Longest monotone subsequences and rare regions of pattern-avoiding permutations

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Abstract

We consider the distributions of the lengths of the longest monotone and alternating subsequences in classes of permutations of size n that avoid a specific pattern or set of patterns, with respect to the uniform distribution on each such class. We obtain exact results for any class that avoids two patterns of length 3, as well as results for some classes that avoid one pattern of length 4 or more. In our results, the longest monotone subsequences have expected length proportional to n for pattern-avoiding classes, in contrast with the $\sqrt{n}$ behaviour that holds for unrestricted permutations.

In addition, for a pattern τ of length k, we scale the plot of a random τ-avoiding permutation down to the unit square and study the "rare region," which is the part of the square that is exponentially unlikely to contain any points. We prove that when $\tau_1 > \tau_k$, the complement of the rare region is a closed set that contains the main diagonal of the unit square. For the case $\tau_1 = k$, we also show that the lower boundary of the part of the rare region above the main diagonal is a curve that is Lipschitz continuous and strictly increasing on $[0, 1]$.

Keywords: pattern-avoiding permutations, longest increasing subsequence problem, longest alternating subsequence, rare region.

1 Introduction

For each integer $n \geqslant 1$, let $[n] := \{1, 2, \cdots, n\}$. A permutation of $[n]$ is a bijection $\sigma : [n] \to [n]$, written as $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i = \sigma(i)$ for $i \in [n]$. The set of all

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permutations on $[n]$ is denoted by S_n. For $\tau \in S_k$, we write $S_n(\tau)$ to denote the set of all permutations in S_n that avoid the pattern τ. (See Section 1.4 for complete definitions.)

We use $\mathbb{P}_n^\tau$ to denote the uniform probability measure on $S_n(\tau)$: for any subset A of $S_n(\tau)$, we have $\mathbb{P}_n^\tau(A) := \frac{|A|}{|S_n(\tau)|}$. We use $\mathbb{E}_n^\tau(X)$ and $\mathbb{SD}_n^\tau(X)$ to denote the expected value and standard deviation of a random variable X on $S_n(\tau)$ under $\mathbb{P}_n^\tau$. More generally, if T is a set of patterns, then $S_n(T)$ denotes the set of permutations in S_n avoiding all the patterns in T, and $\mathbb{P}_n^T$, $\mathbb{E}_n^T$ and $\mathbb{SD}_n^T$ are the corresponding probability operators.

The cardinalities $|S_n(T)|$ have been studied extensively by researchers in the last few decades but have been computed only for some limited cases of sets T.

It took 24 years to prove the 1980 conjecture of Stanley and Wilf, which says that

$$L(\tau) := \lim_{n\to\infty} |S_n(\tau)|^{1/n} \quad \text{exists and is finite for every } \tau \in S_k \tag{1}$$

(existence was proved by Arratia [3], and finiteness by Marcus and Tardos [20]).

For more on pattern-avoiding permutations, see chapters 4 and 5 in [10], [17], and the survey paper [27].

For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, the *complement* of σ is $\sigma^c = \sigma_1^c\sigma_2^c\cdots\sigma_n^c$ where $\sigma_i^c = n+1-\sigma_i$ for $i \in [n]$. The *reverse* of σ is defined to be $\sigma^r = \sigma_n\sigma_{n-1}\cdots\sigma_1$. These operations give rise to bijections among $S_n(\tau), S_n(\tau^c)$ and $S_n(\tau^r)$. In addition, inversion gives a bijection from $S_n(\tau)$ to $S_n(\tau^{-1})$. Therefore,

$$|S_n(\tau)| = |S_n(\tau^c)| = |S_n(\tau^r)| = |S_n(\tau^{-1})|\,.$$

The paper is organized as follows. Sections 1.1, 1.2, and 1.3 present the introduction, background, and overview of our results on the topics of longest monotone subsequences, longest alternating sequences, and rare regions respectively. Section 1.4 lists some basic notation and terminology. In sections 2.1 and 2.2, we will state and prove our results related to the distribution of longest monotone and alternating subsequences respectively of random permutations from $S_n(\tau^{(1)}, \tau^{(2)})$ where $\tau^{(1)}, \tau^{(2)} \in S_3$. Section 2.3 contains some results on patterns of length greater than three. Finally, in section 2.4, we will present our results on the rare regions in permutations' plots.

1.1 Longest monotone subsequences

For a given $\sigma \in S_n$, we say that $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is an *increasing subsequence of length k* in σ if $i_1 < i_2 < \cdots < i_k$ and $\sigma_{i_1} < \sigma_{i_2} < \cdots < \sigma_{i_k}$. Let $\mathrm{LIS}_n(\sigma)$ be the length of the longest increasing subsequence in σ. Similarly, let $\mathrm{LDS}_n(\sigma)$ be the length of the longest decreasing subsequence in σ. In a short and elegant argument, usually called the *Erdős-Szekeres lemma*, Erdős and Szekeres [14] proved that every permutation of length $(r-1)(s-1)+1$ or more contains either a decreasing subsequence of length r or an increasing subsequence of length s; equivalently, $\mathrm{LIS}_n(\sigma)\cdot\mathrm{LDS}_n(\sigma) \geqslant n$ for every $\sigma \in S_n$.

Determining the asymptotic distribution of LIS_n on S_n under the uniform distribution has a rich and interesting history [2]. The efforts of many researchers around this problem culminated in the celebrated result of Baik, Deift and Johansson in 1999 [5] which completely determined the asymptotic distribution of LIS_n. They proved that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LIS}_n - 2\sqrt{n}}{n^{1/6}} \leqslant t\right) = F(t) \quad \text{for all} \quad t \in \mathbb{R},$$

where F is the Tracy-Widom distribution function. This distribution was first obtained by Tracy and Widom [26] in the context of random matrix theory as the distributional limit of the (centered and scaled) largest eigenvalue of the Gaussian unitary ensemble.

The longest increasing subsequence problem in the context of pattern-avoiding permutations was first studied by Deutsch, Hildebrand and Wilf [12] for $S_n(\tau)$ where $\tau \in S_3$. They showed that in the case $\tau = 231$, $\mathbb{E}_n^{231}(\mathrm{LIS}_n) \sim \frac{n}{2}$, $\mathbb{SD}_n^{231}(\mathrm{LIS}_n) \sim \frac{\sqrt{n}}{2}$ and the normalized LIS_n converges to the standard normal distribution. For $\tau = 132$ and $\tau = 321$, we have $\mathbb{E}_n^{132}(\mathrm{LIS}_n) \sim \sqrt{\pi n}$, $\mathbb{SD}_n^{132}(\mathrm{LIS}_n) \sim \sqrt{n}$, and $\mathbb{E}_n^{321}(\mathrm{LIS}_n) \sim \frac{n}{2}$, $\mathbb{SD}_n^{321}(\mathrm{LIS}_n) \sim \sqrt{n}$; for both of them the normalized LIS_n converges to non-normal distributions which are evaluated exactly in [12]. Since $\mathrm{LIS}_n(\sigma) = \mathrm{LDS}_n(\sigma^r) = \mathrm{LIS}_n(\sigma^{rc}) = \mathrm{LIS}_n(\sigma^{-1})$, their results give a complete picture for LIS_n and LDS_n on $S_n(\tau)$ under the uniform measure for every $\tau \in S_3$. We are not aware of any existing work on this problem for other classes of pattern-avoiding permutations.

In section 2.1, we determine the distributions of LIS_n and LDS_n on $S_n(T)$ for $T \subset S_3$ with $|T| = 2$. The results are summarized in Table 1. The operations *reverse, complement,* and *inverse* induce obvious symmetry bijections among the classes $S_n(T)$ for different pairs T, resulting in the five groupings shown in Figure 1. Group (e) is empty for large n, so we shall present our results in terms of groupings (a) through (d).

	LIS_n			LDS_n		
$\{\tau^{(1)}, \tau^{(2)}\}$	mean	standard deviation	asymp. normal?	mean	standard deviation	asymp. normal?
(a) $\{132, 321\}$	$\sim \frac{5n}{6}$	$\sim \frac{5n}{6\sqrt{2}}$	No	$\to 2$	$\to 0$	No
(b) $\{132, 231\}$	$\frac{n+1}{2}$	$\frac{\sqrt{n-1}}{2}$	Yes	$\frac{n+1}{2}$	$\frac{\sqrt{n-1}}{2}$	Yes
(c) $\{132, 123\}$	$\to 2$	$\to 0$	No	$\frac{3n}{4}$	$\sim \frac{\sqrt{n}}{4}$	Yes
(d) $\{132, 213\}$	$\sim \log_2 n$	$\to$ constant	No	$\frac{n+1}{2}$	$\frac{\sqrt{n-1}}{2}$	Yes

Table 1: Summary of asymptotic behaviour of LIS_n and LDS_n on $S_n(\tau^{(1)}, \tau^{(2)})$ for the groupings (a)–(d) of Figure 1. See Theorem 2 and Figure 3. (We write "$\to$" to mean "converges to," and "$a_n \sim b_n$" to mean "$a_n/b_n \to 1$". Otherwise, expressions are exact.)

$$\begin{tikzcd}
(132,321) \arrow[loop above, "i"] \arrow[r, leftrightarrow, "r"] \arrow[d, leftrightarrow, "c"'] & (231,123) \arrow[d, leftrightarrow, "c"] \arrow[dl, leftrightarrow, "i"'] \\
(312,123) \arrow[r, leftrightarrow, "r"] & (213,321) \arrow[loop below, "i"]
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
(132,231) \arrow[loop above, "r"] \arrow[r, leftrightarrow, "c"] \arrow[d, leftrightarrow, "i"'] & (312,213) \arrow[loop above, "r"] \arrow[d, leftrightarrow, "i"] \\
(132,312) \arrow[loop below, "c"] \arrow[r, leftrightarrow, "r"] & (231,213) \arrow[loop below, "c"]
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
(132,123) \arrow[loop above, "i"] \arrow[r, leftrightarrow, "r"] \arrow[d, leftrightarrow, "c"'] & (231,321) \arrow[d, leftrightarrow, "c"] \arrow[dl, leftrightarrow, "i"'] \\
(312,321) \arrow[r, leftrightarrow, "r"] & (213,123) \arrow[loop below, "i"]
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
(132,213) \arrow[loop above, "i"] \arrow[r, leftrightarrow, "{r,c}"] & (231,312) \arrow[loop above, "i"]
\end{tikzcd}$$

(d)

$$\begin{tikzcd}
(123,321) \arrow[loop above, "{r,c,i}"]
\end{tikzcd}$$

(e)

Figure 1: The bijections among the subclasses $S_n(\tau^{(1)}, \tau^{(2)})$ for $\tau^{(1)}, \tau^{(2)} \in S_3$ under the operations i = inverse, c = complement, r = reverse.

Our results are less precise for LIS_n and LDS_n on $S_n(\tau)$ with longer patterns τ. For $\tau \in S_k$ with $\tau_1 = k$, we prove (see Corollary 13) that

$$\liminf_{n\to\infty} \frac{\mathbb{E}_n^\tau(\mathrm{LIS}_n)}{n} \geqslant \frac{1}{L(\tau)}. \tag{2}$$

In the special case $\tau = k(k-1)\cdots 21$, the Schensted correspondence leads to exact asymptotics, and in particular to the result that LIS_n/n converges in probability to $1/(k-1)$ (see Theorem 9). We conjecture that the behaviour of Equation (2) is generic, in the sense that at least one of $\mathbb{E}_n^\tau(\mathrm{LIS}_n)/n$ or $\mathbb{E}_n^\tau(\mathrm{LDS}_n)/n$ is bounded away from 0 for any pattern τ.

1.2 Longest alternating subsequences

In 2006, an analogous theory for alternating subsequences in S_n with the uniform probability measure was developed by Stanley [25] and Widom [28]. For a given $\sigma \in S_n$, we say that $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is an *alternating subsequence of length* k in σ if $i_1 < i_2 < \cdots < i_k$ and $\sigma_{i_1} > \sigma_{i_2} < \sigma_{i_3} > \sigma_{i_4} \cdots \sigma_{i_k}$. Let $\mathrm{LAS}_n(\sigma)$ be the length of the longest alternating subsequence in σ. Stanley proved that $\mathbb{E}(\mathrm{LAS}_n) = \frac{4n+1}{6}$ for $n \geqslant 2$ and $\mathbb{SD}(\mathrm{LAS}_n) = \sqrt{\frac{8}{45}n - \frac{13}{180}}$ for $n \geqslant 4$. Furthermore, Widom proved that LAS_n is asymptotically normal.

In [15], Firro, Mansour and Wilson studied the longest alternating subsequence problem for pattern-avoiding permutations. They showed that for each $\tau \in S_3$, LAS_n on

$S_n(\tau)$ is asymptotically normal with mean $\mathbb{E}_n^\tau(\mathrm{LAS}_n) \sim n/2$ and standard deviation $\mathbb{SD}_n^\tau(\mathrm{LAS}_n) \sim \sqrt{n}/2$. For the exact values of the means and standard deviations, see [15]. We are not aware of any existing work on this problem for other classes of pattern-avoiding permutations.

We consider LAS_n on $S_n(T)$ for $T \subset S_3$ with $|T| = 2$. Some cases such as $T = \{132, 231\}$ force $\mathrm{LAS}_n \leqslant 3$ for every n. Barring such cases, we show that LAS_n is asymptotically normal with mean $n/2$ and standard deviation $\sqrt{n}/2$ (Theorem 7).

For patterns τ of length 4 or more, our results for LAS_n on $S_n(\tau)$ are less complete. In some cases, including 4231 and 2413, we can show that the expected value of LAS_n is at least cn for some positive constant c (see Theorem 16). We conjecture that this is true for every pattern (except 12 and 21).

1.3 Rare regions

A permutation $\sigma \in S_n$ can be visualized via its plot, which is the set $\{(i, \sigma_i) : i \in [n]\}$ of n points in the plane. Plots of randomly generated τ-avoiding permutations are shown in Figure 2 for some patterns τ of length 4. Such plots suggest that for some patterns, there can be large regions of $[n]^2$ that rarely contain any points of randomly generated members of $S_n(\tau)$. Let's first introduce some definitions.

To deal with these concepts, we scale the plot of σ down to the unit square, and consider what parts of the unit square are likely to remain empty. We shall say that a point (x, y) of $[0, 1]^2$ is τ-rare if

$$\text{for every sequence } \{(I_n, J_n)\}_{n\geqslant 1} \text{ such that } (I_n, J_n) \in [n]^2 \text{ and } \lim_{n\to\infty}\left(\frac{I_n}{n}, \frac{J_n}{n}\right) = (x, y),$$
$$\text{we have } \limsup_{n\to\infty} \mathbb{P}_n^\tau(\sigma_{I_n} = J_n)^{1/n} < 1\,;$$

that is, every sequence (I_n, J_n) of grid points that scales down to (x, y) has exponentially decaying probabilities of being in the plot of $S_n(\tau)$. Then the "rare region" $\mathcal{R} \equiv \mathcal{R}(\tau)$ is defined to be the set of all τ-rare points in the unit square. Let $\mathcal{R}^\uparrow = \mathcal{R} \cap \{(x, y) \in [0, 1]^2 : y > x\}$ be the part of the rare region above the diagonal $y = x$. Using this terminology, we have the following basic result from [4] (see Theorems 1.3 and 8.1 and Proposition 3.1).

Theorem 1 ([4]). *Assume $\tau \in S_k$ and $\tau_1 > \tau_k$.*

(a) Assume $\tau_1 = k$. Then there is a $\delta > 0$ such that $[0, \delta] \times [1 - \delta, 1] \subset \mathcal{R}$; that is, all points sufficiently close to the point $(0, 1)$ are τ-rare.

(b) Assume $\tau_1 < k$. Then $\mathcal{R}^\uparrow = \varnothing$; that is, there are no τ-rare points above the diagonal.

Without loss of generality, we shall assume for the rest of this paragraph that $\tau \in S_k$ and $\tau_1 > \tau_k$. We define the "good region" $\mathcal{G}$ to be the complement of the rare region: $\mathcal{G} = [0, 1]^2 \setminus \mathcal{R}$. We prove that the region $\mathcal{G}$ contains the diagonal $y = x$ and is a closed set (Theorems 17 and 18). When $\tau_1 = k$, we prove that the boundary between $\mathcal{R}^\uparrow$ and $\mathcal{G}$ is

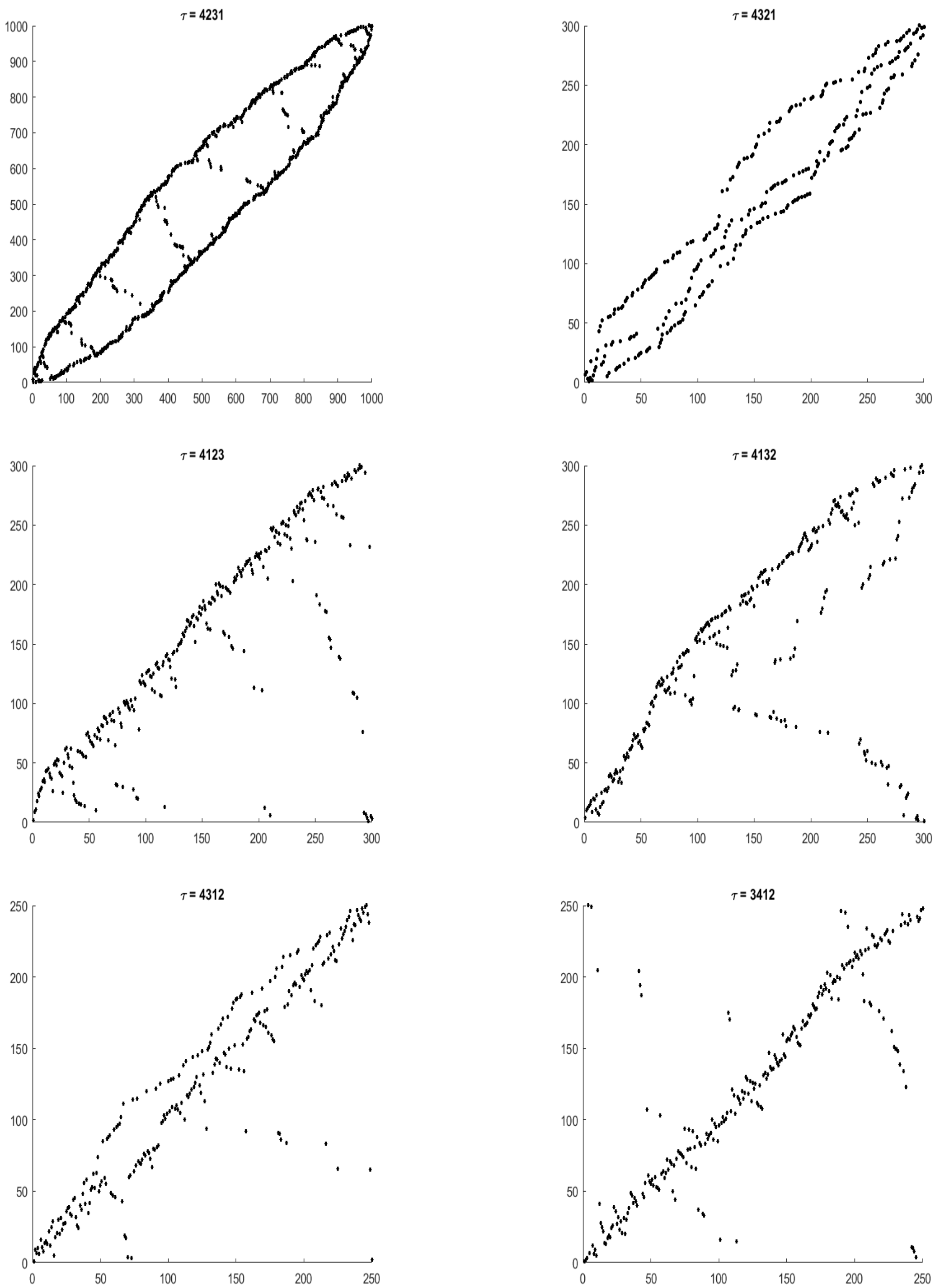


Figure 2: Examples of randomly generated permutations in $S_n(\tau)$ for some patterns $\tau \in S_4$. The top left figure was generated by Yosef Bisk under the supervision of N. Madras, using a modification of the Monte Carlo algorithm of [18].

a curve $y = r^{\uparrow}(x)$ that is Lipschitz continuous and strictly increasing on $[0, 1]$. Moreover, the left and right derivatives of $r^{\uparrow}$ are in the interval $[L(\tau)^{-1}, L(\tau)]$ at every point (see Theorem 17).

1.4 Notation and terminology

This section contains basic notation and terminology used in this paper, much of which is standard in the permutation pattern literature.

For $\tau \in S_k$ and $\sigma \in S_n$, we say that σ *contains the pattern* τ if there is a subsequence $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ of k elements of σ that appears in the same relative order as the pattern τ. For example, the permutation $\sigma = 6431257$ contains the patterns 321 and 3124 (since σ contains the subsequences $\sigma_1\sigma_2\sigma_3 = 643$ or $\sigma_1\sigma_3\sigma_5 = 632$, and $\sigma_2\sigma_4\sigma_5\sigma_7 = 4127$). We say that σ *avoids the pattern* τ if it does not contain τ. The set of all permutations in S_n avoiding τ is denoted by $S_n(\tau)$. For any set T of patterns, we write $S_n(T)$ to denote the set of permutations in S_n avoiding all the patterns in T, that is, $S_n(T) = \cap_{\tau\in T} S_n(\tau)$. For example, the permutation 521346 is in $S_6(132, 2314)$ because it avoids both 132 and 2314.

The *direct sum* of two permutations $\sigma \in S_n$ and $\phi \in S_m$ is the permutation $\sigma \oplus \phi$ in S_{n+m} obtained by concatenating ϕ to the northeast corner of σ,

$$\sigma \oplus \phi \;=\; \sigma_1 \cdots \sigma_n(n + \phi_1) \cdots (n + \phi_m)\,,$$

while the *skew sum* concatenates ϕ to the southeast corner of σ,

$$\sigma \ominus \phi \;=\; (m + \sigma_1) \cdots (m + \sigma_n)\phi_1 \cdots \phi_m\,.$$

A permutation is *layered* if it is the direct sum of one or more decreasing permutations.

For a given permutation σ, we say that σ_i is a *left-to-right maximum* if $\sigma_i > \sigma_j$ for all $j < i$. In this situation, i is referred as the *location* of the left-to-right maximum. (For clarity, we sometimes refer to σ_i as the *height*.) Similarly, we say that σ_i is a *right-to-left maximum* if $\sigma_i > \sigma_j$ for all $j > i$.

For natural numbers n, i, and j, we define $S_n(\tau; \sigma_i = j)$ to be the set of permutations σ in $S_n(\tau)$ such that $\sigma_i = j$. More generally, for any statement $\mathcal{P}$ about a generic permutation σ, we let $S_n(\tau; \mathcal{P})$ be the set of permutations in $S_n(\tau)$ for which $\mathcal{P}$ is true.

The number of elements in a set A is denoted by $|A|$.

For two sequences $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

2 Results and proofs

In this section, we will state and prove our results.

2.1 Longest monotone subsequences of permutations in $S_n(\tau^{(1)}, \tau^{(2)})$ for $\tau^{(1)}$, $\tau^{(2)} \in S_3$

In Table 1, we summarized our results on the mean and standard deviation of LIS_n and LDS_n on $S_n(\tau^{(1)}, \tau^{(2)})$ for $\tau^{(1)}, \tau^{(2)} \in S_3$, and whether they are asymptotically normal or not.

Theorem 2 below gives fuller descriptions of the distributions of the longest monotone subsequences in these cases. To prepare for the statement of the theorem, we first introduce the relevant distributions.

We use $\mathrm{Bin}(n, p)$ to denote a binomial random variable with parameters n and p.

For $n \geqslant 3$, let $\mathcal{D}[n]$ denote the triangular set of lattice points above the diagonal in the square $[n] \times [n]$, together with the origin; that is, $\mathcal{D}[n] = \{(k, m) \in \mathbb{Z}^2 : 1 \leqslant k < m \leqslant n\} \cup \{(0, 0)\}$. Then $|\mathcal{D}[n]| = \binom{n}{2} + 1$. We put uniform probability measure on $\mathcal{D}[n]$ and consider the following random variable:

$$\mathrm{D}_n((k, m)) \;=\; n - \min(m - k, k) \;=\; \max(n - m + k, n - k).$$

Then we have

$$\mathbb{P}(\mathrm{D}_n = n - j) \;=\; \frac{2n - 4j + 1}{\binom{n}{2} + 1} \quad \text{for } 1 \leqslant j \leqslant \frac{n}{2}, \quad \text{and} \quad \mathbb{P}(\mathrm{D}_n = n) \;=\; \frac{1}{\binom{n}{2} + 1}.$$

Here is a way to think about D_n, given that $(k, m) \neq (0, 0)$. Choose two points uniformly without replacement from $[n]$; let k be the smaller number and let m be the larger. Then D_n is the number of points remaining in $[n]$ after discarding the smaller of the intervals $(0, k]$ and $(k, m]$.

Let $X_1, X_2, \cdots$ be an independent sequence of Bernoulli($\frac{1}{2}$) random variables:

$$\mathbb{P}(X_i = 1) \;=\; \mathbb{P}(X_i = 0) \;=\; \frac{1}{2} \qquad \text{for all } i \geqslant 1. \tag{3}$$

Then define the following random variables for $n \geqslant 3$:

$$\mathrm{T}_n = n - 1 - \sum_{i=2}^{n-2} X_i + \sum_{i=2}^{n-1} X_i X_{i-1} \tag{4}$$

$$\mathrm{R}_n = \text{the length of the longest run of zeros in } X_1, X_2, \cdots, X_n. \tag{5}$$

For two random variables X and Y, we write $X \stackrel{\mathrm{d}}{=} Y$ to say that X and Y have the same distribution.

Theorem 2. *Let T denote a 2-element subset of S_3. Consider $S_n(T)$ with the uniform probability measure. Then we have the following:*

(a) For $T = \{132, 321\}$, $\mathrm{LIS}_n \stackrel{\mathrm{d}}{=} \mathrm{D}_n$ and $\mathrm{LDS}_n \leqslant 2$.

(b) For $T = \{132, 231\}$, $\mathrm{LIS}_n \stackrel{\mathrm{d}}{=} \mathrm{LDS}_n \stackrel{\mathrm{d}}{=} \mathrm{Bin}(n - 1, 1/2) + 1$.

(c) For $T = \{132, 123\}$, $\mathrm{LDS}_n \stackrel{\mathrm{d}}{=} \mathrm{T}_n$ and $\mathrm{LIS}_n \leqslant 2$.

(d) For $T = \{132, 213\}$, $\mathrm{LDS}_n \stackrel{\mathrm{d}}{=} \mathrm{Bin}(n - 1, 1/2) + 1$ and $\mathrm{LIS}_n \stackrel{\mathrm{d}}{=} \mathrm{R}_{n-1} + 1$.

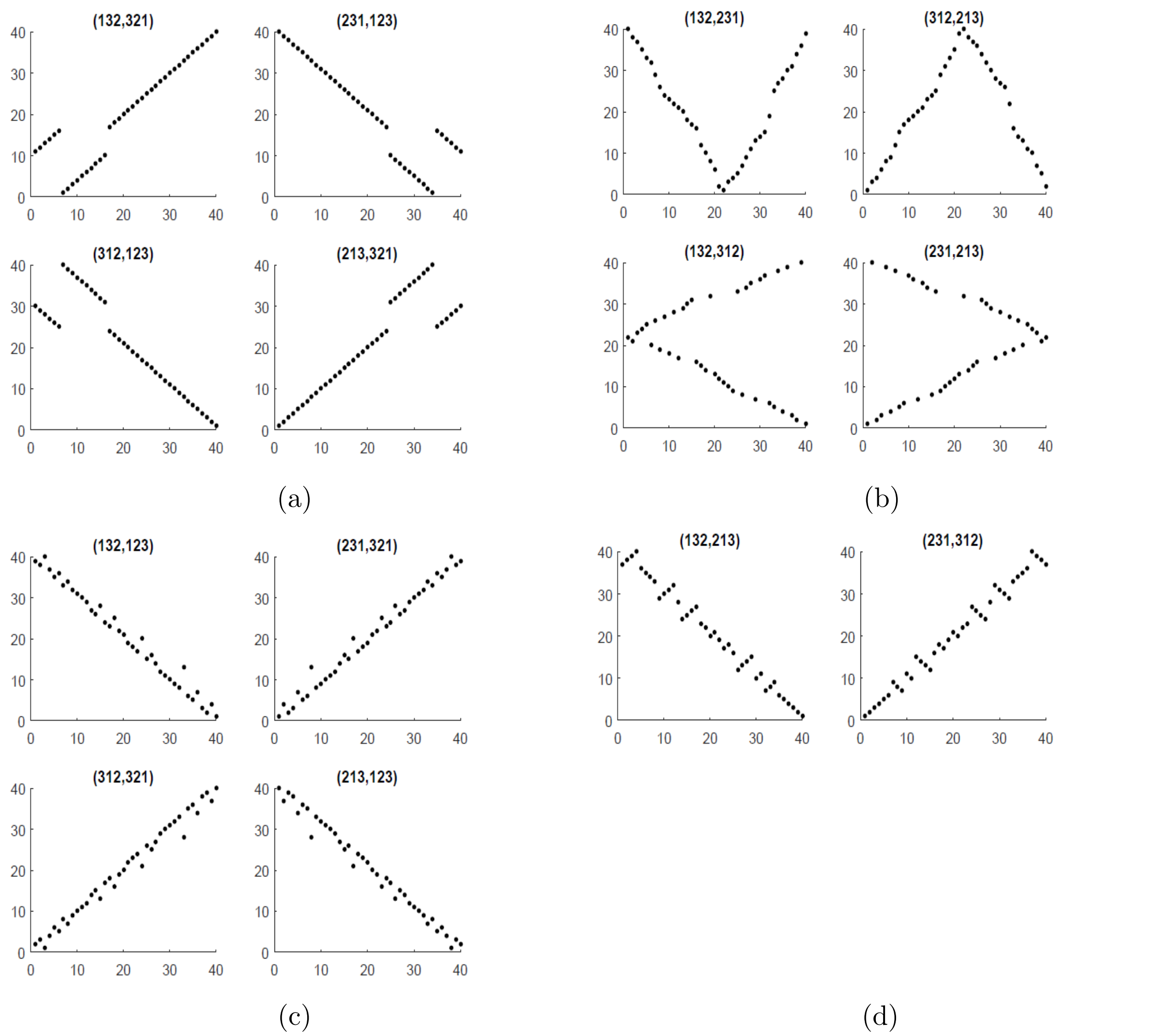


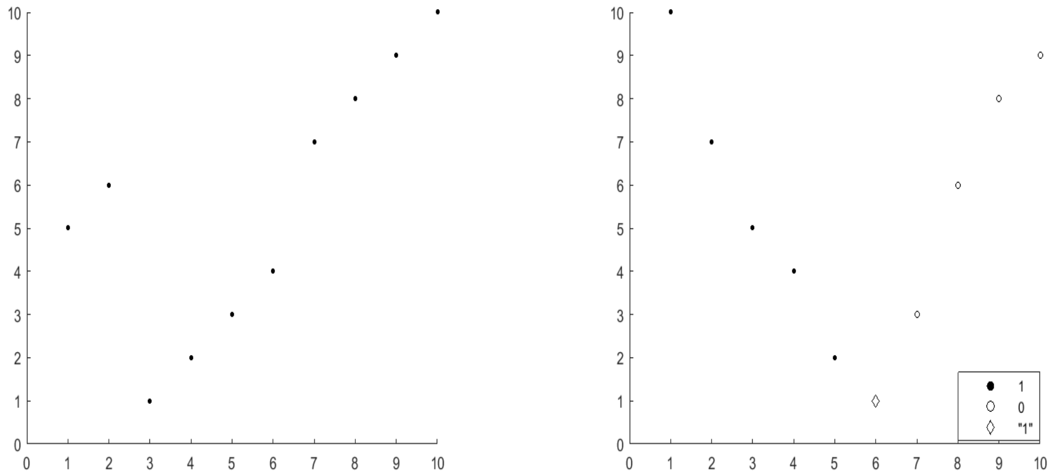
Figure 3: Examples of permutations from $S_n(\tau^{(1)}, \tau^{(2)})$ for $\tau^{(1)}, \tau^{(2)} \in S_3$ and $n = 40$. The permutations from $S_{40}(132, 321)$, $S_{40}(132, 231)$, $S_{40}(132, 123)$ and $S_{40}(132, 213)$ were generated randomly, and the rest were obtained using bijections of Figure 1.

The proof of Theorem 2 exploits the nice structure of each $S_n(T)$. First, we note that Simion and Schmidt [24] showed

$$|S_n(132, 321)| = \binom{n}{2} + 1 \quad \text{and} \quad |S_n(132, 231)| = |S_n(132, 123)| = |S_n(132, 213)| = 2^{n-1}.$$

The proofs of each part of Theorem 2 includes an explicit bijection for $S_n(\tau)$: with $\mathcal{D}[n]$ for part (a), and with binary strings of length $n-1$ for the other parts. We shall use these bijections again in the proof of Theorem 7 in Section 2.2 about longest alternating subsequences.

Proof of Theorem 2. (a) The case $T = \{132, 321\}$: We will follow Propositions 11 and 13



(a) (b)

Figure 4: The left figure is the plot of the permutation $\sigma = 5\,6\,1\,2\,3\,4\,7\,8\,9\,10$ in $S_{10}(132, 321)$ where $k = 2$ and $m = 6$. The right figure is the plot of the permutation $\sigma = 10\,7\,5\,4\,2\,1\,3\,6\,8\,9$ in $S_{10}(132, 231)$ which corresponds to the 0-1 sequence of length 9, 101101001.

in [24] which shows that there exists a bijection between $S_n(132, 321) \setminus \{12 \cdots n\}$ and the 2-element subsets of $[n]$. Let $\sigma \in S_n(132, 321)$.

If σ is not the identity permutation, let m be the largest i such that $\sigma_i \neq i$, and let k be the index such that $\sigma_k = m$. Then we must have $\sigma_1\sigma_2 \cdots \sigma_{k-1} = (m-k+1)(m-k+2) \cdots (m-1)$ and $\sigma_{k+1}\sigma_{k+2} \cdots \sigma_m = 12 \cdots (m-k)$. That is, σ must have the form

$$\sigma = (m-k+1)(m-k+2) \cdots m12 \cdots (m-k)(m+1)(m+2) \cdots n$$

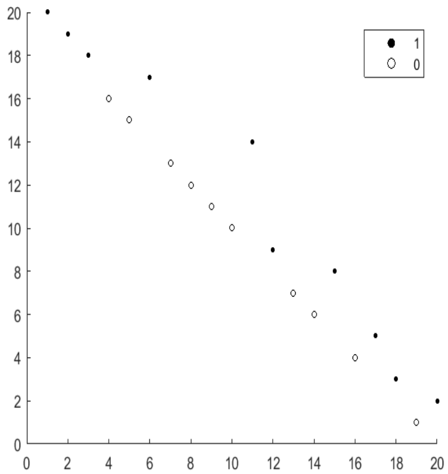
for some $1 \leqslant k \leqslant m \leqslant n$, and $m = k$ iff $\sigma = 12 \cdots n$. See Figure 4(a) for an example.

For $1 \leqslant k < m \leqslant n$, the pair $\{k, m\}$ determines σ uniquely. It follows from this argument that with uniform probability measure on $S_n(132, 321)$, $\mathrm{LIS}_n \stackrel{\mathrm{d}}{=} \mathrm{D}_n$.

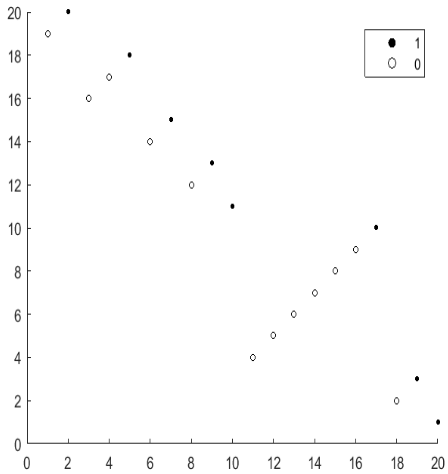
Remark 3. We note that $\sigma \in S_n(132, 321)$ can be represented as the *inflation* of 213 by the increasing permutations $\tau^{(1)}, \tau^{(2)}, \tau^{(3)}$ where $\tau^{(1)} = 12 \cdots k$, $\tau^{(2)} = 12 \cdots (m-k)$, $\tau^{(3)} = 12 \cdots (n-m)$; that is, $\sigma = 213[\tau^{(1)}, \tau^{(2)}, \tau^{(3)}]$. $S_n(132, 321)$ is an example of a 3×3 *monotone grid class* (see section 4 of [27]).

(b) The case $T = \{132, 231\}$: Let $\sigma \in S_n(132, 231)$. Let k be the index such that $\sigma_k = 1$. To avoid the pattern 132, we must have $\sigma_{k+1} < \sigma_{k+2} < \cdots < \sigma_n$; and to avoid the pattern 231, we must have $\sigma_1 > \sigma_2 > \cdots > \sigma_{k-1}$. In the terminology of [7], $S_n(132, 231)$ is a *skinny monotone grid class*, that is, it is the *juxtaposition* of a decreasing and an increasing sequence (see chapter 3 of [7]).

For our permutation σ we obtain a 0-1 sequence of length $n-1$ by labeling the decreasing points in the plot of σ with 1 and the increasing points with 0, and reading



(c)



(d)

Figure 5: The left figure is the plot of the permutation $\sigma = 20\,19\,18\,16\,15\,17\,13\,12\,11\,10\,14\,9\,7\,6\,8\,4\,5\,3\,1\,2$ in $S_{20}(132, 123)$ which corresponds to the 0-1 sequence of length 19, 1110010000110010110. The right figure is the plot of the permutation $\sigma = 19\,20\,16\,17\,18\,14\,15\,12\,13\,11\,4\,5\,6\,7\,8\,9\,10\,2\,3\,1$ in $S_{20}(132, 213)$ which corresponds to the 0-1 sequence of length 19, 0100101011000000101.

them from bottom to top, excluding the point $\sigma_k = 1$. That is, for $j \in [n-1]$, let i be the index such that $\sigma_i = j + 1$, and then set x_j to be 1 (respectively, 0) if $i < k$ (respectively, $i > k$). See Figure 4(b) for an example. It is not hard to see that this gives a bijection between $S_n(132, 231)$ and $\{0, 1\}^{n-1}$. Thus the uniform distribution ensures that every sequence has probability $2^{-(n-1)}$ and hence that $x_1, \ldots, x_{n-1}$ are independent Bernoulli($\frac{1}{2}$) random variables.

Then we have

$$\mathrm{LDS}_n(\sigma) \;=\; k+1 \;=\; \sum_{i=1}^{n-1} 1_{x_i=1} + 1 \quad \text{and} \quad \mathrm{LIS}_n(\sigma) \;=\; l+1 \;=\; \sum_{i=1}^{n-1} 1_{x_i=0} + 1\,,$$

and hence $\mathrm{LDS}_n \overset{\mathsf{d}}{=} \mathrm{LIS}_n \overset{\mathsf{d}}{=} \mathrm{Bin}(n-1, 1/2) + 1$.

(c) The case $T = \{132, 123\}$: Let $\sigma \in S_n(132, 123)$. Suppose $\sigma_k = n$ for some $k \in [n]$. To avoid the pattern 123, we must have $\sigma_1 > \sigma_2 > \cdots > \sigma_{k-1}$; and to avoid 132, we must have $\sigma_i > n - k$ whenever $i < k$. Therefore, recalling notation from Section 1.4, we must have $\sigma = (\delta \oplus 1) \ominus \phi$ where δ is a decreasing permutation (possibly of length 0), 1 is the one-element permutation, and ϕ avoids 123 and 132. Iterating this argument, we see that σ is the skew sum of one or more permutations of the form $\delta \oplus 1$. In the terminology of [1], $\cup_n S_n(132, 123)$ is the class $\ominus(\mathbf{D} \oplus \mathbf{1})$. Observe that each "$\oplus\mathbf{1}$" in this decomposition corresponds to a right-to-left maximum of σ.

We construct a map from $S_n(132, 123)$ to the set of 0-1 sequences $x_1 x_2 \cdots x_{n-1}$ as follows. Given $\sigma \in S_n(132, 123)$, let $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1$ be the locations in $[n-1]$ of its right-to-left maxima. Then define $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 1$ and $x_j = 0$ for $j \notin \{i_1, i_2, \cdots, i_k\}$. It is not hard to see that this map is a bijection. Moreover, fixing $x_n = x_0 = 1$, we have

$$\begin{aligned}
\mathrm{LDS}_n(\sigma) &= \sum_{j=1}^{k+1} \max(1, i_j - i_{j-1} - 1) \\
&= \sum_{j=1}^{k+1} (i_j - i_{j-1} - 1) + \sum_{j=1}^{k+1} 1_{i_j = i_{j-1}+1} \\
&= \sum_{i=1}^{n-1} 1_{x_i=0} + \sum_{i=1}^{n} 1_{x_{i-1}=x_i=1} \\
&= 2 + \sum_{i=2}^{n-2} 1_{x_i=0} + \sum_{i=2}^{n-1} 1_{x_{i-1}=x_i=1} \\
&= 2 + \sum_{i=2}^{n-2} (1 - x_i) + \sum_{i=2}^{n-1} x_i x_{i-1} \\
&= n - 1 - \sum_{i=2}^{n-2} x_i + \sum_{i=2}^{n-1} x_i x_{i-1}.
\end{aligned}$$

Therefore with the uniform probability measure on $S_n(132, 123)$, $\mathrm{LDS}_n \stackrel{\mathsf{d}}{=} \mathrm{T}_n$.

Example 4. For the 0-1 sequence 01101 of length 5, the corresponding permutation of length 6 in $S_6(132, 123)$ is $\sigma = 5\,6\,4\,2\,3\,1$. See also the example in Figure 5(c).

(d) The case $T = \{132, 213\}$*:* Let $\sigma \in S_n(132, 213)$. Suppose $\sigma_k = n$ for some $k \in [n]$. To avoid the pattern 213, we must have $\sigma_1 < \sigma_2 < \cdots < \sigma_k$; and to avoid 132, we must have $\sigma_i > n - k$ whenever $i < k$. Therefore, $\sigma = \lambda \ominus \phi$ where λ is the increasing permutation of length k, and $\phi \in S_{n-k}(132, 213)$. Iterating this argument, we see that σ is the skew sum of one or more increasing permutations. Thus, this class is the reverse (or complement) of the layered permutations.

Observe also that a member of $S_n(132, 213)$ is uniquely determined by the locations of its right-to-left maxima. Therefore, as in the proof of part (c), we obtain a a bijection between $S_n(132, 213)$ and the set of 0-1 sequences $x_1 x_2 \cdots x_{n-1}$ by setting x_i to be 1 (respectively 0) if σ_i is (respectively, is not) a left-to-right maximum of σ. Then we have

$$\mathrm{LDS}_n(\sigma) = \sum_{i=1}^{n-1} 1_{x_i=1} + 1$$

and

$$\mathrm{LIS}_n(\sigma) = 1 + \text{length of the longest run of zeros in } x_1 x_2 \cdots x_{n-1}.$$

Therefore, with the uniform probability measure on $S_n(132, 213)$, we have $\mathrm{LDS}_n \stackrel{\mathsf{d}}{=} \mathrm{Bin}(n-1, 1/2) + 1$ and $\mathrm{LIS}_n \stackrel{\mathsf{d}}{=} \mathrm{R}_{n-1} + 1$. □

Example 5. The permutation $\sigma = 5\,6\,1\,2\,3\,4$ in $S_6(132, 213)$ corresponds to the 0-1 sequence 01000 of length 5. See also the example in Figure 5(d).

Recall that a *composition* of an integer n is a way of writing n as the sum of an ordered sequence of positive integers. Two sequences with the same terms but in different order correspond to different compositions but to the same *partition* of their sum. It is known that each positive integer n has 2^{n-1} distinct compositions, and that there is an explicit bijection between the compositions of n and the set of layered permutations of length n. Probabilistic properties of random integer compositions and partitions have been studied in the literature. From this viewpoint, our results for group (d) correspond to results on page 337 and in Theorem 8.39 of [16].

We conclude this subsection by explaining how the results of Table 1 follow from Theorem 2 and some classical results. The mean and standard deviation of D_n are asymptotically

$$\mathbb{E}(\mathrm{D}_n) \sim \frac{5n}{6} \text{ and } \mathbb{SD}(\mathrm{D}_n) \sim \frac{5n}{6\sqrt{2}}.$$

It is easy to show that

$$\mathbb{E}(\mathrm{T}_n) = \frac{3n}{4} \text{ and } \mathbb{SD}(\mathrm{T}_n) \sim \frac{\sqrt{n}}{4}.$$

Moreover, the fact that the limiting distribution of $(\mathrm{T}_n - \mathbb{E}(\mathrm{T}_n))/\,\mathbb{SD}(\mathrm{T}_n)$ is standard normal is a consequence of the following lemma, which can be found for example in [11].

Lemma 6 ([11], Theorem 7.3.1)**.** *Suppose that $\{Y_n\}$ is a sequence of m-dependent, uniformly bounded random variables such that as $n \to +\infty$*

$$\frac{\mathbb{SD}(S_n)}{n^{1/3}} \to +\infty$$

where $S_n = Y_1 + Y_2 + \cdots + Y_n$. Then $(S_n - \mathbb{E}(S_n))/\,\mathbb{SD}(S_n)$ converges in distribution to the standard normal distribution.

Finally, Erdős and Renyi [13] proved that

$$\lim_{n\to\infty} \frac{\mathrm{R}_n}{\log_2 n} = 1 \text{ a.s.} \tag{6}$$

It is also known that as n tends to infinity $\mathbb{E}(\mathrm{R}_n) \sim \log_2 n$, $\mathbb{SD}(\mathrm{R}_n)$ converges to a positive constant, and $\mathrm{R}_n - \log_2 n$ possesses no limit distribution; for more on this topic see the survey paper [23].

2.2 Longest alternating subsequences

In this section, we will present our results on the asymptotic behaviour of the length of the longest alternating subsequence, LAS_n, on $S_n(\tau^{(1)},\tau^{(2)})$ with $\tau^{(1)},\tau^{(2)} \in S_3$ under the uniform probability distribution. For a result on LAS_n for longer patterns, see Theorem 16 in section 2.3.

Theorem 7. *Let T denote a 2-element subset of S_3. Consider $S_n(T)$ with the uniform probability measure. Then we have the following:*

a) If $T = \{132, 321\}$, then we have $\mathrm{LAS}_n(\sigma) \leqslant 3$ for any $\sigma \in S_n(T)$.

b) If $T = \{132, 231\}$, then we have $\mathrm{LAS}_n(\sigma) \leqslant 3$ for any $\sigma \in S_n(T)$.

If $T = \{132, 312\}$, then LAS_n is asymptotically normal with the mean $\mathbb{E}_n^T(\mathrm{LAS}_n) \sim n/2$ and the standard deviation $\mathbb{SD}_n^T(\mathrm{LAS}_n) \sim \sqrt{n}/2$.

c) If $T = \{132, 123\}$, then LAS_n is asymptotically normal with the mean $\mathbb{E}_n^T(\mathrm{LAS}_n) \sim n/2$ and the standard deviation $\mathbb{SD}_n^T(\mathrm{LAS}_n) \sim \sqrt{n}/2$.

d) If $T = \{132, 213\}$, then LAS_n is asymptotically normal with the mean $\mathbb{E}_n^T(\mathrm{LAS}_n) \sim n/2$ and the standard deviation $\mathbb{SD}_n^T(\mathrm{LAS}_n) \sim \sqrt{n}/2$.

Remark 8. Parts (a) through (d) correspond to the labeling in Figures 1 and 3. Note that since $|\mathrm{LAS}_n(\sigma) - \mathrm{LAS}_n(\sigma^c)| \leqslant 1$ and $|\mathrm{LAS}_n(\sigma) - \mathrm{LAS}_n(\sigma^r)| \leqslant 1$, it suffices to consider one representative pair from each symmetry class in Figure 1 except the symmetry class in Figure 1-(b).

The proof uses the structures exploited in the proof of Theorem 2. The proof of part (b) needs a bit of new work, so we prove it last.

Proof of Theorem 7. (a) Assume $\sigma \in S_n(132, 321)$. Recall from the proof of Theorem 2(a) that σ is either the identity or else it can be represented as $\sigma = 213[\tau^{(1)},\tau^{(2)},\tau^{(3)}]$, the inflation of 213 by increasing permutations $\tau^{(1)},\tau^{(2)},\tau^{(3)}$ (Figure 3(a)). Therefore, $\mathrm{LAS}_n(\sigma) \leqslant 3$.

(c) Recall that the bijection in the proof of Theorem 2(c) defines $x_i = 1$ if σ_i is a right-to-left maximum, and $x_i = 0$ otherwise, for each $i \in [n-1]$. Then $x_1, x_2, \cdots, x_{n-1}$ are independent Bernoulli($\frac{1}{2}$) random variables.

Let $\mathrm{A}_n = \displaystyle\sum_{i=1}^{n-1} 1_{x_i=0}1_{x_{i+1}=1}$. Note that $|\mathrm{LAS}_n - 2\,\mathrm{A}_n| \leqslant 4$. It is easy to show that $\mathbb{E}(\mathrm{A}_n) \sim \frac{n}{4}$ and $\mathbb{SD}(\mathrm{A}_n) \sim \frac{\sqrt{n}}{4}$. Moreover, by Lemma 6, normalized A_n converges to the standard normal distribution. Therefore $\mathbb{E}_n^T(\mathrm{LAS}_n) \sim 2\,\mathbb{E}(\mathrm{A}_n)$ and $\mathbb{SD}_n^T(\mathrm{LAS}_n) \sim 2\,\mathbb{SD}(\mathrm{A}_n)$, and we get the asymptotic normality of LAS_n.

(d) Using the bijection from the proof of Theorem 2(d), the proof will be the same as in part (c) above.

(b) Let $\sigma \in S_n(132, 231)$. As shown in the proof of Theorem 2, σ is the juxtaposition of a decreasing and an increasing subsequence. Therefore $\mathrm{LAS}_n(\sigma) \leqslant 3$.

Let $\sigma \in S_n(132, 312)$. Note that all the entries greater (respectively, smaller) than σ_1 must be in the increasing (respectively, decreasing) order, otherwise we would have a 132 (respectively, 312) pattern.

For $i \in [n-1]$, define x_i to be 1 if $\sigma_{i+1} > \sigma_1$, and 0 otherwise. Then $x_1, x_2, \cdots, x_{n-1}$ are independent Bernoulli($\frac{1}{2}$) random variables.

Set $\mathrm{A}_n = \sum_{i=1}^{n-1} 1_{x_i=1} 1_{x_{i+1}=0}$. Then the rest of the proof follows as for part (c). □

2.3 Results for patterns of length $k \geqslant 4$

In this subsection, we will present our results on longer patterns. Theorem 9 concerns monotone patterns. It yields precise properties because we can exploit the Schensted correspondence, which we review below. The next group of results proves that LIS_n is unlikely to be $o(n)$ on $S_n(\tau)$ when τ is of the form $\tau = k\tau_2 \cdots \tau_k \in S_k$. Finally, Theorem 16 proves an analogous result for LAS_n on $S_n(\tau)$ for some patterns τ.

First we recall some basic definitions used in the algebraic combinatorics of permutations. For more on this topic, see chapter 7 of [10] or the survey paper [2]. A *partition* $\lambda = (\lambda_1, \cdots, \lambda_j)$ of an integer $n \geqslant 1$ is a sequence of integers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_j \geqslant 1$ and $\sum_i \lambda_i = n$. A partition λ can be identified with its associated *Ferrers shape* of n cells with λ_i left justified cells in row i. A (standard) Young Tableau is a Ferrers shape on n boxes in which each number in $[n]$ is assigned to its own box, so that the numbers are increasing within each row and each column (going down). The Schensted [22] correspondence induces a bijection between permutations and pairs (P, Q) of Young Tableaux of the same shape. The number of Young Tableaux of a given shape λ, denoted by d_λ, is given by the celebrated *hook length formula* of Frame and Robinson:

$$d_\lambda = \frac{n!}{\prod_c h_c}.$$

The *hook length* h_c of a cell c in a Ferrers shape is the number of cells to the right of c in its row, the cells below c in its column, and the cell c itself. For illustration, in the following Ferrers shape corresponding to the partition $\lambda = (5, 4, 2)$, each cell c is occupied by its hook length h_c. (Lest confusion arise, we note that it is not a Young Tableau.)

7	6	4	3	1
5	4	2	1	
2	1			

Importantly, if the Schensted correspondence associates the shape λ with a given $\sigma \in S_n$, then $\lambda_1 = \mathrm{LIS}_n(\sigma)$. In [22], Schensted shows also that the length of the decreasing subsequences in σ are encoded by the P-tableau of σ: For any $\sigma \in S_n$, we have

$$P_{\sigma^r} = P_\sigma^t \tag{7}$$

where P_{σ^r} is the P-tableau of the reverse σ^r of σ and P^t_σ is the transpose (reflection through anti-diagonal) of P_σ.

Theorem 9. *Consider $S_n(\tau)$ with $\tau = k(k-1)\cdots 1$. Then whenever $0 < \epsilon < k-2$, we have*

$$\lim_{n\to\infty} \mathbb{P}^\tau_n \left(\mathrm{LIS}_n > \frac{1+\epsilon}{k-1} n\right)^{1/n} = \left[(1+\epsilon)^{(1+\epsilon)} \left(1 - \frac{\epsilon}{k-2}\right)^{k-2-\epsilon}\right]^{-\frac{2}{k-1}} \tag{8}$$

and

$$\mathbb{P}^\tau_n \left(\mathrm{LIS}_n < \frac{n}{k-1}\right) = 0 \quad \textit{for every } n. \tag{9}$$

Therefore

$$\frac{\mathrm{LIS}_n}{n} \to \frac{1}{k-1} \quad \textit{in probability as } n \to \infty.$$

Proof of Theorem 9. Fix the decreasing pattern $\tau = k(k-1)\cdots 1$. For any $\sigma \in S_n(\tau)$, there will be at most $k-1$ rows in the corresponding Ferrers shape because $\mathrm{LDS}_n(\sigma) \leqslant k-1$ and $\mathrm{LDS}_n(\sigma) = \mathrm{LIS}_n(\sigma^r) =$ number of columns in P_{σ^r} = number of rows in P_σ by Equation (7). Note also that it follows from the Erdős-Szekeres lemma that $\mathrm{LIS}_n(\sigma) \geqslant \frac{n}{k-1}$ for all $\sigma \in S_n(\tau)$. This proves Equation (9).

Let $\mathcal{H}_{n,k}$ denote the set of partitions of n with at most $k-1$ rows in their corresponding Ferrers shape. For notational convenience, we will consider

$$\mathcal{H}_{n,k} := \{(\lambda_1, \cdots, \lambda_{k-1}) \in \mathbb{Z}^{k-1} : \lambda_1 \geqslant \cdots \geqslant \lambda_{k-1} \geqslant 0, \sum_{i=1}^{k-1} \lambda_i = n\}.$$

For a given $\epsilon > 0$, let $\mathcal{H}^\epsilon_{n,k} := \{\lambda \in \mathcal{H}_{n,k} : \lambda_1 > n\frac{1+\epsilon}{k-1}\}$ and let

$$\mathcal{C}^\epsilon_k := \left\{(x_1, \cdots, x_{k-1}) \in \mathbb{R}^{k-1} :\ x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{k-1} \geqslant 0,\ \sum_{i=1}^{k-1} x_i = 1,\ x_1 \geqslant \frac{1+\epsilon}{k-1}\right\}.$$

Let Λ_n be a random variable taking values in $\mathcal{H}_{n,k}$ with the distribution $\mathcal{Q} \equiv \mathcal{Q}_{n,k}$ given by

$$\mathcal{Q}(\Lambda_n = \lambda) := \frac{d^2_\lambda}{|S_n(\tau)|}. \tag{10}$$

This distribution can be considered as a "shape distribution" on the Ferrers shapes corresponding to the uniform distribution on permutations in $S_n(\tau)$. Specifically, $\mathbb{P}^\tau_n(\mathrm{LIS}_n(\sigma) = a) = \mathcal{Q}(\Lambda_{n,1} = a)$.

Note that for $\lambda = (\lambda_1, \cdots, \lambda_{k-1}) \in \mathcal{H}_{n,k}$, we have

$$\prod_{i=1}^{k-1} \lambda_i! \ \leqslant\ \prod_c h_c \ \leqslant\ \prod_{i=1}^{k-1} (\lambda_i + k - 1)!,$$

and hence

$$\frac{n!}{\prod_{i=1}^{k-1}(\lambda_i+k-1)!} \;\leqslant\; \frac{n!}{\prod_c h_c} \;\leqslant\; \frac{n!}{\prod_{i=1}^{k-1}\lambda_i!}. \tag{11}$$

If $x \in \mathcal{C}_k^\epsilon$ and the sequence $\{\lambda(n)\}$ (with $\lambda(n) \in \mathcal{H}_{n,k}^\epsilon$) satisfies $\lim_{n\to\infty}\lambda(n)/n = x$, then by Stirling's approximation, $n! \sim n^n e^{-n}\sqrt{2\pi n}$, we see from Equation (11) that

$$\lim_{n\to\infty} d_{\lambda(n)}^{1/n} \;=\; D(x) \;:=\; \frac{1}{x_1^{x_1}\cdots x_{k-1}^{x_{k-1}}} \tag{12}$$

where $0^0 = 1$. Note that the maximum of $D(x)$ over $\mathcal{C}_k^\epsilon$ is attained at $x = \alpha$ defined by $\alpha_1 = \frac{1+\epsilon}{k-1}$ and $\alpha_2 = \cdots = \alpha_{k-1} = \frac{1}{k-1}(1-\frac{\epsilon}{k-2})$, with the value $D(\alpha) = (k-1)/[(1+\epsilon)^{(1+\epsilon)}(1-\frac{\epsilon}{k-2})^{k-2-\epsilon}]^{1/(k-1)}$. (To see this, first note that if we fix $x_1 = A$, then a Lagrange multiplier calculation shows that the maximum is at $x(A) := (A, \frac{1-A}{k-2}, \ldots, \frac{1-A}{k-2})$; then show that $\log D(x(A))$ is a concave function of $A \in (0,1)$ with maximum at $A = \frac{1}{k-1}$.)

For $k \geqslant 2$, Regev [21] proved that the Stanley-Wilf limit of the increasing pattern $\tau = 12\cdots k$ is given by

$$L(12\cdots k) \;=\; (k-1)^2. \tag{13}$$

Using Equations (10) and (13), we see that

$$\liminf_{n\to\infty} \mathcal{Q}(\Lambda_n \in \mathcal{H}_{n,k}^\epsilon)^{1/n} \;\geqslant\; \frac{D(\alpha)^2}{(k-1)^2}. \tag{14}$$

For the complementary inequality, let $\hat\lambda(n) := \operatorname{argmax}_{\lambda\in\mathcal{H}_{n,k}^\epsilon}\mathcal{Q}(\Lambda_n = \lambda)$. Then

$$\mathcal{Q}(\Lambda_n \in \mathcal{H}_{n,k}^\epsilon) \;\leqslant\; |\mathcal{H}_{n,k}^\epsilon|\,\mathcal{Q}(\Lambda_n = \hat\lambda(n)). \tag{15}$$

Since $\mathcal{C}_k^\epsilon$ is a compact set and $\frac{\hat\lambda(n)}{n} \in \mathcal{C}_k^\epsilon$, there exists a subsequence $\hat\lambda(n_l)$ such that $\frac{\hat\lambda(n_l)}{n_l}$ converges to a point $z \in \mathcal{C}_k^\epsilon$ and

$$\lim_{l\to\infty} \mathcal{Q}(\Lambda_{n_l} = \hat\lambda(n_l))^{1/n_l} \;=\; \limsup_{n\to\infty} \mathcal{Q}(\Lambda_n = \hat\lambda(n))^{1/n}. \tag{16}$$

Since $|\mathcal{H}_{n,k}^\epsilon| = O(n^{k-1})$ as $n\to\infty$, we see from Equations (15) and (16) and the argument for Equation (14) that

$$\limsup_{n\to\infty} \mathcal{Q}(\Lambda_n \in \mathcal{H}_{n,k}^\epsilon)^{1/n} \;\leqslant\; \frac{D(z)^2}{(k-1)^2} \;\leqslant\; \frac{D(\alpha)^2}{(k-1)^2}. \tag{17}$$

Equations (14) and (17) imply that Equation (8) holds, and the theorem follows. □

The next group of results concerns LIS_n when $\tau_1 = k$. The key observation is that LIS_n is at least as big as the number of left-to-right maxima (Equation (18)). Theorem 10 then tells us that LIS_n is at least a constant times n, with very high probability.

Let $\mathrm{RL}_n(\sigma)$ be the number of right-to-left maxima in the permutation σ, and $\mathrm{LR}_n(\sigma)$ be the number of left-to-right maxima in σ. Note that the left-to-right maxima (right-to-left maxima) in σ form an increasing (decreasing) subsequence in σ. In particular,

$$\mathrm{LIS}_n(\sigma) \geqslant \mathrm{LR}_n(\sigma) \qquad \text{and} \qquad \mathrm{LDS}_n(\sigma) \geqslant \mathrm{RL}_n(\sigma). \tag{18}$$

Recall also the notation $S_n(\tau; \mathcal{P})$ from Section 1.4.

Theorem 10. *Let $\tau = k\tau_2 \cdots \tau_k \in S_k$. For every real δ such that $0 < \delta < 1/L(\tau)$, the following strict inequality holds:*

$$\limsup_{n\to\infty} |S_n(\tau; \mathrm{LR}_n < \delta n)|^{1/n} < L(\tau).$$

Theorem 10 is an immediate consequence of Theorem 11, which is a stronger (and more technical) result. We will also need Theorem 11 for the proof of Theorem 17(b).

For $\tau \in S_k$, $\delta > 0$, and an interval I, define

$$\begin{aligned} &S_n(\tau;\, \mathrm{LR}_n[I] < \delta n) \\ &\qquad = \{\sigma \in S_n(\tau) : \sigma \text{ has fewer than } \delta n \text{ left-to-right maxima } \sigma_i \text{ such that } \sigma_i \in I\ \}. \end{aligned}$$

Theorem 11. *Let $\tau = k\tau_2 \cdots \tau_k \in S_k$. For $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$, let $I_n = [a_n, b_n]$ where $a_n/n \to \alpha_1$ and $b_n/n \to \alpha_2$. For every $0 < \delta < \frac{\alpha_2 - \alpha_1}{L(\tau)}$, the following strict inequality holds:*

$$\limsup_{n\to\infty} |S_n(\tau; \mathrm{LR}_n[I_n] < \delta n)|^{1/n} < L(\tau).$$

Intuitively, this result says that for a typical member of $S_n(\tau)$, the left-to-right maxima fill $[1, n]$ with a positive density throughout $[1, n]$.

We shall require an *insertion operation* $\mathcal{I}$ which was introduced in [4]. Suppose $\sigma \in S_n$ and $h \in [n]$. Let $J = \min\{j : \sigma_j \geqslant h\}$. Define the permutation $\theta \in S_{n+1}$ as follows:

$$\theta_i = \begin{cases} \sigma_i & \text{if } i < J \\ h & \text{if } i = J \\ \sigma_{i-1} & \text{if } i \geqslant J+1 \text{ and } \sigma_{i-1} < h \\ \sigma_{i-1} + 1 & \text{if } i \geqslant J+1 \text{ and } \sigma_{i-1} \geqslant h \end{cases}$$

We denote θ by $\mathcal{I}(\sigma; h)$. Think of $\mathcal{I}(\sigma; h)$ as inserting a new left-to-right maximum in σ at height h, while preserving all the other left-to-right maxima of σ (perhaps shifting them slightly).

The proof of Theorem 11 is similar to the proof of Proposition 6.2 of [4]. The main idea is that repeated use of $\mathcal{I}$ can transform a permutation σ in $S_n(\tau)$ into many other permutations σ' in $S_{n+r}(\tau)$ with r more left-to-right maxima. By insisting that σ started with few left-to-right maxima, we significantly limit the number of choices of σ that could give rise to a particular σ'. We then estimate how much smaller the number of preimages σ is than the number of images σ', which is at most $L(\tau)^{n+r}$.

The following lemma says that if $\tau_1 = k$, then $\mathcal{I}$ preserves τ-avoidance. For the proof, see the proof of Lemma 6.1 in [4].

Lemma 12 ([4]). *Let $\tau = k\tau_2 \cdots \tau_k \in S_k$. Assume $1 \leqslant h_1 < h_2 < \cdots < h_r \leqslant n$. Let $\sigma^{(r)} = \sigma \in S_n(\tau)$ and let $\sigma^{(l-1)} = \mathcal{I}(\sigma^{(l)}; h_l)$ for $l = r, r-1, \cdots, 2, 1$. We denote $\sigma^{(0)}$ by $\mathcal{I}(\sigma; h_1, h_2, \cdots, h_r)$. Then we have*

$$\sigma^{(0)} \in S_{n+r}(\tau).$$

Observe also that $h_i + i - 1$ is a left-to-right maximum of $\sigma^{(0)}$ for each $i \in [r]$.

Proof of Theorem 11. Assume $0 < \delta < \frac{\alpha_2-\alpha_1}{L(\tau)}$ and choose a positive integer M so that $\frac{L(\tau)}{M} < 1 - \frac{\delta L(\tau)}{\alpha_2-\alpha_1}$. Let $r \equiv r_n := \lfloor \frac{b_n-a_n-2}{M} \rfloor$.

For fixed n, define the *intervals of heights* in I_n as $\mathcal{J}_i := (\lceil a_n \rceil + (i-1)M, \lceil a_n \rceil + iM]$ for each $i \in [r]$. Then $\mathcal{J}_1, \cdots, \mathcal{J}_r$ are disjoint subintervals of I_n.

In the following, we shall use the notation of Lemma 12. Define the function $\Psi \equiv \Psi_n$ as

$$\Psi : S_n(\tau; \mathrm{LR}_n[I_n] < \delta n) \times [M]^r \to S_{n+r}(\tau)$$

such that $\Psi(\sigma, (\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_r)) = \mathcal{I}(\sigma; h_1, h_2, \cdots, h_r)$, where $h_i = \lceil a_n \rceil + \hat{h}_i + (i-1)M$ for $i \in [r]$ (so that $h_i \in \mathcal{J}_i$). Corresponding to the intervals of heights $\mathcal{J}_i$, we define the shifted intervals of heights as follows:

$$\mathcal{J}_i^{\Psi} := \mathcal{J}_i + i - 1 = (\lceil a_n \rceil + (i-1)(M+1), \lceil a_n \rceil + i(M+1) - 1], \quad i = 1, \cdots, r.$$

Then $\mathcal{J}_1^{\Psi}, \ldots, \mathcal{J}_r^{\Psi}$ are disjoint subintervals of $(a_n, b_n + r)$. Given $\sigma' \in \text{Image } \Psi$, we want to find an upper bound on the number of $(\bar{\sigma}, (\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_r))$ in the domain of Ψ such that

$$\Psi(\bar{\sigma}, (\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_r)) = \sigma'. \tag{19}$$

For $i \in [r]$, let b_i be the number of left-to-right maxima of σ' in $\mathcal{J}_i^{\Psi}$. Observe that if Equation (19) holds, then $\lceil a_n \rceil + \bar{h}_i + (i-1)M + i - 1$ is a left-to-right maximum of σ' in $\mathcal{J}_i^{\Psi}$.

Also note that there are at most $\delta n + r$ left-to-right maxima of σ' in $(a_n, b_n + r)$. It follows that

$$|\Psi^{-1}(\sigma')| \leqslant \prod_{i=1}^{r} b_i \leqslant \left(\frac{\sum_{i=1}^{r} b_i}{r}\right)^r \leqslant \left(\frac{\delta n + r}{r}\right)^r.$$

Therefore

$$|S_n(\tau; \mathrm{LR}_n[I_n] < \delta n)| M^r = \sum_{\sigma' \in \text{Image } \Psi} |\Psi^{-1}(\sigma')| \leqslant |S_{n+r}(\tau)| \left(\frac{\delta n + r}{r}\right)^r,$$

and hence

$$|S_n(\tau; \mathrm{LR}_n[I_n] < \delta n)| \leqslant L(\tau)^{n+r} \left(\frac{\delta n + r}{Mr}\right)^r.$$

Since $\dfrac{r_n}{n} \to \dfrac{\alpha_2 - \alpha_1}{M}$ as $n \to \infty$, it follows that

$$\limsup_{n\to\infty} |S_n(\tau; \mathrm{LR}_n[I_n] < \delta n)|^{1/n} \leqslant L(\tau) \left(\frac{\delta L(\tau)}{\alpha_2 - \alpha_1} + \frac{L(\tau)}{M}\right)^{(\alpha_2-\alpha_1)/M} < L(\tau).$$

Then Theorem 11 follows. □

Theorem 10 and Equation (18) imply the following.

Corollary 13. *Consider $S_n(\tau)$ with $\tau = k\tau_2 \cdots \tau_k \in S_k$. Then*

$$\liminf_{n\to\infty} \frac{\mathbb{E}_n^\tau(\mathrm{LIS}_n)}{n} \geqslant \frac{1}{L(\tau)}.$$

Proof of Corollary 13. Recall from Equation (18) that $\mathrm{LIS}_n(\sigma) \geqslant \mathrm{LR}_n(\sigma)$ for every $\sigma \in S_n$. Therefore for any $0 < \delta < \frac{1}{L(\tau)}$,

$$\mathbb{E}_n^\tau(\mathrm{LIS}_n) \;\geqslant\; \mathbb{E}_n^\tau(\mathrm{LR}_n) \;\geqslant\; \mathbb{E}_n^\tau(\mathrm{LR}_n\, 1_{\mathrm{LR}_n \geqslant \delta n}) \;\geqslant\; \delta n\, \frac{|S_n(\tau; \mathrm{LR}_n \geqslant \delta n)|}{|S_n(\tau)|}\,.$$

Then the result follows from Theorem 10. □

The next result provides a sharp contrast with Theorem 10 in the case $\tau = k\tau_2 \cdots \tau_k$. Thus it is of some independent interest, even though it tells us nothing about longest increasing subsequences.

Theorem 14. *Assume $\tau \in S_k$ and satisfies at least one of the following conditions:*

(i) $\tau_1 < \tau_2$, or
(ii) k occurs to the right of $k-1$ in τ.

Then

$$\liminf_{n\to\infty} \mathbb{E}_n^\tau(\mathrm{LR}_n) \leqslant L(\tau). \tag{20}$$

If also $k \leqslant 5$ or if $\tau_1 = 1$ or if $\tau_k = k$, then we can replace the above "lim inf" by "lim sup."

We shall use the following result in the proof of Theorem 14. For its proof, see Theorem 6.4, Remark 2, and Theorem 7.1 of [4]. Presumably the result (21) holds for every τ, but the proof remains elusive.

Theorem 15 ([4])**.** *For every τ in S_k with $k \leqslant 5$ or when τ_1 (or τ_k) equals 1 or k, we have*

$$\lim_{n\to\infty} \frac{|S_{n+1}(\tau)|}{|S_n(\tau)|} = L(\tau)\,. \tag{21}$$

Proof of Theorem 14. First assume that condition (ii) holds, i.e. that k occurs to the right of $k-1$ in τ.

For each $\sigma \in S_n$ and $i = 1, \cdots, n+1$, let σ^i be the permutation in S_{n+1} obtained by inserting $n+1$ between the $(i-1)^{th}$ and i^{th} positions of σ. That is, $\sigma^i = \sigma_1 \cdots \sigma_{i-1}(n+1)\sigma_i \cdots \sigma_n$.

For a given $\sigma \in S_n(\tau)$, define

$$\mathrm{W}_n(\sigma) = \text{ the number of values of } i \text{ for which } \sigma^i \in S_{n+1}(\tau).$$

Let $\sigma \in S_n(\tau)$ and suppose σ_i is a left-to-right maximum. Suppose that σ^i contains a subsequence that forms the pattern τ. Then the following must be true:

- For some $j < i$, σ_j and $n+1$ must be the second-largest and largest entries, respectively, in this subsequence.
- This subsequence cannot include σ_i, because σ_i is a left-to-right maximum and thus $\sigma_j < \sigma_i$.

But then replacing $n+1$ with σ_i in this subsequence of σ^i would produce a subsequence of σ that forms the pattern τ, which is a contradiction. Hence $\sigma^i \in S_{n+1}(\tau)$.

By the above argument, for any $\sigma \in S_n(\tau)$ we have

$$\mathrm{LR}_n(\sigma) \leqslant \mathrm{W}_n(\sigma). \tag{22}$$

As argued in Lemma 2.1 of [18], we note that for each $\rho \in S_{n+1}(\tau)$, there is a unique $\sigma \in S_n(\tau)$ and a unique $i \in [n+1]$ such that $\sigma^i = \rho$. It follows that

$$\sum_{\sigma \in S_n(\tau)} \mathrm{W}_n(\sigma) = S_{n+1}(\tau).$$

Dividing both sides by $|S_n(\tau)|$ gives us

$$\mathbb{E}_n^\tau(\mathrm{W}_n) = \frac{|S_{n+1}(\tau)|}{|S_n(\tau)|}. \tag{23}$$

From [3], we also have

$$L(\tau) := \lim_{n\to\infty} |S_n(\tau)|^{1/n} = \sup_{n \geqslant 1} |S_n(\tau)|^{1/n}. \tag{24}$$

Hence $\liminf_{n\to\infty} \mathbb{E}_n^\tau(\mathrm{W}_n) \leqslant L(\tau)$ by Equations (23) and (24). Together with Equation (22), this proves Equation (20). The final statement of Theorem 14 follows from Theorem 15.

This concludes the proof of Theorem 14 under condition (*ii*). The theorem under condition (*i*) will follow upon applying the bijection $B : \sigma \mapsto (\sigma^{-1})^{rc}$, which corresponds to reflection through the decreasing diagonal. We only require two observations: firstly, that τ satisfies condition (*i*) if and only if $B(\tau)$ satisfies condition (*ii*), and secondly that $\mathrm{LR}_n(\sigma) = \mathrm{LR}_n(B(\sigma))$ for every $\sigma \in S_n$ (indeed, $j = \sigma_i$ is a left-to-right maximum of σ if and only if $(B(\sigma))_{n+1-j} = n+1-i$ is a left-to-right maximum of $B(\sigma)$).

In particular, for $k = 4$, we conclude that $\mathbb{E}_n^\tau(\mathrm{LR}_n)$ is of order n when $\tau_1 = 4$, and is bounded for every other $\tau \in S_4$ except perhaps 2143. $\square$

For all nontrivial cases of pattern-avoidance that we have considered, the longest monotone subsequence (i.e., the maximum of LIS_n and LDS_n) is of order n on average, which contrasts with the order $\sqrt{n}$ that holds for the set of all permutations. We don't yet know whether this order n behaviour holds in general under pattern avoidance, but this hypothesis is consistent with our limited simulation experiments so far. For example, Figure 6 summarizes some simulation results on the length of the longest increasing

n	$\mathbb{E}_n^{2413}(\mathrm{LIS}_n)$	$\mathbb{E}_n^{2413}(\mathrm{LIS}_n)/n$
75	22.3425 ± 0.3783	0.2979 ± 0.0050
100	29.1375 ± 0.5336	0.2914 ± 0.0053
125	35.0050 ± 0.5866	0.2800 ± 0.0047
150	39.9825 ± 0.7321	0.2665 ± 0.0049
175	46.4550 ± 0.8172	0.2655 ± 0.0047
200	51.5350 ± 0.8890	0.2577 ± 0.0044
225	56.3400 ± 1.0091	0.2504 ± 0.0045
235	59.3122 ± 1.0020	0.2524 ± 0.0043
250	62.9425 ± 1.1173	0.2518 ± 0.0045

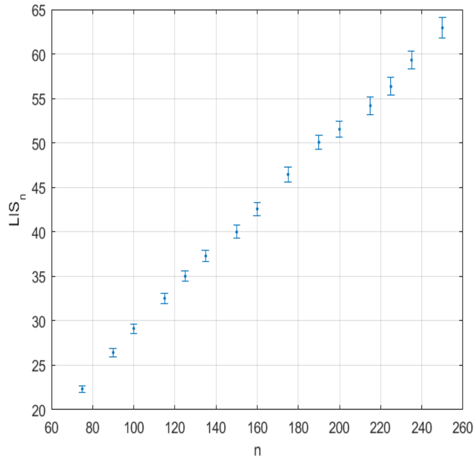


Figure 6: 95% confidence interval on $\mathbb{E}_n^{2413}(\mathrm{LIS}_n)$ and $\mathbb{E}_n^{2413}(\mathrm{LIS}_n)/n$ for permutations avoiding 2413. For each n, we used a sample of 400 (approximately independent) permutations generated by the Markov Chain Monte Carlo method of [18].

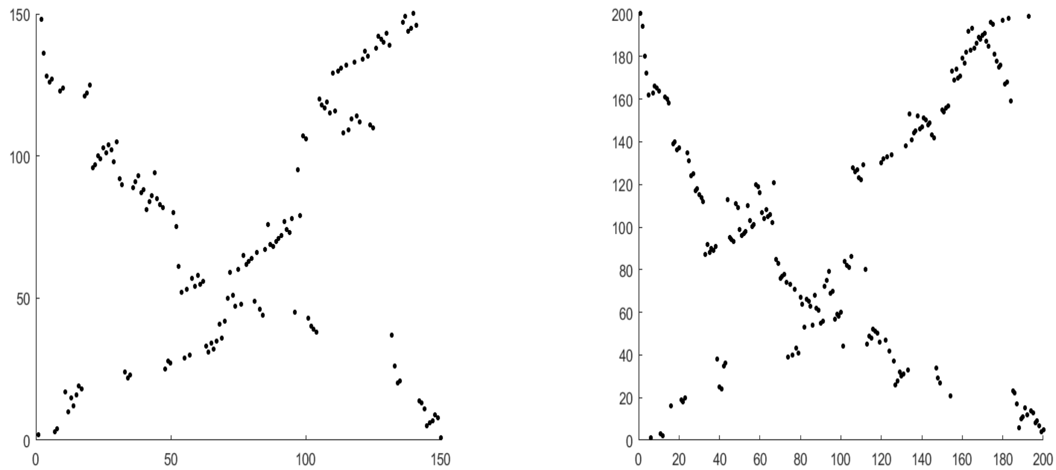


Figure 7: Examples of randomly generated 2413-avoiding permutations in $S_{150}(2413)$ and $S_{200}(2413)$. These are reminiscent of plots given in [6] of separable permutations, $S_n(2413, 3142)$, which appears to be a more tractable class.

subsequence for random permutations in $S_n(2413)$. The data suggest that $\mathbb{E}_n^{2413}(\mathrm{LIS}_n)/n$ converges to a number between 0.2 and 0.25.

As described in Section 2.2, it is known that the expected length of the longest alternating subsequence is either bounded or else asymptotically proportional to $n/2$, for all cases of $S_n(T)$ with T consisting of one or two patterns of length 3. We do not know whether this extends to all longer patterns, but we can prove the following.

Theorem 16. *Consider $S_n(\tau)$ with $\tau \in S_k$ where $k \geqslant 4$. Assume that either* *(a)* $\tau_1 = k$ *and* $\tau_2 \neq k-1$, *or*

(b) $|\tau_i - \tau_{i+1}| > 1$ *for every* $i \in [k-1]$.

Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}_n^{\tau}(\mathrm{LAS}_n)}{n} \;\geqslant\; \frac{2}{L(\tau)^2}\,.$$

In particular, Theorem 16 proves that $\mathbb{E}_n^{\tau}(\mathrm{LAS}_n)/n$ is bounded away from 0 for the patterns $\tau = 4231$ (part (a)) and $\tau = 2413$ (part (b)). Permutations which satisfy the condition in part (b) are called 1-*prolific* in [9].

The strategy of the proof is similar to that of Theorem 11.

Proof of Theorem 16. (a) Assume $\tau \in S_k$ with $\tau_1 = k$ and $\tau_2 \neq k-1$. It suffices to prove that for every positive real $\delta < L(\tau)^{-2}$, we have

$$\limsup_{n\to\infty} |S_n(\tau; \mathrm{LAS}_n < 2\delta n)|^{1/n} \;<\; L(\tau). \tag{25}$$

Suppose $\sigma \in S_n$ and $h \in [n]$. Let $J = \min\{j : \sigma_j \geqslant h\}$. Define the permutation $\theta \in S_{n+2}$ as follows:

$$\theta_i = \begin{cases} \sigma_i & \text{if } i < J \\ h+1 & \text{if } i = J \\ h & \text{if } i = J+1 \\ \sigma_{i-2} & \text{if } i \geqslant J+2 \text{ and } \sigma_{i-2} < h \\ \sigma_{i-2}+2 & \text{if } i \geqslant J+2 \text{ and } \sigma_{i-2} \geqslant h \end{cases}$$

We denote θ by $\mathcal{I}_2(\sigma; h)$. Think of $\mathcal{I}_2(\sigma; h)$ as inserting two new points into σ in a 21 pattern, adjacent in location as well as in height, with the left point being a new left-to-right maximum at height $h+1$.

Assume $1 \leqslant h_1 < h_2 < \cdots < h_r \leqslant n$. Let $\sigma^{(r)} = \sigma$ and let $\sigma^{(l-1)} = \mathcal{I}_2(\sigma^{(l)}; h_l)$ for $l = r, r-1, \cdots, 1$. Then we denote $\sigma^{(0)}$ by $\mathcal{I}_2(\sigma; h_1, h_2, \cdots, h_r)$. It is not hard to see that $\sigma^{(0)}$ avoids τ, and that $\sigma^{(0)}$ has left-to-right maxima with heights $h_i + 2i - 1$ $(i \in [r])$.

Fix $0 < \delta < L(\tau)^{-2}$ and choose a positive integer M such that $\delta + \frac{1}{M} < L(\tau)^{-2}$. Let $r \equiv r_n := \lfloor n/M \rfloor$.

For each $i \in [r]$, we define $\mathcal{J}_i = ((i-1)M, iM]$ (the *"intervals of heights"*).

For each permutation $\sigma \in S_n$, define the set of heights

$$\mathcal{A}^*(\sigma) \;:=\; \{\sigma_t : \; \sigma_t \text{ is a left-to-right maximum and } \sigma_{t+1} = \sigma_t - 1\}\,.$$

E.g., $\mathcal{A}^*(254367981) = \{5, 9\}$. Then it follows that $\mathrm{LAS}_n(\sigma) \geqslant 2|\mathcal{A}^*(\sigma)|$. Next, let

$$S_n^*(\tau) \;:=\; \{\sigma \in S_n(\tau) : |\mathcal{A}^*(\sigma)| < \delta n\}\,.$$

Then we have

$$S_n(\tau; \mathrm{LAS}_n < 2\delta n) \;\subset\; S_n^*(\tau)\,. \tag{26}$$

Define the function $\Psi_2 \equiv \Psi_{2,n}$ as

$$\Psi_2 : S_n^*(\tau) \times [M]^r \to S_{n+2r}(\tau)$$

such that $\Psi_2(\sigma, (\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_r)) = \mathcal{I}_2(\sigma; h_1, h_2, \cdots, h_r)$, where $h_i = \hat{h}_i + (i-1)M$ for $i \in [r]$ (so that $h_i \in \mathcal{J}_i$). We also define the set of shifted intervals of heights as follows:

$$\mathcal{J}_i^\Psi \; := \; \mathcal{J}_i + 2i - 1 \; = \; ((i-1)(M+2)+1, i(M+2)-1], \quad i = 1, \cdots, r.$$

Then $\mathcal{J}_i^\Psi, \ldots, \mathcal{J}_r^\Psi$ are disjoint intervals in $[1, n+2r]$.

Given $\sigma' \in \text{Image } \Psi_2$, we would like to find an upper bound on the number of $(\bar{\sigma}, (\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_r))$ in the domain of Ψ_2 such that

$$\Psi_2(\bar{\sigma}, (\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_r)) = \sigma'. \tag{27}$$

For $i \in [r]$, let $b_i^* = |\mathcal{A}^*(\sigma') \cap \mathcal{J}_i^\Psi|$. Observe that if Equation (27) holds, then $\bar{h}_i + (i-1)M + (2i-1) \in \mathcal{A}^*(\sigma') \cap \mathcal{J}_i^\Psi$. Note that $|\mathcal{A}^*(\sigma')| \leqslant \delta n + r$. Therefore

$$|\Psi_2^{-1}(\sigma')| \; \leqslant \; \prod_{i=1}^r b_i^* \; \leqslant \; \left(\frac{\sum_{i=1}^r b_i^*}{r} \right)^r \; \leqslant \; \left(\frac{\delta n + r}{r} \right)^r.$$

Therefore,

$$|S_n^*(\tau)| \, M^r \; = \sum_{\sigma' \in \text{Image } \Psi_2} |\Psi_2^{-1}(\sigma')| \; \leqslant \; |S_{n+2r}(\tau)| \left(\frac{\delta n + r}{r} \right)^r,$$

and hence

$$|S_n^*(\tau)| \; \leqslant \; L(\tau)^{n+2r} \left(\frac{\delta n + r}{Mr} \right)^r.$$

It follows that

$$\limsup_{n\to\infty} |S_n^*(\tau)|^{1/n} \; \leqslant \; L(\tau) \left(L(\tau)^2 \left(\delta + \frac{1}{M} \right) \right)^{1/M} \; < \; L(\tau).$$

Equation (25) follows from this and Equation (26), and the result of part (a) is proved.

(*b*) Assume $\tau \in S_k$ and $|\tau_i - \tau_{i-1}| > 1$ for every $i \in [k-1]$. Given $\sigma \in S_n(\tau)$ and $h \in [n]$, let $\mathcal{H}^*(\sigma; h)$ be the permutation θ defined as follows. Let J be the index such that $\sigma_J = h$ (note the distinction from part (a) here), and let

$$\theta_i = \begin{cases} \sigma_i & \text{if } i \leqslant J \text{ and } \sigma_i \leqslant h \\ \sigma_i + 2 & \text{if } i \leqslant J \text{ and } \sigma_i > h \\ h+2 & \text{if } i = J+1 \\ h+1 & \text{if } i = J+2 \\ \sigma_{i-2} & \text{if } i > J+2 \text{ and } \sigma_{i-2} \leqslant h \\ \sigma_{i-2}+2 & \text{if } i > J+2 \text{ and } \sigma_{i-2} > h \end{cases}$$

Then $\theta \in S_{n+2}(\tau)$. In effect, $\mathcal{H}^*$ inserts two points into σ to the right of location J, so that σ_J $(= \theta_J)$ and the two new points (θ_{J+1} and θ_{J+2}) form a 132 pattern with three contiguous heights. As in part (a), assume $1 \leqslant h_1 < h_2 < \cdots < h_r \leqslant n$. Let $\sigma^{(r)} = \sigma$ and let $\sigma^{(l-1)} = \mathcal{H}^*(\sigma^{(l)}; h_l)$ for $l = r, r-1, \cdots, 1$. Then we denote $\sigma^{(0)}$ by $\mathcal{H}^*(\sigma; h_1, h_2, \cdots, h_r)$.

For each permutation σ, we define the set of heights

$$\mathcal{B}^*(\sigma) \;:=\; \{\sigma_t \,:\, \sigma_t = \sigma_{t+1} - 2 = \sigma_{t+2} - 1\}\,.$$

Then $\mathrm{LAS}_n(\sigma) \;\geqslant\; 2|\mathcal{B}^*(\sigma)|$. Arguing as in part (a) with the same choice of δ, M, and r, we obtain

$$|S_n(\tau\,;\,|\mathcal{B}^*(\sigma)| < \delta n)| \;\leqslant\; L(\tau)^{n+2r}\left(\frac{\delta n + r}{Mr}\right)^r$$

and the result of part (b) follows. □

2.4 Rare regions for $S_n(\tau)$.

As noted in Section 1.3, plots of random permutations avoiding some patterns have large regions that are usually empty (see Figure 2). The results of this subsection relate to these regions.

Without loss of generality, we shall assume for the rest of this subsection that $\tau \in S_k$ and $\tau_1 > \tau_k$. Recall from Section 1.3 that the "rare region" $\mathcal{R} \equiv \mathcal{R}(\tau)$ is the set

$$\mathcal{R} = \{(x,y) \in [0,1]^2 : \text{ for all sequences } \{(I_n, J_n)\}_{n\geqslant 1} \text{ such that } (I_n, J_n) \in [n]^2 \text{ and}$$
$$\lim_{n\to\infty}\left(\frac{I_n}{n}, \frac{J_n}{n}\right) = (x,y), \text{ we have } \limsup_{n\to\infty} |S_n(\tau; \sigma_{I_n} = J_n)|^{1/n} < L(\tau)\},$$

We also defined $\mathcal{G} = [0,1]^2 \setminus \mathcal{R}$ (the "good region") and $\mathcal{R}^{\uparrow} = \mathcal{R} \cap \{(x,y) \in [0,1]^2 : y > x\}$. For every $x \in [0,1]$, let

$$r^{\uparrow}(x) \;=\; \sup\{y : (x,y) \in \mathcal{G}\} \quad \text{ and } \quad r^{\downarrow}(x) \;=\; \inf\{y : (x,y) \in \mathcal{G}\}\,.$$

(We shall see that $r^{\uparrow}$ and $r^{\downarrow}$ are well-defined functions since the set $\{y : (x,y) \in \mathcal{G}\}$ is never empty.)

By Theorem 1, we know that $\mathcal{R}^{\uparrow} \neq \varnothing$ if and only if $\tau_1 = k$; in particular, when $\tau_1 \neq k$, then $r^{\uparrow}$ is identically 1. (Similarly, $r^{\downarrow}$ is identically 0 when $\tau_k \neq 1$.) The other case, in which $\tau_1 = k$, is addressed in the following theorem.

Theorem 17. *Assume $\tau = k\tau_2 \cdots \tau_k \in S_k$.*

(a) If $(x,y) \in \mathcal{G}$, then the convex hull of $\{(x,y), (0,0), (1,1)\}$ is contained in $\mathcal{G}$. In particular, $\mathcal{G}$ contains the diagonal $\{(x,x) : x \in [0,1]\}$.

(b) $\mathcal{G} \subset \{(x,y) \in [0,1]^2 : y \leqslant L(\tau)x$ and $y \leqslant 1 - (1-x)/L(\tau)\}$.

(c) The function $r^{\uparrow}$ satisfies $r^{\uparrow}(0) = 0$, $r^{\uparrow}(1) = 1$, and $r^{\uparrow}(x) \geqslant x$ for every $x \in (0,1)$.

(d) *The function $r^\uparrow$ is strictly increasing and Lipschitz continuous, with left and right derivatives contained in the interval $[L(\tau)^{-1}, L(\tau)]$ at every point.*

Theorem 17(b) implies that for every $\tau \in S_k$ with $\tau_1 = k$, the closure of $\mathcal{R}^\uparrow(\tau)$ includes the point $(0,0)$. This had been proven in Proposition 9.2 of [4] under an additional assumption (called "TPIP") on $\tau_2 \cdots \tau_k$. Theorem 17(b) shows that this additional assumption is unnecessary.

Proof of Theorem 17. (a) Note that $\{(0,0),(1,1)\} \subset \mathcal{G}$ since $|S_n(\tau; \sigma_1 = 1)| = |S_{n-1}(\tau)|$ and $|S_n(\tau; \sigma_n = n)| = |S_{n-1}(\tau)|$ for all $n \geqslant 1$.

Let $(x,y) \in \mathcal{G}$. Then there exist $I_n, J_n \in [n]$ such that $(\frac{I_n}{n}, \frac{J_n}{n}) \to (x,y)$ and

$$\limsup_{n\to\infty} |S_n(\tau; \sigma_{I_n} = J_n)|^{\frac{1}{n}} \;=\; L(\tau).$$

Let $t \in (0,1)$ and let $m := m_n$ be a sequence of integers such that $\frac{n}{n+m} \to t$. Then it follows that $(\frac{I_n}{n+m}, \frac{J_n}{n+m}) \to (tx, ty)$ and

$$|S_{n+m}(\tau; \sigma_{I_n} = J_n)|^{\frac{1}{n+m}} \;\geqslant\; |S_n(\tau; \sigma_{I_n} = J_n)|^{\frac{1}{n}\frac{n}{n+m}} |S_m(\tau)|^{\frac{1}{m}\frac{m}{m+n}}$$

(the inequality is proved using the injection $S_n(\tau) \times S_m(\tau) \hookrightarrow S_{n+m}(\tau)$ given by the direct sum (defined in Section 1.4)).

Hence we have

$$\limsup_{n\to\infty} |S_{n+m}(\tau; \sigma_{I_n} = J_n)|^{\frac{1}{n+m}} \;\geqslant\; L(\tau)^t L(\tau)^{1-t} \;=\; L(\tau).$$

Therefore $l_1 := \{(0,0) + t(x,y) : 0 < t < 1\} \subset \mathcal{G}$. By a similar argument, we can show that $l_2 := \{(1-t)(1,1) + t(x,y) : 0 < t < 1\} \subset \mathcal{G}$ and $l_3 := \{(0,0) + t(1,1) : 0 < t < 1\} \subset \mathcal{G}$. Since any point in the convex hull of $\{(x,y),(0,0),(1,1)\}$ can be written as a linear combination of $(0,0)$ and a point from l_2, the result follows.

(b) Assume $(x,y) \in [0,1]^2$ and $y > L(\tau)x$. Assume that $\left(\frac{I_n}{n}, \frac{J_n}{n}\right) \to (x,y)$. If $\sigma_I = J$, then every left-to-right maxima σ_i with $\sigma_i < J$ must satisfy $i < I$. Therefore,

$$S_n(\tau; \sigma_{I_n} = J_n) \;\subset\; S_n(\tau; \mathrm{LR}_n([1, J_n]) < I_n).$$

In Theorem 11, let $\alpha_1 = 0, \alpha_2 = y$, and choose δ such that $x < \delta < \frac{y}{L(\tau)}$. Then $S_n(\tau; \sigma_{I_n} = J_n) \subset S_n(\tau; \mathrm{LR}_n([1, J_n]) < \delta n)$ for all sufficiently large n. Therefore, by Theorem 11, we see that $(x,y) \in \mathcal{R}$. Therefore $\mathcal{G} \subset \{(x,y) \in [0,1]^2 : y \leqslant L(\tau)x\}$.

To show the rest of (b), consider the mapping $(x,y) \mapsto (1-y, 1-x)$, which is the reflection through the line $x + y = 1$. This reflection corresponds to the bijection $\sigma \mapsto (\sigma^{-1})^{rc}$. Letting $\tilde{\tau} = (\tau^{-1})^{rc}$, we have $L(\tilde{\tau}) = L(\tau)$ and $\tilde{\tau}_1 = k$. Now suppose $(x,y) \in \mathcal{G}(\tau)$. Then $(1-y, 1-x) \in \mathcal{G}(\tilde{\tau})$, and hence $(1-x) \leqslant L(\tilde{\tau})(1-y)$ by the preceding paragraph. Therefore $y \leqslant 1 - (1-x)/L(\tau)$. This completes the proof of (b).

(c) Since $(0,0) \in \mathcal{G}$, part (b) implies that $r^\uparrow(0) = 0$. The rest of part (c) follows from part (a).

(d) Strict monotonicity follows from the convex hull property of part (a), together with the fact (a consequence of (b)) that $r^{\uparrow}(x) < 1$ for every $x \in (0,1)$. Next, observe that if $0 < u < v < 1$, then the convex hull property implies that the point $(u, r^{\uparrow}(u))$ cannot be below the line segment joining $(0,0)$ to $(v, r^{\uparrow}(v))$. That is, $r^{\uparrow}(v)/v \leqslant r^{\uparrow}(u)/u$, and so

$$\frac{r^{\uparrow}(v) - r^{\uparrow}(u)}{v-u} \;\leqslant\; \frac{r^{\uparrow}(u)\left(\frac{v}{u}\right) - r^{\uparrow}(u)}{v-u} \;=\; \frac{r^{\uparrow}(u)}{u} \;\leqslant\; L(\tau)$$

(where the final inequality follows from part (b)). This proves the upper bound on the derivatives. The lower bound follows using the reflection argument from part (b). □

Theorem 18 below holds for any pattern τ. Together with Theorem 17, it shows that the graph of $r^{\uparrow}$ is precisely the boundary between $\mathcal{R}^{\uparrow}$ and $\mathcal{G}$ (when $\mathcal{R}^{\uparrow}$ is not empty).

Theorem 18. *Let $\tau \in S_k$. Then $\mathcal{G} = \{(x,y) \in [0,1]^2 : r^{\downarrow}(x) \leqslant y \leqslant r^{\uparrow}(x)\}$.*

Proof of Theorem 18. Without loss of generality, assume $\tau_1 > \tau_k$. It suffices to show that for every $(x,y) \in [0,1]^2$ with $y \geqslant x$, we have $(x,y) \in \mathcal{G}$ if and only if $y \leqslant r^{\uparrow}(x)$. We already know this from the discussion preceding Theorem 17 when $\tau_1 \neq k$, so assume $\tau_1 = k$. The desired result will follow from the convex hull property of Theorem 17(a) if we can prove that the point $(x, r^{\uparrow}(x))$ is in $\mathcal{G}$ for every $x \in (0,1)$. We shall accomplish this by proving that $\mathcal{G}$ is closed.

Proving that $\mathcal{G}$ is closed is essentially an exercise in analysis. Here are the details. Consider a point (x,y) in the closure of $\mathcal{G}$. It suffices to construct a strictly increasing sequence of natural numbers $n(1), n(2), \ldots$ and a sequence of pairs of integers $\{(i_{n(m)}, j_{n(m)}) : m \geqslant 1\}$ such that $(i_{n(m)}, j_{n(m)}) \in [n(m)]^2$ for every m, $\lim_{m\to\infty}(i_{n(m)}, j_{n(m)})/n(m) = (x,y)$, and $\lim_{m\to\infty} |S_{n(m)}(\tau; \sigma_{i_{n(m)}} = j_{n(m)})|^{1/n(m)} = L(\tau)$. We shall construct the sequences inductively. Let $n(1) = 1 = i_1 = j_1$. Given $m > 1$ and $n(m-1) \in \mathbb{N}$, choose $(x(m), y(m)) \in \mathcal{G}$ such that $||(x(m), y(m)) - (x,y)||_1 < 1/m$. Then there exists a sequence $\{(I_n(m), J_n(m)) : n \geqslant 1\}$ such that $(I_n(m), J_n(m)) \in [n]^2$ for every n,

$$\lim_{n\to\infty}\left(\frac{I_n(m)}{n}, \frac{J_n(m)}{n}\right) \;=\; (x(m), y(m)), \quad \text{and}$$
$$\limsup_{n\to\infty} |S_n(\tau; \sigma_{I_n(m)} = J_n(m))|^{1/n} \;=\; L(\tau).$$

Therefore we can choose $n' > n(m-1)$ such that

$$|S_{n'}(\tau; \sigma_{I_{n'}(m)} = J_{n'}(m))|^{1/n'} \;\geqslant\; L(\tau) - \frac{1}{m} \quad \text{and}$$
$$\left\| \left(\frac{I_{n'}(m)}{n'}, \frac{J_{n'}(m)}{n'}\right) - (x(m), y(m)) \right\|_1 \;<\; \frac{1}{m}\,.$$

Let $n(m) = n'$, $i_{n(m)} = I_{n'}(m)$, and $j_{n(m)} = J_{n'}(m)$. One can now check that these sequences have the desired properties. Hence $(x,y) \in \mathcal{G}$. Thus $\mathcal{G}$ is closed, and the theorem follows. □

It is known that $r^{\uparrow}$ is the identity function $r^{\uparrow}(x) = x$ for the monotone pattern $\tau = k(k-1)\cdots 1$, as well as for some other patterns [19]. So far, there is no pattern $\tau \in S_k$ with $\tau_1 = k$ for which we can prove that $r^{\uparrow}$ is not the identity function. Simulations are not yet clear about whether 4231 is one such pattern; see Figure 2 as well as [8]. We conjecture that $r^{\uparrow}$ is always a concave function.

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