Covers of D-type Artin groups

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Abstract

We study certain quotients of generalized Artin groups which have a natural map onto D-type Artin groups, where the generalized Artin group $A(T)$ is defined by a signed graph T. Then we find a certain quotient $G(T)$ according to the graph T, which also have a natural map onto $A(D_n)$. We prove that $G(T)$ is isomorphic to a semidirect product of a group $K^{(m,n)}$, with the Artin group $A(D_n)$, where $K^{(m,n)}$ depends only on the number m of cycles and on the number n of vertices of the graph T.

Keywords: Artin groups, signed graphs

1 Introduction

Coxeter and Artin groups are used in many areas in mathematics, such as those dealing with reflections, symmetries, classification of Lie Algebras, associated to Dynkin diagrams

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which also appear in families of finite simple groups, computations in algebraic geometry and many others.

The structure of Coxeter and Artin groups is very interesting and these groups are defined in a very easy way, in terms of generators and relations. These groups can be described easily by diagrams which are called Dynkin diagrams, and the groups have interesting properties in terms of group theory, such as the cancelation property (see [11], [14]).

Our research involving Coxeter and Artin groups began helping to face algebraic difficulties in classification of algebraic surfaces, and the motivation of this paper comes from algebraic geometry. The difficulties appeared initially when it has been tried to classify the surface $\mathcal{T} \times \mathcal{T}$ (for $\mathcal{T}$ a complex torus). The fundamental group $\pi_1(\mathbb{CP}^2 - S)$ of the branch curve S of this surface in $\mathbb{CP}^2$ had 54 generators and admitted roughly 2000 relations. The goal was to compute the fundamental group of the Galois cover of the surface $\mathcal{T} \times \mathcal{T}$ (a certain quotient of the group $\pi_1(\mathbb{CP}^2 - S)$). For this, the presentation of the group $\pi_1(\mathbb{CP}^2 - S)$ has been simplified, and considered its quotient over the squares of the generators and to map it onto the symmetric group S_n. Since this structure was very complicated, the paper [19] offered a nice solution by an insight into the world of Coxeter groups.

Firstly, this solved the problem which has been in the case of the $\mathcal{T} \times \mathcal{T}$ surface, see [8] and [10]. Secondly, the results of the paper [19] were very interesting from the algebraic point of view, therefore there are a series of papers which generalize the results to other types of Coxeter groups and Artin groups (see, e.g., [6], [4]), and extended the results to other algebraic surfaces and applications (see, e.g., [9], [5]). In Section 2 we give a detailed review of all these papers.

Our paper is in fact the fourth in a series of papers (the other three are [19], [6] and [4]). In [4], there is a generalization of [19], A-type Artin groups (braid groups) instead of A-type Coxeter groups (symmetric groups) are considered. In this paper, we generalize [4], i.e., we deal with D-type Artin groups instead of dealing with A-type Artin groups. So, we use the definitions, theorems and results given there.

In our Main Theorem (Theorem 31), we prove the following. Given a signed graph T with n vertices, we define an Artin group $A(T)$. There exists a quotient $G(T)$ of $A(T)$, which depends on the graph T, such that $G(T) \simeq K^{(m,n)} \rtimes A(D_n)$ (the group $A(D_n)$ is the D-type Artin group with n generators), where the group $K^{(m,n)}$ depends only on the number n of vertices and the number m of cycles in T.

Although the structure of $G(T)$ in this paper is similar to the structure of $G(T)$ defined in [4], our paper is innovative, since the use of signed graphs and mapping onto $A(D_n)$ instead of mapping onto Br_n (braid groups with n generators) allows us to deal with a much wider class of simply laced Artin groups than those in [4].

The goal of this paper is to find a structure for certain quotients of Artin groups which has a natural map onto the finite type simply laced Artin group $A(D_n)$, such that through that structure the word problem is solvable in that certain quotient.

The ultimate goal of this series of papers is to find general structures for certain quotients of Coxeter and Artin groups, in a way that will make it easy to solve the word problem through natural maps onto Coxeter or Artin groups.

The paper is divided as follows. Section 2 reviews the scientific background which relates to our paper. We sketch all the results of the related papers. Moreover, we present the setup of our paper and state the main result. In Section 3, we define the group $A(T)$ which we obtain from the signed graph T. Then we define $A_Y(T)$ as a quotient of $A(T)$, which depends on certain configurations of subgraphs of T. We describe the basic properties of $A_Y(T)$. In section 4, we recall the techniques and the main results of the paper [4], which we use in the proofs of the Theorems of this paper. In Section 5, we consider a graph T that contains only one cycle and only one anti-cycle. Then we define $G(T)$ as a quotient of $A_Y(T)$. Using the properties of $A_Y(T)$ from Section 3, we prove that $G(T) \simeq K \rtimes A(D_n)$, where K is a group whose structure is described in the section. Section 6 generalizes Section 5 to a signed graph T with arbitrary number of cycles and anti-cycles. We prove the Main Theorem (Theorem 31): $G(T) \simeq K^{(m,n)} \rtimes A(D_n)$. We conclude that the word problem is solvable in $G(T)$.

2 Scientific background and motivation

The main goal of our work in algebraic geometry is the classification of algebraic surfaces. One of the invariants to classify surfaces is the fundamental group of the Galois cover of a surface. In general, we take a projective surface X, with a generic map of degree n to $\mathbb{CP}^2$, and S is its branch curve in $\mathbb{CP}^2$. There is a natural map from the fundamental group of the complement of the branch curve $\pi_1(\mathbb{CP}^2 - S)$ to the symmetric group S_n. The kernel of $\pi_1(\mathbb{CP}^2 - S) \to S_n$ is the fundamental group of the Galois cover of X over the squares of the generators, see [16].

The first works in this direction were done by Moishezon-Teicher in [16], [17], [18] and [15]. They worked with the $\mathbb{CP}^1 \times \mathbb{CP}^1$ surface, the Veronese surface and the Hirzebruch surface. The next step was to deal with the surface $\mathcal{T} \times \mathcal{T}$ for $\mathcal{T}$ a complex torus. The group $\pi_1(\mathbb{CP}^2 - S)$ was with 54 generators and about 2000 relations, see [7]. The computation of the kernel of $\pi_1(\mathbb{CP}^2 - S) \to S_n$ over the squares of the generators was very complicated, and it was simplified in the breakthrough paper [19], which found an easy description of the kernel which is denoted as $C_Y(T)$. The paper [19] describes the structure of a certain quotient of a Coxeter group $C(T)$, which is defined by a graph T and has a natural map onto S_n. Then there exists a certain quotient $C_Y(T)$ of $C(T)$. This $C_Y(T)$ is isomorphic to $A_{t,n} \rtimes S_n$, where $A_{t,n}$ is a group whose only invariants are t (the number of cycles of T) and n (the number of vertices of T). Since the word problem is solvable in $A_{t,n}$, it is also solvable in $C_Y(T)$.

In [6] we were motivated to generalize [19] to wider class of Coxeter groups $C(T)$. This paper deals with Coxeter groups that can be mapped onto B_n or D_n (the classical Coxeter groups). The graph T is generalized to a signed graph in which every edge is labeled either by $+1$ or by -1, and which may include loops. Similar signed graphs were introduced in [12]. The main theorem of [6] proves that there is a ceratin quotient $C_Y(T)$ of $C(T)$, which is isomorphic to $A_{t,n} \rtimes D_n$ or $A_{t,n} \rtimes B_n$, depending whether T contains loops, or not.

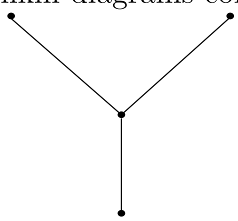
The paper [19] can be generalized in a different algebraic direction. The paper [4] generalizes Coxeter covers to Artin covers. The graph T defines an Artin group $A(T)$ (which means that the generators are not necessarily involutions). The main theorem of [4] proves that there exists a quotient $G(T)$ of $A(T)$ that is isomorphic to $K_{t,n} \rtimes Br_n$, where Br_n is the braid group with n strings, and $K_{t,n}$ is a group which is defined by the graph T and depends on t (the number of cycles in T) and on n (the number of vertices of T). Since the word problem is solvable in $K_{t,n}$, it is solvable in $G(T)$ as well.

The paper [19] gave us solutions in order to proceed in the work in algebraic geometry and to apply the techniques from this paper to algebraic surfaces. Firstly, we could solve the problem which was raised by the surface $\mathcal{T} \times \mathcal{T}$. In [8], we computed the Coxeter quotient of the fundamental group of a Galois cover of $\mathcal{T} \times \mathcal{T}$, and in [10], we computed the fundamental group of the Galois cover of the surface. Secondly, we extended the work to other algebraic surfaces such as $\mathbb{CP}^1 \times \mathcal{T}$ in [2] and [3], and some Hirzebruch surface in [9].

Another application to the above techniques is the paper [5]. In this paper, we survey the fundamental groups of Galois covers of the cubic embedding of the Hirzebruch surface F_1, the Cayley cubic (or a smooth surface in the same family), a quartic surface that degenerates to the union of a triple point and a plane not through the triple point, and a quartic 4-point. We also complete the classification of the surface $\mathbb{CP}^1 \times \mathbb{CP}^1$, which was begun in [16].

Our paper contains a combined generalization of [6] and [4] to Artin groups that have natural maps onto the simply laced finite type Artin group $A(D_n)$. According to [13], $A(D_n) \simeq F^{n-1} \rtimes Br_n$ and $A(B_n) \simeq F^n \rtimes Br_n$. In Coxeter groups, it is known that $D_n \leqslant B_n$, but $A(D_n)$ is not embeddable into $A(B_n)$. Therefore, it is impossible to combine Artin covers $A(B_n)$ and $A(D_n)$ as done in [6]. Hence, in this paper, we deal only with covers of $A(D_n)$.

This paper is in the spirit of [4], and we consider here the definitions, theorems and results given there. This paper deals with a wider class of groups, e.g., groups whose Dynkin diagrams contain subgraphs of the form



(e.g., $A(D_n)$).

In our paper, we define the group $A(T)$ that we obtain from the signed planar graph T. Then we define $A_Y(T)$ as a quotient of $A(T)$, where the configuration of the subgraphs from which the relations of $A_Y(T)$ arise, is similar to the one in [4]. First we consider a planar graph T that contains only one cycle and only one anti-cycle. Then we define $G(T)$ as a quotient of $A_Y(T)$ and prove that $G(T) \simeq K \rtimes A(D_n)$, where the structure of K is similar to the structure defined in [4]. Moreover, we prove that every signed graph T is equivalent to a signed graph $T'(m)$ in Figure 1.
Finally, we prove that $G(T) \simeq K^{(m,n)} \rtimes A(D_n)$, where n is the number of vertices of T and $m+1$ is the number of cycles and anti-cycles in T (T must include at least one anti-cycle).

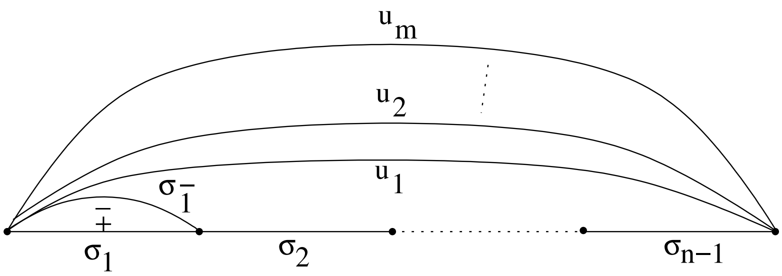


Figure 1

Since $G(T) \simeq K^{(m,n)} \rtimes A(D_n)$, the word problem in $G(T)$ is solvable. Our motivation is that we can deal easily with quotients of specific Artin groups.

3 The quotient $A_Y(T)$

We start with a definition of a signed planar graph T. Then by the graph T we define a generalized Artin group $A(T)$. After it, we construct $A_Y(T)$ which is the quotient of $A(T)$, which plays an important role in the main Theorem 31.

Definition 1. *We call a weighted planar graph T "a signed graph" if every edge of T contained in a cycle is signed by $+1$ or by -1.*

Example 2. *Here we give an example of a signed graph T, in which the edges are signed in each cycle by $+1$ or by -1. See Figure 2.*

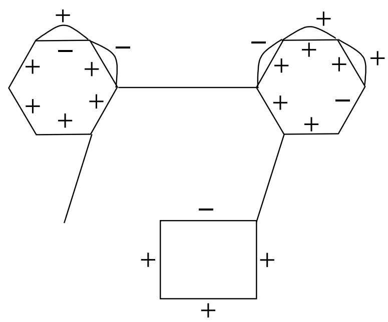
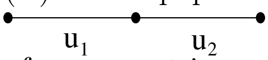


Figure 2

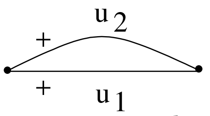
In this paper, the edges that are not contained in any cycle are not signed, and we may assume that the signs of all such edges are $+1$.

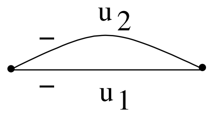
We denote by $s(e)$ the sign of the edge e. Let $A(T)$ be the generalized Artin group that corresponds to the graph T (i.e., $A(T)$ is generated by the edges of T). The relations in $A(T)$ in this paper are:



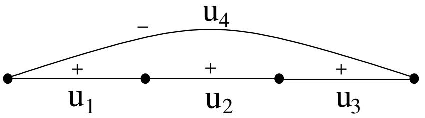
$< u_1, u_2 >= 1$ if u_1 and u_2 meet in a vertex. The signs of u_1 and of u_2 are not important, since the subgraph with the edges u_1 and u_2 does not form a cycle.

$[u_1, u_2] = 1$ if u_1 and u_2 are disjoint. The signs of u_1 and of u_2 are not important, since the subgraph with the edges u_1 and u_2 does not form a cycle.

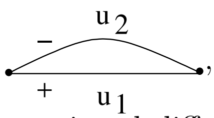




There is no relation between u_1 and u_2 if u_1 and u_2 connect the same two vertices and $s(u_1) = s(u_2)$.

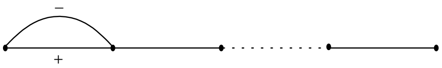


In the case of a cycle with odd number of negative signs, we have an additional relation: $[u_1^{-1} \dots u_{n-2}^{-1} u_{n-1} u_{n-2} \dots u_1, u_n] = 1$. We call such a cycle an *anti-cycle* (see [6]).

Note that an anti-cycle of length two has the form 

, where u_1 and u_2 are two edges that connect the same two vertices but are signed differently. Hence the induced relation is $[u_1, u_2] = 1$. Note also that the relation associated to an anti-cycle of length n does not depend on the enumeration of the edges.

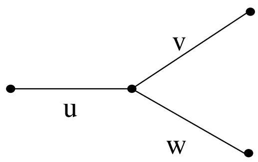
Remark 3. *In [4], each edge in a graph is considered as a positive signed edge. In this paper, there is a generalization to signed graphs, where each edge can be signed by + or -.*

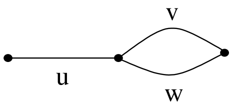
Remark 4. *The graph* 

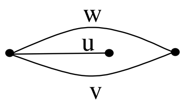
represents the finite type Artin group $A(D_n)$.

Now we define the quotient $A_Y(T)$.

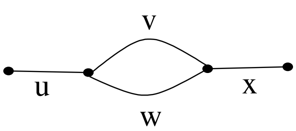
Definition 5. *Let T be a planar graph, $A_Y(T)$ is the quotient of $A(T)$ by the following relations (similar to the relations in [4] with an additional case):*

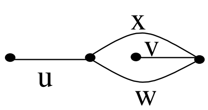
1. $[w^{-1}uw, v] = 1$ *if* u, v, w *as in* 

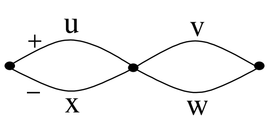
2. $\langle w^{-1}uw, v \rangle = 1$ *if* u, v, w *as in* 

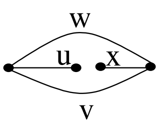


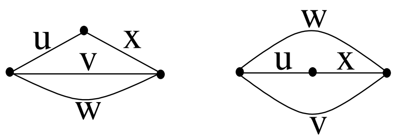
3. $[w^{-1}uw, v^{-1}xv] = 1$ *if* u, v, w, x *as in*









4. $\langle w^{-1}uw, v^{-1}xv \rangle = 1$ *if* u*,* v*,* w*,* x *as in* 

Now we define virtual edges.

Definition 6. *Let x and y be paths in a signed planar graph T, such that x and y intersect in no more than one point. Then we define a new edge $x \cdot y$, called a* virtual edge*, as:*

1. $x \cdot y = y$ *if x and y do not intersect,*
2. $x \cdot y$ *is a virtual edge if x and y intersect in one vertex (See Fig. 3).*

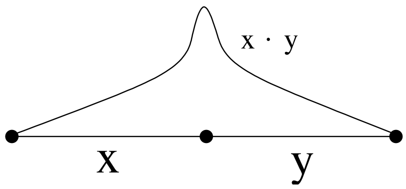


Figure 3

The sign of $x \cdot y$ is $+1$ in the case when x and y have the same sign (both $+1$ or both -1). The sign of $x \cdot y$ is -1 if x and y are signed differently (one of them $+1$ and the other -1).

We remind the reader that these edges originally do not appear in the given graph T. We construct them and use them throughout the paper.

We note that the definition of $x \cdot y$ is similar to Definition 3.5 in [4], but we have also introduced signs for virtual edges.

Definition 7. *Let T be a planar graph. We define $\hat{T}$ as a graph with the same vertices as those of T. The edges of $\hat{T}$ are either actual or virtual, and for every ordered pair of edges $x, y \in T$, we have the virtual edge $x \cdot y$ in $\hat{T}$ with the corresponding sign. (See [4], Definition 3.7).*

In the following theorem, we connect the group $A(\hat{T})$ with the quotient $A_Y(T)$. This gives us an algebraic meaning for the expression $x \cdot y$.

Theorem 8. *Let T be a planar graph. There is a well-defined map $A(\hat{T}) \to A_Y(T)$, that maps each actual edge $x \in \hat{T}$ to $x \in T$ and each virtual edge $x \cdot y$ to $x^{-1}yx$.*

Proof. The proof is the similar to the proof of Theorem 3.8 of [4], where, we compare the relations in the two groups. In case the edges x, and y are not intersect, $x \cdot y = y$, and $x^{-1}yx = y$ too, therefore the map is trivial in that case. Thus, we assume that x and y are intersect, and therefore $x^{-1}yx \neq y$. Let t be an edge in T, then the following holds:

- In case the edge t has no intersection vertex with both edges x and y: $[t, x \cdot y] = 1$ in $\hat{T}$, and $[t, x^{-1}yx] = 1$ in $A_Y(T)$;

- In case the three edges x, y, and t are intersect in a common vertex: $[t, x \cdot y] = 1$ in $\hat{T}$, and $[t, x^{-1}yx] = 1$ in $A_Y(T)$;
- In case the edges t and x are intersect in a vertex, but the edges t and y are not intersect: $\langle t, x \cdot y\rangle = 1$ in $\hat{T}$, and $\langle t, x^{-1}yx\rangle = 1$ in $A_Y(T)$;
- In case the edges t and y are intersect in a vertex, but the edges t and x are not intersect: $\langle x \cdot y, t\rangle = 1$ in $\hat{T}$, and $\langle x^{-1}yx, t\rangle = 1$ in $A_Y(T)$;
- In case the edge t intersect the edges x and y in two different vertices (i. e. t connects the same two vertices which the edge $x \cdot y$ connects in $\hat{T}$), and the three edges x, y, and t are form an anti-cycle (i.e. $s(t) = -s(x)s(y)$), then we have: $[t, x^{-1}yx] = 1$ in $A_Y(T)$. In this case, the edges t and $x \cdot y$ are two differently signed edges which connect the same two vertices in $\hat{T}$, therefore $[t, x \cdot y] = 1$ in $\hat{T}$. □

We denote by $L(x, y)$ the element $x^{-1}yx$, and more generally, denote by $L(x_1, x_2, \ldots, x_n)$ the element $x_1^{-1}x_2^{-1} \cdots x_{n-1}^{-1}x_nx_{n-1} \cdots x_2x_1$. Then, in Theorem 8 the virtual edge $x \cdot y$ in $\hat{A}$ maps to $L(x, y)$ in $A_Y(T)$, and the virtual edge $x_1 \cdot x_2 \cdots x_n$ in $\hat{(A)}$ maps to $L(x_1, x_2, \ldots, x_n)$ in $A_Y(T)$.

Remark 9. *We mentioned that we consider signs of edges that are contained in a cycle of the planar graph T, and we assume that all the edges of T which are not contained in any cycle are positively signed. Moreover, all the relations of $A(T)$ and of $A_Y(T)$ which depends on the signs of the edges, are relations which come from an anti-cycle of T (i.e., A cycle of T, with an odd number of negatively signed edges). Therefore, there are two cases*

1. *If the graph T does not contain any anti-cycle, then there are no relations of $A(T)$ and of $A_Y(T)$ which depends on the signs of the edges of T. Thus, we may assume in this case that the edges of T are not signed (Which is the same like all the edges of T are positively signed). Moreover, the relations of $A(T)$ and of $A_Y(T)$ are the same as it is for $A(T)$ and $A_Y(T)$ in the paper [4]. Thus, all the theorems of the paper [4] holds in this case. Therefore, there is a homomorphism:*
$$A(T) \to A_Y(T) \to Br_n \simeq A(S_n).$$
2. *If the graph T contains at least one anti-cycle, then there is at least one negatively signed edge in T (which is in the anti-cycle). In this case there are relations of $A(T)$ and of $A_y(T)$ which comes from the anti-cycle, and which depend on the signs of the edges in T. These relations are not appeared in the paper [4]. In this case, the groups $A(T)$ and $A_Y(T)$ are quotients of $A(\tilde{T})$ and of $A_Y(\tilde{T})$ respectively, where $\tilde{T}$ is a planar graph which we get from T by omitting the signs of T. In this paper we will show that in a case of signed planar graph T which is just one anti-cycle connected to a path (The analogue of a tree in the case of non-signed planar graphs), $A_Y(T) \simeq A(D_n)$, where n is the number of vertices of T. We will show also that in every case of T which contains at least one anti-cycle, there is a homomorphism:*
$$A(T) \to A_Y(T) \to A(D_n)$$

4 Graphs without anti-cycles

Let T be a signed planar graph which does not contain anti-cycles, and let $\tilde{T}$ be the non-signed planar graph which is obtained from T by omitting the signs of the edges in T. By Remark 9, the groups $A(T)$ and $A_Y(T)$ are isomorphic to the groups $A(\tilde{T})$ and to $A_Y(\tilde{T})$ respectively. The paper [4] deals with the case of non-signed graphs. Since, the results of the current paper (where we consider the second case of Remark 9) are a generalization of the results of the paper [4] (i.e. The first case of Remark 9), therefore, in this section we recall the main results of the paper [4].

Let T be a non-signed planar graph, and Let $A(T)$ be the corresponding Artin group which is generated by the edges of T, and with all the relations which is described in Section 3, apart from the relations which involves anti-cycles. Let $A_Y(T)$ be a quotient of $A(T)$ where we use the same relations of $A_Y(T)$ which is described in Section 3, again apart from the relation which involves two differently signed edges. Then, the following holds:

- In case where the graph T does not contain a cycle, $A_Y(T) \simeq Br_n$, where n is the number of vertices of T;
- The group $A_Y(T)$ depends only on the number of cycles of T. i.e. If two planar graphs T_1 and T_2 have te same number of cycles, then $A_Y(T_1) \simeq A_Y(T_2)$;
- Let T be a planar graph with n vertices which contains a single cycle, then $A_Y(T) \simeq A_Y(T^{(1)})$, where $T^{(1)}$ is a cycle with n vertices $p_1, p_2, \ldots, p_n$, and n edges $\sigma_1, \sigma_2, \ldots \sigma_{n-1}, u$, where the edge σ_r connects the vertices p_r and p_{r+1} for $1 \leqslant r \leqslant n-1$, and the edge u connects the vertices p_1 and p_n. Then, there is defined the following specific elements in $T^{(1)}$:
  1. α to be $\sigma_1^{-1}\sigma_2^{-1} \ldots \sigma_{n-2}^{-1}\sigma_{n-1}\sigma_{n-2} \ldots \sigma_2\sigma_1$;
  2. $x_{\vec{\alpha}}$ to be $u\alpha_{-1}$, and $x_{\overleftarrow{\alpha}}$ to be $\alpha^{-1}u$;
  3. The quotient G of $A_Y(T^{(1)})$, with the additional relations $[x_{\vec{\alpha}}, \alpha^2] = 1$, and $[x_{\vec{\alpha}}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1$.
- The quotient G of $A_Y(T^{(1)})$ is isomorphic to $K \rtimes Br_n$, where K is generated by $n-1$ conjugates of $x_{\vec{\alpha}}$, namely: $x_{\overrightarrow{\sigma_1}}, x_{\overrightarrow{\sigma_2}}, \ldots, x_{\overrightarrow{\sigma_{n-1}}}$, where:
  1. $[x_{\overrightarrow{\sigma_r}}, x_{\overrightarrow{\sigma_t}}] = 1$, for $|r-t| \geqslant 2$;
  2. $[x_{\overrightarrow{\sigma_r}}, x_{\overrightarrow{\sigma_{r+1}}}] = z$, where $z^2 = 1$, and z is a central element of G.

  The action of Br_n (which is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$) on K is the following: $\sigma_t(z) = z$, and

$$\sigma_t(x_{\vec{\sigma}_r}) = \begin{cases} zx_{\vec{\sigma}_r}^{-1} & r = t; \\ x_{\vec{\sigma}_t}x_{\vec{\sigma}_{t+1}} & r = t+1; \\ x_{\vec{\sigma}_t}x_{\vec{\sigma}_{t-1}} & r = t-1; \\ x_{\vec{\sigma}_r} & |r-t| \geqslant 2. \end{cases}$$

- Then, there is a generalization to T with more than one cycle. Let T be a planar graph which contains n vertices and m cycles, then $A_Y(T) \simeq A_Y(T^{(m)})$,
where $T^{(m)}$ is a planar graph with n vertices $p_1, p_2, \ldots, p_n$, and $n-1+m$ edges $\sigma_1, \sigma_2 \ldots \sigma_{n-1}, u_1, u_2, \ldots u_m$.
The edge σ_r connects the vertices p_r and p_{r+1} for $1 \leqslant r \leqslant n-1$, and the edge u_j connects the vertices p_1 and p_n, for $1 \leqslant j \leqslant m$. Then, similarly to the definitions of specific elements in $T^{(1)}$ there is defined the following specific elements in $A_(T^{(m)})$:

  1. α to be $\sigma_1^{-1}\sigma_2^{-1}\ldots\sigma_{n-2}^{-1}\sigma_{n-1}\sigma_{n-2}\ldots\sigma_2\sigma_1$;
  2. $x^{(j)}_{\overrightarrow{\alpha}}$ to be $u_j\alpha_{-1}$, and $x^{(j)}_{\overleftarrow{\alpha}}$ to be $\alpha^{-1}u_j$, for $1 \leqslant j \leqslant m$;
  3. The quotient G of $A_Y(T^{(m)})$, with the additional relations $[x^{(j)}_{\overrightarrow{\alpha}}, \alpha^2] = 1$, and $[x^{(j)}_{\overrightarrow{\alpha}}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1$, for $1 \leqslant j \leqslant m$.

- The quotient G of $A_Y(T^{(m)})$ is isomorphic to $K \rtimes Br_n$, where K is generated by $x^{(j)}_{\overrightarrow{\sigma_1}}, x^{(j)}_{\overrightarrow{\sigma_2}}, \ldots, x^{(j)}_{\overrightarrow{\sigma_{n-1}}}$, for $1 \leqslant j \leqslant m$, such that $x^{(j)}_{\overrightarrow{\sigma_r}}$ is a conjugate of $x^{(j)}_{\overrightarrow{\alpha}}$ for $1 \leqslant r \leqslant n-1$ and $1 \leqslant j \leqslant m$. The group K satisfies the following relations:

  1. $[x^{(i)}_{\overrightarrow{\sigma_r}}, x^{(j)}_{\overrightarrow{\sigma_t}}] = 1$, for $|r-t| \geqslant 2$, $1 \leqslant i, j \leqslant m$;
  2. $[x^{(i)}_{\overrightarrow{\sigma_r}}, x^{(i)}_{\overrightarrow{\sigma_{r+1}}}] = z_i$, where $z_i^2 = 1$, and z_i is a central element of G, for $1 \leqslant i \leqslant m$;
  3. $[x^{(i)}_{\overrightarrow{\sigma_r}}x^{(i)}_{\overrightarrow{\sigma_{r-1}}}, x^{(j)}_{\overrightarrow{\sigma_r}}x^{(j)}_{\overrightarrow{\sigma_{r+1}}}] = 1$, for $1 \leqslant i, j \leqslant m$;
  4. $[x^{(i)}_{\overrightarrow{\sigma_r}}, x^{(-j)}_{\overrightarrow{\sigma_{r+1}}}] = [x^{(-i)}_{\overrightarrow{\sigma_{r+1}}}, x^{(j)}_{\overrightarrow{\sigma_r}}]$, for $1 \leqslant i, j \leqslant m$.

  The action of Br_n (which is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$) on K is the following: $\sigma_t(z_i) = z_i$, and

$$\sigma_t(x^{(i)}_{\overrightarrow{\sigma}_r}) = \begin{cases} z_i x^{-(i)}_{\overrightarrow{\sigma}_r} & r = t; \\ x^{(i)}_{\overrightarrow{\sigma}_t} x^{(i)}_{\overrightarrow{\sigma}_{t+1}} & r = t+1; \\ x^{(i)}_{\overrightarrow{\sigma}_t} x^{(i)}_{\overrightarrow{\sigma}_{t-1}} & r = t-1; \\ x^{(i)}_{\overrightarrow{\sigma}_r} & |t-r| \geqslant 2. \end{cases}$$

- Then, simplifying the presentation of K by defining the element $a_r^{(i)}$ (where $1 \leqslant i \leqslant m$, $1 \leqslant r \leqslant n-1$) to be:
$a^{(i)}_{n-1} = x^{(i)}_{\overrightarrow{\sigma}_{n-1}}$ and $a^{(i)}_r = x^{(i)}_{\overrightarrow{\sigma}_{n-1}} x^{(i)}_{\overrightarrow{\sigma}_{n-2}} \cdots x^{(i)}_{\overrightarrow{\sigma}_r}$, for $1 \leqslant r \leqslant n-2$.

- By using the generators $a^{(i)}_1, \ldots a^{(i)}_{n-1}$ instead of the generators $x^{(i)}_{\overrightarrow{\sigma}_1}, \ldots x^{(i)}_{\overrightarrow{\sigma}_{n-1}}$, the presentation of K is simplified to be:

  1. $[a^{(i)}_r, a^{(i)}_s] = z_i$, for $r \neq s$, and $1 \leqslant i \leqslant m$;
  2. $[a^{(i)}_r, a^{(j)}_s] = [a^{(i)}_{r'}, a^{(j)}_{s'}]$ for any $r \neq s$, and $r' \neq s'$, and $1 \leqslant i \leqslant m$.

- Finally, it is concluded that the word problem is solvable in G, since it is solvable in both K and Br_n.

5 Graphs with a single cycle

From now on we consider signed planar graphs T, which contains at least one negatively signed edge. By Remark 9, the graph T contains at least one anti-cycle. In this section we describe the basic case of a signed planar graph T where in addition to an anti-cycle, there is at most one more cycle or anti-cycle in T (i.e. The graph $\tilde{T}$ which we get from T by omitting the signs of the edges, contains one or two cycles).

Definition 10. *$(pq \to pr)$ − operation in a planar graph T:*

Let T_1 be a planar graph and let x and y be two edges in T_1 that have a common vertex q, where:

- *x connecting the vertices p and q;*
- *y connecting the vertices q and r.*

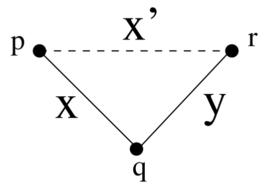


Figure 4

Then we obtain a new planar graph T_2 by $(pq \to pr)$ − operation in a graph T_1 where we omit the edge x, and add a new edge x' (The virtual edge $x \cdot y$) which connects the vertices p and r (these vertices are not connected in T_1 by an edge). The sign of x' is the product of the signs of x and y, i.e., if x and y have the same sign then the sign of x' is $+1$, and if x and y have different signs then the sign of x' is -1.

Definition 11. *The map $f_{(pq\to pr)}$ from $A(T_2)$ to $A(T_1)$:*

Let T_1 and T_2 be two planar graphs such that we obtain T_2 from T_1 by $(pq \to pr)$ − operation on T_1. Then we define a map

$$f_{(pq\to pr)}$$

from $A(T_2)$ to $A(T_1)$ as follows:

$$f_{(pq\to pr)}(x') = x^{-1}yx$$

, where x' is the path connecting p and r in T_2, x is the edge connecting p and q in T_1, and y is the edge connecting q and r in T_1, and

$$f_{(pq\to pr)}(v) = v$$

for every $v \in T_2$, such that $v \neq x'$.

The map $f_{(pq\to pr)}$ defines an isomorphism between $A(T_2)$ and $A(T_1)$ and also between $A_Y(T_2)$ and $A_Y(T_1)$, as described in Section 8 of [4].

Definition 12. *Equivalence in graphs:*

Two graphs T_1 and T_2 are considered equivalent, if we can obtain T_2 from T_1 by a several number of $(p_iq_i \to p_ir_i)$ − operations. for some vertices p_i, q_i, and r_i.

First, we consider the simplest case of signed planar graph T, where T is a single anti-cycle connected to a path.

Theorem 13. *Let T be an anti-cycle of length n. Then T is equivalent to a graph T' that contains an anti-cycle of length two connected to a path. See Figure 5.*

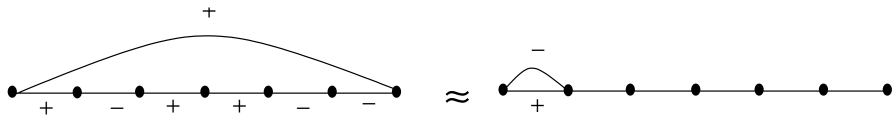


Figure 5

Proof. Let T be an anti-cycle with edges: $\sigma_1, \sigma_2, \ldots \sigma_n$ such that $\prod_{j=1}^{n} s(\sigma_i) = -1$, and vertices $p_1, p_2, \ldots, p_n$, such that for $1 \leqslant i \leqslant n-1$, the edge σ_i connects between the vertices p_i and p_{i+1}, and σ_n connects between p_1 and p_n. Then, first performing $(p_1p_n \to p_1p_{n-1})$-operation on T, we get a graph T_2, with a new edge σ_n', where $s(\sigma_{n'}) = s(\sigma_{n-1})s(\sigma_n)$ and $f_{(p_1p_n\to p_1p_{n-1})}(\sigma_{n'}) = \sigma_{n-1}^{-1}\sigma_n\sigma_{n-1}$. Therefore, $\prod_{j=1}^{n-2} s(\sigma_i)\cdot s(\sigma_{n'}) = \prod_{j=1}^{n} s(\sigma_i) = -1$, and then the edges $\sigma_1, \sigma_2, \ldots, \sigma_{n-2}, \sigma_{n'}$ form an anti-cycle in T_2. Now, performing $(p_1p_{n-1} \to p_1p_{n-2})$-operation on T_2, then we get a graph T_3, with a new edge σ_n'' instead of the edge σ_n', where $s(\sigma_{n''}) = s(\sigma_{n-2})s(\sigma_{n'})$ and $f_{(p_1p_{n-1}\to p_1p_{n-2})}(\sigma_n'') = \sigma_{n-2}^{-1}\sigma_{n'}\sigma_{n-2}$, and therefore $f_{(p_1p_n\to p_1p_{n-1})} \circ f_{(p_1p_{n-1}\to p_1p_{n-2})}(\sigma_{n''}) = \sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n\sigma_{n-1}\sigma_{n-2}$, and $\prod_{j=1}^{n-3} s(\sigma_i)\cdot s(\sigma_{n''}) = \prod_{j=1}^{n} s(\sigma_i) = -1$, and then the edges $\sigma_1, \sigma_2, \ldots, \sigma_{n-3}, \sigma_{n''}$, form an anti-cycle in T_3. For $i \geqslant 3$, continue performing $(p_1p_i \to p_1p_{i-1})$-operation on T_{n-i+1}, then we get the graph T_{n-i+2}, with a new edge $\sigma_n^{(n-i+1')}$, where $f_{(p_1p_n\to p_1p_{n-1})} \circ f_{(p_1p_{n-1}\to p_1p_{n-2})} \circ \cdots \circ f_{(p_1p_i\to p_1p_{i-1})}(\sigma_n^{(n-i+1')}) = \sigma_{i-1}^{-1}\sigma_i^{-1}\ldots\sigma_{n-1}^{-1}\sigma_n\sigma_{n-1}\ldots\sigma_i\sigma_{i-1}$, and $\prod_{j=1}^{i-2} s(\sigma_j)\cdot s(\sigma_{n''}) = \prod_{j=1}^{n} s(\sigma_j) = -1$, and then the edges $\sigma_1, \sigma_2, \ldots, \sigma_{i-2}, \sigma_n^{(n-i+1)'}$, form an anti-cycle in T_{n-i+1}. In the last step, where $i = 3$, e get the graph T_{n-1} by performing $(p_1p_3 \to p_1p_2)$-operation on T_{n-2}. Then we get a new edge $\sigma_n^{(n-2')}$, which connects the vertices p_1 and p_2. Denote by T' the graph T_{n-1}, and denote by $\sigma_{\bar{1}}$ be the edge $\sigma_n^{(n-i+1')}$ in the graph T'. Then $f_{(p_1p_n\to p_1p_{n-1})} \circ f_{(p_1p_{n-1}\to p_1p_{n-2})} \circ \cdots \circ f_{(p_1p_3\to p_1p_2)}(\sigma_{\bar{1}}) = \sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{n-1}^{-1}\sigma_n\sigma_{n-1}\ldots\sigma_2$. Then $s(\sigma_{\bar{1}}) = \prod_{j=2}^{n} s(\sigma_i)$ and since $\prod_{j=1}^{n} s(\sigma_i) = -1$, $s(\sigma_{\bar{1}}) = -s(\sigma_1)$. Hence we get T' which includes the anti-cycle of length two σ_1 and $\sigma_{\bar{1}}$, where they are connected by a path. Since the only edges of T that are involved in a cycle or in an anti-cycle are σ_1 and $\sigma_{\bar{1}}$, we can omit the signs from $\sigma_2, \ldots, \sigma_{n-1}$. ☐

Corollary 14. *All the anti-cycles of length n are equivalent, and therefore, $A_Y(T) \simeq A(D_n)$, where T is an anti-cycle of length n.*

Proof. By Theorem 13, every anti-cycle of length n is equivalent to an anti-cycle of length two connected to a path. All the anti-cycles are equivalent to the same graph T'. By Remerk 4, The group $A_Y(T')$ is isomorphic to $A(D_n)$. The group $A(D_n)$ is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_{\bar{1}}$, where the following holds:

- For $1 \leqslant i \leqslant n-2$, $\langle \sigma_i, \sigma_{i+1} \rangle = 1$;
- For $1 \leqslant i, j \leqslant n-1$, such that $|i-j| \geqslant 2$, $[\sigma_i, \sigma_j] = 1$;
- $\langle \sigma_{\bar{1}}, \sigma_2 \rangle = 1$;
- For $3 \leqslant i \leqslant n-1$, $[\sigma_{\bar{1}}, \sigma_i] = 1$;

Therefore, $A_Y(T') \simeq A(D_n)$. Then for every equivalent graph T, $A_Y(T) \simeq A_Y(T') \simeq A(D_n)$. $\square$

Theorem 15. *Let T be a planar graph consisting of two anti-cycles connected to a path. Then T is equivalent to T', where T' is a cycle with an additional negatively signed edge connected to two adjacent vertices of the cycle. See Figure 6.*

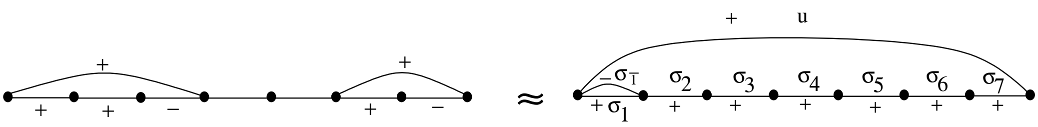
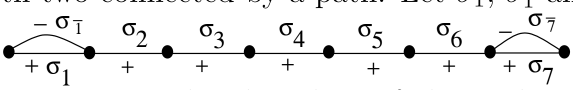


Figure 6

Proof. By Theorem 13, T is equivalent to T'', where T'' consists of two anti-cycles of length two connected by a path. Let σ_1, $\sigma_{\bar{1}}$ and σ_n, $\sigma_{\bar{n}}$



be the edges of the two anti-cycles in T'' and $\sigma_2, \ldots, \sigma_{n-1}$ be the edges of the path connecting them. By a several $(pq \to pr)$-operations, it is possible to get a graph T' which is equivalent to T'', such that $\sigma_{\bar{n}}$ is replaced by $u = \sigma_{\bar{1}}^{-1}\sigma_2^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{n}}\sigma_{n-1} \ldots \sigma_2\sigma_{\bar{1}}$ (Similarly to the proof of Theorem 13). $\square$

Now we classify the graphs T that include a single cycle connected by a path to an anti-cycle or graphs that include only two anti-cycles connected to a path.

Theorem 16. *Let T be a planar graph consisting of an anti-cycle C connected to a cycle by a path. Then T is equivalent to T', where T' is a cycle with an additional negatively signed edge, which is connected to two adjacent vertices of the cycle.*

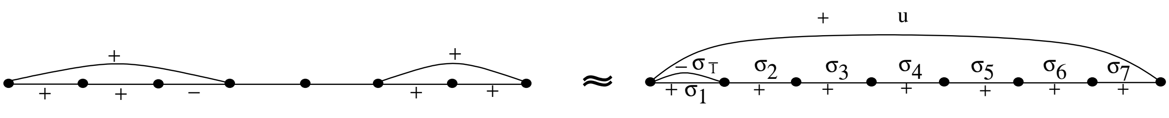


Figure 7

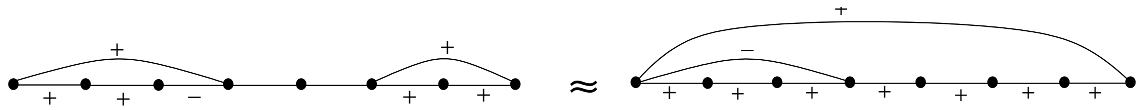


Figure 8

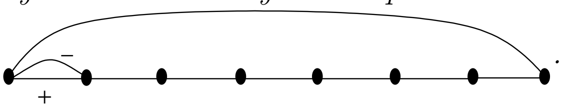
Proof. By the same proof as in Lemma 9.3 in [4], we have Figure 8.

Notice, that the signs of edges which are used by the relations of $A_Y(T)$ are just signs of the edges in the cycles of T. Therefore, where we consider T with two cycles $C1$ and $C2$, we may consider just the parity of the negatively signed edges in the cycles $C1$ and $C2$. Thus, every graph T with two given cycles $C1$ and $C2$, such that the number of negatively signed edges of $C1$ is odd, and number of negatively signed edges of $C2$ is even, gives the same $A_Y(T)$. In this case, an anti-cycle $C1$ is connected to a cycle $C2$ by a path. Thus, we may assume that all the edges of the graph T apart from one edge in the anti-cycle $C1$ are positively signed, as it appears in Figure 8. Then using Theorem 15, the anti-cycle $C1$ is equivalent to an anti-cycle of length two connected to a path. Then, combining this with the cycle, we get T' as in Figure 9.

Figure 9

□

Corollary 17. *Every planar graph T with a path connecting either a cycle with an anti-cycle or two anti-cycles is equivalent to T' of a form:*



Proposition 18. *The length of the cycle in T' is one less than the number of edges in T.*

Proof. T' contains a cycle and an additional edge signed by -1. Hence the length of the cycle in T' is one less than the length of the number of edges in T'. Since we get T' from T by triangulation, and triangulation preserves the number of the edges, the proposition follows. □

Now we define the group $G(T)$ for a planar graph T, where T consists of either two anti-cycles connected by a path or an anti-cycle connected to a cycle by a path.

By Theorems 15 and 16, T is equivalent to T', where T' is a cycle and an additional edge signed by -1 that is connected to two adjacent vertices in the cycle in T', as in Figure 10. Hence $A_Y(T') \simeq A_Y(T)$.

The edges of T' are labelled by σ_i, $1 \leqslant i \leqslant n-1$, $\sigma_{\bar{1}}$ and u, the generators of $A(T')$. Without loss of generality, we may assume $s(u) = (+1)$, $s(\sigma_i) = (+1)$ for $1 \leqslant i \leqslant n-1$, and $s(\sigma_{\bar{1}}) = (-1)$.

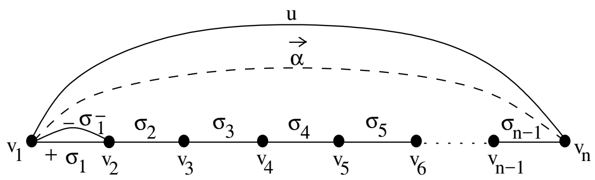


Figure 10

We denote by α the element

$$\alpha = L(\sigma_1, \ldots, \sigma_{n-1}) = \sigma_1^{-1}\sigma_2^{-1} \ldots \sigma_{n-2}^{-1}\sigma_{n-1}\sigma_{n-2} \ldots \sigma_2\sigma_1.$$

Similarly to [4, Section 6], we define the path $\overrightarrow{\alpha}$ as a directed signed path with the starting point at the vertex v_1 and the ending point at the vertex v_n, with $\overrightarrow{\alpha} = (+1)$. We notice that the path $\overrightarrow{\sigma}_i$ is a positively signed path with the starting point at v_i and the ending point at v_{i+1}, where $\overleftarrow{\sigma}_i$ is the positively signed path with opposite direction to $\overrightarrow{\sigma}_i$, which means that the starting point of $\overleftarrow{\sigma}_i$ is v_{i+1}, and the ending point of $\overleftarrow{\sigma}_i$ is v_i.

Now we define negatively signed paths for the action of $A(D_n)$ in T'. $\overrightarrow{\sigma}_{\bar{1}}$ is the negatively signed path with starting point v_1 and ending point v_2, and similarly, $\overleftarrow{\sigma}_{\bar{1}}$ is the negatively signed path with starting point v_2 and ending point v_1.

By Definition 5, $A_Y(T')$ is the group generated by $\sigma_1, \ldots \sigma_{n-1}, \sigma_{\bar{1}}$ and u with the relations

$[\sigma_i, \sigma_j] = 1$ for $|i - j| > 1$
$[u, \sigma_i] = 1$ for $2 \leqslant i \leqslant n - 2$
$[\sigma_{\bar{1}}, \sigma_j] = 1$ for $j \neq 2$
$\langle \sigma_i, \sigma_{i+1} \rangle = 1$ for $1 \leqslant i \leqslant n - 2$
$\langle \sigma_{\bar{1}}, \sigma_2 \rangle = 1$
$\langle \sigma_1, u \rangle = 1$
$\langle \sigma_{\bar{1}}, u \rangle = 1$
$\langle \sigma_{n-1}, u \rangle = 1$
$[\sigma_{\bar{1}}^{-1} u \sigma_{\bar{1}}, \sigma_1^{-1}\sigma_2\sigma_1] = 1$
$[\sigma_1^{-1} u \sigma_1, \sigma_{\bar{1}}^{-1}\sigma_2\sigma_{\bar{1}}] = 1.$

Note that $A_Y(T') = \langle A(D_n), u \rangle$, where $A(D_n)$ is the parabolic subgroup of $A_Y(T')$ generated by σ_i, $1 \leqslant i \leqslant n - 1$ and $\sigma_{\bar{1}}$.

As in [4, Section 6], we define $x_{\overrightarrow{\alpha}}$, and $x_{\overleftarrow{\alpha}}$ in the following way:

- $x_{\overrightarrow{\alpha}} = u\alpha^{-1}$;
- $x_{\overleftarrow{\alpha}} = \alpha^{-1} x_{\overrightarrow{\alpha}} \alpha = \alpha^{-1} u.$

The stabilizer of $\overrightarrow{\alpha}$ in the action of $A(D_n)$ is generated by $\sigma_2, \ldots, \sigma_{n-2}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}, \alpha^2$ as in [4, Remark 6.2], and moreover, $\sigma_{n-1}\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_{n-1}^{-1}$, $\sigma_{\bar{1}}\sigma_1^{-1}\alpha\sigma_1\sigma_{\bar{1}}^{-1}$ and $\sigma_1\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_1^{-1}$ (three elements that involve $\sigma_{\bar{1}}$).

In the spirit of [4, Section 6], we define the group $G(T')$. Then we show the structure of this group, given a graph with a single cycle, see Theorem 27.

Definition 19. *$G(T') = \langle A(D_n), x_{\overrightarrow{\alpha}} \rangle$ with the relations*
$\langle \sigma_1, x_{\overrightarrow{\alpha}}\alpha \rangle = 1,$
$\langle \sigma_{\bar{1}}, x_{\overrightarrow{\alpha}}\alpha \rangle = 1,$
$\langle \sigma_{n-1}, x_{\overrightarrow{\alpha}}\alpha \rangle = 1,$
$[x_{\overrightarrow{\alpha}}, \sigma_i] = 1$, *for* $2 \leqslant i \leqslant n-2$
which hold also in $A(T')$, and five additional relations concerning the stabilizer of $\overrightarrow{\alpha}$:
$[x_{\overrightarrow{\alpha}}, \alpha^2] = 1,$
$[x_{\overrightarrow{\alpha}}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1,$
$[x_{\overrightarrow{\alpha}}, \sigma_{\bar{1}}\sigma_1^{-1}\alpha\sigma_1\sigma_{\bar{1}}^{-1}] = 1,$
$[x_{\overrightarrow{\alpha}}, \sigma_1\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_1^{-1}] = 1,$
$[x_{\overrightarrow{\alpha}}, \sigma_{n-1}\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_{n-1}^{-1}] = 1.$

Assume we get T' from T by obtaining k-times $(p_jq_j \to p_jr_j)-$*operations*. Let $T_0 = T$, and let T_j be the graph we get from T_{j-1} by $(p_jq_j \to p_jr_j) -$ *operation*. Then $T_k = T'$. Let $f^{(j)}_{(p_jq_j \to p_jr_j)}$ be the isomorphism from $A(T_j)$ to $A(T_{j-1})$. Then, we define $G(T)$ in a similar way as we defined $G(T')$, where we take:

$$f^{(1)}_{(p_1q_1 \to p_1r_1)} \circ f^{(2)}_{(p_2q_2 \to p_2r_2)} \circ \cdots \circ f^{(k)}_{(p_kq_k \to p_kr_k)}(\sigma_i)$$

instead of σ_i, and we take:

$$f^{(1)}_{(p_1q_1 \to p_1r_1)} \circ f^{(2)}_{(p_2q_2 \to p_2r_2)} \circ \cdots \circ f^{(k)}_{(p_kq_k \to p_kr_k)}(u)$$

instead of u.

Since, $f^{(j)}_{(p_jq_j \to p_jr_j)}$ is an isomorphism from $A_Y(T_i)$ onto $A_Y(T_{i-1})$, we get $G(T)$ is isomorphic to $G(T')$, for every graph T which is equivalent to the graph T'.

Proposition 20.
$[u, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1,$
$[u, \sigma_{\bar{1}}\sigma_1^{-1}\alpha\sigma_1\sigma_{\bar{1}}^{-1}] = 1,$
$[u, \sigma_1\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_1^{-1}] = 1,$
$[u, \sigma_{n-1}\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_{n-1}^{-1}] = 1.$

Proof. Since $x_{\overrightarrow{\alpha}} = u\alpha^{-1}$, the relation $[x_{\overrightarrow{\alpha}}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1$ implies $[u\alpha^{-1}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1$. Now, $[\alpha\sigma_{n-1}\alpha^{-1}, \alpha\sigma_1\alpha^{-1}] = 1$, which is equivalent to $[\sigma_{n-1}^{-1}\alpha\sigma_{n-1}, \sigma_1^{-1}\alpha\sigma_1] = 1$ which implies $[\alpha, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}]$.
Then we get $[u, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1$. The proof for the other three commutation relations is very similar. ☐

Definition 21. *For $1 \leqslant i \leqslant n-1$, and for $i = \bar{1}$ define the elements $x_{\overrightarrow{\sigma_i}}$, and $x_{\overleftarrow{\sigma_i}}$ in the following way.*

- $x_{\overrightarrow{\sigma_i}} = u^{(\sigma_i)}\sigma_i^{-1}$ *where* $u^{(\sigma_i)} = L(\sigma_{i+1}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1})$, *and* $u^{(\sigma_{\bar{1}})} = L(\sigma_2, \ldots, \sigma_{n-1}, \sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}})$;
- $x_{\overleftarrow{\sigma_i}} = \sigma_i^{-1}x_{\overrightarrow{\sigma_i}}\sigma_i = \sigma_i^{-1}u^{(\sigma_i)}$.

Proposition 22. *The n elements of the form $x_{\overrightarrow{\sigma_{\bar{1}}}}, x_{\overrightarrow{\sigma_1}}, x_{\overrightarrow{\sigma_2}}, \ldots, x_{\overrightarrow{\sigma_{n-1}}}$ are conjugate elements in $G(T')$, and these elements are conjugate to $x_{\overrightarrow{\alpha}}$.*

Proof. Since,

$$\alpha = L(\sigma_1, \ldots, \sigma_{n-1}) = \sigma_{n-1} \ldots \sigma_2 \sigma_1 \sigma_2^{-1} \ldots \sigma_{n-1}^{-1}, \sigma_1 = \sigma_2^{-1} \ldots \sigma_{n-1}^{-1} \alpha \sigma_{n-1} \ldots \sigma_2.$$

It thus follows that σ_1 is the conjugate of α by the element $\sigma_{n-1} \ldots \sigma_2$.
Hence $u^{(\sigma_1)} = \sigma_2^{-1} \ldots \sigma_{n-1}^{-1} u \sigma_{n-1} \ldots \sigma_2$, which by definition is $L(\sigma_2, \ldots, \sigma_{n-1}, u)$.

Now assume, by induction on i, that $x_{\overrightarrow{\sigma_r}}$ is a conjugate of $x_{\overrightarrow{\sigma_1}}$, for $1 \leqslant r \leqslant i$, and we prove that te elements $x_{\overrightarrow{\sigma_{i+1}}}$ and $x_{\overrightarrow{\sigma_i}}$. First notice, $\sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1}$. We get the following:

$$\begin{aligned}
\sigma_i \sigma_{i+1} u^{(\sigma_i)} \sigma_{i+1}^{-1} \sigma_i^{-1} &= \sigma_i \sigma_{i+1} L(\sigma_{i+1}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1}) \sigma_{i+1}^{-1} \sigma_i^{-1} \\
&= \sigma_i \sigma_{i+1} \sigma_{i+1}^{-1} \ldots \sigma_{n-1}^{-1} u^{-1} \sigma_1^{-1} \ldots \sigma_{i-2}^{-1} \sigma_{i-1} \sigma_{i-2} \ldots \sigma_1 u \sigma_{n-1} \ldots \sigma_{i+1} \sigma_{i+1}^{-1} \sigma_i^{-1} \\
&= \sigma_{i+2}^{-1} \ldots \sigma_{n-1}^{-1} u^{-1} \sigma_1^{-1} \ldots \sigma_{i-2}^{-1} \sigma_i \sigma_{i-1} \sigma_i^{-1} \sigma_{i-2} \ldots \sigma_1 u \sigma_{n-1} \ldots \sigma_{i+2} \\
&= \sigma_{i+2}^{-1} \ldots \sigma_{n-1}^{-1} u^{-1} \sigma_1^{-1} \ldots \sigma_{i-2}^{-1} \sigma_{i-1}^{-1} \sigma_i \sigma_{i-1} \sigma_{i-2} \ldots \sigma_1 u \sigma_{n-1} \ldots \sigma_{i+2} \\
&= L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_i) = u^{(\sigma_{i+1})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
x_{\overrightarrow{\sigma_{i+1}}} &= u^{(\sigma_{i+1})} \sigma_{i+1}^{-1} \\
&= \sigma_i \sigma_{i+1} (u^{(\sigma_i)} \sigma_i^{-1}) \sigma_{i+1}^{-1} \sigma_i^{-1} \\
&= \sigma_i \sigma_{i+1} x_{\overrightarrow{\sigma_i}} \sigma_{i+1}^{-1} \sigma_i^{-1}
\end{aligned}$$

Now we prove the expression for $u^{(\sigma_{\bar{1}})}$. First, notice that $\sigma_{\bar{1}} = \sigma_2^{-1} \sigma_{\bar{1}}^{-1} \sigma_2 \sigma_{\bar{1}} \sigma_2$. Hence

$$\begin{aligned}
\sigma_2^{-1} \sigma_{\bar{1}}^{-1} u^{(\sigma_2)} \sigma_{\bar{1}} \sigma_2 &= \sigma_2^{-1} \sigma_{\bar{1}}^{-1} L(\sigma_3, \ldots, \sigma_{n-1}, u, \sigma_1) \sigma_{\bar{1}} \sigma_2 \\
&= \sigma_2^{-1} \sigma_{\bar{1}}^{-1} \sigma_3^{-1} \ldots \sigma_{n-1}^{-1} u^{-1} \sigma_1 u \sigma_{n-1} \ldots \sigma_3 \sigma_{\bar{1}} \sigma_2 \\
&= \sigma_2^{-1} \sigma_3^{-1} \ldots \sigma_{n-1}^{-1} \sigma_{\bar{1}}^{-1} u^{-1} \sigma_1 u \sigma_{\bar{1}} \sigma_{n-1} \ldots \sigma_3 \sigma_2 \\
&= \sigma_2^{-1} \sigma_3^{-1} \ldots \sigma_{n-1}^{-1} (\sigma_{\bar{1}}^{-1} \sigma_1 u \sigma_1^{-1} \sigma_{\bar{1}}) \sigma_{n-1} \ldots \sigma_3 \sigma_2 \\
&= L(\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \sigma_{\bar{1}}^{-1} \sigma_1 u \sigma_1^{-1} \sigma_{\bar{1}}) \\
&= u^{(\sigma_{\bar{1}})}.
\end{aligned}$$

□

Now we prove that $x_{\overrightarrow{\sigma}_i}$ and $x_{\overrightarrow{\sigma}_{\bar{1}}}$ satisfy similar relations as in [4, Section 6].

Proposition 23.

1. $[x_{\overrightarrow{\sigma}_i}, \sigma_j] = 1$ *for* $|i - j| > 1$,
2. $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_j}] = 1$ *for* $|i - j| > 1$,
3. $[x_{\overrightarrow{\sigma}_{\bar{1}}}, \sigma_j] = 1$ *for* $j \neq 2$,

4. $[x_{\vec{\sigma}_j}, \sigma_{\bar{1}}] = 1$ *for* $j \neq 2$,

5. $[x_{\vec{\sigma}_{\bar{1}}}, x_{\vec{\sigma}_j}] = 1$ *for* $j \neq 2$.

Proof. For the proof of (1) and (2), see[4].

We now prove (3). First we prove $[x_{\vec{\sigma}_{\bar{1}}}, \sigma_j] = 1$ for $j \geqslant 4$. Since $x_{\vec{\sigma}_{\bar{1}}} = u^{(\vec{\sigma}_{\bar{1}})}\vec{\sigma}_{\bar{1}}^{-1}$, and $[\sigma_{\bar{1}}^{-1}, \sigma_j] = 1$, it is enough to prove that $[u^{(\sigma_{\bar{1}})}, \sigma_j] = 1$.

$$\begin{aligned}
[u^{(\sigma_{\bar{1}})}, \sigma_j] &= [L(\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}), \sigma_j] \\
&= [\sigma_2^{-1}\sigma_3^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_3\sigma_2, \sigma_j] \\
&= [\sigma_{j-1}^{-1}\sigma_j^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_j\sigma_{j-1}, \sigma_j] \\
&= [\sigma_j^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_j, \sigma_{j-1}\sigma_j\sigma_{j-1}^{-1}] \\
&= [\sigma_j^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_j, \sigma_j^{-1}\sigma_{j-1}\sigma_j] \\
&= [\sigma_{j+1}^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_{j+1}, \sigma_{j-1}],
\end{aligned}$$

since $3 \leqslant j-1 \leqslant n-2$.

Now we prove that $[u^{(\sigma_{\bar{1}})}, \sigma_1] = 1$.

$$\begin{aligned}
[u^{(\sigma_{\bar{1}})}, \sigma_1] &= [\sigma_2^{-1}\sigma_3^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_3\sigma_2, \sigma_1] \\
&= [\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}, \sigma_{n-1} \ldots \sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1} \ldots \sigma_{n-1}^{-1}] \\
&= [\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}, \alpha] = [u, \sigma_{\bar{1}}\sigma_1^{-1}\alpha\sigma_1\sigma_{\bar{1}}^{-1}]
\end{aligned}$$

by Proposition 20.

Now we prove that $[u^{(\sigma_{\bar{1}})}, \sigma_3] = 1$.

$$\begin{aligned}
[u^{(\sigma_{\bar{1}})}, \sigma_3] &= [\sigma_2^{-1}\sigma_3^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_3\sigma_2, \sigma_3] \\
&= [\sigma_3^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_3, \sigma_2\sigma_3\sigma_2^{-1}] \\
&= [\sigma_3^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_3, \sigma_3^{-1}\sigma_2\sigma_3] \\
&= [\sigma_4^{-1} \ldots \sigma_{n-1}^{-1}\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}\sigma_{n-1} \ldots \sigma_4, \sigma_2] \\
&= [\sigma_{\bar{1}}^{-1}\sigma_1 u \sigma_1^{-1}\sigma_{\bar{1}}, \sigma_2] = [\sigma_{\bar{1}}^{-1} u \sigma_{\bar{1}}, \sigma_1^{-1}\sigma_2\sigma_1] = 1
\end{aligned}$$

according to Relation (3) in Definition 5 (see Figure 11). Hence case (3) is proved.

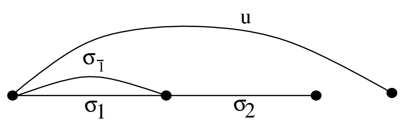


Figure 11

Now we prove case (4), $[x_{\vec{\sigma}_i}, \sigma_{\bar{1}}] = 1$ for $i \neq 2$. Since $x_{\vec{\sigma}_i} = u^{(\sigma_i)}\sigma_i^{-1}$, and $[\sigma_i^{-1}, \sigma_{\bar{1}}] = 1$ for $i \neq 2$, it is enough to prove that $[u^{(\sigma_i)}, \sigma_{\bar{1}}] = 1$ for $i \neq 2$.

For the case $i = 1$, $[u^{(\sigma_i)}, \sigma_{\bar{1}}] = 1$ holds by [4, Lemma 3.9], since $u^{(\sigma_1)} = L(\sigma_2, \ldots, \sigma_{n-1}, u)$ and $\sigma_{\bar{1}}$ is disjoint from the virtual edge $L(\sigma_2, \ldots, \sigma_{n-1}, u)$.

If $i \geqslant 3$,

$$\begin{aligned}[u^{(\sigma_i)}, \sigma_{\bar{1}}] &= [\sigma_{i+1}^{-1} \dots \sigma_{n-1}^{-1} u^{-1} \sigma_1^{-1} \sigma_2^{-1} \dots \sigma_{i-1} \dots \sigma_2 \sigma_1 u \sigma_{n-1} \dots \sigma_{i+1}, \sigma_{\bar{1}}] \\ &= [u^{-1} \sigma_1^{-1} \sigma_2^{-1} \dots \sigma_{i-1} \dots \sigma_2 \sigma_1 u, \sigma_{\bar{1}}] = [\sigma_{i-1} \dots \sigma_2 \sigma_1 u \sigma_1^{-1} \sigma_2^{-1} \dots \sigma_{i-1}^{-1}, \sigma_{\bar{1}}] \\ &= [\sigma_2 \sigma_1 u \sigma_1^{-1} \sigma_2^{-1}, \sigma_{\bar{1}}] = [\sigma_1 u \sigma_1^{-1}, \sigma_2^{-1} \sigma_{\bar{1}} \sigma_2] = [\sigma_1 u \sigma_1^{-1}, \sigma_{\bar{1}} \sigma_2 \sigma_{\bar{1}}^{-1}] \\ &= [\sigma_{\bar{1}}^{-1} u \sigma_{\bar{1}}, \sigma_1^{-1} \sigma_2 \sigma_1] = 1\end{aligned}$$

in $A_Y(T')$. Hence (4) holds.

It remains to prove (5), $[x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_i}]$ for $i \neq 2$. Since $x_{\overrightarrow{\sigma}_{\bar{1}}} = u^{(\sigma_{\bar{1}})} \sigma_{\bar{1}}^{-1}$ and $x_{\overrightarrow{\sigma}_i} = u^{(\sigma_i)} \sigma_i^{-1}$, and $[\sigma_{\bar{1}}^{-1}, \sigma_i^{-1}] = 1$ for $i \neq 2$, and by (3) and (4), $[u^{(\sigma_{\bar{1}})}, \sigma_i^{-1}] = 1$, $[u^{(\sigma_i)}, \sigma_{\bar{1}}^{-1}] = 1$, it is enough to prove: $[u^{(\sigma_{\bar{1}})}, u^{(\sigma_i)}] = 1$ for $i \neq 2$.

In the case $i = 1$,

$$\begin{aligned}[u^{(\sigma_{\bar{1}})}, u^{(\sigma_1)}] &= [\sigma_2^{-1} \dots \sigma_{n-1}^{-1} \sigma_{\bar{1}}^{-1} \sigma_1 u \sigma_1^{-1} \sigma_{\bar{1}} \sigma_{n-1} \dots \sigma_2, \sigma_2^{-1} \dots \sigma_{n-1}^{-1} u \sigma_{n-1} \dots \sigma_2] \\ &= [\sigma_{\bar{1}}^{-1} \sigma_1 u \sigma_1^{-1} \sigma_{\bar{1}}, u] = [\sigma_1 u \sigma_1^{-1}, \sigma_{\bar{1}} u \sigma_{\bar{1}}^{-1}] = [u^{-1} \sigma_1 u, u^{-1} \sigma_{\bar{1}} u] = 1.\end{aligned}$$

In the case $i \geqslant 4$,

$$\begin{aligned}[u^{(\sigma_{\bar{1}})}, u^{(\sigma_i)}] &= [\sigma_2^{-1} \dots \sigma_{n-1}^{-1} \sigma_1 \sigma_{\bar{1}}^{-1} u \sigma_{\bar{1}} \sigma_1^{-1} \sigma_{n-1} \dots \sigma_2, u^{(\sigma_i)}] \\ &= [(\sigma_2^{-1} \dots \sigma_{n-1}^{-1} \sigma_1 \sigma_{\bar{1}}^{-1} \sigma_{n-1} \dots \sigma_2) u^{(\sigma_1)} (\sigma_2^{-1} \dots \sigma_{n-1}^{-1} \sigma_1^{-1} \sigma_{\bar{1}} \sigma_{n-1} \dots \sigma_2), u^{(\sigma_i)}] \\ &= [\sigma_2^{-1} \sigma_1 \sigma_{\bar{1}}^{-1} \sigma_2 u^{(\sigma_1)} \sigma_2^{-1} \sigma_1^{-1} \sigma_{\bar{1}} \sigma_2, u^{(\sigma_i)}] = [u^{(\sigma_1)}, u^{(\sigma_i)}].\end{aligned}$$

Since $i \geqslant 4$, $[u^{(\sigma_1)}, u^{(\sigma_i)}] = 1$ by [4].

In the case $i = 3$,

$$\begin{aligned}[u^{(\sigma_{\bar{1}})}, u^{(\sigma_3)}] &= [\sigma_2^{-1} \sigma_1 \sigma_{\bar{1}}^{-1} \sigma_2 u^{(\sigma_1)} \sigma_2^{-1} \sigma_1^{-1} \sigma_{\bar{1}} \sigma_2, u^{(\sigma_3)}] = [\sigma_1 \sigma_{\bar{1}}^{-1} \sigma_2 u^{(\sigma_1)} \sigma_2^{-1} \sigma_{\bar{1}} \sigma_1^{-1}, \sigma_2 u^{(\sigma_3)} \sigma_2^{-1}] \\ &= [\sigma_1 \sigma_{\bar{1}}^{-1} \sigma_1^{-1} u^{(\sigma_2)} \sigma_1 \sigma_{\bar{1}} \sigma_1^{-1}, \sigma_3^{-1} u^{(\sigma_2)} \sigma_3] = [\sigma_{\bar{1}}^{-1} u^{(\sigma_2)} \sigma_{\bar{1}}, \sigma_3^{-1} u^{(\sigma_2)} \sigma_3] \\ &= [u^{(\sigma_2)} \sigma_{\bar{1}} (u^{(\sigma_2)})^{-1}, u^{(\sigma_2)} \sigma_3 (u^{(\sigma_2)})^{-1}] = [\sigma_{\bar{1}}, \sigma_3] = 1.\end{aligned}$$

□

Proposition 24.

1. $\sigma_i x_{\overrightarrow{\sigma}_{i+1}} \sigma_i^{-1} = \sigma_{i+1}^{-1} x_{\overrightarrow{\sigma}_i} \sigma_{i+1} = x_{\overrightarrow{\sigma}_{i+1}} x_{\overrightarrow{\sigma}_i}$, $1 \leqslant i \leqslant n-2$,

2. $\sigma_i^{-1} x_{\overrightarrow{\sigma}_{i+1}} \sigma_i = \sigma_{i+1} x_{\overrightarrow{\sigma}_i} \sigma_{i+1}^{-1} = x_{\overrightarrow{\sigma}_i} x_{\overrightarrow{\sigma}_{i+1}}$, $1 \leqslant i \leqslant n-2$,

3. $\sigma_2 x_{\overrightarrow{\sigma}_{\bar{1}}} \sigma_2^{-1} = \sigma_{\bar{1}}^{-1} x_{\overrightarrow{\sigma}_2} \sigma_{\bar{1}} = x_{\overrightarrow{\sigma}_{\bar{1}}} x_{\overrightarrow{\sigma}_2}$,

4. $\sigma_2^{-1} x_{\overrightarrow{\sigma}_{\bar{1}}} \sigma_2 = \sigma_{\bar{1}} x_{\overrightarrow{\sigma}_2} \sigma_{\bar{1}}^{-1} = x_{\overrightarrow{\sigma}_2} x_{\overrightarrow{\sigma}_{\bar{1}}}$.

Proof. The conjugation $x_{\overrightarrow{\sigma}_{i+1}} = \sigma_i \sigma_{i+1} x_{\overrightarrow{\sigma}_i} \sigma_{i+1}^{-1} \sigma_i^{-1}$ for $1 \leqslant i \leqslant n-2$ has been shown in the proof of Proposition 22. Then, by conjugating both sides by σ_i, we get $\sigma_i^{-1} x_{\overrightarrow{\sigma}_{i+1}} \sigma_i = \sigma_{i+1} x_{\overrightarrow{\sigma}_i} \sigma_{i+1}^{-1}$. Similarly, $x_{\overrightarrow{\sigma}_i} = \sigma_{i+1} \sigma_i x_{\overrightarrow{\sigma}_{i+1}} \sigma_i^{-1} \sigma_{i+1}^{-1}$ for $1 \leqslant i \leqslant n-2$, which is by conjugating both sides by σ_{i+1} implies $\sigma_i x_{\overrightarrow{\sigma}_{i+1}} \sigma_i^{-1} = \sigma_{i+1}^{-1} x_{\overrightarrow{\sigma}_i} \sigma_{i+1}$.
The conjugation $x_{\overrightarrow{\sigma}_{\bar{1}}} = \sigma_2^{-1} \sigma_{\bar{1}}^{-1} x_{\overrightarrow{\sigma}_2} \sigma_{\bar{1}} \sigma_2$ has been shown in the proof of Proposition 22

too. Then, by conjugating both sides by σ_2^{-1}, we have $\sigma_2 x_{\overrightarrow{\sigma}_{\bar{1}}}\sigma_2^{-1} = \sigma_{\bar{1}}^{-1} x_{\overrightarrow{\sigma}_2}\sigma_{\bar{1}}$. Similarly, it is easy to show, $x_{\overrightarrow{\sigma}_2} = \sigma_{\bar{1}}^{-1}\sigma_2^{-1} x_{\overrightarrow{\sigma}_{\bar{1}}}\sigma_2\sigma_{\bar{1}}$, which is by conjugating both sides by $\sigma_{\bar{1}}^{-1}$ implies $\sigma_{\bar{1}} x_{\overrightarrow{\sigma}_2}\sigma_{\bar{1}}^{-1} = \sigma_2^{-1} x_{\overrightarrow{\sigma}_{\bar{1}}}\sigma_2$.

Now, we prove $\sigma_i^{-1} x_{\overrightarrow{\sigma}_{i+1}}\sigma_i = x_{\overrightarrow{\sigma}_i} x_{\overrightarrow{\sigma}_{i+1}}$ for $1 \leqslant i \leqslant n-2$. First, recall that

$$\begin{aligned} &L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1}) \\ &= \sigma_{i+2}^{-1}\sigma_{i+3}^{-1}\cdots\sigma_{n-1}^{-1}\sigma_u^{-1}\sigma_1^{-1}\cdots\sigma_{i-2}^{-1}\sigma_{i-1}\sigma_{i-2}\cdots\sigma_1\sigma_u\sigma_{n-1}\cdots\sigma_{i+2}. \end{aligned}$$

Then,

$$\begin{aligned} &\langle \sigma_{i+1}, L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1})\rangle \\ &= \langle \sigma_{i+1}, \sigma_{i+2}\rangle = 1. \end{aligned}$$

Since

$$U^{(\sigma_i)} = \sigma_{i+1}^{-1} L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1})\sigma_{i+1},$$

and,

$$\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_i = L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1}),$$

Thus,

$$\begin{aligned} &L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1})\sigma_{i+1} L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1})^{-1} \\ &= \sigma_{i+1}^{-1} L(\sigma_{i+2}, \ldots, \sigma_{n-1}, u, \sigma_1, \ldots, \sigma_{i-1})\sigma_{i+1} \end{aligned}$$

implies,

$$\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_i\sigma_{i+1} = U^{(\sigma_i)}\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_i.$$

Thus,

$$\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = U^{(\sigma_i)}\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_{i+1}^{-1},$$

and therefore,

$$\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_{i+1}^{-1}\sigma_i = U^{(\sigma_i)}\sigma_i^{-1} U^{(\sigma_{i+1})}\sigma_{i+1}^{-1}$$

Then substituting $x_{\overrightarrow{\sigma}_i} = U^{(\sigma_i)}\sigma_i^{-1}$, we get

$$\sigma_i^{-1} x_{\overrightarrow{\sigma}_{i+1}}\sigma_i = x_{\overrightarrow{\sigma}_i} x_{\overrightarrow{\sigma}_{i+1}}.$$

In a very similar way it can been shown the following identities

$$\begin{aligned} \sigma_{i+1}^{-1} x_{\overrightarrow{\sigma}_i}\sigma_{i+1} &= x_{\overrightarrow{\sigma}_{i+1}} x_{\overrightarrow{\sigma}_i}, \\ \sigma_{\bar{1}}^{-1} x_{\overrightarrow{\sigma}_2}\sigma_{\bar{1}} &= x_{\overrightarrow{\sigma}_{\bar{1}}} x_{\overrightarrow{\sigma}_2}, \\ \sigma_2^{-1} x_{\overrightarrow{\sigma}_{\bar{1}}}\sigma_2 &= x_{\overrightarrow{\sigma}_2} x_{\overrightarrow{\sigma}_{\bar{1}}}. \end{aligned}$$

□

Proposition 25. $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}] = [x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_2}] = z$, *where* $z^2 = 1$ *and* $z \in C(G(T'))$.

Proof. By [4, Proposition 6.8], $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}] = [x_{\overrightarrow{\sigma}_j}, x_{\overrightarrow{\sigma}_{j+1}}]$ for each $1 \leqslant i \leqslant n-2$, $1 \leqslant j \leqslant n-2$.

Let $G'(T')$ be the subgroup of $G(T')$ generated by $\sigma_{\bar{1}}, \sigma_2, \dots, \sigma_{n-1}$ and u. By Propositions 23 and 24 : $[\sigma_{\bar{1}}, u^{(\sigma_i)}] = 1$ for $i \neq 2$ and $\sigma_{\bar{1}} u^{(\sigma_2)} \sigma_{\bar{1}}^{-1} = \sigma_2^{-1} u^{(\sigma_{\bar{1}})} \sigma_2$, $\sigma_{\bar{1}}^{-1} u^{(\sigma_2)} \sigma_{\bar{1}} = \sigma_2 u^{(\sigma_{\bar{1}})} \sigma_2^{-1}$. Hence $G'(T')$ is isomorphic to the group G that is defined in [4, Section 6] as a quotient of $A(T^{(1)})$, where $T^{(1)}$ is a single cycle.

The map $\varphi : G'(T') \to G$ is an isomorphism, where $\varphi(\sigma_{\bar{1}}) = \sigma_1$, $\varphi(\sigma_i) = \sigma_i$ for $2 \leqslant i \leqslant n-1$ and $\varphi(x_{\overrightarrow{\sigma}_{\bar{1}}}) = x_{\overrightarrow{\sigma}_1}$. Since $[x_{\overrightarrow{\sigma}_1}, x_{\overrightarrow{\sigma}_2}] = [x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}] = z$ in G, by the isomorphism: $[x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_2}] = [x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}]$ in $G'(T')$. Hence $[x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_2}] = [x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}]$ in $G(T')$. Since $[x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_2}]$ is a central element in $G'(T')$ and $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}] = [x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_2}]$, $[[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}], \sigma_{\bar{1}}] = 1$, and from [4, Section 6], $[[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}], \sigma_i] = 1$ for $1 \leqslant i \leqslant n-1$, we have $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_{i+1}}] = [x_{\overrightarrow{\sigma}_1}, x_{\overrightarrow{\sigma}_2}] = z$, where $z \in C(G(T'))$ and $z^2 = 1$. □

Proposition 26. $x_{\overleftarrow{\sigma}_i} = z x_{\overrightarrow{\sigma}_i}^{-1}$ *for* $1 \leqslant i \leqslant n-1$ *and* $x_{\overleftarrow{\sigma}_{\bar{1}}} = z x_{\overrightarrow{\sigma}_{\bar{1}}}^{-1}$.

Proof. $x_{\overleftarrow{\sigma}_i} = z x_{\overrightarrow{\sigma}_i}^{-1}$ is proved in [4, Proposition 6.8]. $x_{\overleftarrow{\sigma}_{\bar{1}}} = z x_{\overrightarrow{\sigma}_{\bar{1}}}^{-1}$ can also be deduced from [4, Proposition 6.8] by considering $G'(T')$, which was defined in the proof of Proposition 25. □

Finally, we prove the theorem which describes the structure of the group $G(T')$ in the case of a single cycle. Then, in Section 6, we generalize this theorem to a graph with an arbitrary number of cycles.

Theorem 27. $G(T') \simeq K \rtimes A(D_n)$*, where* K *is the group generated by* $x_{\overrightarrow{\sigma}_i}$, $1 \leqslant i \leqslant n-1$ *and* $x_{\overrightarrow{\sigma}_{\bar{1}}}$ *with the relations*

1. $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_j}] = 1$, $|i-j| > 1$;
2. $[x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_j}] = 1$, $j \neq 2$;
3. $[x_{\overrightarrow{\sigma}_i}, x_{\overrightarrow{\sigma}_j}] = z$, $|i-j| = 1$;
4. $[x_{\overrightarrow{\sigma}_{\bar{1}}}, x_{\overrightarrow{\sigma}_2}] = z$*, where*
   z *is a central element and* $z^2 = 1$.

where the action of $A(D_n)$ *on* $x_{\overrightarrow{\sigma}_r}$, $1 \leqslant r \leqslant n-1$ *and on* $x_{\overrightarrow{\sigma}_{\bar{1}}}$ *is the following:*
$\sigma_t(z) = z$,

$$\sigma_t(x_{\overrightarrow{\sigma}_r}) = \begin{cases} z x_{\overrightarrow{\sigma}_r}^{-1} & r = t; \\ x_{\overrightarrow{\sigma}_t} x_{\overrightarrow{\sigma}_r} & (r = t+1) \quad \textit{or} \quad (t = \bar{1} \quad \textit{and} \quad r = 2); \\ x_{\overrightarrow{\sigma}_t} x_{\overrightarrow{\sigma}_r} & (r = t-1) \quad \textit{or} \quad (t = 2 \quad \textit{and} \quad r = \bar{1}); \\ x_{\overrightarrow{\sigma}_r} & (|i-t| \geqslant 2) \quad \textit{or} \quad (t = \bar{1} \quad \textit{and} \quad r \neq 2) \quad \textit{or} \quad (t \neq 2 \quad \textit{and} \quad r = \bar{1}). \end{cases}$$

Proof. The subgroup generated by $\sigma_1, \dots, \sigma_{n-1}$ and $\sigma_{\bar{1}}$ is $A(D_n)$. Then using Propositions 23 – 26, we get the result from the same argument as in [4, Theorems 6.11]. □

6 The general case

In this section, we generalize the graph with a single cycle from Section 5 to a graph with an arbitrary number of cycles. Moreover, we generalize the result of Theorem 27, see Theorem 31.

In the following theorem, we show that all graphs with the same number of cycles are equivalent.

Theorem 28. *Every graph T that includes at least one anti-cycle is equivalent to a graph $T'^{(m)}$, where $T'^{(m)}$ consists of m cycles including the edges $\sigma_1, \ldots, \sigma_{n-1}$, u_i for $1 \leqslant i \leqslant m$ and a negatively signed edge $\sigma_{\bar{1}}$ that connects the vertices v_1 and v_2 (see Figure 12).*

Remark. *Here $m+1$ is the number of the cycles and anti-cycles in T, i.e., $m+1$ is the number of cycles in $\bar{T}$, where $\bar{T}$ is the graph obtained from T by omitting the signs.*

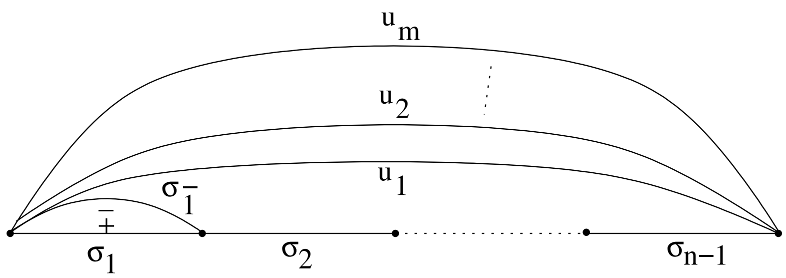
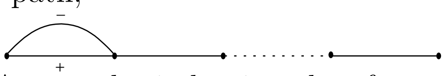


Figure 12

Proof. The proof is by induction on m. In the case $m = 0$, T contains an anti-cycle. Hence the number of negative signs in T is odd. By Theorem 9, $\bar{T}$ is equivalent by triangulation to a cycle since $\bar{T}$ contains just one cycle. Since triangulation can only convert an anti-cycle to another anti-cycle and not to a cycle, T is equivalent to an anti-cycle connected to a path,


which is $T'^{(0)}$.

Assume by induction that for $m \leqslant k$ the theorem holds. Then T, with $k+1$ cycles and anti-cycles, is equivalent to $T'^{(m)}$. Now assume $m = k+1$. If we consider only k cycles (i.e., the subgraph $\tilde{T}$ obtained by omitting one edge from one of the cycles or one of the anti-cycles of T), $\tilde{T}$ is equivalent to $T'^{(k)}$ by the induction hypothesis. Since triangulation preserves the number of the edges of T, T contains one more edge e which does not appear in $T'^{(k)}$. The edge e forms one more cycle or one more anti-cycle, since triangulation preserves the number of cycles. Hence T is equivalent to the graph $T'^{(k),e}$ (see Figure 13), where the edge e connects two vertices v_i and v_j.

Then we look at the subgraph $T'^{(0),e}$ of $T'^{(k),e}$ that contains the edges σ_i, $1 \leqslant i \leqslant n-1$, $\sigma_{\bar{1}}$ and e (i.e. $T'^{(0),e}$ we get from $T'^{(k),e}$ by omitting u_i for $1 \leqslant i \leqslant k$). By Theorems 15 and 16, $T'^{(0),e}$ is equivalent to $T'^{(1)}$. Hence $T'^{(k),e}$ is equivalent to $T'^{(k+1)}$ (adding the edges u_i to $T'^{(1)}$). ☐

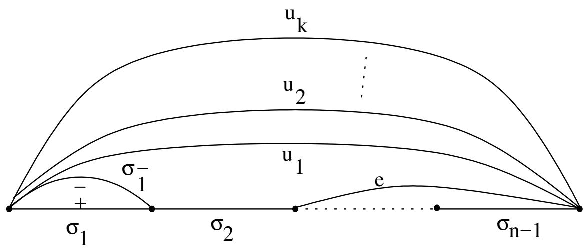


Figure 13

Proposition 29. *In $A_Y(T'^{(m)})$, the following relations hold for $1 \leqslant i < j \leqslant m$:*

1. $\langle \sigma_1 u_i \sigma_1^{-1}, u_j \rangle = 1$,
2. $\langle \sigma_{\bar{1}} u_i \sigma_{\bar{1}}^{-1}, u_j \rangle = 1$,
3. $\langle \sigma_{n-1} u_i \sigma_{n-1}^{-1}, u_j \rangle = 1$,
4. $[\sigma_1 u_i \sigma_1^{-1}, \sigma_{n-1} u_j \sigma_{n-1}^{-1}] = 1$,
5. $[\sigma_{\bar{1}} u_i \sigma_{\bar{1}}^{-1}, \sigma_{n-1} u_j \sigma_{n-1}^{-1}] = 1$,
6. $[\sigma_1 u_i \sigma_1^{-1}, \sigma_{\bar{1}} u_j \sigma_{\bar{1}}^{-1}] = 1$,
7. $[\sigma_1 u_j \sigma_1^{-1}, \sigma_{\bar{1}} u_i \sigma_{\bar{1}}^{-1}] = 1$.

Proof. The proof is derived directly from the definition of $A_Y(T'^{(m)})$, see Figure 13. $\square$

In the spirit of [4], we define $x^{(j)}_{\overrightarrow{\alpha}} = u_j \alpha^{-1}$ and $x^{(j)}_{\overleftarrow{\alpha}} = \alpha^{-1} u_j$ for $1 \leqslant j \leqslant m$.

Now we define the group $G(T'^{(m)})$, which is a quotient of $A_Y(T'^{(m)})$. In Theorem 31, we describe its structure precisely.

Definition 30. *Let $G(T'^{(m)})$ be a quotient of $A_Y(T'^{(m)})$ by the following relations:*

1. $[x^{(j)}_{\overrightarrow{\alpha}}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1$;
2. $[x^{(j)}_{\overrightarrow{\alpha}}, \sigma_{\bar{1}}\sigma_1^{-1}\alpha\sigma_1\sigma_{\bar{1}}^{-1}] = 1$;
3. $[x^{(j)}_{\overrightarrow{\alpha}}, \sigma_1\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_1^{-1}] = 1$;
4. $[x^{(j)}_{\overrightarrow{\alpha}}, \sigma_{n-1}\sigma_{\bar{1}}^{-1}\alpha\sigma_{\bar{1}}\sigma_{n-1}^{-1}] = 1$;
5. $[x^{(j)}_{\overrightarrow{\alpha}}, \alpha^2] = 1$, *for* $1 \leqslant j \leqslant m$.

Finally we present our main result, that is, the structure of the needed group $G(T'^{(m)})$.

Theorem 31. *$G(T'^{(m)}) \simeq K^{(m,n)} \rtimes A(D_n)$, where $K^{(m,n)}$ is the group generated by $x^{(i)}_{\overrightarrow{\sigma}_k}$ and $x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}$, $1 \leqslant i, j \leqslant m$, $1 \leqslant r, t \leqslant n-1$ with the following relations:*

1. $[x^{(i)}_{\overrightarrow{\sigma}_r}, x^{(j)}_{\overrightarrow{\sigma}_t}] = 1$, $|r-t|>1$;

2. $[x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{(j)}_{\overrightarrow{\sigma}_t}] = 1$, $t \neq 2$;

3. $[x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{(i)}_{\overrightarrow{\sigma}_2}] = [x^{(i)}_{\overrightarrow{\sigma}_r}, x^{(i)}_{\overrightarrow{\sigma}_t}] = z_i$, $|r-t|=1$;

4. $[x^{(i)}_{\overrightarrow{\sigma}_r} x^{(i)}_{\overrightarrow{\sigma}_{r-1}}, x^{(j)}_{\overrightarrow{\sigma}_r} x^{(j)}_{\overrightarrow{\sigma}_{r+1}}] = [x^{(i)}_{\overrightarrow{\sigma}_2} x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{(j)}_{\overrightarrow{\sigma}_2} x^{(j)}_{\overrightarrow{\sigma}_1}] = [x^{(i)}_{\overrightarrow{\sigma}_2} x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{(j)}_{\overrightarrow{\sigma}_2} x^{(j)}_{\overrightarrow{\sigma}_3}] = 1$;

5. $[x^{(i)}_{\overrightarrow{\sigma}_r}, x^{-(j)}_{\overrightarrow{\sigma}_{r+1}}] = [x^{-(i)}_{\overrightarrow{\sigma}_{r+1}}, x^{(j)}_{\overrightarrow{\sigma}_r}]$;

6. $[x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{-(j)}_{\overrightarrow{\sigma}_2}] = [x^{-(i)}_{\overrightarrow{\sigma}_2}, x^{(j)}_{\overrightarrow{\sigma}_{\bar{1}}}]$;

7. z_i *are central elements for* $1 \leqslant i \leqslant m$, *and* $z_i^2 = 1$.

where similarly to Theorem 27, the action of $A(D_n)$ *on* $x^{(i)}_{\overrightarrow{\sigma}_r}$, $1 \leqslant r \leqslant n-1$ *and on* $x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}$ *is the following:*

$\sigma_t(z_i) = z_i$, *for* $1 \leqslant i \leqslant m$,

$$\sigma_t(x^{(i)}_{\overrightarrow{\sigma}_r}) = \begin{cases} z_i x^{-(i)}_{\overrightarrow{\sigma}_r} & r=t; \\ x^{(i)}_{\overrightarrow{\sigma}_t} x^{(i)}_{\overrightarrow{\sigma}_r} & (r=t+1) \quad or \quad (t=\bar{1} \quad and \quad r=2); \\ x^{(i)}_{\overrightarrow{\sigma}_t} x^{(i)}_{\overrightarrow{\sigma}_r} & (r=t-1) \quad or \quad (t=2 \quad and \quad r=\bar{1}); \\ x^{(i)}_{\overrightarrow{\sigma}_r} & (|i-t| \geqslant 2) \quad or \quad (t=\bar{1} \quad and \quad r \neq 2) \quad or \quad (t \neq 2 \quad and \quad r=\bar{1}). \end{cases}$$

Proof. The relations in $K^{(m,n)}$ that do not involve $x_{\overrightarrow{\sigma}_{\bar{1}}}$ were proved in [4, Section 10]. Now look at the subgroup $K'^{(m,n)}$ of $K^{(m,n)}$, where $K'^{(m,n)}$ is generated by $x_{\overrightarrow{\sigma}_{\bar{1}}}$ and $x_{\overrightarrow{\sigma}_t}$ where $2 \leqslant t \leqslant n-1$ (i.e., without the generator $x_{\overrightarrow{\sigma}_1}$). Then $K'^{(m,n)}$ is isomorphic to $\tilde{K}^{(m,n)}$, where $\tilde{K}^{(m,n)}$ is the group $K^{(m,n)}$ from [4, Section 10], where $\varphi(x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}) = x^{(i)}_{\sigma_1}$, $\varphi(x^{(i)}_{\overrightarrow{\sigma}_t}) = x^{(i)}_{\sigma_t}$ for $2 \leqslant t \leqslant n-1$. Hence, by [4, Section 10], $[x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{(j)}_{\overrightarrow{\sigma}_t}] = 1$ for $t \geqslant 3$, $[x^{(i)}_{\overrightarrow{\sigma}_2} x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{(j)}_{\overrightarrow{\sigma}_2} x^{(j)}_{\overrightarrow{\sigma}_3}] = 1$, and $[x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}, x^{-(j)}_{\overrightarrow{\sigma}_2}] = [x^{-(i)}_{\overrightarrow{\sigma}_2}, x^{(j)}_{\overrightarrow{\sigma}_{\bar{1}}}]$. It remains to prove the relation between $x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}$ and $x^{(i)}_{\overrightarrow{\sigma}_1}$. By Proposition 29 ((5) and (6)), $[\sigma_1 u_i \sigma_1^{-1}, \sigma_{\bar{1}}^{-1} u_j \sigma_{\bar{1}}] = 1$ implies that $[\sigma_1 u_i \alpha^{-1} \sigma_1^{-1}, \sigma_{\bar{1}}^{-1} u_j \alpha^{-1} \sigma_{\bar{1}}] = 1$, since $[\sigma_1 \alpha \sigma_1^{-1}, \sigma_{\bar{1}} \alpha \sigma_{\bar{1}}^{-1}] = 1$.
Hence $[\sigma_1 x^{(i)}_{\overrightarrow{\alpha}} \sigma_1^{-1}, \sigma_{\bar{1}} x^{(j)}_{\overrightarrow{\alpha}} \sigma_{\bar{1}}^{-1}] = 1$. Then, from Proposition 24, $[\alpha^{-1} x^{(i)}_{\overrightarrow{\sigma}_1} \alpha, \alpha^{-1} x^{(j)}_{\overrightarrow{\sigma}_{\bar{1}}} \alpha] = 1$ for each i and j. This completes the proof. ☐

Now, for $1 \leqslant i \leqslant m$, $1 \leqslant r \leqslant n-1$, and $r = \bar{1}$ defining the element $a^{(i)}_r$ to be:

- $a^{(i)}_{n-1} = x^{(i)}_{\overrightarrow{\sigma}_{n-1}}$;
- $a^{(i)}_r = x^{(i)}_{\overrightarrow{\sigma}_{n-1}} x^{(i)}_{\overrightarrow{\sigma}_{n-2}} \cdots x^{(i)}_{\overrightarrow{\sigma}_r}$, for $1 \leqslant r \leqslant n-2$;
- $a^{(i)}_{\bar{1}} = x^{(i)}_{\overrightarrow{\sigma}_{n-1}} x^{(i)}_{\overrightarrow{\sigma}_{n-2}} \cdots x^{(i)}_{\overrightarrow{\sigma}_2} x^{(i)}_{\overrightarrow{\sigma}_{\bar{1}}}$.

Theorem 32. *By using the generators* $a_{\bar{1}}^{(i)}, a_1^{(i)}, \dots a_{n-1}^{(i)}$ *instead of the generators* $x_{\vec{\sigma}_{\bar{1}}}^{(i)}, x_{\vec{\sigma}_1}^{(i)}, \dots x_{\vec{\sigma}_{n-1}}^{(i)}$*, the presentation of* $K^{(m,n)}$ *can be simplified in the following way,*

- $[a_{\bar{1}}^{(i)}, a_1^{(j)}] = 1$*, for any* $1 \leqslant i, j \leqslant m$*;*
- $[a_r^{(i)}, a_s^{(i)}] = z_i$*, for* $r \neq s$*,* $\bar{r} \neq s$*, and* $1 \leqslant i \leqslant m$*;*
- $[a_r^{(i)}, a_s^{(j)}] = [a_{r'}^{(i)}, a_{s'}^{(j)}]$ *for any* $r \neq s$*,* $\bar{r} \neq s$*, and* $r' \neq s'$*,* $\bar{r'} \neq s'$*, and* $1 \leqslant i, j \leqslant m$*.*

Proof. The proof of the first relation:

$$\begin{aligned}
[a_{\bar{1}}^{(i)}, a_1^{(j)}] &= [x_{\vec{\sigma}_{n-1}}^{(i)} x_{\vec{\sigma}_{n-2}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(i)} x_{\vec{\sigma}_{\bar{1}}}^{(i)}, x_{\vec{\sigma}_{n-1}}^{(i)} x_{\vec{\sigma}_{n-2}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(j)} x_{\vec{\sigma}_1}^{(j)}] \\
&= [\sigma_{n-1}(x_{\vec{\sigma}_{n-1}}^{(i)} x_{\vec{\sigma}_{n-2}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(i)} x_{\vec{\sigma}_{\bar{1}}}^{(i)}), \sigma_{n-1}(x_{\vec{\sigma}_{n-1}}^{(i)} x_{\vec{\sigma}_{n-2}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(j)} x_{\vec{\sigma}_1}^{(j)})] \\
&= [x_{\vec{\sigma}_{n-2}}^{(i)} x_{\vec{\sigma}_{n-3}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(i)} x_{\vec{\sigma}_{\bar{1}}}^{(i)}, x_{\vec{\sigma}_{n-2}}^{(i)} x_{\vec{\sigma}_{n-3}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(j)} x_{\vec{\sigma}_1}^{(j)}] \\
&= [\sigma_{n-2}(x_{\vec{\sigma}_{n-2}}^{(i)} x_{\vec{\sigma}_{n-3}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(i)} x_{\vec{\sigma}_{\bar{1}}}^{(i)}), \sigma_{n-2}(x_{\vec{\sigma}_{n-2}}^{(i)} x_{\vec{\sigma}_{n-3}}^{(i)} \cdots x_{\vec{\sigma}_2}^{(j)} x_{\vec{\sigma}_1}^{(j)})] \\
&= \dots \\
&= [x_{\vec{\sigma}_2}^{(i)} x_{\vec{\sigma}_{\bar{1}}}^{(i)}, x_{\vec{\sigma}_2}^{(j)} x_{\vec{\sigma}_1}^{(j)}] = 1.
\end{aligned}$$

The proof of the other two relations are the same to the proof of [4, Colloraly 10.7]. ☐

Note that by similar arguments as in [4], the word problem is solvable in $K^{(m,n)}$ and solvable in $A(D_n)$ too, we get

Corollary 33. *The word problem is solvable in* $G(T)$*.*

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