Minimum cuts of distance-regular digraphs

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Abstract

In this paper, we investigate the structure of minimum vertex and edge cuts of distance-regular digraphs. We show that each distance-regular digraph Γ, different from an undirected cycle, is super edge-connected, that is, any minimum edge cut of Γ is the set of all edges going into (or coming out of) a single vertex. Moreover, we will show that except for undirected cycles, any distance regular-digraph Γ with diameter $D = 2$, degree $k \leqslant 3$ or $\lambda = 0$ (λ is the number of 2-paths from u to v for an edge uv of Γ) is super vertex-connected, that is, any minimum vertex cut of Γ is the set of all out-neighbors (or in-neighbors) of a single vertex in Γ. These results extend the same known results for the undirected case with quite different proofs.

Keywords: Distance-regular digraphs; Strongly regular digraphs; Minimum cuts; Connectivity

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1 Introduction

A *digraph* (or a *directed graph*) is an ordered pair $\Gamma = (V, E)$, where V is a set whose elements are called *vertices* or *nodes*, and E is a set of ordered pairs of vertices, called *arcs* or *directed edges*. In contrast, a graph in which the edges are bidirectional is called an *undirected graph*. In order to simplify our notations, we will use the word "edge" instead of the word "directed edge" in this paper. A digraph with no multiple edges or loops (corresponding to a binary adjacency matrix with 0's on the diagonal) is called *simple*. Here we only consider finite simple graphs and digraphs. A digraph Γ is called *regular* with degree (valency) k, if the in-degree and the out-degree at each vertex of Γ are equal to k. We will denote by $\partial_\Gamma(x, y)$ (or briefly $\partial(x, y)$) the distance from a vertex x to a vertex y in a digraph Γ. For every vertex x, we define the *directed shell* $\Gamma_k^+(x)$ (resp. $\Gamma_k^-(x)$) to be the set of vertices at distance k from x (resp. the set of vertices from which x is at distance k). For distinct vertices u and v of Γ, we say that u is *adjacent* to v if there is an edge (directed edge) from u to v. For every $x \in V(\Gamma)$ and $A \subseteq V(\Gamma)$, by $d_A^+(x)$ and $d_A^-(x)$ we mean the number of out-neighbors and in-neighbors of x in A, respectively. The maximum (directed) distance between distinct pairs of vertices is called the *diameter* of Γ and is denoted by D. The *girth* g is the smallest length of a cycle in Γ. In this paper, by a *walk*, *path* or *cycle*, we mean a directed walk, path or cycle. A digraph is (*strongly*) *connected* if there is a path between every pair of vertices. For a connected digraph Γ, a set of edges $F \subseteq E(\Gamma)$ (resp. a set of vertices $F \subseteq V(\Gamma)$) is called an *edge-cut* (resp. a *vertex-cut*) if $\Gamma - F$ is disconnected. The sizes of the minimum edge-cut and the minimum vertex-cut of a connected digraph Γ are called the *edge connectivity* and the *vertex connectivity* of Γ, respectively. A digraph is called *super edge-connected* if any minimum edge cut of Γ is the set of all edges going into (or coming out of) a single vertex. Also, a digraph is called *super vertex-connected* if any minimum vertex cut of Γ is the set of all out-neighbors (or in-neighbors) of a single vertex in Γ. All of the above concepts can be defined for undirected graphs in a natural way, it only suffices to consider an undirected graph as a directed graph whose edges are bidirected. For more information on digraphs, we refer the reader to [2]. Throughout this paper, let $\Gamma = (V, E)$ be a connected simple digraph of order n and diameter D.

A *distance-regular graph* is a regular graph such that for any two vertices v and w at distance i, the number of vertices adjacent to w and at distance j from v only depends on i and j. Distance-regular graphs with diameter two are precisely the strongly regular graphs, which have been studied by several mathematicians [3]. For more background on different concepts of distance-regularity in graphs see [5, 6, 11, 16]. The concept of *"distance-regular digraphs"* was introduced by Damerell [12]. A digraph Γ with diameter D is distance-regular if for every two vertices u and v with $\partial(u, v) = k$ for $0 \leqslant k \leqslant D$, the numbers $a_{i1}^k = |\Gamma_i^+(u) \cap \Gamma_1^+(v)|$ for each i with $0 \leqslant i \leqslant k+1$, are independent of the choices of u and v. Trivial examples of distance-regular digraphs are the directed cycles (the distance-regular digraphs of degree 1). Moreover, distance-regular digraphs with girth $g = 2$ are precisely the distance-regular graphs. We refer the reader to [14, 18, 19] and the references therein, for more information on the distance-regular digraphs.

If we replace $\Gamma_1^+(v)$ by $\Gamma_1^-(v)$, in the above definition, we get a new family of digraphs called *"weakly distance-regular digraphs"*. This concept was introduced by Comellas et al. [10], as a generalization of distance-regular digraphs. In fact, distance-regular digraphs are precisely weakly distance-regular digraphs with *normal* adjacency matrices (a matrix $\mathbf{A}$ is normal if $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$). Also, in [10] it has been shown that a digraph Γ of diameter D is weakly distance-regular if for each nonnegative integer $\ell \leqslant D$, the number of walks of length ℓ from a vertex u to a vertex v only depends on ℓ and their distance $\partial(u, v)$. Note that in [20], Suzuki and Wang suggested that a "weakly distance-regular digraph" is a digraph with the following property: for all vertices u and v with $(\partial(u, v), \partial(v, u)) = (k_1, k_2)$, the number of vertices w satisfying $(\partial(u, w), \partial(w, u)) = (i_1, i_2)$ and $(\partial(v, w), \partial(w, v)) = (j_1, j_2)$ depends only on the values $k_1, k_2, i_1, i_2, j_1, j_2$. In this paper, we do not assume the Suzuki and Wang's definition of weakly distance-regular digraphs and we only consider the mentioned definition that was introduced by Comellas et al. in [10].

The weakly distance-regular digraphs with diameter two are the same as the strongly regular digraphs introduced by Duval in [13] as an extension of strongly regular graphs to the directed case. A k-regular digraph on n vertices is called a *strongly regular digraph* with parameters (n, k, t, λ, μ) if the number of walks of length two between two vertices is t, λ or μ when these vertices are the same, adjacent, or not adjacent, respectively. The case $t = k$ corresponds to the undirected case. On the other extreme, the case $t = 0$, we have tournaments. For more details, we refer to Brouwer's website [4].

In [9] Brouwer and Mesner showed that all strongly regular graphs are super vertex-connected. Also, in [7] Brouwer and Haemers proved that the edge connectivity of a distance-regular graph of degree k is equal to k and, for $k > 2$, the graph is super edge-connected. The same result for minimum vertex cuts of distance-regular graphs was obtained by Brouwer and Koolen in [8]. In fact they showed that the vertex-connectivity of a distance-regular graph Γ of degree k equals k. Moreover, Γ is super vertex-connected for $k > 2$. Note that here we assume that the graph on one vertex with no edge is disconnected and so each complete graph is super vertex-connected, while in [8] these graphs are the only exception among the distance-regular graphs that are not super vertex-connected (since the authors considered a complete graph on one vertex as a connected graph). Note that distance-regular graphs with degree 2 are precisely the (undirected) cycles. Therefore, except undirected cycles, each distance-regular graph is super edge-connected as well as super vertex-connected. The eigenvalue methods are the main tools to obtain the most of the above results. In this paper, we investigate to the structure of minimum edge and vertex cuts in distance-regular and strongly regular digraphs and we only use the combinatorial techniques to extend the mentioned results on the minimum cuts for directed case.

The paper is organized as follows. In the next section, we show that each distance-regular digraph, different from an undirected cycle, is super edge-connected. In Section 3, we will investigate to the problem of super vertex-connectivity of distance-regular digraphs and we will show that except undirected cycles, all distance regular-digraphs with diameter $D = 2$, degree $k \leqslant 3$ or $\lambda = 0$ are super vertex-connected, where λ is the number of 2-paths from u to v for an edge uv of Γ. Based on these results we conjecture that (similar to the main result in Section 2) except undirected cycles, each distance-regular digraph is super vertex-connected. In Section 4, we will investigate to the minimum cuts of strongly regular digraphs. A well-known result in [15] implies that each strongly regular digraph with degree $k \geqslant 3$ is super edge-connected. Hence by only looking at the strongly regular digraphs with degree $k = 1, 2$ we will characterize those that are super edge-connected. Moreover, we will give a strongly regular digraph with degree $k = 3$ that is not super vertex-connected. This example shows that the statement of our conjecture in Section 3 (Conjecture 9) does not hold for strongly regular digraphs. Note that distance-regular digraphs are precisely weakly distance-regular digraphs with normal adjacency matrices. Also, weakly distance-regular digraphs with diameter 2 are precisely strongly regular digraphs. Based on the above results, in the last section we conjecture that each weakly distance-regular digraph with degree $k > 2$ is super edge-connected. In contrast, a strongly regular digraph with degree $k = 3$ given in Section 4 (see Figure 1) shows that the same conjecture for the super vertex-connectivity of weakly distance-regular digraphs is incorrect.

2 Minimum edge cuts of distance-regular digraphs

In this section, we investigate to the minimum edge cuts of distance-regular digraphs and we show that if a distance-regular digraph Γ is not an undirected cycle, then Γ is super edge-connected. Note that a distance-regular digraph Γ with girth $g = 2$ is a distance-regular graph and, due to a result of Brouwer and Haemers in [7], Γ is super edge-connected, unless it is an undirected cycle. Hence here we only focus on distance-regular digraphs with girth $g \geqslant 3$. First we give a useful known result that will be used later on.

Lemma 1. ([17]) *In any edge cut $[A, V - A]$ of a regular digraph, the number of edges from A to $V - A$ equals the number of edges from $V - A$ to A.*

We remind that for every two vertices u and v with $0 \leqslant \partial(u, v) = k \leqslant D$ of a distance-regular digraph Γ with diameter D, the numbers $a_{i1}^k = |\Gamma_i^+(u) \cap \Gamma_1^+(v)|$ and $b_{i1}^k = |\Gamma_i^+(u) \cap \Gamma_1^-(v)|$ for each i with

$0 \leqslant i \leqslant k+1$, are independent of the choices of u and v. Since, the adjacency matrix $\mathbf{A}$ of a distance-regular digraph Γ is normal, that is, the matrix satisfying $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$, we have $a_{11}^k = |\Gamma_1^+(u) \cap \Gamma_1^+(v)| = |\Gamma_1^-(u) \cap \Gamma_1^-(v)|$ for two vertices u and v with $0 \leqslant \partial(u,v) = k \leqslant D$. Finally we remind that the notations b_{11}^1 and b_{11}^2 are usually denoted by λ and μ, respectively. In fact in a distance-regular digraph Γ, the parameters λ and μ are the numbers of 2-paths from u to v when $\partial(u,v)=1$ and $\partial(u,v)=2$, respectively.

Now we introduce a family $\mathcal{D}$ of distance-regular digraphs as an extension of trivial examples (the directed cycles). Assume that $A \times B = \{(a,b) | a \in A, b \in B\}$ for two sets A and B. For $t \geqslant 3$, we denote by $C[X_1, X_2, \ldots, X_t]$ a digraph with vertex set $V = \bigcup_{i=1}^t X_i$ and edge set $E = \bigcup_{i=1}^{t-1}(X_i \times X_{i+1}) \bigcup (X_t \times X_1)$. If $t \geqslant 3$ and $\rho = |X_i|$ is constant, then $\Gamma = C[X_1, X_2, \ldots, X_t]$ is a distance-regular digraph with $\lambda = 0$. We denote this family of distance-regular digraphs by $\mathcal{D}$. In the following we will see that the family $\mathcal{D}$ are exactly all distance-regular digraphs with $\lambda = 0$ that are not undirected. Since distance regular digraphs with girth $g = 2$ are precisely distance-regular graphs, we can say that the family $\mathcal{D}$ are exactly all distance-regular digraphs with $g \geqslant 3$ and $\lambda = 0$.

In [12], Damerell showed that every distance-regular digraph Γ with girth g is stable; that is, $\partial(x,y) + \partial(y,x) = g$ for every two vertices x and y at distance $0 < \partial(x,y) < g$. Consequently, every distance-regular digraph with girth $g \geqslant 3$ has diameter $D = g$ (known as long type) or $D = g-1$ (known as short type). Also, he showed that every distance-regular digraph of long type is obtained from a distance-regular digraph of short type by a known construction as follows:

Let Ω be a distance-regular digraph of short type and $m > 1$ be an integer. Now let Γ be a digraph, where

$$V(\Gamma) = V(\Omega) \times \{1, 2, \ldots, m\}$$

and

$$E(\Gamma) = \{(u,i)(v,j) | uv \in E(\Omega), 1 \leqslant i, j \leqslant m\}.$$

It is easy to see that Γ is a distance-regular digraph of long type with the same girth as Ω. As you see in the following theorem, Damerell showed that the converse is true.

Theorem 2. ([12]) *Every distance-regular digraph Γ of long type is obtained from a distance-regular digraph Ω of short type, of the same girth, by the construction described above. Starting from a distance-regular digraph Γ of long type, a distance-regular digraph Ω of short type is obtained by identifying all antipodal vertices of Γ.*

In [18], it is shown that for every non-trivial distance-regular digraph of short type we have $\lambda > 0$. Therefore, the only distance-regular digraphs of short type with $\lambda = 0$ are the directed cycles. Now let Γ be the distance-regular digraph of long type that is obtained from a distance-regular digraph Ω of short type by the Damerell's construction described above. Clearly the parameter λ for Γ is m times of the same parameter for Ω. Therefore, every distance-regular digraph of long type with $\lambda = 0$ is obtained from a directed cycle by the Damerell's construction and thus it is a member of $\mathcal{D}$. Hence $\mathcal{D}$ are precisely all distance-regular digraphs that are not undirected (or equivalently $g \geqslant 3$) with $\lambda = 0$.

In fact the family $\mathcal{D}$ of digraphs is a type of the so-called *generalized cycles*. A *generalized t-cycle* is a digraph whose set of vertices can be partitioned into t parts that are cyclically ordered in such a way that the vertices in one part are adjacent only to vertices in the next part.

In order to study the connectivity of digraphs, we need a new parameter related to the number of shortest paths that was used in [15]. For a given digraph Γ with diameter D, assume that $l = l(\Gamma)$, $1 \leqslant l \leqslant D$, is the greatest integer such that for every two vertices u and v in $V(\Gamma)$, the shortest path from u to v is unique when $\partial(u,v) \leqslant l$. Moreover, there is no path of length $\partial(u,v)+1$ from u to v if $\partial(u,v) < l$.

Lemma 3. ([1]) *Let Γ be a generalized t-cycle, $t \geqslant 3$, with parameter $l = l(\Gamma)$, diameter D and minimum degree $\delta \geqslant 2$. If $D \leqslant 2l + t - 3$, then Γ is super vertex-connected. Moreover, Γ is super edge-connected if $D \leqslant 2l + t - 2$.*

Clearly each $\Gamma \in \mathcal{D}$ is a generalized t-cycle with $t \geqslant 3$, parameter $l = l(\Gamma) \geqslant 1$, diameter $D = t-1$ and minimum degree $\delta \geqslant 2$ and so by Lemma 3, Γ is super vertex-connected and super edge-connected.

The statement of the following lemma about a_{11}^l was shown in [18] for non-trivial distance-regular digraphs of short type. The proof is not correct as stated, although the statement remains valid as we demonstrate. Here we give an alternative way to prove this result for all distance-regular digraphs with $g \geqslant 3$ and $\lambda \neq 0$.

Lemma 4. *Let Γ be a distance-regular digraph with diameter D and $\Gamma \notin \mathcal{D}$. Then for every $2 \leqslant l \leqslant D$, we have $a_{11}^l \geqslant 1$.*

Proof. Note that $\Gamma \notin \mathcal{D}$, so we have $g \geqslant 3$ and $\lambda \neq 0$. First let $l = g$ (this case can happen only when Γ is a distance-regular digraph of long type). Then consider a path $u_1u_2\dots u_{g+1}$ of minimum length between two vertices $u = u_1$ and $v = u_{g+1}$ at distance $\partial(u,v) = g$. Clearly $\partial(u_2, v) = D-1 = g-1$. Using the fact that Γ is stable (which means that $\partial(x,y) + \partial(y,x) = g$ for every two vertices x and y at distance $0 < \partial(x,y) < g$) we have $\partial(v,u_2) = 1$ and so $\Gamma_1^+(u) \cap \Gamma_1^+(v) \neq \emptyset$. This fact implies that $a_{11}^l \geqslant 1$. Now let $2 \leqslant l \leqslant g-1$. Assume that $w \in V(\Gamma)$, $u \in \Gamma_1^+(w)$ and H is the digraph induced by $\Gamma_1^+(w)$. Clearly $|H_1^-(u)| = \lambda$ and for each $x \in H_1^-(u)$ we have $g-1 = \partial_\Gamma(u,x) \leqslant \partial_H(u,x)$, where $\partial_\Gamma(u,x)$ and $\partial_H(u,x)$ are the distances from u to x in Γ and H, respectively. Now let $x \in H_1^-(u)$ and $P = u_0u_1\dots u_t$ be a minimum path in H from $u_0 = u$ to $u_t = x$. We have $t \geqslant g-1$, since $g-1 = \partial_\Gamma(u,x) \leqslant \partial_H(u,x)$. First assume that $\partial_\Gamma(u,u_i) = \partial_H(u,u_i)$ for each $2 \leqslant i \leqslant t$. Then $t = g-1$ and $u_l \in \Gamma_l^+(u)$ for every $2 \leqslant l \leqslant g-1$. Since $w \in \Gamma_1^-(u) \cap \Gamma_1^-(u_l)$, we have $\Gamma_1^+(u) \cap \Gamma_1^+(u_l) \neq \emptyset$, which implies that $a_{11}^l \geqslant 1$, this follows from the fact that the adjacency matrix A of Γ is normal. Now let $2 \leqslant i_1 \leqslant t$ be the minimum number i with $\partial_\Gamma(u,u_i) < \partial_H(u,u_i)$. Let $S_{j,k} = \{\partial_\Gamma(u,u_i) | j \leqslant i < k\}$ for every $0 \leqslant j < k \leqslant t+1$ and $S = \{\partial_\Gamma(u,u_i) | 0 \leqslant i \leqslant t\}$. Our goal is to show that $\{0,1,2,\dots,g-1\} \subseteq S$. Therefore, for every $2 \leqslant l \leqslant g-1$, we have $\Gamma_l^+(u) \cap V(P) \neq \emptyset$. On the other hand, for each $v \in \Gamma_l^+(u) \cap V(P)$ we have $w \in \Gamma_1^-(u) \cap \Gamma_1^-(v)$ and so $\Gamma_1^+(u) \cap \Gamma_1^+(v) \neq \emptyset$, which implies that $a_{11}^l \geqslant 1$. To show that $\{0,1,2,\dots,g-1\} \subseteq S$, consider the integers $i_0 = 0 < i_1 < i_2 < \dots < i_m \leqslant i_{m+1} = t$ with maximum m such that for each $1 \leqslant j \leqslant m$, an integer $i_j \in (i_{j-1}, t]$ is the minimum number with $\partial_\Gamma(u_{i_{j-1}}, u_{i_j}) < \partial_H(u_{i_{j-1}}, u_{i_j})$. Clearly $m \geqslant 1$.

Claim 5. *For each $0 \leqslant j \leqslant m$, we have $\{0,1,2,\dots,\partial_\Gamma(u,u_{i_j}) + i_{j+1} - i_j - 1\} \subseteq S_{0,i_{j+1}}$.*

Proof. We give a proof for Claim 5 by induction on j. For each $0 \leqslant i < i_1$, we have $\partial_\Gamma(u,u_i) = \partial_H(u,u_i) = i$ and so $S_{0,i_1} = \{0,1,2,\dots,i_1-1\}$. Hence our claim holds for $j = 0$. Now assume that the statement of Claim 5 holds for an integer $j = k \leqslant m-1$. We are going to show that the statement of this claim holds for $j = k+1$. That follows from the equality $S_{0,i_{k+1}} = S_{0,i_k} \cup \{\partial_\Gamma(u,u_i) | i_k \leqslant i < i_{k+1}\}$ and the fact that $\partial_\Gamma(u,u_i) = \partial_\Gamma(u,u_{i_k}) + \partial_H(u_{i_k},u_i) = \partial_\Gamma(u,u_{i_k}) + i - i_k$ for every $i_k \leqslant i < i_{k+1}$. □

Now using Claim 5 for $j = m$, we have $\{0,1,2,\dots,\partial_\Gamma(u,u_{i_m}) + t - i_m - 1\} \subseteq S_{0,i_{m+1}}$. On the other hand $g-1 = \partial_\Gamma(u,x) \leqslant \partial_\Gamma(u,u_{i_m}) + t - i_m$. Therefore $\{0,1,2,\dots,g-1\} \subseteq S_{0,i_{m+1}} \cup \{\partial_\Gamma(u,x)\} = S$ and we are done. □

The following theorem is the main result of this section.

Theorem 6. *A distance-regular digraph Γ is super edge-connected, unless it is an undirected cycle.*

Proof. Assume that Γ is a distance-regular digraph with degree k and it is not an undirected cycle. Note that $g = 2$ implies that Γ is an undirected graph and so we are done due to a result of Brouwer and Haemers in [7]. Hence we may assume that $g \geqslant 3$. We are going to show that Γ is super edge-connected. If $k = 1$, then Γ is a directed cycle and clearly any minimum edge cut is an edge. Now, assume that $k > 1$. First let $\Gamma \in \mathcal{D}$ and $\Gamma = C[X_1, X_2, \dots, X_t]$, where $t \geqslant 3$ and $|X_i| = k$ for each $1 \leqslant i \leqslant t$. In this case Γ is a generalized t-cycle with $t \geqslant 3$, parameter $l = l(\Gamma) \geqslant 1$, diameter $D = t-1$ and minimum degree $\delta \geqslant 2$ and so by Lemma 3, Γ is super edge-connected.

Now let $\Gamma \notin \mathcal{D}$. Suppose that $F = [A,B]$ is a minimum edge cut of Γ. Since the set of all edges going into (or coming out of) a single vertex is an edge cut, we have $|F| \leqslant k$. Set $r = \max\{d_A^+(x) | x \in A\}$, where $d_A^+(x) = |\Gamma_1^+(x) \cap A|$. Clearly every vertex $x \in A$ has at least $k - r$ out-neighbors in B. It follows that there are at least $(k - r)|A|$ edges from A to B and so, we have $r + 1 \leqslant |A| \leqslant \frac{k}{k-r}$. Therefore $r \in \{0, k-1, k\}$.

If $r = 0$, then $|A| = 1$ and $F = [A, B]$ is a set of all edges coming out of a single vertex in A and we are done. Now suppose that $r = k - 1$. So $|A| = k$ and since $|F| \leqslant k$, every vertex $x \in A$ has exactly one out-neighbor in B and $k - 1$ out-neighbors in A. This implies that $g = 2$ and so Γ is a distance-regular graph, a contradiction to our assumptions. Hence we may assume that $r = k$.

Since we do not have an undirected edge (note that $g \geqslant 3$), we have

$$k|A| - \binom{|A|}{2} \leqslant |F| \leqslant k.$$

Therefore $|A| = 1$ or $|A| \geqslant 2k$. If $|A| = 1$, then $F = [A, B]$ is a set of all edges coming out of a single vertex in A and there is no thing to prove. Hence we assume that $|A| \geqslant 2k$. Let X_1 be the set of all vertices $x \in A$, where $\Gamma_1^+(x) \subseteq A$ and $X_2 = A - X_1$. Clearly $|X_2| \leqslant |F| \leqslant k$ and so $|X_1| \geqslant k$. Similarly assume that Y_1 is the set of all vertices $y \in B$, where $\Gamma_1^+(y) \subseteq B$ and $Y_2 = B - Y_1$. With the same argument, we have $|Y_2| \leqslant k$ and $Y_1 \geqslant k$.

Now choose two vertices $x \in X_1$ and $y \in Y_1$. Set $l = \partial(x, y)$. Clearly $l \geqslant 2$. Since $\Gamma \notin \mathcal{D}$, using Lemma 4 we have $a_{11}^l \geqslant 1$ and so $\Gamma_1^+(x) \cap \Gamma_1^+(y) \neq \emptyset$, a contradiction to the fact that $\Gamma_1^+(x) \subseteq A$ and $\Gamma_1^+(y) \subseteq B$. □

3 Minimum vertex cuts of distance-regular digraphs

As we mentioned in the first section, Brouwer and Koolen in [8] showed that every distance-regular graph of degree $k > 2$ is super vertex-connected. In Section 4, we will give an example (see Figure 1) that shows that the same result is not correct for strongly regular digraphs (and so for weakly distance-regular digraphs). We could not find such an example for distance-regular digraphs. An interesting research problem in this direction is to deduce whether the statement of Brouwer and Koolen's result is correct for distance-regular digraphs. In general this problem seems to be non-trivial and here we only consider some special cases.

Theorem 7. *Let Γ be a distance-regular digraph with diameter 2. Then Γ is super vertex-connected.*

Proof. Suppose that Γ is a distance-regular digraph on n vertices with degree k and diameter $D = 2$. Then the girth g of Γ is either 2 or 3. The case $g = 2$ implies that Γ is a strongly regular graph and so the assertion holds by a result due to Brouwer and Mesner in [9]. Now let $g = 3$. Clearly Γ is a strongly regular digraph with parameters $(n, k, 0, \lambda, \mu)$. Since Γ is stable, for every $x \in V(\Gamma)$, we have $\Gamma_2^+(x) = \Gamma_1^-(x)$ and so $|\Gamma_2^+(x)| = k$ and $n = 2k + 1$. One can easily see that Γ is a tournament and so $|E(\langle\Gamma_1^+(x)\rangle)| = k\lambda = \dfrac{k(k-1)}{2}$, this implies that $\lambda = \dfrac{k-1}{2}$. Therefore for each vertex x, the digraph $\langle\Gamma_1^+(x)\rangle$ is regular with degree $\dfrac{k-1}{2}$. The fact that Γ is a tournament implies that for each vertex x and for each $y \in \Gamma_2^+(x)$ we have $|\Gamma_1^+(y) \cap \Gamma_1^+(x)| = k - \mu$ (note that $|\Gamma_1^-(y) \cap \Gamma_1^+(x)| = \mu$). On the other hand, since $|\Gamma_2^+(x)| = k$, the digraph $\langle\Gamma_2^+(x)\rangle$ is regular with degree $\dfrac{k-1}{2}$. Therefore $\mu = \dfrac{k+1}{2}$ and so Γ is a strongly regular digraph with parameters $(n, k, t, \lambda, \mu) = (2k + 1, k, 0, \dfrac{k-1}{2}, \dfrac{k+1}{2})$.

Suppose that $S \subseteq V(\Gamma)$ is a vertex cut of Γ with minimum size that separates two non-empty sets A and B of $V(\Gamma)$, this means that $V(\Gamma) \setminus S = A \cup B$ and there is no edge from A to B. Without loss of generality we may assume that $|B| \geqslant |A|$. Clearly $|S| \leqslant k$ and so $|A| + |B| \geqslant k + 1$. Set $r := \min\{d_A^+(x) | x \in A\}$ and choose $x \in A$ such that it has r out-neighbors in A and $k - r$ out-neighbors in S. If $r = 0$, then $S = \Gamma_1^+(x)$ and there is no-thing to prove. Hence we may assume that $r \geqslant 1$. Now set $S_1 := \Gamma_1^+(x) \cap S$, $S_2 := S \setminus S_1$, $A_1 := (\Gamma_1^+(x) \cap A) \cup \{x\}$ and $A_2 := A \setminus A_1$. Clearly every vertex

$y \in A$ has at most $(k-r)$ out-neighbors in S. Hence by counting the number of edges coming out of the vertices of A in Γ we have

$$k|A| - \frac{|A|(|A|-1)}{2} \leqslant |A|(k-r),$$

and so $|A| \geqslant 2r+1$ and $|A_2| \geqslant r$. Moreover, $|S| \leqslant k$ implies that $|S_2| \leqslant r$ and hence $|A_2| \geqslant |S_2|$. Since Γ is a tournament and $|E(A_2, B)| = 0$ (note that S is a vertex cut that separates two non-empty sets A and B and $E(A_2, B)$ is the set of all edges from A_2 to B), we have $zy \in E(\Gamma)$ for every $y \in A_2$ and $z \in B$. On the other hand, $\langle \Gamma_2^+(x) \rangle$ is a regular digraph with degree $\frac{k-1}{2}$ and $A_2 \cup B \subseteq \Gamma_2^+(x)$. Hence $|A_2| \leqslant \frac{k-1}{2}$ and each vertex $y \in A_2$ has exactly $\frac{k-1}{2}$ out-neighbors in $A_2 \cup S_2$. Consequently, by counting the number of edges going into the vertices of S_2 in $\Gamma_2^+(x)$ we have the following inequality

$$e = |E(A_2, S_2)| + |E(\langle S_2 \rangle)| \leqslant \frac{k-1}{2}|S_2|.$$

On the other hand, $|E(\langle S_2 \rangle)| = \dfrac{|S_2|(|S_2|-1)}{2}$ and by counting the number of edges coming out of the vertices of A_2 in $\Gamma_2^+(x)$, we have

$$|E(A_2, S_2)| = \frac{(k-1)}{2}|A_2| - \frac{|A_2|(|A_2|-1|}{2}.$$

Therefore

$$e = \frac{(k-1)}{2}|A_2| - \frac{|A_2|(|A_2|-1)}{2} + \frac{|S_2|(|S_2|-1)}{2} \leqslant \frac{k-1}{2}|S_2|.$$

Since $f(t) = \dfrac{(k-1)}{2}t - \dfrac{t(t-1)}{2}$ is an increasing function when $t < k/2$, the fact $|S_2| \leqslant r \leqslant |A_2| \leqslant \dfrac{k-1}{2}$ implies that $\dfrac{k-1}{2}|S_2| \leqslant e$. Therefore $e = \dfrac{k-1}{2}|S_2|$ and so $|S_2| = |A_2| = r$, $|S| = k$ and $|B| = k-2r$. We know that the set of all vertices at distance 2 from x is $\Gamma_2^+(x) = A_2 \cup S_2 \cup B$ and the digraph induced by these vertices is $\dfrac{k-1}{2}$-regular. This fact implies that in the digraph $\langle B \rangle$ (the digraph induced by the vertices in B), each vertex has in-degree at least $(k-2r-1)/2$. Since otherwise, if there is a vertex $z \in B$ with in-degree less than $(k-2r-1)/2$ in $\langle B \rangle$, then

$$\frac{k-1}{2} = d^-_{\Gamma_2^+(x)}(z) \leqslant |S_2| + d^-_B(z) < r + \frac{k-2r-1}{2},$$

a contradiction. Now since Γ is a tournament, one can easily see that the digraph $\langle B \rangle$ is $\frac{k-2r-1}{2}$-regular and for every $y \in S_2$ and $z \in B$, yz is an edge of Γ. Therefore, for a vertex $y \in S_2$ with maximum out-degree in the digraph $\langle S_2 \rangle$, we have

$$\frac{k-1}{2} = d^+_{\Gamma_2^+(x)}(y) \geqslant \frac{|S_2|-1}{2} + |B| = \frac{r-1}{2} + k - 2r.$$

Hence $k \leqslant 3r$ and so $r \geqslant |B| \geqslant |A| \geqslant 2r+1$, a contradiction to the fact that $r \geqslant 1$. □

Now, we focus on another special case when the degree is at most 3.

Theorem 8. *Assume that Γ is a distance-regular digraph with degree $k \leqslant 3$ and Γ is not an undirected cycle. Then Γ is super vertex-connected.*

Proof. The case $D = 2$ follows from Theorem 7. Also the case $g = 2$ implies that Γ is a distance-regular graph and so the assertion holds due to a result of Brouwer and Koolen in [8]. Now assume that $D, g \geqslant 3$. For each vertex x, one can easily see that the digraph induced by $\Gamma_1^+(x)$ is λ-regular and so $\lambda \leqslant \frac{k-1}{2} \leqslant 1$. Hence $\lambda = 0, 1$. First suppose that $\lambda = 0$. Then $\Gamma \in \mathcal{D}$ is a generalized t-cycle with $t \geqslant 3$ (note that

$D, g \geqslant 3$) and hence by Lemma 3, Γ is super vertex-connected. So we may assume the case $\lambda = 1$. In this case we have $k = 3$. Consider two vertices u and v in $\Gamma_1^+(x)$ for some vertex x. The normality of the adjacency matrix of Γ implies that the number of vertices coming out of the vertices u and v is equal to the number of vertices going into that vertices. This implies that $\mu \geqslant 2$. Since $\mu \leqslant k = 3$, we have $\mu = 2, 3$. The case $\mu = 2$ implies that $|\Gamma_2^+(x)| = \frac{k(k-\lambda)}{\mu} = 3$. Now since $\lambda = 1$, the digraph $\langle \Gamma_1^+(x) \rangle$ for each vertex x is a directed triangle. This implies that for each vertex x, the digraph $\langle \Gamma_2^+(x) \rangle$ is a directed triangle and $D = 2$, which is not true based on our assumptions. Therefore, $\mu = 3$ and so the fact that the digraph $\langle \Gamma_1^+(x) \rangle$ is a directed triangle for each vertex x implies that $|\Gamma_2^+(x)| = 2$. Hence for each vertex x, there are at most two directed path of length 2 from y to z for every $y \in \Gamma_1^+(x)$ and $z \in \Gamma_3^+(x)$, which is a contradiction to $\mu = 3$. □

As we mentioned in Section 2, using Lemma 3, each $\Gamma \in \mathcal{D}$ is super vertex-connected. Therefore except the undirected cycles, each distance-regular digraph with $\lambda = 0$ is super vertex-connected. Based on the obtained result in this section we pose the following conjecture.

Conjecture 9. *Every distance-regular digraph is super vertex-connected, unless it is an undirected cycle.*

Now we summarize all the results in this section in the following corollary.

Corollary 10. *Conjecture 9 holds for distance-regular digraphs with $D = 2$, $k \leqslant 3$ or $\lambda = 0$.*

4 Strongly regular digraphs

We know that strongly regular graphs are precisely distance regular graphs with diameter 2 and the only strongly regular graph with degree 2 is the pentagon graph. Hence the known results about the minimum cuts of distance regular graphs (that we mentioned in Section 1) imply that each strongly regular graph is super edge-connected as well as super vertex-connected. Here we are going to look at the same problem for directed case. In fact we will use a result in [15] to see that each strongly regular digraph Γ is super edge-connected unless Γ is either an undirected cycle or the strongly regular digraph with parameters $(n, k, t, \lambda, \mu) = (6, 2, 1, 0, 1)$. Moreover, we will give an example of strongly regular digraphs with degree 3 that is not super vertex-connected. This example shows that the statement of Conjecture 9 does not occur for strongly regular digraphs.

Lemma 11. ([15]) *Let Γ be a digraph with no loop, minimum degree $\delta \geqslant 3$, parameter $l = l(\Gamma)$ and diameter D. If $D \leqslant 2l$, then Γ is super edge-connected. Moreover, Γ is super vertex-connected when $D \leqslant 2l - 2$.*

Using Lemma 11, each strongly regular digraph Γ with degree $k \geqslant 3$ is super edge-connected. Hence in order to study the super edge-connectivity of strongly regular digraphs we only focus on those with degree $k = 1, 2$. The only non-trivial strongly regular digraph with degree $k \leqslant 2$ is the strongly regular digraph with parameters $(n, k, t, \lambda, \mu) = (6, 2, 1, 0, 1)$ and this digraph is not super edge-connected. Therefore we can characterize all strongly regular digraphs that are super edge-connected as follows.

Theorem 12. *The edge connectivity of a strongly regular digraph Γ equals its degree. Moreover, a strongly regular digraph Γ is super edge-connected, unless Γ is either an undirected cycle with four or five vertices or the strongly regular digraph with parameters $(n, k, t, \lambda, \mu) = (6, 2, 1, 0, 1)$.*

As we know, each strongly regular graph is super vertex-connected. The digraph shown in Figure 1 shows that the same result is not correct for strongly regular digraphs (and so for weakly distance-regular digraphs). As you see in Figure 1, this digraph is a strongly regular digraph with parameters $(8, 3, 2, 1, 1)$ and vertex cut $U = \{u_1, u_4\}$ of size 2 (less than its degree).

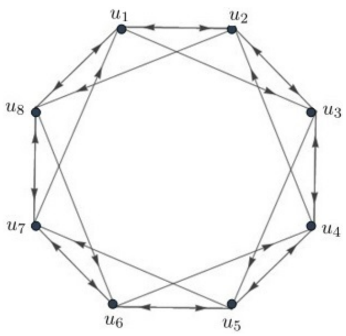

Figure 1: A strongly regular digraph with parameters $(8, 3, 2, 1, 1)$.

5 Concluding remarks and an open problem

The concept of weakly distance-regular digraphs is an extension of two concepts: distance-regular digraphs and strongly regular digraphs. Based on the above results, the investigation of the minimum edge cuts in weakly distance-regular digraphs is an interesting problem. In general, undirected cycles are the weakly distance-regular digraphs that are not super edge-connected. As we mentioned in Sections 4 besides two small undirected cycles, the strongly regular digraph with parameters $(n, k, t, \lambda, \mu) = (6, 2, 1, 0, 1)$ is a nice exception in strongly regular digraphs that is not super edge-connected. Therefore, besides undirected cycles it is natural to think about the family of infinite weakly distance-regular digraphs, each has a minimum edge cut that is not the set of all edges going into (or coming out of) a single vertex. Here we show that such a family exists. In fact, for every positive integer n, we construct a 2-regular weakly distance-regular digraph Γ_n with $2n$ vertices, diameter $D = [n/2] + 1$ such that Γ_n is not super edge-connected. To do this, add the edges $v_i u_i$ and $u_i v_i$ for $1 \leqslant i \leqslant n$ to two disjoint directed cycles $C_1 = v_1 v_2 v_3 \ldots v_{n-1} v_n v_1$ and $C_2 = u_1 u_n u_{n-1} \ldots u_3 u_2 u_1$, to get a 2-regular weakly distance-regular digraph Γ_n with the desired properties. Now, based on the previous results and the above discussion we pose the following conjecture about weakly distance-regular digraphs:

Conjecture 13. *For every weakly distance-regular digraph Γ with degree k, the edge connectivity equals k. Moreover, Γ is super edge-connected if $k > 2$.*

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