $\{\alpha \in X : z_k \in \alpha\} = R_x(k)$. □

Definition 3.6. Let Δ be a homology d-manifold with boundary and let $\sigma = \{z_1, \ldots, z_d\}$ be an interior $(d-1)$-face of Δ with $\partial\overline{\sigma} \subseteq \partial\Delta$. Let R, $R_x(k)$, $R_y(k)$, X and Y be as above. Let $z_1^+, \ldots, z_d^+$ be new vertices and $\sigma^+ = \{z_1^+, \ldots, z_d^+\}$. For $\alpha = \alpha' \cup \{z_{i_1}, \ldots, z_{i_\ell}\} \in X$ with $\alpha' \cap \sigma = \emptyset$, define $\alpha^+ = \alpha' \cup \{z_{i_1}^+, \ldots, z_{i_\ell}^+\}$. Consider the simplicial complex

$$\widetilde{\Delta}^\sigma = \{\alpha \in \Delta : \alpha \notin X\} \cup \{\alpha^+ : \alpha \in X\} \cup \overline{\sigma^+}.$$

The above construction is a simplified version of the construction in [2, Lemma 3.3]. Also, a similar construction for manifolds without boundary was considered by Walkup [18]. Note that if Δ is a homology ball, then $\widetilde{\Delta}^\sigma$ is the disjoint union of its x-component and y-component. We now prove Theorem 3.3.

Proof of Theorem 3.3. We claim that $\widetilde{\Delta} = \widetilde{\Delta}^\sigma$ satisfies the desired conditions by induction on the dimension d. It is clear that we have $\Delta = (\widetilde{\Delta}^\sigma)^{\overline{\psi}}$ if $\widetilde{\Delta}^\sigma$ is a homology manifold and we define $\psi : \sigma^+ \to \sigma$ by $\sigma(z_i^+) = z_i$ for all i. Thus it is enough to prove that $\widetilde{\Delta}^\sigma$ is a homology manifold.

The assertion is clear when $d = 1$. Suppose $d > 1$. We first prove that the x-component and the y-component of $\mathrm{lk}_\Delta(z_k)$ are homology balls for all $1 \leqslant k \leqslant d$, by applying the induction hypothesis to $\mathrm{lk}_\Delta(z_k)$. Let $\Gamma = \mathrm{lk}_\Delta(z_k)$, and let B_1 and B_2 be the x-component and the y-component of the homology $(d-1)$-ball $\Gamma = \mathrm{lk}_\Delta(z_k)$ with respect to $\sigma \setminus \{z_k\}$, respectively. They are homology $(d-1)$-manifolds since their disjoint union $\widetilde{\Gamma}^{\sigma\setminus\{z_k\}}$ is a homology $(d-1)$-manifold by the induction hypothesis. Also, since $B_1 \cup B_2 = \mathrm{lk}_\Delta(z_k)$ and $B_1 \cap B_2 = \overline{\sigma \setminus \{z_k\}}$, the exactness of the Mayer-Vietoris sequence says that they have trivial homology groups. Then, since B_1 and B_2 are orientable as they are subcomplexes of the orientable homology $(d-1)$-manifold $\mathrm{lk}_\Delta(z_k)$, B_1 and B_2 are homology balls by Lemma 2.1.

We now prove that $\widetilde{\Delta}$ is a homology manifold. What we must prove is that each vertex link of $\widetilde{\Delta}$ is either a homology $(d-1)$-sphere or a $(d-1)$-ball. Suppose $v \notin \sigma^+ \cup \sigma$. Define the map $\varphi : \Delta \to \widetilde{\Delta}$ by $\varphi(\alpha) = \alpha$ if $\alpha \notin X$ and $\varphi(\alpha) = \alpha^+$ if $\alpha \in X$. Then φ gives a bijection between $\Delta \setminus \overline{\sigma}$ and $\widetilde{\Delta} \setminus (\overline{\sigma^+} \cup \overline{\sigma})$, in particular, it gives a bijection between $\{\alpha : v \in \alpha \in \Delta\}$ and $\{\alpha : v \in \alpha \in \widetilde{\Delta}\}$. Thus $\mathrm{lk}_{\widetilde{\Delta}}(v)$ is combinatorially isomorphic to $\mathrm{lk}_{\Delta}(v)$, which implies the desired property. Suppose $v = z_k^+$ for some k. Then

$$\mathrm{lk}_{\widetilde{\Delta}}(v) = \mathrm{lk}_{\widetilde{\Delta}}(z_k^+) = \overline{(\sigma \setminus \{z_k\})^+} \cup \{(\alpha \setminus \{z_k\})^+ : z_k \in \alpha \in X\}$$

is combinatorially isomorphic to the x-component of $\mathrm{lk}_{\Delta}(z_k)$ by Lemma 3.5, and therefore is a homology $(d-1)$-ball. Finally, suppose $v = z_k$ for some k. Since $X \cap Y = \emptyset$,

$$\begin{aligned} \mathrm{lk}_{\widetilde{\Delta}}(v) &= \overline{\sigma \setminus \{v\}} \cup \{\alpha \setminus \{v\} : v \in \alpha \in R_y(k) \setminus X\} \\ &= \overline{\sigma \setminus \{z_k\}} \cup \{\alpha \setminus \{z_k\} : \alpha \in R_y(k)\} \end{aligned}$$

is the y-component of $\mathrm{lk}_{\Delta}(v)$ w.r.t. $\sigma \setminus \{v\}$. Thus $\mathrm{lk}_{\widetilde{\Delta}}(v)$ is a homology $(d-1)$-ball. $\square$

4 A characterization of stacked manifolds

In this section, we present a characterization of stacked manifolds. We first define an analogue of Walkup's class for manifolds with boundary.

Definition 4.1. Let $d \geqslant 2$ be an integer. We recursively define $\overline{\mathcal{H}^d}(k)$ as follows.

(a) $\overline{\mathcal{H}^d}(0)$ is the set of stacked triangulations of d-balls.

(b) Δ is a member of $\overline{\mathcal{H}^d}(k+1)$ if it is obtained from a member of $\overline{\mathcal{H}^d}(k)$ by a simplicial handle addition.

Let $\overline{\mathcal{H}^d} = \bigcup_{k \geqslant 0} \overline{\mathcal{H}^d}(k)$.

Note that every stacked triangulation of the d-ball is obtained from a d-simplex by taking a simplicial connected union with a d-simplex repeatedly. See [8, Lemma 2.1]. The classes $\overline{\mathcal{H}^d}$ and $\mathcal{H}^d$ have the following simple relation.

Lemma 4.2. *For all integers $d \geqslant 3$ and $k \geqslant 0$, one has $\mathcal{H}^d(k) = \{\partial\Delta : \Delta \in \overline{\mathcal{H}^d}(k)\}$.*

Proof. The case when $k = 0$ and the inclusion $\mathcal{H}^d(k) \supseteq \{\partial\Delta : \Delta \in \overline{\mathcal{H}^d}(k)\}$, for all $k \geqslant 0$, are obvious. For $k > 0$, the converse inclusion follows by induction on k. Indeed, if $\Gamma \in \mathcal{H}^d(k)$, then by induction we may assume that there is a $\Delta \in \overline{\mathcal{H}^d}(k-1)$ such that $\Gamma = (\partial\Delta)^\psi$ for some admissible bijection $\psi : \sigma \to \tau$ in $\partial\Delta$. Since, $\partial\Delta$ and Δ have the same 1-faces by Lemma 3.2(iii), the bijection ψ is also admissible for Δ, and $\Gamma = \partial(\Delta^{\overline{\psi}}) \in \{\partial\Delta : \Delta \in \overline{\mathcal{H}^d}(k)\}$. $\square$

Lemma 4.3. *If $\Delta \in \overline{\mathcal{H}^d}(k)$ and $\Gamma \in \overline{\mathcal{H}^d}(\ell)$ then their simplicial connected union belongs to $\overline{\mathcal{H}^d}(k+\ell)$.*

Proof. We may assume $k \leqslant \ell$. We use induction on $k+\ell$. If $k+\ell=0$ then the assertion follows from Lemma 3.2(iii). Suppose $k+\ell>0$. Then $\Gamma = \Sigma^{\overline{\varphi}}$ for some $\Sigma \in \overline{\mathcal{H}^d}(\ell-1)$ and for some admissible bijection φ between facets of $\partial\Sigma$. Let ψ be a bijection from a facet of $\partial\Delta$ to a facet of $\partial\Gamma$. Then $\Delta \cup_\psi \Gamma$ is $(\Delta \cup_\psi \Sigma)^\varphi$ (by an appropriate identification of the vertices). By induction hypothesis, we have $\Delta \cup_\psi \Sigma \in \overline{\mathcal{H}^d}(k+\ell-1)$ and hence $\Delta \cup_\psi \Gamma \in \overline{\mathcal{H}^d}(k+\ell)$. □

Remark 4.4. A similar result for $\mathcal{H}^d$ was proved by Walkup [18, Proposition 4.4].

Theorem 4.5. *For $d \geqslant 2$, let Δ be a connected homology d-manifold with boundary. Then Δ is stacked if and only if $\Delta \in \overline{\mathcal{H}^d}$.*

Proof. It is clear any member of $\overline{\mathcal{H}^d}$ is stacked. We prove the 'only if part'. The assertion is obvious if Δ has one facet. Suppose that Δ has more than one facet. Then Δ has an interior $(d-1)$-face σ. Since Δ is stacked, it has no interior faces of dimension $\leqslant d-2$. Thus we have $\partial\overline{\sigma} \subseteq \partial\Delta$. Then by Theorem 3.3, Δ is a simplicial connected union of two connected stacked manifolds or is obtained from a connected stacked manifold having a smaller first Betti number by a simplicial handle addition. Then the assertion follows from Lemma 4.3 by induction on the number of interior $(d-1)$-faces. □

By Lemma 4.2 and Theorem 4.5 we obtain the following.

Corollary 4.6. *Let Δ be a connected, closed, homology manifold of dimension $d \geqslant 2$. Then Δ is stacked if and only if $\Delta \in \mathcal{H}^{d+1}$.*

Finally, we discuss a connection between Corollary 4.6 and a question posed by Novik and Swartz [16]. Novik and Swartz [16, Theorem 5.2] gave the following interesting characterization of members of Walkup's class $\mathcal{H}^{d+1}$ for $d \geqslant 4$.

Proposition 4.7 (Novik–Swartz)**.** *Let Δ be a connected, closed, orientable, homology manifold of dimension $d \geqslant 3$. Then*

$$f_1(\Delta) - (d+1)f_0(\Delta) + \binom{d+2}{2} \geqslant \binom{d+2}{2}\beta_1(\Delta).$$

Further, if $d \geqslant 4$ then $f_1(\Delta) - (d+1)f_0(\Delta) + \binom{d+2}{2} = \binom{d+2}{2}\beta_1(\Delta)$ if and only if $\Delta \in \mathcal{H}^{d+1}$.

It was asked by Novik and Swartz [16, Problem 5.3] if the equality case of the last statement in Proposition 4.7 also holds in dimension 3. Corollary 4.6 and the next result of Bagchi [1, Theorem 1.14] answer this question.

Proposition 4.8 (Bagchi)**.** *Let Δ be a connected, closed, homology 3-manifold. Then $f_1(\Delta) - 4f_0(\Delta) + 10 = 10\beta_1(\Delta)$ if and only if Δ is stacked.*

Corollary 4.9. *Let Δ be a connected, closed, homology 3-manifold. The following conditions are equivalent.*

(i) *Δ is a member of $\mathcal{H}^4$.*

(ii) Δ *is stacked.*

(iii) $f_1(\Delta) - 4f_0(\Delta) + 10 = 10\beta_1(\Delta)$.

Remark 4.10. It is known that the topological type of a member of $\mathcal{H}^{d+1}$ is one of the following: (i) the d-sphere S^d (ii) connected sums of sphere product $S^{d-1} \times S^1$ (iii) connected sums of twisted sphere product $S^{d-1} \underline{\times} S^1$. See [13, Section 3]. Thus Corollary 4.6 also gives a new restriction on the topological types of stacked manifolds in dimensions 2 and 3.

Remark 4.11. Recently, Proposition 4.7 was extended to non-orientable homology manifolds (and even to normal pseudomanifolds) by the second author. See [14, Theorem 5.3].

5 Tight triangulations and stackedness

In this section, we study stackedness of tight triangulations. For a simplicial complex Δ with vertex set V, a subset $\sigma \subseteq V$ of $k+1$ elements is called a *missing k-face* of Δ if $\sigma \notin \Delta$ and all proper subsets of σ are faces of Δ. If σ is a missing k-face of Δ, then we have $\widetilde{H}_{k-1}(\Delta[\sigma]) \cong \mathbb{F}$. The following lemma follows from the definition of tightness.

Lemma 5.1. *Let Δ be a tight simplicial complex on the vertex set V. Then*

(i) *for all subsets $U \subseteq W$ of V, the natural map $\widetilde{H}_i(\Delta[U]) \to \widetilde{H}_i(\Delta[W])$ induced by the inclusion is injective, and*

(ii) *if $\beta_{k-1}(\Delta) = 0$ then Δ has no missing k-faces.*

For a simplicial complex Δ, we identify its 1-*skeleton* $\mathrm{Skel}_1(\Delta) = \{\sigma \in \Delta : \dim(\sigma) \leqslant 1\}$ with the simple graph whose vertex set is the set of the vertices of Δ and whose edge set is the set of the edges (1-simplices) in Δ. We say that a simple graph G is *chordal* if it has no induced cycle of length $\geqslant 4$. The following result is due to Kalai [10, Theorem 8.5].

Proposition 5.2 (Kalai)**.** *Let Δ be a homology $(d-1)$-sphere with $d \geqslant 3$. Then Δ is stacked if and only if the 1-skeleton of Δ is chordal and Δ has no missing k-faces for $1 < k < d-1$.*

Let Δ be a closed homology manifold of dimension $d \geqslant 3$. Recall that Δ is tight-neighborly if $\binom{f_0(\Delta)-d-1}{2} = \binom{d+2}{2}\beta_1(\Delta;\mathbb{F})$. Since $\binom{f_0}{2} - (d+1)f_0 + \binom{d+2}{2} = \binom{f_0-d-1}{2}$, Δ is tight-neighborly if and only if Δ is stacked and neighborly by Proposition 4.7 (see Remark 4.11 for the non-orientable case). Here we prove the following.

Theorem 5.3. *Let Δ be a tight, connected, closed, homology manifold of dimension $d \geqslant 4$ such that $\beta_i(\Delta) = 0$ for $1 < i < d-1$. Then Δ is locally stacked.*

Proof. Let v be a vertex of Δ. We prove that $\mathrm{lk}_\Delta(v)$ is stacked.

We first claim that no induced subcomplex of $\mathrm{lk}_\Delta(v)$ can be a 1-dimensional simplicial complex which forms a cycle. Suppose to the contrary that $\mathrm{lk}_\Delta(v)[W]$ is a cycle for some W. Let $C = \mathrm{lk}_\Delta(v)[W]$ and $v * C = C \cup \{\{v\} \cup \sigma : \sigma \in C\}$. Then we have $\Delta[W \cup \{v\}] = \Delta[W] \cup (v * C)$ and $\Delta[W] \cap (v * C) = C$. Consider the Mayer–Vietoris exact sequence

$$\widetilde{H}_2(\Delta[W \cup \{v\}]) \longrightarrow \widetilde{H}_1(C) \longrightarrow \widetilde{H}_1(\Delta[W]) \oplus \widetilde{H}_1(v * C) \xrightarrow{\varphi} \widetilde{H}_1(\Delta[W \cup \{v\}]).$$

Since Δ is tight and $\beta_2(\Delta) = 0$, we have $\widetilde{H}_2(\Delta[W \cup \{v\}]) = 0$. Then since $\widetilde{H}_1(C) \neq 0$, the map φ has a non-trivial kernel. However, since $\widetilde{H}_1(v * C) = 0$, this contradicts the tightness of Δ as it implies that φ is injective by Lemma 5.1(i). Hence no induced subcomplex of $\mathrm{lk}_\Delta(v)$ can be a cycle.

Now we prove the statement. By Lemma 5.1(ii), Δ has no missing k-faces for $2 < k < d$. This implies that $\mathrm{lk}_\Delta(v)$ has no missing k-faces for $2 < k < d-1$. Also, $\mathrm{lk}_\Delta(v)$ has no missing 2-faces since if it has a missing 2-face σ then $\mathrm{lk}_\Delta(v)[\sigma]$ is a cycle of length 3. Similarly, the 1-skeleton of $\mathrm{lk}_\Delta(v)$ is a chordal graph since if it has an induced cycle of length $\geqslant 4$ with the vertex set W, then $\mathrm{lk}_\Delta(v)[W]$ is a cycle. Thus, by Proposition 5.2, $\mathrm{lk}_\Delta(v)$ is stacked. □

For any field, a tight (connected) homology manifold is orientable and neighborly (cf. [4]). From Theorem 5.3 and all the known results, we have the following.

Corollary 5.4. *Let Δ be a closed, orientable, homology manifold of dimension $d \geqslant 4$. Then the following are equivalent.*

(i) Δ *is tight-neighborly.*

(ii) Δ *is a neighborly member of* $\mathcal{H}^{d+1}$.

(iii) Δ *is neighborly and stacked.*

(iv) Δ *is neighborly and locally stacked.*

(v) Δ *is tight and* $\beta_i(\Delta) = 0$ *for* $1 < i < d-1$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 4.7, (ii) $\Leftrightarrow$ (iii) follows from Corollary 4.6, and (ii) $\Leftrightarrow$ (iv) follows from Kalai's result [10, Corollary 8.4]. Now, (v) $\Rightarrow$ (iv) follows from Theorem 5.3. Since $\Delta \in \mathcal{H}^{d+1}$ implies $\beta_i(\Delta) = 0$ for $1 < i < d-1$, (ii) & (iv) $\Rightarrow$ (v) follows from [4, Theorem 3.11]. This completes the proof. □

From the equivalence of (i) and (v) in Corollary 5.4 it follows that tight triangulations of connected sums of S^{d-1}-bundles over S^1 are tight-neighborly for $d \geqslant 4$. This answers a question asked by Effenberger [9, Question 4.5].

It would be natural to ask if the results in this section hold in dimension 3. Very recently, Bagchi, Spreer and the first author [6] proved the following result (which answers a question asked in a previous version of this paper).

Proposition 5.5 (Bagchi–Datta–Spreer)**.** *A closed triangulated* 3*-manifold* M *is* $\mathbb{F}$*-tight if and only if* M *is* $\mathbb{F}$*-orientable, neighborly and stacked.*

Since every closed triangulated manifold is $(\mathbb{Z}/2\mathbb{Z})$-orientable, as a consequence of Proposition 5.5 and Corollary 4.9 we get the following (compare Corollaries 1.4 and 1.5 of [6]).

Corollary 5.6. *Let* Δ *be a closed triangulated* 3*-manifold. The following conditions are equivalent.*

(i) Δ *is tight-neighborly, that is,* $(f_0(\Delta)-4)(f_0(\Delta)-5)=20\beta_1(\Delta;\mathbb{F})$.

(ii) Δ *is a neighborly member of* $\mathcal{H}^4$.

(iii) Δ *is neighborly and stacked.*

(iv) Δ *is tight.*

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