A Step Towards Yuzvinsky's Conjecture

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Abstract

An intercalate matrix M of type $[r, s, n]$ is an $r \times s$ matrix with entries in $\{1, 2, \ldots, n\}$ such that all entries in each row are distinct, all entries in each column are distinct, and all 2×2 submatrices of M have either 2 or 4 distinct entries. Yuzvinsky's Conjecture on intercalate matrices claims that the smallest n for which there is an intercalate matrix of type $[r, s, n]$ is the Hopf-Stiefel function $r \circ s$. In this paper we prove Yuzvinsky's Conjecture is asymptotically true for $\frac{5}{6}$ of integer pairs (r, s). We prove the conjecture for $r \leqslant 8$, and we study it in the range $r, s \leqslant 32$.

Keywords: Yuzvinsky's Conjecture; Intercalate matrices; Hopf-Stiefel function.

1 Introduction

Let M be an $r \times s$ matrix with entries in a set of *colors* $\{1, 2, \ldots, n\}$. Let $M_{i,j}$ be the (i, j)-entry of M. The matrix M is said to be *intercalate* of type $[r, s, n]$ if it satisfies the following two conditions:

Latinicity The colors along each row and each column are distinct.

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Intercalacy If $M_{i,j} = M_{i',j'}$, then $M_{i,j'} = M_{i',j}$.

These conditions imply that every 2×2 submatrix of M involves an even number of distinct colors. Those involving exactly two colors are called *intercalations*; in this case each color forms a *half-intercalation*. Observe that the set of intercalate matrices is closed under transposition, taking submatrices, and permutation of rows, columns, or colors. Intercalate matrices appeared in the context of the classical problem of sums of squares formulae [21]. For more details see [19] and for a thorough exposition see [17].

In the 1940's, Hopf [6] and Stiefel [18] introduced the function

$$r \circ s = \min\left\{ n \in \mathbb{N} \mid \binom{n}{k} \equiv 0 \bmod 2 \text{ for all } k \text{ in the range } n - r < k < s \right\}$$

in their work on real division algebras by topological methods. This function, currently known as the Hopf-Stiefel function, plays an important role in several other subjects such as algebraic topology [8, 11, 19], quadratic forms [12, 14], sums of squares formulae [15, 16, 19, 20, 21], and additive number theory [4, 5, 7, 14]. The Hopf-Stiefel function can also be computed recursively by the following formula due to Pfister [13]:

$$r \circ s = \begin{cases} s \circ r & \text{if } r > s \\ s & \text{if } r = 1 \\ 2^k & \text{if } 2^{k-1} < r \leqslant s \leqslant 2^k \\ 2^k + r \circ (s - 2^k) & \text{if } r \leqslant 2^k < s. \end{cases}$$

For an extensive description of reformulations of $r \circ s$ we refer the reader to [5].

Stated in Yiu's standard terminology [19], Yuzvinsky showed in 1981 [21] that for every r and s there is an intercalate matrix of type $[r, s, r \circ s]$. For a simple proof see Shapiro [17, p. 274]. Additionally, Yuzvinsky posed the following conjecture:

Conjecture 1 (Yuzvinsky's Conjecture)**.** Let $f(r, s)$ be the smallest value of n for which there exists an intercalate matrix of type $[r, s, n]$. Then $f(r, s) = r \circ s$.

Since every $r \times s$ matrix whose entries are all distinct is intercalate, it follows that $f(r, s) \leqslant rs$, therefore $f(r, s)$ is well defined. Furthermore, Yuzvinsky's result implies $f(r, s) \leqslant r \circ s$. Since intercalate matrices are closed under transposition, it follows that $f(r, s) = f(s, r)$. Furthermore, intercalate matrices are closed under taking submatrices, hence the function f has the following property:

Monotonicity If $r \leqslant R$ and $s \leqslant S$, then $f(r, s) \leqslant f(R, S)$.

Yiu [20] verified Yuzvinsky's Conjecture whenever $r, s \leqslant 16$ and Lam [11] proved it for square intercalate matrices, together with some other special cases. In the same paper Lam gave a result implying the conjecture is asymptotically true for $\frac{2}{3}$ of integer pairs (r, s). In this paper we improve Lam's asymptotic bound to $\frac{5}{6}$ of integer pairs (r, s). We prove the conjecture for $r \leqslant 8$, and we study it in the range $r, s \leqslant 32$.

2 Dyadic intercalate matrices

There is a well-known general distinction between two main classes of intercalate matrices: dyadic and non-dyadic. Dyadic intercalate matrices are an important class whose construction comes from group theory. Let $D = \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}$ be an Abelian 2-group with cardinality 2^k. The addition table of D gives an intercalate matrix of type $[2^k, 2^k, 2^k]$, whose color set is D. A dyadic matrix is a submatrix of the addition table of group D. Following Yiu [19, 20], we can construct a canonical family of dyadic intercalate matrices. Starting with $D_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ define inductively:

$$D_{t+1} = \begin{pmatrix} D_t & 2^t + D_t \\ 2^t + D_t & D_t \end{pmatrix} \text{ for } t = 1, 2, \ldots$$

Here, we consider D_t as a matrix of integers, and obtain $2^t + D_t$ by adding 2^t to each entry of D_t. The intercalate matrix D_t is of type $[2^t, 2^t, 2^t]$ and every submatrix of D_t is an intercalate matrix. Given $r, s \leqslant 2^t$, let $D_{r,s}$ be the principal $r \times s$ submatrix in the upper left hand corner of D_t. Observe that $D_{r,s}$ does not depend on t. Note that $D_i = D_{2^i,2^i}$, and an intercalate matrix is dyadic if it is a submatrix of some D_t. Yuzvinsky proved in [21] that the submatrices of D_t with type $[r, s, n]$ satisfy $n \geqslant r \circ s$, hence Yuzvinsky's Conjecture is true for dyadic matrices. A conceptually elegant proof of this result was obtained by Eliahou and Kervaire [4], using the polynomial method extensively developed by Alon and Tarsi [1]. For different methods to decide when an intercalate matrix is dyadic see [2], for further properties see [3, 22].

In contrast to the dyadic case, there are no theoretical methods to obtain non-dyadic intercalate matrices, although there exist simple examples such as

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 5 & 6 & 7 & 8 \\ 6 & 5 & 9 & 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 3 & 2 & 7 & 8 \\ 9 & 10 & 11 & 8 & 7 \\ 10 & 9 & 12 & 13 & 14 \\ 15 & 16 & 13 & 12 & 17 \end{pmatrix}.$$

Yuzvinsky's Conjecture remains open for non-dyadic intercalate matrices.

3 Constructions relative to color frequencies

There are several known methods to obtain new intercalate matrices from a given intercalate matrix. These have been described mostly in the context of sums of squares formulae [8, 15, 16, 17, 19, 20]. In this section we introduce some constructions based on the frequency of the colors appearing in an intercalate matrix to obtain new intercalate matrices. The procedure given in Theorem 2 was originally introduced by Lam and Yiu [9, 10] while studying sums of squares formulae from a topological perspective, and later by Yiu [19] in combinatorial terms.

Let M be an intercalate matrix of type $[r, s, n]$ and let a be a color of frequency t in M. By permuting rows and columns, we can assume color a appears in the first t rows and columns along the main diagonal of M, to obtain

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A a $t \times t$ submatrix. In this case M is said to be in *block form* with respect to color a.

Theorem 2 ([9, 10, 19])**.** *Let M be an intercalate matrix of type $[r, s, n]$ in block form with respect to color a of frequency t. Then the matrix*

$$M' = \begin{pmatrix} A & B & C^T \end{pmatrix}$$

is intercalate of type $[t, s + r - t, n']$ with $n' \leqslant n$. In particular $n \geqslant f(t, s + r - t)$.

The matrix M' of Theorem 2 is called the intercalate matrix *hidden behind* color a [20]. Analogous to Theorem 2, in the following result we construct a new intercalate matrix M'' when $t = r - 1$.

Theorem 3. *Let M be an intercalate matrix of type $[r, s, n]$ in block form with respect to color a of frequency $t = r - 1$. Then the matrix*

$$M'' = \begin{pmatrix} A & B & C^T \\ C & D & a \end{pmatrix}$$

is intercalate of type $[r, s + 1, n]$. In particular $n \geqslant f(r, s + 1)$.

Proof. Since a does not appear in C nor D, it follows that M'' satisfies the latinicity condition. Now, we prove that each 2×2 submatrix N of M'' also satisfies the intercalacy condition. By Theorem 2 we can assume $N = \begin{pmatrix} M''_{i,j} & M''_{i,s+1} \\ M''_{r,j} & a \end{pmatrix}$. It follows that $M''_{i,j} = a$ if and only if $i = j$ if and only if $M''_{i,s+1} = M''_{r,j}$. Finally, M'' is intercalate of type $[r, s+1, n]$ since M'' is an $r \times (s + 1)$ intercalate matrix with the same number of colors as M. $\square$

It is not possible to generalize Theorem 3 for all $t \leqslant r - 1$. For example, consider the intercalate matrix

$$M = \left(\begin{array}{cc|cc} 1 & 2 & 7 & 8 \\ 2 & 1 & 9 & 10 \\ \hline 3 & 4 & 6 & 5 \\ 5 & 6 & 4 & 3 \end{array}\right)$$

for which there are no colors y and z such that the matrix

$$\left(\begin{array}{cc|cc|cc} 1 & 2 & 7 & 8 & 3 & 5 \\ 2 & 1 & 9 & 10 & 4 & 6 \\ \hline 3 & 4 & 6 & 5 & 1 & z \\ 5 & 6 & 4 & 3 & y & 1 \end{array}\right)$$

is intercalate. This is because the intercalation $\begin{pmatrix} 8 & 5 \\ 5 & z \end{pmatrix}$ implies $z = 8$, the intercalation $\begin{pmatrix} 9 & 4 \\ 4 & y \end{pmatrix}$ implies $y = 9$, but the intercalation $\begin{pmatrix} 1 & z \\ y & 1 \end{pmatrix}$ implies $y = z$.

The following result deals with intercalate matrices M with only two distinct color frequencies and allows us to construct two new intercalate matrices, one of them contained in M.

Theorem 4. *If M is an intercalate matrix of type $[r, s, n]$ with r colors of frequency $t < r$ and all the others with frequency r, then there exist an intercalate matrix of type $[t, n, n]$ and an intercalate submatrix of type $[r - t, s - t, s - t]$.*

Proof. We have $rs = rt + (n - r)r$, hence $n - r = s - t$. Since there are $n - r = s - t$ colors of frequency r and each one of them occurs in every row, it follows that each row has exactly t colors of frequency t. We can assume M is in block form with respect to color a of frequency t. Let c be a color in the submatrix C. If c has frequency r, then it appears in every row of the submatrix $\begin{pmatrix} A & B \end{pmatrix}$. A contradiction, since by Theorem 2 the matrix M' is intercalate. Hence, c has frequency t. Thus, C contains only t colors of frequency t and D contains only colors of frequency r. Therefore, M' is of type $[t, r + s - t, n] = [t, n, n]$ and D is of type $[r - t, s - t, s - t]$. $\square$

For example, the intercalate matrix

$$M = \left(\begin{array}{cc|cccc} 1 & 2 & 5 & 6 & 7 & 8 \\ 2 & 1 & 6 & 5 & 8 & 7 \\ \hline 3 & 4 & 7 & 8 & 5 & 6 \\ 4 & 3 & 8 & 7 & 6 & 5 \end{array}\right)$$

is in block form with respect to color 1 and by Theorem 4 the matrices

$$M' = \left(\begin{array}{cc|cccc|cc} 1 & 2 & 5 & 6 & 7 & 8 & 3 & 4 \\ 2 & 1 & 6 & 5 & 8 & 7 & 4 & 3 \end{array}\right) \quad \text{and} \quad D = \begin{pmatrix} 7 & 8 & 5 & 6 \\ 8 & 7 & 6 & 5 \end{pmatrix}$$

are intercalate.

4 Asymptotic results

In this section we prove Yuzvinsky's Conjecture is asymptotically true for $\frac{5}{6}$ of integer pairs (r, s).

Lemma 5. *If $r = 2^k$ and $s = t2^k$, then $f(r, s) = r \circ s$.*

Proof. By the monotonicity of f, we have $f(r, s) \geqslant f(1, s) = t2^k$. We also have $f(r, s) \leqslant r \circ s = t2^k$. Hence equality holds. $\square$

The next result due to Yiu [20] restricts the size of certain intercalate matrices.

Theorem 6. *If M is an intercalate matrix of type $[r,n,n]$ with $n = 2^p(2\ell+1)$, then $r \leqslant 2^p$.*

For our purposes we include proofs of some results by Lam found in [11].

Lemma 7. *If $r+s = 2^k+1$, then $r \circ s = 2^k$.*

Proof. We may assume $r \leqslant s$. By induction on k. If $k=0$, then $r=s=1$ and $r \circ s = 2^0$. It follows that $s > 2^{k-1} \geqslant r$. Thus, Pfister's formula implies $r \circ s = 2^{k-1} + r \circ (s - 2^{k-1})$. Furthermore, $r+(s-2^{k-1}) = 2^{k-1}+1$ and, by the induction hypothesis, $r \circ (s-2^{k-1}) = 2^{k-1}$. Therefore, $r \circ s = 2^k$. □

Theorem 8 ([11])**.** *If $r+s = 2^k+1$, then $f(r,s) = r \circ s$.*

Proof. By Lemma 7, we have $r \circ s = 2^k$. We assume without loss of generality that $r < s$. By induction on r. If $r=1$, then $f(1,s) = 1 \circ s$. Now assume $r \geqslant 2$. Let M be an intercalate matrix of type $[r,s,n]$. Suppose each color of M has frequency r, hence $n = \frac{rs}{r} = s$ and M is of type $[r,s,s]$. Theorem 6 implies $s \geqslant r \circ s = 2^k$. Consequently, $r = 2^k + 1 - s \leqslant 1$. A contradiction, hence M has a color of frequency $t < r$. Thus, by Theorem 2, there is an intercalate matrix M' of type $[t, r+s-t, n']$ with $n' \leqslant n$. Furthermore, $t + (r+s-t) = r+s = 2^k+1$ and, by the induction hypothesis, $n' \geqslant t \circ (r+s-t) = 2^k$. Therefore, $n \geqslant 2^k = r \circ s$. □

Corollary 9 ([11])**.** *If $r, s \leqslant 2^k$ and $r+s \geqslant 2^k+1$, then $f(r,s) = r \circ s = 2^k$.*

Proof. By Lemma 5 and Pfister's formula we have $f(2^k,2^k) = 2^k \circ 2^k = 2^k$. Furthermore, there are $r' \leqslant r$ and $s' \leqslant s$ such that $r'+s' = 2^k+1$, consequently the monotonicity of f together with Theorem 8 imply $2^k = f(r',s') \leqslant f(r,s) \leqslant f(2^k,2^k) = 2^k$. Also, $2^k = r' \circ s' \leqslant r \circ s \leqslant 2^k \circ 2^k = 2^k$. Therefore, $f(r,s) = 2^k = r \circ s$. □

Let ℓ_k be the number of integer pairs (r,s) guaranteed to satisfy Yuzvinsky's Conjecture by Corollary 9, namely such that $1 \leqslant r,s \leqslant 2^i$ and $r+s \geqslant 2^i+1$ for some $1 \leqslant i \leqslant k$. For example, each dot in Figure 1 corresponds to a known value of $f(r,s)$ for $1 \leqslant r,s \leqslant 64$.

Corollary 10. *The ratio between ℓ_k and the size of the interval $[1,2^k] \times [1,2^k]$ converges asymptotically to $\frac{2}{3}$. That is, $\lim_{k\to\infty} \frac{\ell_k}{2^k \times 2^k} = \frac{2}{3}$.*

Proof. It is easy to see that Corollary 9 implies Yuzvinsky's Conjecture is true for $\ell_k = \frac{1}{2}\sum_{i=0}^{k} 2^k(2^k+1) = \frac{2}{3}(4^k-1)+2^k$ integer pairs (r,s) with $1 \leqslant r,s \leqslant 2^k$. The ratio between ℓ_k and the size of the interval is $\frac{\ell_k}{2^k \times 2^k} = \frac{2}{3}\left(1 - \frac{1}{4^k}\right) + \frac{1}{2^k}$ and hence $\lim_{k\to\infty} \frac{\ell_k}{2^k \times 2^k} = \frac{2}{3}$. □

As a consequence of Theorem 8 and Corollary 9, Lam obtained that Yuzvinsky's Conjecture is true for square matrices.

Corollary 11 ([11])**.** *If $r = s$, then $f(r,s) = r \circ s$.*

We call the pair (r,s) *Lam favorable* if $r+s = 2^{k_h} + \cdots + 2^{k_1} + 1$ for some integers $k_h > \cdots > k_1 > 0$ and $r \leqslant \min(s, 2^{k_1})$.

Figure 1: Integer pairs for which Corollary 9 implies Yuzvinsky's Conjecture.

Lemma 12. *If (r, s) is Lam favorable, then $r \circ s = r + s - 1$.*

Proof. By induction on h. If $h = 1$ then, by Lemma 7, $r \circ s = 2^k = r + s - 1$. Now assume $h > 1$. In this case, $r + (s - 2^{k_h}) = 2^{k_{h-1}} + \cdots + 2^{k_1} + 1$. Thus, by the induction hypothesis, $r \circ (s - 2^{k_h}) = r + s - 2^{k_h} - 1$. Hence, Pfister's formula implies $r \circ s = 2^{k_h} + r \circ (s - 2^{k_h}) = r + s - 1$. ☐

Theorem 13. *If (r, s) is Lam favorable, then $f(r, s) = r \circ s$.*

Proof. By induction on r. If $r = 1$, then $f(1, s) = 1 \circ s$. Now assume $r \geqslant 2$. Let M be an intercalate matrix of type $[r, s, n]$. Suppose all colors of M have frequency r, hence $n = \frac{rs}{r} = s$ and M is of type $[r, s, s]$. Theorem 6 implies $s \geqslant r \circ s$. Furthermore, by Lemma 12, $r \circ s = r + s - 1$, which implies $r \leqslant 1$, a contradiction. Hence, there is a color of frequency $t < r$. Consequently, by Theorem 2 there is an intercalate matrix M' of type $[t, r + s - t, n']$ with $n' \leqslant n$. Since $t + (r + s - t) = r + s$ and $t < r$, it follows that $(t, r + s - t)$ is Lam favorable. Thus, by the induction hypothesis $n' \geqslant t \circ (r + s - t)$. By Lemma 12, $t \circ (r + s - t) = r + s - 1$. Therefore, $n \geqslant n' \geqslant r + s - 1$. ☐

Corollary 14. *Let $k_h > \cdots > k_1 > 0$ and $m = 2^{k_h} + \cdots + 2^{k_1}$. If $r \leqslant 2^{k_1}$, $s \leqslant m$, and $r + s \geqslant m + 1$, then $f(r, s) = r \circ s = m$.*

Proof. Lemma 5 implies $f(2^{k_1}, m) = 2^{k_1} \circ m$. By Pfister's formula we obtain $2^{k_1} \circ m = 2^{k_h} + 2^{k_1} \circ (2^{k_{h-1}} + \cdots + 2^{k_1}) = 2^{k_h} + 2^{k_{h-1}} + 2^{k_1} \circ (2^{k_{h-2}} + \cdots + 2^{k_1}) = \cdots = 2^{k_h} + \cdots + 2^{k_2} + 2^{k_1} \circ 2^{k_1} = 2^{k_k} + \cdots + 2^{k_2} + 2^{k_1} = m$. Furthermore, there are $s' \leqslant s$ and $r' \leqslant r$ such that $s' + r' = m + 1$. Consequently, (s', r') is Lam favorable, then Lemma 12 together with Theorem 13 imply $f(r', s') = r' \circ s' = m$. Hence, $m = f(r', s') \leqslant f(r, s) \leqslant f(2^{k_1}, m) = m$ and $m = r' \circ s' \leqslant r \circ s \leqslant 2^{k_1} \circ m = m$. Therefore, $f(r, s) = m = r \circ s$. ☐

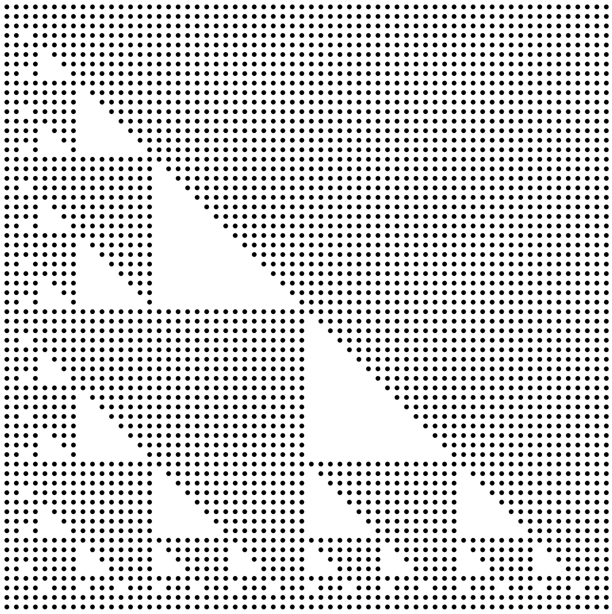

Figure 2: Integer pairs for which Corollary 14 implies Yuzvinsky's Conjecture.

Similarly to ℓ_k, we define ℓ'_k as the number of integer pairs (r, s) guaranteed to satisfy Yuzvinsky's Conjecture by Corollary 14 such that $1 \leqslant r, s \leqslant 2^k$. For example, each dot in Figure 2 corresponds to a known value of $f(r, s)$ for $1 \leqslant r, s \leqslant 64$.

Theorem 15. *The ratio between ℓ'_k and the size of the interval $[1, 2^k] \times [1, 2^k]$ converges asymptotically to $\frac{5}{6}$. That is, $\lim_{k\to\infty} \frac{\ell'_k}{2^k \times 2^k} = \frac{5}{6}$.*

Proof. It is easy to see that Corollary 14 implies Yuzvinsky's Conjecture is true for $\ell'_k = 4^k - \sum_{i=1}^{k-2}(2^{k-i-1} - 1)2^i(2^i - 1) = \frac{1}{6}(5 \cdot 4^k + 4) + (k-1)2^{k-1}$ integer pairs (r, s) with $1 \leqslant r, s \leqslant 2^k$. The ratio between ℓ'_k and the size of the interval is $\frac{\ell'_k}{2^k \times 2^k} = \frac{1}{6}\left(5 + \frac{1}{4^{k-1}}\right) + \frac{k-1}{2^{k+1}}$ and hence $\lim_{k\to\infty} \frac{\ell'_k}{2^k \times 2^k} = \frac{5}{6}$. □

We obtain a generalization of Lemma 5:

Corollary 16. *If $b = 2^k$, then $f(a, b) = a \circ b$ for all $a \geqslant 1$.*

Proof. If $a \leqslant b$, then the claim follows from Corollary 14 with $r = a$ and $s = m = b$. Now, if $a > b$, then let n be the unique integer with $(n-1)b < a \leqslant nb$. Let $s_1 = (n-1)b + 1$ and $s_2 = nb$. By the monotonicity of f, we have $f(s_1, b) \leqslant f(a, b) \leqslant f(s_2, b)$. Using Corollary 14 with $m = nb$, $r = b$ and $s \in \{s_1, s_2\}$, we obtain $f(s, b) = s \circ b = m$. Therefore, $f(a, b) = m = a \circ b$ since $s_1 \circ b \leqslant a \circ b \leqslant s_2 \circ b$. □

We close this section with a short proof of the following result of Lam (private communication):

Theorem 17. *Let $a, b \geqslant 1$. If $r + s = 2^a + 2^b + 1$, then $f(r, s) = r \circ s$.*

Proof. By Corollary 9, we can assume $a > b > 0$. We can also assume $r < s$. If $r \leqslant 2^b$, then (r, s) is Lam favorable and we are done by Theorem 13. Now, if $2^b < r < s \leqslant 2^a$ then, by Corollary 14 taking $m = 2^a$, it follows that $f(r, s) = r \circ s$. □

5 The conjecture for matrices with eight rows or less

Yuzvinsky's Conjecture is known to be true for intercalate matrices with four or less rows [20]. In addition, Corollary 16 implies Yuzvinsky's Conjecture is true for intercalate matrices whose number of rows is a power of two. Hence, a natural step would be to give a proof of Yuzvinsky's Conjecture for intercalate matrices with eight or less rows. This is our next main result.

Theorem 18. *If $r \leqslant 8$, then $f(r, s) = r \circ s$ for all $s \geqslant 1$.*

To prove Theorem 18, we only need to give proofs for matrices with five, six, and seven rows (Theorems 22, 23, and 25). The main idea in these proofs is as follows: First, since we know the upper bound $f(r, s) \leqslant r \circ s$, we can determine which matching lower bounds we have to prove. Second, we assume a color of a given frequency t exists and we use Theorems 2 and 3 to prove that the lower bound holds. Third, we prove that the remaining cases are impossible.

Let us illustrate this method with a proof of $f(3, s) = 3 \circ s$. For $1 \leqslant i \leqslant 4$, let M_i be an intercalate matrix of type $[3, 4k + i, n_i]$, and let a be a color of frequency $1 \leqslant t \leqslant 3$. We assume M_i is in block form with respect to color a. By the monotonicity of f we have:

1. $4k + 4 = 3 \circ (4k + 4) \geqslant f(3, 4k + 4) \geqslant f(3, 4k + 3) \geqslant f(3, 4k + 2)$, and
2. $4k + 3 = 3 \circ (4k + 1) \geqslant f(3, 4k + 1)$.

Therefore, it is enough to prove that $n_2 \geqslant 4k + 4$ and $n_1 \geqslant 4k + 3$.

1. If $t = 1$ then, by Theorem 2, we have $n_2 \geqslant f(1, 4k + 4) = 4k + 4$. If $t = 2$ then, by Theorem 2, we have $n_2 \geqslant f(2, 4k + 3) = 4k + 4$. Now, we can assume M_2 only has colors of frequency 3, hence $n_2 = \frac{3(4k+2)}{3} = 4k + 2$. Thus, M_2 is of type $[3, 4k + 2, 4k + 2]$, contradicting Theorem 6.
2. If $t = 1$ then, by Theorem 2, we have $n_1 \geqslant f(1, 4k + 3) = 4k + 3$. If $t = 2$ then, by Theorem 3, we have $n_1 \geqslant f(3, 4k + 2) = 4k + 4$. Now, we can assume M_1 only has colors of frequency 3, hence $n_1 = \frac{3(4k+1)}{3} = 4k + 1$. Thus, M_1 is of type $[3, 4k + 1, 4k + 1]$, contradicting Theorem 6.

Lemma 19. *The following table gives $r \circ (8k+i)$ for $1 \leqslant r \leqslant 8$:*

r	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$	$8k+8$
1	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$	$8k+8$
2	$8k+2$	$8k+2$	$8k+4$	$8k+4$	$8k+6$	$8k+6$	$8k+8$	$8k+8$
3	$8k+3$	$8k+4$	$8k+4$	$8k+4$	$8k+7$	$8k+8$	$8k+8$	$8k+8$
4	$8k+4$	$8k+4$	$8k+4$	$8k+4$	$8k+8$	$8k+8$	$8k+8$	$8k+8$
5	$8k+5$	$8k+6$	$8k+7$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$
6	$8k+6$	$8k+6$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$
7	$8k+7$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$
8	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$	$8k+8$

Proof. Using Pfister's formula. □

Lemma 20. *Let M be an intercalate matrix of type $[r,s,n]$. For each $1 \leqslant i \leqslant r$, let t_i be the number of colors of M with frequency i. Then the following hold:*

1. $t_1 + 2t_2 + 3t_3 + \cdots + rt_r = rs$.
2. $t_2\binom{2}{2} + t_3\binom{3}{2} + \cdots + t_r\binom{r}{2} \equiv 0 \pmod 2$.
3. $t_2 + t_3 + t_6 + t_7 + \cdots \equiv 0 \pmod 2$.

Proof. The first identity counts the number of entries of M in two different ways. The left-hand side of the second identity counts the number of half-intercalations of M. Since each intercalation consists of two half-intercalations, this number is even. The third identity follows from the second, noting that $\binom{i}{2}$ is odd if and only if $i \equiv 2, 3 \pmod 4$. □

Lemma 21. *There are no $r \times s$ intercalate matrices M with $3 \leqslant r \leqslant s$ such that:*

1. *There are at most $2(r-2)$ colors of frequency less than r.*
2. *There is a color of frequency 2 that intercalates with a color of frequency r.*

Proof. By contradiction, assume there is such a matrix M. By Point *2*, we can assume color c has frequency r, color b has frequency 2, and they intercalate. By permuting rows and columns, and filling its first two columns, M looks like:

$$\begin{pmatrix} c & b & 3 & 4 & \cdots & r & r+1 & \cdots \\ b & c & a_3 & a_4 & \cdots & a_r & \cdot & \cdots \\ 3 & a_3 & c & & & & d & \cdots \\ \vdots & \vdots & & \ddots & & & & \cdots \\ r & a_r & & & & c & & \cdots \end{pmatrix}.$$

We claim that all $2(r-2)$ colors $\{3,\ldots,r,a_3,\ldots,a_r\}$ are pairwise different. Indeed, if $j = a_i$ with $i,j \in \{3,\ldots,r\}$, then $j \neq i$ and $M_{i,2} = M_{1,j}$. Hence, $b = M_{1,2} = M_{i,j}$, a contradiction, since color b has frequency 2. Now, by Point *1*, at least one of those colors (say a_3) must have frequency r and must appear in all rows of M. By permuting columns, we can assume $a_3 = r+1$. Consequently, $M_{1,r+1} = r+1 = a_3 = M_{3,2}$ which implies $d = M_{3,r+1} = M_{1,2} = b$, a contradiction. □

Theorem 22. *$f(5, 8k+i) = 5 \circ (8k+i)$ for all $k \geqslant 0$ and $1 \leqslant i \leqslant 8$.*

Proof. Let M_i be an intercalate matrix of type $[5, 8k+i, n_i]$ and let a be a color of frequency $1 \leqslant t \leqslant 5$. We assume M_i is in block form with respect to color a. By the monotonicity of f and Lemma 19 we have:

1. $8k+8 = 5 \circ (8k+8) \geqslant f(5, 8k+8) \geqslant \cdots \geqslant f(5, 8k+4)$,
2. $8k+7 = 5 \circ (8k+3) \geqslant f(5, 8k+3)$,
3. $8k+6 = 5 \circ (8k+2) \geqslant f(5, 8k+2)$, and
4. $8k+5 = 5 \circ (8k+1) \geqslant f(5, 8k+1)$.

Therefore, it is enough to prove that $n_4 \geqslant 8k+8$, $n_3 \geqslant 8k+7$, $n_2 \geqslant 8k+6$, and $n_1 \geqslant 8k+5$.

1. If $1 \leqslant t \leqslant 4$ then, by Theorem 2, we have $n_4 \geqslant f(t, 8k+9-t) = 8k+8$. Now, we can assume M_4 only has colors of frequency 5, hence $n_4 = \frac{5(8k+4)}{5} = 8k+4$. Consequently, M_4 is of type $[5, 8k+4, 8k+4]$, contradicting Theorem 6.

2. If $t = 1$ then, by Theorem 2, we have $n_3 \geqslant f(1, 8k+7) = 8k+7$. If $t = 3$ then, by Theorem 2, we have $n_3 \geqslant f(3, 8k+5) = 8k+7$. If $t = 4$ then, by Theorem 3, we have $n_3 \geqslant f(5, 8k+4) = 8k+8$. Now, we can assume M_3 has t_2 colors of frequency 2 and t_5 colors of frequency 5, hence $2t_2 + 5t_5 = 5(8k+3)$. For a contradiction, we can assume $n_3 \leqslant 8k+6$, therefore:

   $$8k+6 \geqslant n_3 = t_2 + t_5 \geqslant f(5, 8k+3) \geqslant f(4, 8k+3) = 8k+4.$$

   The unique solution to this Diophantine system is $t_2 = 5$ and $t_5 = 8k+1$. Since t_2 is odd, this contradicts Lemma 20.

3. If $t = 1$ then, by Theorem 2, we have $n_2 \geqslant f(1, 8k+6) = 8k+6$. If $t = 2$ then, by Theorem 2, we have $n_2 \geqslant f(2, 8k+5) = 8k+6$. If $t = 4$ then, by Theorem 3, we have $n_2 \geqslant f(5, 8k+3) = 8k+7$. Now, we can assume M_2 has t_3 colors of frequency 3 and t_5 colors of frequency 5, hence $3t_3 + 5t_5 = 5(8k+2)$. For a contradiction, we can assume $n_2 \leqslant 8k+5$, therefore:

   $$8k+5 \geqslant n_2 = t_3 + t_5 \geqslant f(5, 8k+2) \geqslant f(4, 8k+2) = 8k+4.$$

   The unique solution to this Diophantine system is $t_3 = 5$ and $t_5 = 8k-1$. By Theorem 4, there exists an intercalate matrix of type $[5-3, (8k+2)-3, (8k+2)-3] = [2, 8k-1, 8k-1]$, contradicting Theorem 6.

4. If $t = 1$ then, by Theorem 2, we have $n_1 \geqslant f(1, 8k+5) = 8k+5$. If $t = 4$ then, by Theorem 3, we have $n_1 \geqslant f(5, 8k+2) = 8k+6$. Now, we can assume M_1 has

t_2 colors of frequency 2, t_3 colors of frequency 3, and t_5 colors of frequency 5, hence $2t_2+3t_3+5t_5 = 5(8k+1)$. For a contradiction, we can assume $n_1 \leqslant 8k+4$, therefore:

$$8k + 4 \geqslant n_1 = t_2 + t_3 + t_5 \geqslant f(5, 8k + 1) \geqslant f(4, 8k + 1) = 8k + 4.$$

There are exactly three solutions to this Diophantine system:

(a) The solution $t_2 = 5$, $t_3 = 0$, and $t_5 = 8k-1$. Since t_2+t_3 is odd, this contradicts Lemma 20.

(b) The solution $t_2 = 3$, $t_3 = 3$, and $t_5 = 8k - 2$. Since t_2 is odd, a color of frequency 2, say c, must intercalate with color a of frequency $t \in \{3, 5\}$. By Lemma 21, we can assume $t = 3$. Recall that M_i is in block form with respect to color a. Hence, A has no colors of frequency 5. If C contains a color of frequency 5, then a appears in D, a contradiction. Therefore, A and C form an intercalate matrix of type $[5, 3, 6]$, a contradiction.

(c) The solution $t_2 = 1$, $t_3 = 6$, and $t_5 = 8k-3$. Since t_2+t_3 is odd, this contradicts Lemma 20. □

Theorem 23. *$f(6, 8k + i) = 6 \circ (8k + i)$ for all $k \geqslant 0$ and $1 \leqslant i \leqslant 8$.*

Proof. Let M_i be an intercalate matrix of type $[6, 8k + i, n_i]$ and let a be a color of frequency $1 \leqslant t \leqslant 6$. We assume M_i is in block form with respect to color a. By the monotonicity of f and Lemma 19 we have:

1. $8k + 8 = 6 \circ (8k + 8) \geqslant f(6, 8k + 8) \geqslant \cdots \geqslant f(6, 8k + 3)$, and
2. $8k + 6 = 6 \circ (8k + 2) \geqslant f(6, 8k + 2) \geqslant f(6, 8k + 1)$.

Therefore, it is enough to prove that $n_3 \geqslant 8k + 8$ and $n_1 \geqslant 8k + 6$.

1. If $1 \leqslant t \leqslant 5$ then, by Theorem 2, we have $n_3 \geqslant f(t, 8k + 9 - t) = 8k + 8$. Now, we can assume M_3 only has colors of frequency 6, hence $n_3 = \frac{6(8k+3)}{6} = 8k + 3$. Thus, M_3 is of type $[6, 8k + 3, 8k + 3]$, contradicting Theorem 6.

2. If $1 \leqslant t \leqslant 2$ then, by Theorem 2, we have $n_1 \geqslant f(t, 8k+7-t) = 8k+6$. If $t = 4$ then, by Theorem 3 (applied to the first five rows of M_1) we have $n_1 \geqslant f(5, 8k+2) = 8k+6$. If $t = 5$ then, by Theorem 3, we have $n_1 \geqslant f(6, 8k + 2) = 8k + 6$. Now, we can assume M_1 has t_3 colors of frequency 3 and t_6 colors of frequency 6, hence $3t_3 + 6t_6 = 6(8k + 1)$. For a contradiction, we can assume $n_1 \leqslant 8k + 5$, therefore:

$$8k + 5 \geqslant n_1 = t_3 + t_6 \geqslant f(6, 8k + 1) \geqslant f(5, 8k + 1) = 8k + 5.$$

The unique solution to this Diophantine system is $t_3 = 8$ and $t_6 = 8k - 3$. Since $t_3 + t_6$ is odd, this contradicts Lemma 20. □

Lemma 24. *There is no intercalate matrix of type* $[3, 4, 5]$.

Proof. By contradiction, suppose M is an intercalate matrix of type $[3,4,5]$. Assume first that there is a color of frequency 3, say color 1. Hence, we may assume that M has the form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & x & \cdot \\ 3 & x & 1 & \cdot \end{pmatrix}.$$

By latinicity of columns 2 and 3, we have $x \notin \{1,2,3\}$. If $x = 4$, then by intercalacy $M_{2,4} = 3$ and $M_{3,4} = 2$, a contradiction since M would have only 4 colors. It follows that $x = 5$. By latinicity of row 2 and column 4, we have $M_{2,4} = 3$. A contradiction, by the intercalacy condition on the upper-right 2×2 submatrix. Thus, each color has frequency at most 2. We can assume color 1 has frequency 2 and M has the form

$$\begin{pmatrix} 1 & 2 & \cdot & \cdot \\ 2 & 1 & a & \cdot \\ c & b & 3 & \cdot \end{pmatrix}.$$

By latinicity and intercalacy in the last two rows, it follows that $1, 2, 3, a, b, c$ are pairwise different, a contradiction. □

Theorem 25. *$f(7, 8k + i) = 7 \circ (8k + i)$ for all $k \geqslant 0$ and $1 \leqslant i \leqslant 8$.*

Proof. Let M_i be an intercalate matrix of type $[7, 8k + i, n_i]$ and let a be a color of frequency $1 \leqslant t \leqslant 7$. We assume M_i is in block form with respect to color a. By the monotonicity of f and Lemma 19 we have:

1. $8k + 8 = 7 \circ (8k + 8) \geqslant f(7, 8k + 8) \geqslant \cdots \geqslant f(7, 8k + 2)$, and
2. $8k + 7 = 7 \circ (8k + 1) \geqslant f(7, 8k + 1)$.

Therefore, it is enough to prove that $n_2 \geqslant 8k + 8$ and $n_1 \geqslant 8k + 7$.

1. If $1 \leqslant t \leqslant 6$ then, by Theorem 2, we have $n_2 \geqslant f(t, 8k + 9 - t) = 8k + 8$. Now, we can assume M_2 only has colors of frequency 7, hence $n_2 = \frac{7(8k+2)}{7} = 8k + 2$. Thus, M_2 is of type $[7, 8k + 2, 8k + 2]$, contradicting Theorem 6.
2. If $t \in \{1, 3, 5\}$ then, by Theorem 2, we have $n_1 \geqslant f(t, 8k + 8 - t) = 8k + 7$. If $t = 6$ then, by Theorem 3, we have $n_1 \geqslant f(7, 8k + 2) = 8k + 8$. Now, we can assume M_1 has t_2 colors of frequency 2, t_4 colors of frequency 4, and t_7 colors of frequency 7, hence $2t_2 + 4t_4 + 7t_7 = 7(8k + 1)$. For a contradiction, we can assume $n_1 \leqslant 8k + 6$, therefore:

$$8k + 6 \geqslant n_1 = t_2 + t_4 + t_7 \geqslant f(7, 8k + 1) \geqslant f(6, 8k + 1) = 8k + 6.$$

   There are exactly three solutions to this Diophantine system:

   (a) The solution $t_2 = 7$, $t_4 = 0$, and $t_7 = 8k - 1$. Since t_2 is odd, there exist a color of frequency 2 that intercalates with a color of frequency 7. A contradiction by Lemma 21.

(b) The solution $t_2 = 4$, $t_4 = 5$, and $t_7 = 8k-3$. Since t_2+t_7 is odd, this contradicts Lemma 20.

(c) The solution $t_2 = 1$, $t_4 = 10$ and $t_7 = 8k - 5$. By contradiction, suppose M is an intercalate matrix of type $[7, 8k+1, 8k+6]$. Let b be the color of frequency 2, and let c be the color that intercalates with b. Observe that c must have frequency either 4 or 7.

We assume first c has frequency 7. By the proof of Lemma 21, we obtain a contradiction if at least one element of $W = \{3, \dots, 7, a_3, \dots, a_7\}$ has frequency 7 or if at least two of them are equal. This implies these 10 colors have frequency 4. Hence, M has the form:

$$M = \begin{pmatrix} c & b & 3 & \cdots & 7 & \cdots \\ b & c & a_3 & \cdots & a_7 & \cdots \\ 3 & a_3 & c & \cdots & \cdot & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 7 & a_7 & \cdot & \cdots & c & \cdots \end{pmatrix}.$$

If the other two appearances of these 10 colors are on the first 7 columns of M, then the rest of M would be an intercalate matrix of type $[7, 8k-6, 8k-6]$, contradicting Theorem 6. Consequently, at least one color in W does not appear in the first 7 columns. So, we can assume color 3 appears in column 8. Since the frequency of color 3 is 4, we can assume 3 appears in column 9. Furthermore, by the intercalacy condition, we have $a_{ij} = a_{ji}$ for $3 \leqslant i \leqslant 7$, $3 \leqslant j \leqslant 7$. Hence, by the intercalations induced by colors 3 and c, it follows that M has the form:

$$\left(\begin{array}{cc|ccccc|ccc} c & b & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ b & c & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdots \\ \hline 3 & a_3 & c & 8 & 9 & \cdot & \cdot & 4 & 5 & \cdots \\ 4 & a_4 & 8 & c & \cdot & \cdot & \cdot & 3 & \cdot & \cdots \\ 5 & a_5 & 9 & \cdot & c & \cdot & \cdot & \cdot & 3 & \cdots \\ 6 & a_6 & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdots \\ 7 & a_7 & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdots \end{array}\right).$$

Now, suppose $M_{4,9} = M_{5,8} = d$, then by latinicity of columns 8, 9 and latinicity of rows 4, 5, we have $d \notin \{3, 4, 5, 8, 9, a_4, a_5, a_8, a_9\}$. If $d = a_3$, then $a_4 = 5$. But the colors in W are pairwise different, hence $d \neq a_3$. If $d = 6$, then by the intercalations induced by colors 6 and c, M would look like:

$$\begin{pmatrix} c & b & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ b & c & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdots \\ 3 & a_3 & c & 8 & 9 & \cdot & \cdot & 4 & 5 & \cdots \\ 4 & a_4 & 8 & c & \cdot & 9 & \cdot & 3 & 6 & \cdots \\ 5 & a_5 & 9 & \cdot & c & 8 & \cdot & 6 & 3 & \cdots \\ 6 & a_6 & \cdot & 9 & 8 & c & \cdot & 5 & 4 & \cdots \\ 7 & a_7 & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdots \end{pmatrix}.$$

Consequently, by the intercalations induced by colors 9 and c, we have $M_{3,6} = M_{4,5} = M_{5,4} = M_{6,3} = u$. Since color 7 has frequency 4, it follows that $u \neq 7$ and we can assume $M_{i,10} = 7$ for some $i \in \{2,3,4,5,6\}$. Thus, $M_{7,10} \in \{b,3,4,5,6\}$. But these colors have frequency 2 or 4. Hence $d \neq 6$. Now, if $d = a_6$, then by the intercalations induced by colors a_6 and c, M would have the following form:

$$\begin{pmatrix} c & b & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ b & c & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdots \\ 3 & a_3 & c & 8 & 9 & \cdot & \cdot & 4 & 5 & \cdots \\ 4 & a_4 & 8 & c & \cdot & a_9 & \cdot & 3 & a_6 & \cdots \\ 5 & a_5 & 9 & \cdot & c & a_8 & \cdot & a_6 & 3 & \cdots \\ 6 & a_6 & \cdot & a_9 & a_8 & c & \cdot & a_5 & a_4 & \cdots \\ 7 & a_7 & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdots \end{pmatrix}.$$

We assume $M_{4,5} = e$, then by intercalacy $M_{5,4} = e$. By latinicity of columns 4 and 5, we have $e \notin \{4,5,8,9,a_4,a_5,a_8,a_9\}$. Furthermore, by latinicity of row 4, we have $e \notin \{3,a_6\}$. If $e = 6$ or $e = a_3$, then by intercalacy $a_9 = 5$ or $a_5 = 8$. A contradiction by latinicity of columns 9 and 8, respectively. Consequently $e \notin \{6,a_3\}$. Now, if $e \in \{7,a_7\}$, then $M_{7,4} = M_{4,7} \in \{5,a_5\}$. A contradiction, since a_5 and 5 have frequency 4. Hence, $e \notin W \cup \{8,9,a_8,a_9\}$. In particular e has frequency 7, so we can assume $M_{1,10} = e$. Thus, by the intercalacy condition on color e we have $M_{4,10} = 5$ implying $M_{3,10} = a_6$. A contradiction, since a_6 has frequency 4. Consequently, we can assume $d \notin \{6,a_6\}$. Similarly $d \notin \{7,a_7\}$. Therefore, $d \notin W$, implying d has frequency 7. So, we can assume $M_{1,10} = d$, and M has the form:

$$\begin{pmatrix} c & b & 3 & 4 & 5 & 6 & 7 & 8 & 9 & d & \cdots \\ b & c & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdot & \cdots \\ 3 & a_3 & c & 8 & 9 & \cdot & \cdot & 4 & 5 & f & \cdots \\ 4 & a_4 & 8 & c & f & \cdot & \cdot & 3 & d & 9 & \cdots \\ 5 & a_5 & 9 & f & c & \cdot & \cdot & d & 3 & 8 & \cdots \\ 6 & a_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ 7 & a_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix},$$

where f is the color that intercalates with c in columns 4 and 5. Consequently, $f \notin W$ since these colors have frequency 4. Furthermore, by latinicity of row

4, we have $f \notin \{8, 9, d\}$. If $f = a_8$, then $a_5 = 3$. A contradiction, since the elements of W are pairwise different. Hence $f \neq a_8$. Similarly we have $f \neq a_9$. Therefore, f has frequency 7 and we can suppose M looks like:

$$\begin{pmatrix} c & b & 3 & 4 & 5 & 6 & 7 & 8 & 9 & d & f & \cdots \\ b & c & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdot & \cdot & \cdots \\ 3 & a_3 & c & 8 & 9 & \cdot & \cdot & 4 & 5 & f & d & \cdots \\ 4 & a_4 & 8 & c & f & \cdot & \cdot & 3 & d & 9 & 5 & \cdots \\ 5 & a_5 & 9 & f & c & \cdot & \cdot & d & 3 & 8 & 4 & \cdots \\ 6 & a_6 & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ 7 & a_7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}.$$

If $M_{i,j} = 7$ for some $i \in \{3, 4, 5\}$, then by intercalacy $M_{7,j} = M_{i,1} = i$. A contradiction, since i has frequency 4. If $M_{2,i} = 7$, then by intercalacy $M_{7,i} = b$, a contradiction. So, 7 can only appear in rows 1, 6 and 7. This is a contradiction, since 7 has frequency 4. Hence, our assumption that c had frequency 7 cannot hold.

Therefore, c must have frequency 4. Hence, we can assume

$$M = \left(\begin{array}{cccc|c} c & b & z & w & \\ b & c & y & v & \\ z & y & c & x & Q \\ w & v & x & c & \\ \hline & & & & \\ & & P & & R \end{array}\right).$$

If $v = z$, then $x = b$ by intercalacy with $M_{3,1} = z$. A contradiction, so $v \neq z$. This implies $w \neq y$. Therefore, by latinicity c, b, z, w, v, y, x are pairwise different colors. If there is a color u in P whose frequency is 7, then we can suppose $M_{5,i} = u$ for some $i \in \{1, 2, 3, 4\}$. Also, we can assume $M_{i,5} = u$, hence $M_{5,5} = c$. A contradiction, hence each color of P has frequency 4 in M. If $M_{i,j}$ has frequency 7 with $i \in \{1, 2\}$ and $j \in \{3, 4\}$, then we can assume $M_{i',5} = M_{i,j}$ with $\{i, i'\} = \{1, 2\}$. Consequently, $M_{j,5} = b$, since $M_{j,i} = M_{i,j} = M_{i',5}$ and $M_{i,i'} = b$. This is a contradiction. Therefore, z, y, w, v have frequency 4. Now, suppose x has frequency 7, then x is not a color of P. By the intercalations induced by color x, M would look like:

$$\left(\begin{array}{cccc|cccc} c & b & z & w & x & \cdot & \cdot & \cdots \\ b & c & y & v & \cdot & x & \cdot & \cdots \\ z & y & c & x & w & v & \cdot & \cdots \\ w & v & x & c & z & y & \cdot & \cdots \\ \hline & & & & \cdot & \cdot & x & \cdots \\ & & P & & \cdot & \cdot & \cdot & \cdots \\ & & & & \cdot & \cdot & \cdot & \cdots \end{array}\right).$$

If $M_{5,1} \in \{y, v\}$ or $M_{5,2} \in \{z, w\}$, then $M_{5,3} = b$ or $M_{5,4} = b$. Furthermore, if $M_{5,3} \in \{w, v\}$ or $M_{5,4} \in \{z, y\}$, then $M_{5,5} = c$ or $M_{5,6} = c$. A contradiction, hence $\{w, v, z, y, c\}$ do not appear in P. Consequently, P has at most 5 colors, since these colors have frequency 4 in M. We take the frequencies f_1, f_2, f_3, f_4, f_5 of these colors in P, then $0 \leqslant f_i \leqslant 3$ and $f_1+f_2+f_3+f_4+f_5 = 12$. We can suppose $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_5$, then $f_5 = 3$. By Lemma 24 we have $(f_5, f_4, f_3, f_2, f_1) = (3, 3, 3, 3, 0)$. We take d, e, f, g as the colors of P, then M has the form:

$$\left(\begin{array}{cccc|ccccc} c & b & z & w & x & \cdot & \cdot & \cdot & \cdots \\ b & c & y & v & \cdot & x & \cdot & \cdot & \cdots \\ z & y & c & x & w & v & g & \cdot & \cdots \\ w & v & x & c & z & y & f & \cdot & \cdots \\ \hline d & e & f & g & \cdot & \cdot & x & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{array}\right).$$

If $M_{6,3} = g$ or $M_{6,4} = f$, then $M_{6,7} = c$. A contradiction, since c has frequency 4. Consequently, $M_{6,1} = f$ or $M_{6,1} = g$. This implies $M_{6,7} = w = M_{5,8}$ or $M_{6,7} = z = M_{5,8}$. This is a contradiction, since w and z have frequency 4. Therefore, x has frequency 4. Then there exists a row in P without color x. Assume it is row 5, hence

$$M = \left(\begin{array}{cccc|c} c & b & z & w & \\ b & c & y & v & B \\ z & y & c & x & \\ w & v & x & c & \\ \hline d & e & f & g & \\ \cdot & \cdot & \cdot & \cdot & C \\ \cdot & \cdot & \cdot & \cdot & \end{array}\right)$$

with $x \notin \{d, e, f, g\}$. By latinicity of column 1, we have $d \notin \{c, b, z, w\}$. Since $b \neq f$ and $g \neq b$, it follows that $d \notin \{y, v\}$ and $e \notin \{z, w\}$. Furthermore, e, c, y, v are pairwise different by latinicity of column 2. If $f \in \{v, w\}$ or $g \in \{y, z\}$, then $e = x$ or $d = x$. A contradiction, therefore $z, y, c, x, w, v, d, e, f, g$ are the 10 pairwise different colors of frequency 4. If color x appears in P, then we can assume $M_{6,1} = x$ and M would have the form:

$$\left(\begin{array}{cccc|c} c & b & z & w & \\ b & c & y & v & B \\ z & y & c & x & \\ w & v & x & c & \\ \hline d & e & f & g & \\ x & h & w & z & C \\ \cdot & \cdot & \cdot & \cdot & \end{array}\right)$$

where $M_{6,2} = h$. By latinicity of column 2 and row 6, we have $h \notin \{c, y, v, e, x, w, z\}$. Furthermore, since $e \notin \{x, w, z\}$, then by intercalacy $h \notin \{d, f, g\}$. A contradiction, since h has frequency 4. Hence, x is not in P and d, e, f, g

are the colors of P. Since P is a 3×4 matrix, it follows that d, e, f, g have frequency 3 in P. Furthermore, there is a color of P that appears in columns 1 and 2, since P has 3 rows. Thus, without loss of generality, we can assume $M_{5,1} = M_{6,2} = M_{7,3} = d$ and

$$M = \left(\begin{array}{cccc|c} c & b & z & w & \\ b & c & y & v & B \\ z & y & c & x & \\ w & v & x & c & \\ \hline d & e & f & g & \\ e & d & g & f & C \\ f & g & d & e & \end{array}\right),$$

since every row of P has the colors d, e, f, g. Furthermore, d has frequency 4, and we can suppose $M_{i,5} = d$ for some $i \in \{1, 2, 3, 4\}$. If $1 \leqslant i \leqslant 3$, then $M_{4+i,5} = c$. A contradiction, so $M_{4,5} = d$. Hence, $M_{6,5} = v$ and $M_{7,5} = x$. Thus, $M_{3,5} = e$ since $M_{7,4} = e$. Consequently, $M_{6,5} = z$ by intercalacy of $M_{3,5}$ with $M_{3,1}$ and $M_{6,1}$. This is a contradiction because $M_{6,5} = v$ and $z \neq v$. Hence, c cannot have frequency 4.

Since c cannot have frequency neither 4 nor 7, this concludes the proof of the theorem. □

6 The conjecture in the range 32×32

As mentioned in the introduction, Yiu [20] verified Yuzvinsky's Conjecture whenever $r, s \leqslant 16$. In this section we extend his result to $r, s \leqslant 32$, except for 19 integer pairs (r, s) in this range.

Lemma 26. *If $2^{k-1} < r \leqslant s \leqslant 2^k$ for some k, then $r \circ s = 2^k = r \circ (s-1)$.*

Proof. If $s \neq 2^{k-1}+1$, then $s-1 > 2^{k-1}$ and by Pfister's formula, $r \circ s = 2^k = r \circ (s-1)$. Now, if $s = 2^{k-1}+1$, then $r = 2^{k-1}+1$ and $r \circ s = 2^k$. Thus, by Pfister's formula, $r \circ (s-1) = 2^{k-1} \circ r = 2^{k-1} + 2^{k-1} \circ (r - 2^{k-1}) = 2^k$. Hence, $r \circ s = 2^k = r \circ (s-1)$. □

Lemma 27. *Let r, s be integers such that $1 \leqslant r \leqslant s$, then*

$$\max\{(r-1) \circ s,\ r \circ (s-1)\} \geqslant (r \circ s) - 1.$$

Proof. By induction on $r + s$. If $r = 1$, then $1 \circ (s-1) = s - 1 = (1 \circ s) - 1$. Thus, by Lemma 26, we can assume $2 \leqslant r \leqslant 2^k < s$ for some integer k. Hence, by Pfister's formula, $r \circ s = 2^k + r \circ (s - 2^k)$. By the induction hypothesis

$$\max\big\{(r-1) \circ (s-2^k),\ r \circ (s - 2^k - 1)\big\} \geqslant r \circ (s - 2^k) - 1.$$

If $s \neq 2^k + 1$, then Pfister's formula implies $(r-1) \circ s = 2^k + (r-1) \circ (s - 2^k)$ and $r \circ (s-1) = 2^k + r \circ (s - 2^k - 1)$. Hence, $\max\{(r-1) \circ s,\ r \circ (s-1)\} \geqslant r \circ s - 1$.

Now, we suppose $s = 2^k + 1$. By Pfister's formula, $r \circ s = 2^k + r \circ 1 = 2^k + r$ and $(r-1) \circ s = 2^k + (r-1) \circ 1 = 2^k + r - 1$. Therefore, $(r-1) \circ s = (r \circ s) - 1$. □

We give a shorter proof of the following lemma found in [22].

Lemma 28. *If M is an intercalate matrix of type $[r, s, n]$ which is a counterexample to Yuzvinsky's Conjecture with minimal $r + s$, then $n = r \circ s - 1 = r + s - 2$.*

Proof. Since M is a counterexample to the conjecture, $r \circ s > n$. By the minimality of $r+s$, it follows that $f(r, s-1) = r \circ (s-1)$ and $f(r-1, s) = (r-1) \circ s$. Consequently, the monotonicity of f together with Lemma 27, imply $f(r, s) \geqslant \max\{f(r, s-1), f(r-1, s)\} \geqslant r \circ s - 1$. Furthermore, $n \geqslant f(r, s)$, hence $n = r \circ s - 1 = r + s - 2$. □

Finally, we obtain the following result in the range 32×32.

Theorem 29. *If M is an $r \times s$ intercalate matrix which is a counterexample to Yuzvinsky's Conjecture with minimal $r + s$ and $r \leqslant s \leqslant 32$, then one of the following conditions holds:*

1. $r = 9$ *and* $s \in \{17, 18, 19, 20, 21, 22, 23\}$.
2. $r = 10$ *and* $s \in \{17, 19, 21\}$.
3. $r = 11$ *and* $s \in \{17, 18, 21\}$.
4. $r \in \{12, 14, 15\}$ *and* $s = 17$.
5. $r = 13$ *and* $s \in \{17, 18, 19\}$.

Proof sketch. By Theorem 18, there are no counterexamples with $r \leqslant 8$. By Yiu's result [20], there are no counterexamples with $r \leqslant s \leqslant 16$. Hence, if M is a minimal counterexample, then $17 \leqslant s \leqslant 32$ and $9 \leqslant r \leqslant s$. The only values of r and s in this range satisfying the conditions of Lemma 28 and not covered by Theorem 13 nor Corollary 14 are those listed above. □

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