Strengthening (a, b)-Choosability Results to (a, b)-Paintability

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Submitted: Jul 15, 2017; Accepted: Oct 11, 2017; Published: Oct 20, 2017

Abstract

Let $a, b \in \mathbb{N}$. A graph G is (a, b)-*choosable* if for any list assignment L such that $|L(v)| \geqslant a$, there exists a coloring in which each vertex v receives a set $C(v)$ of b colors such that $C(v) \subseteq L(v)$ and $C(u) \cap C(w) = \emptyset$ for any $uw \in E(G)$. In the online version of this problem, on each round, a set of vertices allowed to receive a particular color is marked, and the coloring algorithm chooses an independent subset of these vertices to receive that color. We say G is (a, b)-*paintable* if when each vertex v is allowed to be marked a times, there is an algorithm to produce a coloring in which each vertex v receives b colors such that adjacent vertices receive disjoint sets of colors.

We show that every odd cycle C_{2k+1} is (a, b)-paintable exactly when it is (a, b)-chosable, which is when $a \geqslant 2b + \lceil b/k \rceil$. In 2009, Zhu conjectured that if G is $(a, 1)$-paintable, then G is (am, m)-paintable for any $m \in \mathbb{N}$. The following results make partial progress towards this conjecture. Strengthening results of Tuza and Voigt, and of Schauz, we prove for any $m \in \mathbb{N}$ that G is $(5m, m)$-paintable when G is planar. Strengthening work of Tuza and Voigt, and of Hladky, Kral, and Schauz, we prove that for any connected graph G other than an odd cycle or complete graph and any $m \in \mathbb{N}$, G is $(\Delta(G)m, m)$-paintable.

1 Introduction

Let G be a graph, and let $g : V(G) \to \mathbb{N}$. A *g-fold coloring* of G assigns to each vertex v a set of $g(v)$ distinct colors such that adjacent vertices have disjoint sets of colors. When $g(v) = m$ for all v, we call this an *m-fold coloring*. When all colors come from $\{1, \ldots, k\}$, we call this a *g-fold k-coloring* and say that G is *(k, g)-colorable*. When G is (k, g)-colorable and $g(v) = m$ for all v, we say that G is *(k, m)-colorable*. An ordinary proper k-coloring is also a 1-fold k-coloring.

More generally, let L be a *list assignment* specifying for each vertex v a set $L(v)$ of available colors. A *g-fold L-coloring* of G is a g-fold coloring ϕ of G such that $\phi(v) \subseteq L(v)$ for each vertex v. A graph G is *(f, g)-choosable* if there is a g-fold L-coloring for any list assignment L such that $|L(v)| \geqslant f(v)$ for all $v \in V(G)$. Introduced by Erdős, Rubin, and Taylor [2], when $f(v) = k$ and $g(v) = m$ for all $v \in V(G)$ and G is (f, g)-choosable, we say that G is *(k, m)-choosable*. When $g(v) = 1$ for every $v \in V(G)$, we shorten (f, g)-choosable to f-choosable. Erdős, Rubin, and Taylor [2] conjectured that if G is k-choosable, then G is (km, m)-choosable for all $m \in \mathbb{N}$.

Schauz [5] introduced an online version of choosability, and Zhu [9] generalized (f, g)-choosability. Vertices are colored in rounds, where the coloring algorithm must decide on round i which vertices will receive color i, without knowing which colors will appear on any vertices later in the lists. This concept is formalized in the following game.

Definition 1.1. Let G be a graph where each vertex v is assigned a nonnegative number $f(v)$ of tokens and a nonnegative number $g(v)$ specifying how many times v must be colored. The *(f, g)-paintability game* is played by two players: Lister and Painter. Each round, Lister marks a nonempty subset M of vertices that have been colored fewer than $g(v)$ times; every vertex in M loses one token. Painter responds by coloring an independent subset D of M; every vertex of D gains a color distinct from those used on earlier rounds. Lister wins the game by marking a vertex that has no tokens remaining, and Painter wins by coloring each vertex v on $g(v)$ distinct rounds.

We say G is *(f, g)-paintable* when Painter has a winning strategy on G in the (f, g)-paintability game. If $f(v) = a$ and $g(v) = b$ for every $v \in V(G)$ and Painter has a winning strategy, then we say G is *(a, b)-paintable*. We say G is *degree-m-paintable* if G is (f, m)-paintable where $f(v) = d(v)m$ for all v. When $m = 1$, we simply say "degree-paintable".

Always, if G is not (f, g)-choosable, then G is not (f, g)-paintable since Lister can mimic a bad list assignment L by marking in round i the set $\{v \in V(G) : i \in L(v)\}$. Thus (f, g)-paintability implies (f, g)-choosability. In [9], Zhu made the following conjecture.

Conjecture 1.2 ([9])**.** *If G is a-paintable, then G is (am, m)-paintable for all $m \in \mathbb{Z}$.*

Mahoney, Meng, and Zhu [4] proved that Conjecture 1.2 is true for all 2-paintable graphs.

Thomassen [6] proved that every planar graph is 5-choosable. Tuza and Voigt [7] strengthed this result by proving that planar graphs are $(5m, m)$-choosable for all $m \in \mathbb{N}$. Schauz [5] strengthened Thomassen's result in a different way by proving that planar graphs are 5-paintable.

In Section 2, we prove that planar graphs are $(5m, m)$-paintable for all $m \geqslant 1$, which strengthens the previous results and makes partial progress towards Conjecture 1.2.

Let G be a connected graph other than an odd cycle or a complete graph, and let $\Delta(G)$ denote the maximum degree of G. Brooks' Theorem [1] states that G is $\Delta(G)$-colorable. Stronger versions of Brooks' Theorem are proved by Tuza and Voigt [7] and by Hladky, Kral, and Schauz [3]. In Section 3, we prove that G is $(\Delta(G)m, m)$-paintable, strengthing both results and making partial progress towards Conjecture 1.2.

In [8], Voigt proved that if C_{2k+1} is (a, b)-choosable, then $a \geqslant 2b + \lceil b/k \rceil$. We conclude the introduction by strengthening this result, characterizing the (a, b)-paintability and (a, b)-choosability of odd cycles.

Theorem 1.3. *For $k \geqslant 1$, the following are equivalent:*

(a) C_{2k+1} is (a, b)-paintable.

(b) C_{2k+1} *is* (a,b)*-choosable.*

(c) $a \geqslant 2b + \lceil b/k \rceil$.

Proof. $(a) \Rightarrow (b)$ Always (a,b)-paintability implies (a,b)-choosability.

$(b) \Rightarrow (c)$ Originally proved in [8], we provide a short proof for completeness. Suppose C_{2k+1} is (a,b)-choosable. Consider the list assignment where $L(v) = \{1,\dots,a\}$ for each vertex v. Each color can be used on at most k vertices. Since each vertex must receive b colors, we have that the lists must have size at least $(2k+1)b/k$.

$(c) \Rightarrow (a)$ Give the cycle a consistent orientation, and label the vertices $v_0,\dots,v_{2k}$. Consider all indices modulo $2k+1$. Let M be the set Lister marks. If $|M| < 2k+1$, then the graph induced by the marked set is a linear forest. Painter colors vertices greedily along each path starting at the tail.

If $|M| = 2k+1$, then we keep track of how many times moves of this type have occurred in the game. If a move of this type has been played i times before (mod $2k+1$), then Painter colors $\{v_i, v_{i+2}, \dots, v_{i+2k-2}\}$. There are exactly $2k+1$ distinct independent sets of size k for C_{2k+1}. In this strategy, Painter balances which of these independent sets is colored by cycling through all possible choices.

Suppose Lister can win against this particular Painter strategy when each vertex has $2b + \lceil b/k \rceil$ tokens. Let Lister's marked sets be $M_1,\dots,M_t$, where Lister wins on round t, and let Painter's responses be $D_1,\dots,D_{t-1}$. Note that when Lister marks the set M_t, some vertex will have been marked $2b+\lceil b/k \rceil+1$ times. In particular $t \geqslant 2b+\lceil b/k \rceil$. If a vertex v_i is marked in the set M_j, then Painter strategy implies that v_{i-1} was also marked in M_j. Since v_{i-1} is colored at most b times, there must be at least $\lceil b/k \rceil + 1$ rounds where both v_i and v_{i-1} are marked and not colored. This only happens once every $2k+1$ rounds when both vertices are marked. Thus the number of rounds where all vertices are marked is at least $\lceil b/k \rceil (2k+1)+1$, which is greater than $2b + \lceil b/k \rceil$. So there are at elast $2b + \lceil b/k \rceil + 1$ rounds in which all vertices are marked. If any of these rounds were preceded by a round in which not all vertices were marked, then one vertex would be marked $2b + \lceil b/k \rceil + 1$ times earlier than round t. Thus the first $2b + \lceil b/k \rceil + 1$ marked sets must all be $V(C_{2k+1})$. After marking all vertices just $2b + \lceil b/k \rceil$ times, Painter's strategy ensures that every vertex is colored b times. Thus there is no vertex for Lister to mark on round $2b + \lceil b/k \rceil + 1$, and thus there is no way for Lister to win the game against this Painter strategy. □

2 Planar Graphs

The following lemma is a generalization of Lemma 2.2 in [5].

Lemma 2.1 (Edge Lemma). *If G is (f,g)-paintable and $uv \notin E(G)$, then $G \cup uv$ is (f',g)-paintable where* $f'(w) = \begin{cases} f(v)+f(u), & \text{if } w = v \\ f(w), & \text{otherwise} \end{cases}$.

Proof. Let $\mathcal{S}$ be a winning strategy for Painter in the (f,g)-paintability game on G. In the (f',g')-paintability game on $G \cup uv$, whenever Lister marks u, we sacrifice a token on v by having Painter respond to the marked set $M - v$. At most $f(u)$ tokens are sacrificed on v. In rounds when u is not marked, Painter may respond according to $\mathcal{S}$ because any response in G is an independent set in $G \cup uv$. At least $f(v)$ tokens are available for moves of this type, so $g(v)$ colors will be assigned to v by playing according to $\mathcal{S}$. □

The following lemma is a generalization of Lemma 2.5 in [5].

Lemma 2.2 (Merge Lemma). *Let $G = G_1 \cup G_2$, and let $T = V(G_1) \cap V(G_2)$. If G_i is (f_i, g_i)-paintable and $f_2(v) = g_2(v) = g_1(v)$ for all $v \in T$, then G is (f, g)-paintable where* $f(v) = \begin{cases} f_1(v), & \text{if } v \in V(G_1) \\ f_2(v), & \text{otherwise} \end{cases}$ *and* $g(v) = \begin{cases} g_1(v), & \text{if } v \in V(G_1) \\ g_2(v), & \text{otherwise} \end{cases}$.

Proof. We use induction on $\sum g(v)$. For the basis step, if $\sum g(v) = 0$, then G is trivially (f, g)-paintable. Now consider $\sum g(v) > 0$.

Let M be the set marked by Lister. For $i \in \{1, 2\}$, let $\mathcal{S}_i$ be a winning strategy for Painter in G_i under token assignment f_i, and let $M_i = M \cap V(G_i)$. Let D_1 be the response to M_1 in G_1 according to $\mathcal{S}_1$. In G_2, Painter responds to the marked set $M_2 - (T - D_1)$ according to $\mathcal{S}_2$. We interpret vertices of $(M - D_1) \cap T$ as having lost a token in G_1 but not in G_2. Because $f_2(v) = g_2(v)$ for all $v \in T$, it must be the case that $(D_1 \cap T) \subseteq D_2$. Thus $D_1 \cup D_2$ is an independent set; Painter now colors $D_1 \cup D_2$.

Let $G^* = G - (D_1 \cup D_2)$. To make use of the induction hypothesis, we define the following functions:

$$f_1'(v) = \begin{cases} f_1(v) - 1, & \text{if } v \in M \\ f_1(v), & \text{otherwise} \end{cases}$$

$$f_2'(v) = \begin{cases} f_2(v) - 1, & \text{if } v \in M_2 - (T - D_1) \\ f_2(v), & \text{otherwise} \end{cases}$$

$$\text{For } i \in \{1, 2\},\ g_i'(v) = \begin{cases} g_i(v) - 1, & \text{if } v \in D_i \\ g_i(v), & \text{otherwise} \end{cases}$$

Because D_1 and D_2 were chosen according to a winning strategies in G_1 and in G_2, we have that G_i is (f_i', g_i')-paintable for $i \in \{1, 2\}$ and $f_2'(v) = g_2'(v) = g_1'(v)$ for all $v \in T$. Since $M \neq \emptyset$, we may assume that $D_1 \cup D_2 \neq \emptyset$. Thus $\sum g(v)$ decreases, and by induction this yields G^* is (f^*, g^*)-paintable where $f^*(v) = \begin{cases} f_1'(v), & \text{if } v \in V(G_1) \\ f_2'(v), & \text{otherwise} \end{cases}$ and $g^*(v) = \begin{cases} g_1'(v), & \text{if } v \in V(G_1) \\ g_2'(v), & \text{otherwise} \end{cases}$. This proves that Painter's response to M is a winning move, and thus G is (f, g)-paintable. □

We now prove the main theorem of this section.

Theorem 2.3. *Planar graphs are $(5m, m)$-paintable for all $m \in \mathbb{N}$.*

Proof. We proceed using an argument mirroring that of Thomassen [6] and of Schauz [5]. First, we restrict our attention to weak triangulations of planar graphs since adding edges only makes coloring the graph more difficult for Painter. Let G be a planar graph of order n with vertices $v_1, \ldots, v_p$ in clockwise order on the unbounded face. By induction on n, we prove a stronger result:

$$G \text{ is } (f, m)\text{-paintable when } f(v) = \begin{cases} m, & \text{if } v = v_p \\ 2m, & \text{if } v = v_1 \\ 3m, & \text{if } v = v_i \text{ for } 1 < i < p \\ 5m, & \text{otherwise} \end{cases}.$$

Case 1: There is a chord v_iv_j connecting two vertices on the unbounded face.
Let G_1 be the graph induced by the vertices of the cycle containing v_1 and v_p and by the vertices on the interior of this cycle. Let G_2 have vertex set $(V(G) - V(G_1)) \cup \{v_i, v_j\}$ and edge set $E(G) - E(G_1)$. $f_1(v) = f(v)$ for all $v \in V(G_1)$, and $f_2(v) = \begin{cases} f(v), & \text{if } v \in V(G_2) - \{v_i, v_j\} \\ m, & \text{if } v \in \{v_i, v_j\} \end{cases}$.

By the induction hypothesis G_1 is (f_1, m)-paintable, and G_2 is (f_2, m)-paintable by first applying Lemma 2.1 to the edge v_iv_j and then using the induction hypothesis. Lemma 2.2 then implies that G is (f, m)-paintable.

Case 2: The unbounded face is chordless.
Consider $N(v_2)$. Since all bounded faces are triangles, there exists a path $v_1, u_1, \dots, u_t, v_3$. Let $U = \{u_1, \dots, u_t\}$, and let $G' = G - v_2$. Applying the induction hypothesis to G', we show that if each $u \in U$ is given $2m$ additional tokens, then we can extend a winning strategy for Painter on G' to a winning strategy on G.

Suppose Lister marks a set M, and let $\mathcal{S}$ be a winning strategy for Painter in G'. Let D be Painter's response to the marked set $M - \{v_2\}$ according to $\mathcal{S}$. If $v_2 \notin M$, then Painter colors D. If $v_2 \in M$ and $v_1 \in D$, then Painter colors D and sacrifices a token on v_2. When $v_2 \in M$, and $v_1 \notin D$, then Painter obtains the response D' to the marked set $(M - \{v_2\}) - U$ according to $\mathcal{S}$ and colors v_2 if $v_3 \notin D'$. Each vertex of U loses at most $2m$ tokens from moves of this type. Also, v_2 is marked and not colored at most m times because of $v_3 \in D'$. Finally, v_3 never loses tokens because of v_2. Therefore, every vertex is colored m times before it runs out of tokens. □

3 Brooks' Theorem

Brooks' Theorem [1] states that a connected graph G is $\Delta(G)$-colorable except when G is an odd cycle or a complete graph. Tuza and Voigt [7] strengthened this result by proving that such a graph G is $(\Delta(G)m, m)$-choosable. Hladky, Kral, and Schauz [3] proved that such a graph G is $\Delta(G)$-paintable. The following theorem strengthens both results.

Theorem 3.1. *If G is a connected graph other than an odd cycle or a complete graph, then G is $(\Delta(G)m, m)$-paintable for all $m \in \mathbb{N}$.*

Let $N_G[v]$ denote the closed neighborhood $N_G(v) \cup \{v\}$. We will make use of a degeneracy argument of Zhu [9] and a well-known structural lemma of Erdős, Rubin, and Taylor [2]. A *block* in a graph is a maximal 2-connected subgraph or a cut-edge.

Proposition 3.2 ([9])**.** *Let G be a graph with token-color assignments f and g. If $f(v) \geqslant \sum_{u \in N[v]} g(u)$, then G is (f, g)-paintable if and only if $G - v$ is (f', g')-paintable where f' and g' are the restrictions of f and g to $G - v$.*

Lemma 3.3 ([2])**.** *If G is a 2-connected graph that is not an odd cycle or a complete graph, then G contains an induced even cycle having at most one chord.*

We now show that the induced subgraph obtained from the conclusion of Lemma 3.3 is degree-m-paintable for all $m \in \mathbb{N}$.

Lemma 3.4. *An even cycle with at most one chord is degree-m-paintable for all $m \in \mathbb{N}$.*

Proof. **Case 1:** Let G be a chordless even cycle. Zhu [9] proved that C_{2n} is $(2m, m)$-paintable for $n \geqslant 2, m \in \mathbb{N}$.

Case 2: Let G be an even cycle with exactly one chord. Let $v_1, \ldots, v_n$ be the vertices of this cycle in clockwise order, and suppose $v_1 v_i$ is the chord. Consider the graph G' obtained from G by removing the edge $v_n v_1$. Let f' be a token assignment obtained from f by removing $2m$ tokens from v_1. By Lemma 2.1, if G' is (f', g)-paintable, then G is degree-m-paintable. In G', we repeatedly apply Proposition 3.2 to $V(G')$ in the order $v_n, v_{n-1}, \ldots, v_1$. At each step, the vertex being removed has at least as many tokens as the number of times it and its neighbors must be colored, therefore G' is (f', g)-paintable. □

The next lemma allows us to extend good strategies on an induced subgraph to a larger graph.

Lemma 3.5. *Given a connected graph G, if there exists an induced subgraph H that is degree-m-paintable, then G is degree-m-paintable for all $m \in \mathbb{N}$.*

Proof. If $H = G$, there is nothing to show, so suppose $V(G) - V(H) \neq \emptyset$, and let $U = \{u_1, \ldots, u_t\} = V(G) - V(H)$. Let $\mathcal{S}$ be a winning degree-m-paintability strategy for Painter on H.

Let M be the set that Lister marks. Let D be the independent subset of $M \cap U$ chosen greedily with respect to the ordering $u_1, \ldots, u_t$. According to $\mathcal{S}$, Painter obtains a response D' in H to the marked set $(M \cap V(H)) - N(D)$. We sacrifice a token on each vertex of $M \cap V(H) \cap N(D)$, and Painter colors $D \cup D'$. Note that $D \cup D'$ is an independent set because we forbid coloring any neighbors of vertices in D.

Each $v \in V(H)$ sacrifices at most m tokens for any neighbor outside of H, which guarantees that at least $d_H(v)m$ tokens are available for the strategy $\mathcal{S}$. Each $u \in U$ is marked and not colored at most m times for each earlier neighbor, which always leaves at least m tokens available to color u when it has no more incomplete earlier neighbors. Therefore G is degree-m-paintable. □

Lemmas 3.3, 3.4, and 3.5 imply that every block of a non-degree-m-paintable connected graph must be an odd cycle or a clique. A connected graph in which every block is an odd cycle or a clique is called a *Gallai tree.*

Theorem 3.6. *Given $m \in \mathbb{N}$, a connected graph G is degree-m-paintable if and only if G is not a Gallai tree.*

Proof. If G is a Gallai tree, it is not degree-m-choosable [7], and hence, not degree-m-paintable.

When G is not a Gallai tree, there exists a block B that is not a complete graph or an odd cycle. By Lemma 3.3, B contains an induced even cycle with at most one chord. Lemma 3.4 implies that B is degree-m-paintable. Lastly, Lemma 3.5 implies that G is degree-m-paintable. □

We conclude by proving Theorem 3.1.

Proof of Theorem 3.1. If G is not a Gallai tree, then Theorem 3.6 implies $(\Delta(G)m, m)$-paintability. We may assume that G is a Gallai tree with at least two blocks. Thus G is not $\Delta(G)$-regular, and every vertex of maximum degree is a cut-vertex. Thus every subgraph of G contains a vertex of degree at most $\Delta(G) - 1$, so Proposition 3.2 implies G is $(\Delta(G)m, m)$-paintable. □

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