

A Major-Index Preserving Map on Fillings

Per Alexandersson

Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm, Sweden

per.w.alexandersson@gmail.com

Mehtaab Sawhney

Department of Mathematics
Massachusetts Institute of Technology
Cambridge MA, U.S.A.

msawhney@mit.edu

Submitted: Mar 15, 2017; Accepted: Sep 18, 2017; Published: Oct 6, 2017
Mathematics Subject Classifications: 05E10, 05E05

Abstract

We generalize a map by S. Mason regarding two combinatorial models for key polynomials, in a way that accounts for the major index. Furthermore we define a similar variant of this map, that regards alternative models for the modified Macdonald polynomials at $t = 0$, and thus partially answers a question by J. Haglund. These maps together imply a certain uniqueness property regarding inversion- and coinversion-free fillings. These uniqueness properties allow us to generalize the notion of charge to a non-symmetric setting, thus answering a question by A. Lascoux and the analogous question in the symmetric setting proves a conjecture by K. Nelson.

Keywords: Macdonald polynomials; Hall–Littlewood polynomials; charge; major index; Demazure characters; key polynomials.

1 Introduction

The area of Macdonald polynomials and related combinatorics has been very active in the last 10 years. In 2006, James Haglund, Mark Haiman and Nick Loehr described a combinatorial formula for the non-symmetric Macdonald polynomials, [7]. This combinatorial model specializes to a model for Demazure characters, or *key polynomials* and *Demazure atoms*, which is studied in *e.g.* [16, 14].

This model includes a *basement*, a certain parameter $\sigma \in S_n$, which can be modeled combinatorially, or via Demazure–Lusztig operators, see [4, 1]. A generalization to other types using the Ram–Yip combinatorial model [17] can be found in [3]. The fillings we consider in this paper are the combinatorial objects that generate the specialization $E_\alpha^\sigma(\mathbf{x}; q, 0)$ of *permuted-basement non-symmetric Macdonald polynomials* and the closely related $\tilde{H}_\alpha^\sigma(\mathbf{x}; q, 0)$ which are a specialization of *permuted-basement modified Macdonald polynomials*.

Note that the non-symmetric Macdonald polynomial $E_\alpha(\mathbf{x}; q, t)$ specializes to the *key polynomial* (or Demazure character) $\mathcal{K}_\alpha(\mathbf{x})$ at $q = t = 0$, so the specialization $E_\alpha(\mathbf{x}; q, 0)$ can be considered as a q -deformation of key polynomials. These can be seen as non-symmetric extensions of the modified Hall–Littlewood polynomials $\tilde{H}_\lambda(\mathbf{x}; q)$ in the following sense: we have that $\omega E_\lambda(\mathbf{x}; q, 0) = q^{*(\lambda)} \tilde{H}_\lambda(x_1, \dots, x_n; q^{-1})$ whenever λ is a partition of length n , and $*(\lambda)$ is an appropriate integer¹. This identity follows from properties of LLT polynomials, see *e.g.*, [5].

We also need to mention two relevant articles. First, [8], which concerns key polynomials and Demazure atoms, without any extra parameters. Haglund, Luoto and Mason provide a non-symmetric version of the Littlewood–Richardson rule, by decomposing a product of a Schur polynomial and a key polynomial into key polynomials, as well as the corresponding statement for Demazure atoms. This is done via a non-symmetric analogue of the Robinson–Schensted–Knuth algorithm. In a later article, [9], this algorithm is extended to *permuted basement fillings*, which are similar to the non-attacking fillings with a basement (see further down), except they demand an additional *B-increasing condition*, in order for their RSK-algorithm to work. This extra condition has the disadvantage that for fixed σ , the set of *permuted basement polynomials* $\hat{E}_\alpha^\sigma(\mathbf{x})$ they consider is not in general a basis for the polynomial ring in n variables as α ranges over weak compositions with n parts, see [9, Prop. 14]. In our setting, the families $E_\alpha^\sigma(\mathbf{x}; q, 0)$ and $E_\alpha^\sigma(\mathbf{x}; 0, 0)$ are both bases in the corresponding setting.

1.1 Overview of results

We construct maps between certain fillings that give the evaluation of non-symmetric Macdonald polynomials at $t = 0$. These maps preserve the major index statistic. The existence of such maps are implied by relations given by Demazure operators, but have not been constructed explicitly. We construct such maps with particularly nice properties not implied by the operators themselves — in particular, we show that the maps can be made to preserve column-sets. These maps and their properties allow us to solve several problems in this area:

- In [14], two models for key polynomials are given, with a column-set preserving bijection showing that these are equal. We generalize this map to incorporate a q -parameter, corresponding to major index. It is worth noting that even when $q = 0$, the bijection between the sets we consider is nontrivial — this case was treated in [10] and the map given in the present paper specializes to the one in [10] when dealing with $q = 0$.
- We explicitly construct a biword for coinversion-free fillings and extend the biword given for modified Macdonald polynomials in [5]. In the first case, the charge of the biword is shown to be equal to the major index of the filling, while in the second case the cocharge of the biword is equal to the major index of the filling. The extension of

¹The maximum number of inversion triples in the diagram of shape λ .

the second biword beyond the partition case proves a generalization of a conjecture given in [15].

- We demonstrate a bijective proof that $\tilde{H}_{\sigma_\lambda}^\sigma(\mathbf{x}; q, 0) = \tilde{H}_\lambda(\mathbf{x}; q, 0)$, for any fixed basement $\sigma \in S_n$. In particular, we show that the bijection between the corresponding fillings can be taken to be *column-set-preserving*, a property that uniquely defines this bijection.

Our proof method amounts to first constructing the maps for fillings with two rows, and then proving that these maps are compatible with a larger filling. The second part of this proof is partially done through computer verification, due to the large (but finite) number of cases that need to be considered.

The paper is structured as follows: In Section 2, we introduce the necessary terminology regarding the combinatorial model for non-symmetric Macdonald polynomials. In Section 3 and Section 4, we define and prove properties of the maps. Finally, in Section 5, we examine the various applications and consequences of the maps.

2 Non-symmetric Macdonald polynomials and fillings

In this section, we review the necessary terminology regarding the combinatorial model for non-symmetric Macdonald polynomials, and modified Macdonald polynomials. We use the same notation as in [1], which differs slightly from the one used in [6, 7]. In particular, we use *English notation*, and not the “skyline” way of presenting fillings.

An *augmented diagram* of shape α is a Young diagram where the length of row i from the top is given by $1 + \alpha_i$. The leftmost column is considered special and is referred to as the *basement*. An *augmented filling* is an assignment of natural numbers to the boxes in the diagram. We specify the entries in the basement by listing them from top to bottom — in most places, the basement is a permutation expressed in the one-line notation. The *weight* of a filling is the multiset of entries that are not part of the basement. We let $\mathbf{x}^F := \prod_{u \in F} x_u$, where u ranges over all non-basement entries in F .

Example 1. Below is an augmented filling with shape $(2, 4, 0, 3, 2)$, basement given by $(4, 5, 1, 3, 2)$ and $\mathbf{x}^F = x_1^2 x_2^2 x_3^3 x_4 x_5^2$.

4	1	2			
5	5	4	1	3	
1					
3	3	3	5		
2	2				

Let F be an augmented filling. Two boxes a, b , are said to be *attacking* if $F(a) = F(b)$ and the boxes are either in the same column, or they are in adjacent columns with the rightmost box in a row strictly below the other box. A filling is *non-attacking* if there are no attacking pairs of boxes. The filling in Example 1 is non-attacking.

2.1 Inversions, coinversions and descents

A *triple of type A* is an arrangement of boxes, a, b, c , located such that a is immediately to the left of b , and c is somewhere below b , and the row containing a and b is at least as long as the row containing c . In a similar fashion, a *triple of type B* is an arrangement of boxes, a, b, c , located such that a is immediately to the left of b , and c is somewhere above a , and the row containing a and b is *strictly* longer than the row containing c .

A type A triple is an *inversion triple* if the entries ordered increasingly, form a *counter-clockwise* orientation. Similarly, a type B triple is an inversion triple if the entries ordered increasingly form a *clockwise* orientation. In the case of equal entries, the one with largest subscript in Eq. (1) is considered to be largest.

$$\text{Type A: } \begin{array}{c} \boxed{a_3} \boxed{b_1} \\ \vdots \\ \boxed{c_2} \end{array} \quad \text{Type B: } \begin{array}{c} \boxed{c_2} \\ \vdots \\ \boxed{a_3} \boxed{b_1} \end{array} \quad (1)$$

A triple which is not an inversion triple is called a *coinversion triple*.

Let F be an augmented filling and suppose b is a non-basement box, and a is the box immediately to the left of b . We say that b is a *descent* of F if $F(a) < F(b)$. The set of descents of F is denoted $\text{Des}(F)$.

Boxes in the diagram are indexed according to row and column, with row index increasing further down in the diagram. The *leg* of a box u , $\text{leg}(u)$, in an augmented diagram is the number of boxes to the right of u . The *arm*, $\text{arm}(u)$, of $u = (r, c)$ in an augmented diagram α is the total cardinality of the sets

$$\{(r', c) \in \alpha : r < r' \text{ and } \alpha_{r'} \leq \alpha_r\} \text{ and} \\ \{(r', c - 1) \in \alpha : r' < r \text{ and } \alpha_{r'} < \alpha_r\}.$$

Given an augmented filling F , the *major index*, $\text{maj}(F)$, is defined as

$$\text{maj}(F) := \sum_{u \in \text{Des}(F)} \text{leg}(u) + 1,$$

and the number of inversions, $\text{inv}(F)$ is the number of inversion triples in F . Similarly, $\text{coinv}(F)$ is the number of coinversion triples in F .

Let $\sigma \in S_n$ and let α be a composition with n parts, and let $\text{NAF}(\alpha, \sigma)$ denote the set of non-attacking fillings of the augmented diagram of shape α and basement σ , with entries in $1 \dots, n$. The *non-symmetric permuted basement Macdonald polynomial* $E_\alpha^\sigma(\mathbf{x}; q, t)$ is defined as

$$E_\alpha^\sigma(\mathbf{x}; q, t) := \sum_{F \in \text{NAF}(\alpha, \sigma)} \mathbf{x}^F q^{\text{maj } F} t^{\text{coinv } F} \prod_{\substack{u \in F \\ F(u_-) \neq F(u)}} \frac{1 - t}{1 - q^{1 + \text{leg } u} t^{1 + \text{arm } u}}, \quad (2)$$

where u_- denotes the box to the left of u , and we consider $F(u_-)$ not to be equal to $F(u)$ if u is a box in the basement.

The ordinary non-symmetric Macdonald polynomial $E_\alpha(\mathbf{x}; q, t)$ considered in [7] is recovered when taking $\sigma = w_0 = (n, \dots, 2, 1)$, that is, the unique longest permutation in S_n . The basement w_0 is often referred to as *the key basement* — the reason will be evident further down.

Example 2. The set $\text{NAF}(\alpha, \sigma)$ for $\alpha = (1, 0, 2, 2)$, $\sigma = (2, 1, 3, 4)$ consists of the following augmented fillings:

<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">1</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">2</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:0 maj:0</p>	2	1	1		3	3	2	4	4	4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">1</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">2</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:0 maj:0</p>	2	1	1		3	3	3	4	4	2	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">1</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:0 maj:0</p>	2	1	1		3	3	3	4	4	4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">1</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">4</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">2</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:2 maj:1</p>	2	1	1		3	3	4	4	4	2
2	1																																										
1																																											
3	3	2																																									
4	4	4																																									
2	1																																										
1																																											
3	3	3																																									
4	4	2																																									
2	1																																										
1																																											
3	3	3																																									
4	4	4																																									
2	1																																										
1																																											
3	3	4																																									
4	4	2																																									
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">1</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:1 maj:0</p>	2	2	1		3	3	1	4	4	4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">1</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:1 maj:0</p>	2	2	1		3	3	3	4	4	1	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:0 maj:0</p>	2	2	1		3	3	3	4	4	4	<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td style="padding: 2px 5px;">2</td><td style="padding: 2px 5px;">2</td></tr> <tr><td style="padding: 2px 5px;">1</td><td style="padding: 2px 5px;"></td></tr> <tr><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">3</td><td style="padding: 2px 5px;">4</td></tr> <tr><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">4</td><td style="padding: 2px 5px;">1</td></tr> </table> <p style="text-align: center; margin: 0;">coinv:3 maj:1</p>	2	2	1		3	3	4	4	4	1
2	2																																										
1																																											
3	3	1																																									
4	4	4																																									
2	2																																										
1																																											
3	3	3																																									
4	4	1																																									
2	2																																										
1																																											
3	3	3																																									
4	4	4																																									
2	2																																										
1																																											
3	3	4																																									
4	4	1																																									

Furthermore the *modified Macdonald polynomials* are a class of symmetric functions, defined² via

$$\tilde{H}_\alpha(\mathbf{x}; q, t) = \sum_{F \in \text{FIL}(\alpha, w_0)} \mathbf{x}^F q^{\text{maj } F} t^{\text{inv } F} \quad (3)$$

where $\text{FIL}(\alpha, w_0)$ is the set of fillings of *partition shape* α with no restriction whatsoever. The basement w_0 is now a *big basement* — a decreasing sequence of “infinities”, that is, $w_0 = (\infty_n, \infty_{n-1}, \dots, \infty_1)$, where we consider $\infty_i < \infty_j$ if $i < j$, and $\infty_i > k$ for all natural numbers i, k in the definition of major index and inversion triples, see [5] for details.

It is possible to generalize the modified Macdonald polynomials to *permuted basement modified Macdonald Polynomials* by considering a big basement σ as the sequence $(\infty_{\sigma_1}, \infty_{\sigma_2}, \dots, \infty_{\sigma_n})$. Then define

$$\tilde{H}_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{F \in \text{FIL}(\alpha, \sigma)} \mathbf{x}^F q^{\text{maj } F} t^{\text{inv } F} \quad (4)$$

where $\text{FIL}(\alpha, \sigma)$ is the set of fillings of *composition shape* α with no restriction whatsoever. However, one can show that these polynomials (up to a multiple of a power of t) only depend on the parts of α , see *e.g.* Eq. (21) further down. This fact is not obvious to see from the definition here.

²This is the same definition as in [5], by using the fact that $\tilde{H}_\alpha(\mathbf{x}; q, t) = \tilde{H}_{\alpha'}(\mathbf{x}; t, q)$.

2.2 Specializations of Macdonald polynomials

The main topic of this paper is the specialization of (2) and (4) at $t = 0$. In the first case we have the simplified expression

$$E_\alpha^\sigma(\mathbf{x}; q, 0) = \sum_{F \in \text{CoInvFree}(\alpha, \sigma)} \mathbf{x}^F q^{\text{maj } F} \quad (5)$$

where $\text{CoInvFree}(\alpha, \sigma)$ is the set of coinversion-free fillings of shape α and basement σ . Note that a coinversion-free filling is also automatically non-attacking. We define the *key polynomial* $\mathcal{K}_\alpha(\mathbf{x})$ as the specialization $E_\alpha^{w_0}(\mathbf{x}; 0, 0)$ — note the use of the key basement w_0 .

Caution: our notation differs slightly from other literature: $\kappa_\alpha = \mathcal{K}_{\text{rev}(\alpha)}$, where κ_α is the notation for key polynomials in *e.g.* [18]. Our notation fulfills the relation $\mathcal{K}_\lambda(x_1, \dots, x_n) = s_\lambda(x_1, \dots, x_n)$ whenever λ is a partition with n parts.

Similarly, we have the simplified expression

$$\tilde{H}_\alpha^\sigma(\mathbf{x}; q, 0) = \sum_{F \in \text{InvFree}(\alpha, \sigma)} \mathbf{x}^F q^{\text{maj } F} \quad (6)$$

where $\text{InvFree}(\alpha, \sigma)$ is the set of inversion-free fillings of shape α and basement σ . The standard *modified Hall–Littlewood polynomials*, $\tilde{H}_\alpha(\mathbf{x}; q, 0)$ are recovered by letting $\sigma = w_0$ and α be a partition.

2.3 Brief background on Demazure operators

We only use the following operators briefly in Proposition 5 below, and this subsection is present largely to provide some context.

The non-symmetric Macdonald polynomials and the more general permuted basement Macdonald polynomials can also be recursively defined using Demazure–Lusztig operators, see *e.g.*, [7, 1] for details.

Let s_i act by simple transposition on the indices of the x_i , and define

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}, \quad \pi_i = \partial_i x_i, \quad \theta_i = \pi_i - 1.$$

It is straightforward to see that ∂_i , π_i and θ_i are operators on $\mathbb{C}[\mathbf{x}]$. The operators π_i and θ_i are used to define the key polynomials and Demazure atoms, see *e.g.* [12, 14]. We define the following t -deformations of the above operators.

$$\tilde{\pi}_i(f) = (1 - t)\pi_i(f) + ts_i(f) \quad \tilde{\theta}_i(f) = (1 - t)\theta_i(f) + ts_i(f). \quad (7)$$

The $\tilde{\theta}_i$ are the *Demazure–Lusztig operators*, and generators of the affine Hecke algebra. The $\tilde{\pi}_i$ and $\tilde{\theta}_i$ both satisfy the braid relations, and $\tilde{\pi}_i \tilde{\theta}_i = \tilde{\theta}_i \tilde{\pi}_i = t$. These operators act on the basement of permuted basement Macdonald polynomials, as well as the indexing composition:

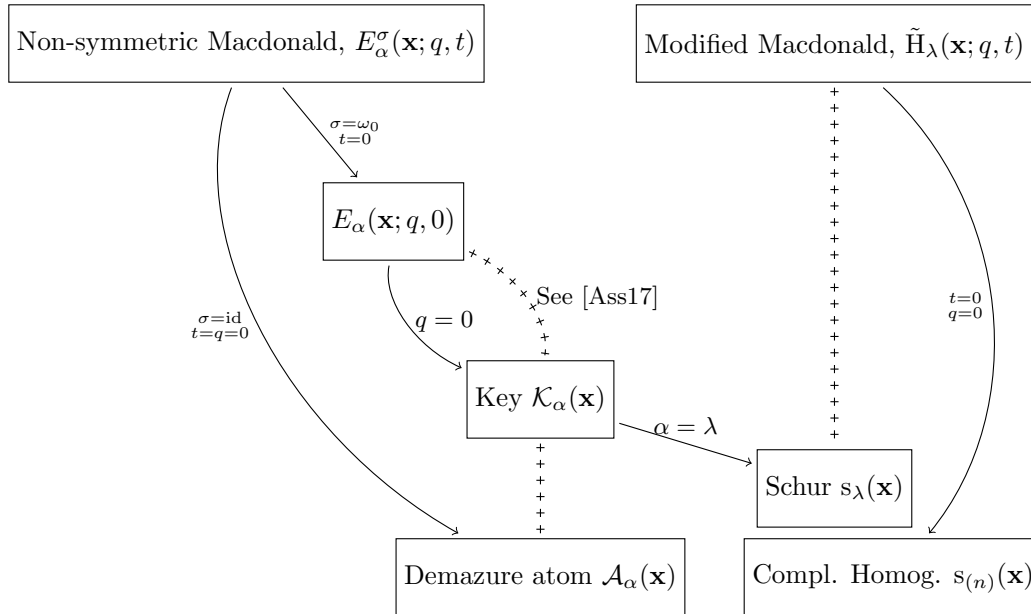


Figure 1: Here is an overview of the polynomials. Arrows down indicate the relation *specializes to*, and plus-arrows down indicate *expands positively into*. Note that one relation is Conjecture 18, which we discuss further down. Here, λ is a partition, and α is a composition.

Proposition 3 (See [1]). *Let α be a composition and let σ be a permutation. Furthermore, let γ_i be the length of the row with basement label i , that is, $\gamma_i = \alpha_{\sigma_i^{-1}}$. If $\ell(\sigma s_i) < \ell(\sigma)$, then*

$$\tilde{\theta}_i E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^{\sigma s_i}(\mathbf{x}; q, t) \times \begin{cases} t & \text{if } \gamma_i \leq \gamma_{i+1} \\ 1 & \text{otherwise.} \end{cases} \quad (8)$$

Similarly, if $\ell(\sigma s_i) > \ell(\sigma)$, then

$$\tilde{\pi}_i E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^{\sigma s_i}(\mathbf{x}; q, t) \times \begin{cases} t & \text{if } \gamma_i < \gamma_{i+1} \\ 1 & \text{otherwise.} \end{cases} \quad (9)$$

Proposition 4 ([See [1]). *If $\alpha_j < \alpha_{j+1}$, $\sigma_j = i + 1$ and $\sigma_{j+1} = i$ for some i, j , then*

$$E_{s_j \alpha}^\sigma(\mathbf{x}; q, t) = \left(\tilde{\theta}_i + \frac{1-t}{1-q^{1+\text{leg } u} t^{\text{arm } u}} \right) E_\alpha^\sigma(\mathbf{x}; q, t), \quad (10)$$

where $u = (j + 1, \alpha_j + 1)$ in the diagram of shape α .

For a reference on these operators and how they relate to Macdonald polynomials, see e.g. [13, 7]. In this literature, $\tilde{\theta}_i$ is denoted T_i , but we prefer to view $\tilde{\theta}_i$ as a deformation of θ_i , which is the notation used by Lascoux and Schützenberger in their study of key polynomials and Demazure atoms, [12].

3 A column-set preserving map

By using the operators in Section 2.3 together with the properties proved in [1], it is possible to show the existence of a weight preserving and major index preserving bijection between two sets of coinversion-free fillings:

Proposition 5. *Let $\text{CoInvFree}(\alpha, \sigma)$ be the set of coinversion-free fillings with shape α and basement σ . Suppose $\sigma_i = \sigma_{i+1} + 1$ and $\alpha_i < \alpha_{i+1}$. Then there exists a bijection*

$$\phi : \text{CoInvFree}(\alpha, \sigma) \longleftrightarrow \text{CoInvFree}(s_i\alpha, \sigma) \sqcup \text{CoInvFree}(s_i\alpha, s_i\sigma) \quad (11)$$

with the property that ϕ preserves major index.

Proof. By letting $t = 0$ in the propositions in the previous section — note in particular that the parenthesis in (10) simplifies to π_i — we may conclude that

$$\begin{aligned} E_\alpha^\sigma(\mathbf{x}; q, 0) &= \pi_i E_{s_i\alpha}^\sigma(\mathbf{x}; q, 0) \\ &= (1 + \theta_i) E_{s_i\alpha}^\sigma(\mathbf{x}; q, 0) \\ &= E_{s_i\alpha}^\sigma(\mathbf{x}; q, 0) + E_{s_i\alpha}^{s_i\sigma}(\mathbf{x}; q, 0), \end{aligned} \quad (12)$$

where s_i acts via simple transpositions on the parts of α .

Since $E_\alpha^\sigma(\mathbf{x}; q, 0)$ is the weighted sum over the elements in $\text{CoInvFree}(\alpha, \sigma)$, and both $E_{s_i\alpha}^\sigma(\mathbf{x}; q, 0)$ and $E_{s_i\alpha}^{s_i\sigma}(\mathbf{x}; q, 0)$ are the weighted sums over $\text{CoInvFree}(s_1\alpha, \sigma)$ and $\text{CoInvFree}(s_1\alpha, \sigma)$ respectively, it follows that a weight-preserving bijection must exist. \square

Given a filling, its *column-sets* is simply the list of (multi)sets of entries in each column. For example, the filling

$$\begin{array}{|c|c|c|} \hline 2 & 2 & \\ \hline 1 & & \\ \hline 3 & 3 & 1 \\ \hline 4 & 4 & 4 \\ \hline \end{array} \text{ has column-sets } (\{1, 2, 3, 4\}, \{2, 3, 4\}, \{1, 4\}).$$

The purpose of this paper is to explicitly construct a bijection ϕ with the additional property that it is *column-set preserving*, that is, F and $\phi(F)$ have the same column-sets. Note that it is not clear a priori that we can impose such a strong condition on ϕ .

In order to construct such a ϕ — which turns out to be unique — we first prove the following statement:

Proposition 6. *Let $\text{CoInvFree}(\alpha, \sigma)$ be the set of coinversion-free fillings with shape α and basement σ . Suppose $\sigma_i > \sigma_{i+1}$ and $\alpha_i > \alpha_{i+1}$. Then there is an injection*

$$\phi : \text{CoInvFree}(\alpha, \sigma) \hookrightarrow \text{CoInvFree}(s_i\alpha, \sigma) \sqcup \text{CoInvFree}(s_i\alpha, s_i\sigma)$$

with the property that ϕ preserves column-sets and major index. Furthermore, ϕ is a bijection whenever $\sigma_i = \sigma_{i+1} + 1$.

The first part of this proposition together with Proposition 5 implies that ϕ is a bijection whenever $\sigma_i = \sigma_{i+1} + 1$.

The proof of Proposition 6 is broken into two major parts: we first construct ϕ for fillings with two rows in Lemma 7, and then show that the result is compatible with the remainder of the filling in Lemma 9.

Lemma 7. *Let $\text{CoInvFree}(\alpha, \sigma)$ be the set of coinversion-free two-row fillings with shape $\alpha = (\alpha_1, \alpha_2)$ and basement $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 > \sigma_2$ and $\alpha_1 > \alpha_2$. Then there is a column-set and major-index preserving injection ϕ*

$$\phi : \text{CoInvFree}(\alpha, \sigma) \hookrightarrow \text{CoInvFree}(s_1\alpha, \sigma) \sqcup \text{CoInvFree}(s_1\alpha, s_1\sigma).$$

Proof. We define a *filling rule* to transform a filling F in $\text{CoInvFree}(\alpha, \sigma)$ to some $F' \in \text{CoInvFree}(s_1\alpha, \sigma) \sqcup \text{CoInvFree}(s_1\alpha, s_1\sigma)$: Start at the end of the first row of F and map that entry to the bottom row (and corresponding column) in F' . Label this entry C and consider the adjacent column to the left.

If this adjacent column in F has one entry, map that entry to the bottom row of F' and let C now denote this entry.

Otherwise, this column has entries $\{A, B\}$. If either entry is greater than or equal to C , choose the least element greater than or equal to C to be in the bottom row. If neither entry is greater than or equal to C , choose the smallest entry to be in the bottom row. Repeat this procedure by letting C be the bottom entry in F' in the column to the right of the column being processed. It suffices to show that the resulting F' is both coinversion-free and that it has the same major index as F .

The first part is easy — note that $A \neq B$ since otherwise, F would contain a coinversion. Then exactly one of the arrangements of A and B can produce a coinversion together with C , and it is straightforward to verify that the filling rule gives the coinversion-free choice.

The main difficulty therefore is demonstrating that the filling rule is major index preserving. We proceed by strong induction on the length of the shortest row α_2 to demonstrate that the map is major index preserving. For the remainder of the proof, let $<$ indicates the presence of a descent between two entries while \geq indicates the lack of such a descent. For the base case, suppose that the shorter row has length 0.

Suppose F has the form

$$\begin{array}{cccc} a_1 & c_1 & c_2 & \cdots & c_m \\ & b_1 & & & \end{array}$$

where a_1 and b_1 are the basement entries. There are two separate cases to consider.

Base case a: $c_1 > a_1$. Since $a_1 > b_1$ it follows that F' has the form

$$\begin{array}{cccc} & a_1 & & & \\ & b_1 & < & c_1 & c_2 & \cdots & c_m \end{array}$$

and therefore both the initial and final fillings have a descent between the first and second columns and the major index is preserved.

Base case b: $c_1 \leq a_1$. Then F' is of the form

$$\begin{array}{c} a_1 \\ b_1 \geq c_1 \end{array} \begin{array}{c} c_2 \\ \dots \\ c_m \end{array} \quad \text{or} \quad \begin{array}{c} b_1 \\ a_1 \geq c_1 \end{array} \begin{array}{c} c_2 \\ \dots \\ c_m \end{array}$$

and in either case, there is no descent between the first and second column. Hence, the major index is preserved and we have proved the base case.

For all the general cases below, we use filling

$$F = \begin{array}{cccccc} a_1 & a_2 & \dots & a_k & c_1 & \dots & c_m \\ b_1 & b_2 & \dots & b_k & & & \end{array}$$

to represent an $F \in \text{CoInvFree}(\alpha, \sigma)$ where a_1, b_1 is the basement with $a_1 > b_1$. As before, F' is the filling obtained from F via the filling rule. We proceed by casework based on how the filling rule was applied.

The main outline for this casework is as follows: There are a few cases to consider depending on if there is some (smallest) index ℓ , $1 < \ell \leq k$ such that $a_\ell > b_\ell$, and if so, how this column appears in F' . Note that we can use the induction hypothesis on all descents to the right of such a column, and conclude that these contribute the same amount to the major index in the two fillings.

Case 1: There exist some $\ell \geq 3$, such that $a_\ell > b_\ell$, $a_i < b_i$ for $2 \leq i \leq \ell - 1$ and a_ℓ appears on top of b_ℓ in F' . Remember, F is of the form

$$\begin{array}{cccccc} a_1 & \dots & a_\ell & \dots & a_k & c_1 & c_2 & \dots & c_m \\ b_1 & \dots & b_\ell & \dots & b_k & & & & \end{array}$$

By the inductive hypothesis, we have that the major index is preserved after the ℓ^{th} column.³

Subcase 1a: Suppose $b_{\ell-1} < b_\ell$. It follows that $a_{\ell-1} < b_{\ell-1} < b_\ell < a_\ell$. Since F has no coinversions it follows that $b_{i+1} > a_i \geq a_{i+1}$ for $2 \leq i \leq \ell - 2$ and since $b_i > a_i$ over the same indices it follows that $b_i > a_i \geq a_{i+1}$. Therefore — using the filling rule — it follows that F' is of the form

$$\begin{array}{c} \sigma(a_1, b_1) b_2 \dots b_{\ell-1} < a_\ell \dots \\ \sigma(a_1, b_1) a_2 \dots a_{\ell-1} < b_\ell \dots c_1 \dots c_m \end{array}$$

and the major index is clearly preserved from the second column onwards. To prove that the entries in the first and second column preserves major index, there are two subsubcases to consider. If $b_1 \geq a_2$, then $b_2 > a_1 > b_1 \geq a_2$ since F has no coinversions. Thus,

$$F = \begin{array}{ccc} a_1 & \geq & a_2 \dots \\ b_1 & < & b_2 \dots \end{array} \quad \text{and} \quad F' = \begin{array}{ccc} a_1 & < & b_2 \dots \\ b_1 & \geq & a_2 \dots \end{array}$$

³Note that we are essentially treating the ℓ^{th} column as a basement in order to invoke the inductive hypothesis. This is done throughout the proof when invoking the inductive hypothesis.

Since the descent in both fillings is in the shortest row, it is clear that in both cases it has the same contributes to the major index.

In the second subsubcase, we have $a_2 > b_1$, and since F has no coinversions this implies that $b_2 > a_1 \geq a_2 > b_1$. Now we have

$$F = \begin{array}{cccc} a_1 & \geq & a_2 & \cdots \\ b_1 & < & b_2 & \cdots \end{array} \quad \text{and} \quad F' = \begin{array}{cccc} b_1 & < & b_2 & \cdots \\ a_1 & \geq & a_2 & \cdots \end{array}$$

and major index is preserved again.

Subcase 1b: Otherwise suppose $b_{\ell-1} \geq b_\ell$.

Subcase 1b.i: If $a_{\ell-1} \geq b_\ell$ it follows that $b_{\ell-1} > a_{\ell-1} \geq b_\ell$. Since F is coinversion-free, it follows that $b_\ell < a_\ell \leq a_{\ell-1} < b_{\ell-1}$. Since F is coinversion-free it also implies that $b_{i+1} > a_i \geq a_{i+1}$ for $2 \leq i \leq \ell - 2$. Furthermore, since $b_i > a_i$ over the same indices, it follows that $b_i > a_i \geq a_{i+1}$ and F' is of the form

$$\begin{array}{ccccccc} \sigma(a_1, b_1) & b_2 & \cdots & b_{\ell-1} & < & a_\ell & \cdots \\ \sigma(a_1, b_1) & a_2 & \cdots & a_{\ell-1} & < & b_\ell & \cdots & c_1 & c_2 & \cdots & c_m. \end{array}$$

Using the same treatment as in Subcase 1a, it follows that $\text{maj } F = \text{maj } F'$.

Subcase 1b.ii: Otherwise, $a_\ell > b_\ell > a_{\ell-1}$ and $b_{\ell-1} \geq b_\ell$.

- If $b_{\ell-1} > b_{\ell-2}$, it follows that $b_{\ell-1} > b_{\ell-2} > a_{\ell-2} \geq a_{\ell-1}$ and F has the form

$$\begin{array}{ccccccc} \cdots & a_{\ell-2} & \geq & a_{\ell-1} & < & a_\ell & \cdots & c_1 & c_2 & \cdots & c_m \\ \cdots & b_{\ell-2} & < & b_{\ell-1} & \geq & b_\ell & \cdots \end{array}$$

while F' is of the form

$$\begin{array}{ccccccc} \cdots & b_{\ell-2} & \geq & a_{\ell-1} & < & a_\ell & \cdots \\ \cdots & a_{\ell-2} & < & b_{\ell-1} & \geq & b_\ell & \cdots & c_1 & c_2 & \cdots & c_m. \end{array}$$

The major index is preserved before the $(\ell - 2)^{\text{nd}}$ column by the reasoning in Subcase 1a. Hence, the major index is preserved on the entire filling and statement follows.

- Otherwise, $b_{\ell-1} \leq b_{\ell-2}$. Since F is coinversion-free, it follows that $a_{\ell-1} \leq a_{\ell-2} < b_{\ell-1}$. In this case, both F and F' have two rows which are of the form

$$\begin{array}{ccccccc} \cdots & a_{\ell-2} & \geq & a_{\ell-1} & < & a_\ell & \cdots \\ \cdots & b_{\ell-2} & \geq & b_{\ell-1} & \geq & b_\ell & \cdots \end{array}$$

Continuing left in such a manner, we either reach an index such that $b_{j-1} < b_j$ for $3 \leq j \leq \ell - 1$ or there is no such index. If $b_{j-1} < b_j$ is the greatest such j , it follows using logic identical to above that F has the form

$$\begin{array}{ccccccc} \cdots & a_{j-1} & \geq & a_j & \geq & a_{j+1} & \cdots & \geq & a_{\ell-1} & < & a_\ell & \cdots & c_1 & c_2 & \cdots & c_m \\ \cdots & b_{j-1} & < & b_j & \geq & b_{j+1} & \cdots & \geq & b_{\ell-1} & \geq & b_\ell & \cdots \end{array}$$

while F' has the form

$$\begin{aligned} \cdots b_{j-1} \geq a_j \geq a_{j+1} \cdots \geq a_{\ell-1} < a_\ell \cdots \\ \cdots a_{j-1} < b_j \geq b_{j+1} \cdots \geq b_{\ell-1} \geq b_\ell \cdots c_1 c_2 \cdots c_m \end{aligned}$$

and the major index is preserved from column j through column ℓ . The remaining columns have major index preserved due to Subcase 1a, and thus major index is preserved overall. If there is no such j , note that $b_2 > a_1 \geq a_2$ since F is coinversion-free with $a_1 > b_1$. Therefore, F is of the form

$$\begin{aligned} a_1 \geq a_2 \cdots \geq a_{\ell-1} < a_\ell \cdots c_1 c_2 \cdots c_m \\ b_1 < b_2 \cdots \geq b_{\ell-1} \geq b_\ell \cdots \end{aligned}$$

while F' is of the form

$$\begin{aligned} a_1 \geq a_2 \cdots \geq a_{\ell-1} < a_\ell \cdots \\ b_1 < b_2 \cdots \geq b_{\ell-1} \geq b_\ell \cdots c_1 c_2 \cdots c_m \end{aligned}$$

and major index is preserved as the sum of leg lengths are the same.

Case 2: There exist some $\ell \geq 3$, such that $a_\ell > b_\ell$, $a_i < b_i$ for $2 \leq i \leq \ell - 1$ and a_ℓ appears below b_ℓ in F' .

Subcase 2a: If $a_{\ell-1} \geq a_\ell$ then it follows that $a_{\ell-1} \geq a_\ell > b_\ell$. Therefore, F' is of the form

$$\begin{aligned} \cdots b_{\ell-1} b_\ell \cdots a_k \\ \cdots a_{\ell-1} a_\ell \cdots b_k c_1 c_2 \cdots c_m \end{aligned}$$

and the major index is preserved using Subcase 1a.

Subcase 2b: Otherwise $a_\ell > a_{\ell-1}$.

Subcase 2b.i: Suppose that $a_\ell > b_{\ell-1}$. Then note that $a_\ell > b_{\ell-1} > a_{\ell-1}$ and F of the form

$$\begin{aligned} \cdots a_{\ell-1} a_\ell \cdots a_k c_1 c_2 \cdots c_m \\ \cdots b_{\ell-1} b_\ell \cdots b_k \end{aligned}$$

while F' is of the form

$$\begin{aligned} \cdots b_{\ell-1} b_\ell \cdots a_k \\ \cdots a_{\ell-1} a_\ell \cdots b_k c_1 c_2 \cdots c_m \end{aligned}$$

and thus the major index is preserved using the logic of Subcase 1a.

Subcase 2b.ii: Otherwise, it follows that $b_{\ell-1} \geq a_\ell > b_\ell$. Since F is coinversion-free, it follows that $a_{\ell-1} < b_\ell < a_\ell$ and F is of the form

$$\begin{aligned} \cdots a_{\ell-1} < a_\ell \cdots a_k c_1 c_2 \cdots c_m \\ \cdots b_{\ell-1} \geq b_\ell \cdots b_k \end{aligned}$$

which yields an F' of the form

$$\begin{aligned} \cdots a_{\ell-1} < b_{\ell} \cdots a_k \\ \cdots b_{\ell-1} \geq a_{\ell} \cdots b_k c_1 c_2 \cdots c_m. \end{aligned}$$

The major index is preserved in this case by following Subcase 1b.ii.

Case 3: Case $\ell = 2$. This means $a_2 > b_2$. By inductive hypothesis, the major index is preserved from the second column onward. We only need to verify that major index is preserved among descents between first and second column.

Subcase 3a: Suppose that $b_2 > a_1$ then it follows that $a_2 > b_2 > a_1 > b_1$. In this case, F' is of the form

$$\begin{array}{ccc} a_1 < a_2 & \cdots & \\ b_1 < b_2 & \cdots & \end{array} \quad \text{or} \quad \begin{array}{ccc} a_1 < b_2 & \cdots & \\ b_1 < a_2 & \cdots & \end{array}$$

and

$$F = \begin{array}{ccc} a_1 < a_2 & \cdots & \\ b_1 < b_2 & \cdots & \end{array}$$

so the major index is preserved, since the descents between first and second columns appear in both fillings.

Subcase 3b: Otherwise $a_1 \geq b_2$ and then there are two possible cases.

Subcase 3b.i: a_2 appear below b_2 in F' :

- Case $b_1 \geq a_2$: It follows that we have the indicated non-descents:

$$F = \begin{array}{ccc} a_1 \geq a_2 & \cdots & \\ b_1 \geq b_2 & \cdots & \end{array} \quad \text{and} \quad F' = \begin{array}{ccc} a_1 \geq b_2 & \cdots & \\ b_1 \geq a_2 & \cdots & \end{array}$$

and thus the major index is preserved.

- Case $b_1 < a_2$: With F being coinversion-free, we have $a_1 \geq a_2 > b_2$, so

$$F = \begin{array}{ccc} a_1 \geq a_2 & \cdots & \\ b_1 * b_2 & \cdots & \end{array} \quad \text{and} \quad F' = \begin{array}{ccc} b_1 * b_2 & \cdots & \\ a_1 \geq a_2 & \cdots & \end{array}$$

where $*$ contributes in the same way to both fillings if it is a descent.

Subcase 3b.ii: a_2 appear above b_2 in F' : There are again two possible cases.

- Case $b_1 \geq b_2$: This condition together with F being coinversion-free implies that $b_2 < a_2 \leq a_1$ and we must have

$$F = \begin{array}{ccc} a_1 \geq a_2 & \cdots & \\ b_1 \geq b_2 & \cdots & \end{array} \quad \text{and} \quad F' = \begin{array}{ccc} a_1 \geq a_2 & \cdots & \\ b_1 \geq b_2 & \cdots & \end{array}$$

- Case $b_1 < b_2$: This gives $a_1 \geq a_2 > b_2 > b_1$ and it follows that

$$F = \begin{matrix} a_1 & \geq & a_2 & \dots \\ b_1 & < & b_2 & \dots \end{matrix} \quad \text{and} \quad F' = \begin{matrix} a_1 & < & a_2 & \dots \\ b_1 & \geq & b_2 & \dots \end{matrix}$$

In both these subcases, major index is preserved.

Case 4: The last case is occurs whenever $a_1 > b_1$ and $a_i < b_i$ for $2 \leq i \leq k$.

Subcase 4a: If $c_1 \leq a_k < b_k$ or $a_k < b_k \leq c_1$ then F' is

$$\begin{matrix} \dots & b_2 & \dots & b_k \\ \dots & a_2 & \dots & a_k & c_1 & c_2 & \dots & c_m \end{matrix}$$

and using the logic in Subcase 1a the result follows.

Subcase 4b: Otherwise $a_k < c_1 \leq b_k$. Then we have

$$F = \begin{matrix} \dots & a_k & < & c_1 & \dots \\ \dots & b_k & & & \dots \end{matrix} \quad \text{and} \quad F' = \begin{matrix} \dots & a_k \\ \dots & b_k & \geq & c_1 & \dots \end{matrix}$$

Using the reasoning from Subcase 1b.ii, there exists a $j \geq 1$ such that

$$F = \begin{matrix} \dots & a_j & \geq & a_{j+1} & \geq & \dots & a_k & < & c_1 & \dots \\ \dots & b_j & < & b_{j+1} & \geq & \dots & b_k & & \end{matrix}$$

and

$$F' = \begin{matrix} \dots & a_j & \geq & a_{j+1} & \geq & \dots & a_k \\ \dots & b_j & < & b_{j+1} & \geq & \dots & b_k & \geq & c_1 & \dots \end{matrix}$$

The two marked descents in F has leg lengths that sum to the leg length of the single marked descent in F' . Furthermore the major index is preserved before the j^{th} using the logic of Subcase 1a and the therefore the major index is preserved overall.

All cases have now been covered, and this concludes the proof. \square

Example 8. As examples of the previous lemma, ϕ acts as follows. Here, we are in the Subcase 3.b.ii, and

$$\phi : \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 3 & 2 & 1 & 3 \\ \hline 3 & 2 & 2 & 1 & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 3 & 2 & & \\ \hline 3 & 2 & 2 & 1 & 1 & 3 \\ \hline \end{array} \quad (13)$$

In the following case, Subcase 3b.i, gives

$$\phi : \begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 3 & 2 & 1 \\ \hline 2 & 1 & 4 & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & 4 & & \\ \hline 4 & 3 & 3 & 2 & 1 \\ \hline \end{array} \quad (14)$$

One noteworthy takeaway from even these small examples is that upon inspection it is not clear how to relate the original and final fillings other than the algorithmic procedure described.

The remainder of the proof is the verification that this two row filling rule is compatible with the remainder of the filling — that is, applying the filling rule on two rows in a larger filling, no coinversions are introduced.

Since the presence of a coinversion is a local condition, we can reduce the proof to a finite list of possible configurations. We verify these via computer verification and the procedure is described in the detail in the following proof.

Lemma 9. *The filling rule in Lemma 7 produces fillings that are compatible with entries in a larger filling, i.e., no coinversions are introduced.*

Proof. Since a coinversion only depends on elements in two separate rows, it suffices to verify the algorithm for augmented fillings with precisely three rows — the two rows affected by the filling rule, and a fixed third row. We may then assume that these three rows are adjacent.

The general strategy is to consider a 2×3 sub-grid of the diagram in F , and then conclude that there are no coinversions in F' . Furthermore, it suffices to restrict the entries between 1 and 6, since only the relative order among the entries determines the presence of coinversions. Let the two by three grid be as below

$$\begin{array}{cc} a & d \\ b & e \\ c & f \end{array}$$

Since inversions depend on the relative order of the row lengths, we need to take that into consideration, by assigning each row a *rank*, a number between 1 and 3, indicating the relative order of the row lengths. In case two rows have the same length, the bottom row is assigned a lower rank. The rank together with the up to six entries in the grid allow us to determine which triples in the grid that are coinversions, and we only consider such possible grids in F which are coinversion-free.

Some of the entries in the grid might not be present in the case we are examining the far right of a filling. However, it is straightforward to see that there is no loss of generality to assume that a , b and c are always present, as the other cases can easily be verified by hand.

The filling rule processes the two rows being swapped from right to left, and we need to verify that there is no coinversion in F' being produced involving the third fixed row. Note that there might locally be *two possible images*, F' and F'' given a local grid F , depending on what the filling rule implied in the second column — in this example, the top two rows are swapped:

$$\begin{array}{ccc} 3 : a & d & 1 : * & d & 1 : * & e \\ F = 1 : b & e & F' = 3 : * & e & F'' = 3 : * & d \\ 2 : c & f & 2 : c & f & 2 : c & f \end{array} \tag{15}$$

The entries marked $*$ are permutations of a and b , and the positions are determined by the filling rule. By construction, there are no coinversions involving only the rows being swapped, so it suffices to check coinversions involving the third, fixed row.

There are exactly three things that can occur locally in the grid. We verify this using the computer.

The diagram is degenerate. One of the rows being swapped has an element missing in the grid. If this is the case, the position of the grid entries in F' are uniquely determined by the filling rule and there is only one case in (15). By checking all such local cases, we see that all corresponding grids are coinversion-free.

Both possibilities are valid.

The non-fixed entries in the second column can either be swapped or not by the filling rule, and *both* these possibilities yield a coinversion-free grid using the filling rule. This is verified by computer. As an example of this situation, we might have

$$\begin{array}{ccc}
 3 : 4 & 2 & 1 : 4 & 2 & 1 : 1 & 1 \\
 F = 1 : 1 & 1 & F' = 3 : 1 & 1 & F'' = 3 : 4 & 2 \\
 2 : 6 & 5 & 2 : 6 & 5 & 2 : 6 & 5
 \end{array}$$

and in either case, the filling rule produces valid (coinversion-free) grid.

Only one of the grids is coinversion-free.

This situation requires a more careful analysis, and we need to do a non-local analysis to prove that F' is indeed of the form that produce a coinversion-free filling.

Computer check verifies that the event that only one of the two grids are valid occurs only under the conditions in the following claim, which then determines that we are in the case that produces a valid grid:

Claim 10. Suppose we swap the longest and shortest row in the 2×3 grid, as in

$$\text{swap} \begin{cases} 3 : a & d \\ 1 : b & e \\ 2 : c & f \end{cases} \quad \text{or} \quad \text{swap} \begin{cases} 2 : a & d \\ 3 : b & e \\ 1 : c & f \end{cases} \quad (16)$$

and one of $e > d > f$ or $d > f > e$ or $f > e > d$ hold (down-increasing condition). Then the corresponding grid in F' *must* be of the respective forms

$$\begin{array}{ccc}
 1 : * & e & 2 : a & d \\
 3 : * & d & 1 : * & f \\
 2 : c & f & 3 : * & e
 \end{array} \quad \text{and} \quad (17)$$

Proof. Suppose that the entries in the second column of F' are not as in Eq. (17), that is, we assume they did not “flip”.

If the adjacent column to the right of the second column in F also has all three entries present, it follows (via computer verification) that these entries also have the down-increasing condition. This third column in F then also appears identically in F' .

The down-increasing condition is therefore an invariant, present in all further columns to the right, via induction.

Eventually, we reach the end one of the shortest row, where last complete column satisfies the down-increasing condition and is identical in both F and F' . Finally, an exhaustive search on the computer shows that it is impossible for F and F' to be of these specified forms and simultaneously be coinversion-free, a contradiction.

Hence, the entries in F' must be arranged as in Eq. (17). Computer verification on the local situation in Eq. (17) verifies that the filling rule produces no coinversions. \square

To conclude the proof, here is a summary of properties that need to be verified via computer:

- Only the three local cases listed above appear among all possible 2×3 -grids; degenerate, two valid possibilities, one valid possibility of specified form.
- The filling rule in Claim 10 preserve the down-increasing condition.
- Preserving the down-increasing condition eventually leads to a contradiction — that is, once we reach the end of the shortest row, F and F' cannot simultaneously be coinversion-free. This forces F to be of the expected form.

We have done this in Mathematica and Java with two different implementations, by checking each possible 2×3 -grid and relative ordering of the rows.

To give a sense of complexity, there are $6^6 \times 3! \approx 280,000$ non-degenerate configurations, but the non-attacking condition and coinversion-free condition reduce the actual number of diagrams quite a bit. Given this relatively small search space, and many configurations which can be dismissed immediately, it follows that a brute force verification of the above claims is relatively quick. \square

4 Another column-set preserving map

It is possible to mimic the above proposition in the case of modified Hall–Littlewood polynomials — fillings without inversions.

Proposition 11. *Let $\text{InvFree}(\alpha, \sigma)$ be the set of inversion-free fillings with shape α and big basement σ . Suppose $\sigma_i > \sigma_{i+1}$ and $\alpha_i \geq \alpha_{i+1}$. Then there is a bijection*

$$\varphi : \text{InvFree}(\alpha, \sigma) \longleftrightarrow \text{InvFree}(s_i \alpha, s_i \sigma)$$

with the property that φ preserves column-sets and major index.

First note that the above proposition is trivial when $a_i = a_{i+1}$ as we can simply interchange the two basement entries and preserve the remainder of the filling, since this action does not introduce inversions and it clearly preserves the major index. In the case when $a_i > a_{i+1}$, we proceed as before and first establish a bijection in the two row case, followed by proving that this is compatible with a larger filling.

Lemma 12. Let $\text{InvFree}(\alpha, \sigma)$ be the set of inversion-free two-row fillings with shape $\alpha = (\alpha_1, \alpha_2)$ and big basement $\sigma = (\sigma_1, \sigma_2)$, with $\sigma_1 > \sigma_2$ and $\alpha_1 > \alpha_2$. Then there is a bijection φ

$$\varphi : \text{InvFree}(\alpha, \sigma) \longleftrightarrow \text{InvFree}(s_1\alpha, s_1\sigma)$$

with the property that φ preserves column-sets and major index.

Proof. We first consider the case when $\alpha_2 = 0$; in this case we can simply interchange the rows and the result follows trivially. Otherwise for the remainder of the proof, treat the column immediately after the basement to be the first column and note the entries this column are in non-increasing order in order for the filling to have no inversion.

We now define a filling rule to transform a filling F in $\text{InvFree}(\alpha, \sigma)$ to a $F' \in \text{InvFree}(s_1\alpha, s_1\sigma)$: Start at the end of the first row and map that entry into to the bottom row in F' . Label this entry C and consider the adjacent column to the left.

If this adjacent column in F has one entry, map that entry to the bottom row of F' and let C denote this entry.

Otherwise, the adjacent column in F has entries $\{A, B\}$. If both entries are greater than or equal to C or less than C , map the larger entry to be in the bottom row of F' . Otherwise map the lesser entry to be in the bottom row of F' . Repeat this procedure with C being the bottom entry in the adjacent column to the right of the column being processed. It suffices to demonstrate that F' is inversion-free and that it has the same major index as F . The first part is note that given a particular $\{A, B\}$ there is exactly one arrangement of A and B that does not give a coinversion with C , and it follows upon a straightforward verification that the filling rule gives the inversion-free choice.

We now proceed by strong induction on the length of the shorter row α_2 to demonstrate that the filling rule is major index preserving. As in the previous proof let $<$ indicate the presence of a descent between two entries while \geq indicates the lack of a descent. For the base case, suppose that the shorter row has length 1. Suppose that F has the form

$$\begin{array}{ccccccc} a_1 & c_1 & c_2 & \dots & c_m \\ & b_1 & & & \end{array}$$

where a_1 and b_1 are the first entries after the basement. Then there are two separate cases to consider.

Base case a: $a_1 \geq c_1$. Since $a_1 \leq b_1$ it follows that F' has the form

$$\begin{array}{ccccccc} & & & & & & a_1 \\ & & & & & & b_1 \geq c_1 & c_2 & \dots & c_m \end{array}$$

and thus both F and F' lack a descent between the first and second columns and the major index is preserved.

Base case b: $a_1 < c_1$. The F' is therefore

$$\begin{array}{ccccccc} & & & & & & a_1 \\ & & & & & & b_1 < c_1 & c_2 & \dots & c_m \end{array} \quad \text{or} \quad \begin{array}{ccccccc} & & & & & & b_1 \\ & & & & & & a_1 < c_1 & c_2 & \dots & c_m \end{array}$$

and in both cases there is a descent between the first and second column as in the original filling. Hence the major index is preserved and the base case is proved.

For all the cases below let original filling F be represented by

$$F = \begin{array}{cccccc} a_1 & a_2 & \dots & a_k & c_1 & \dots & c_m \\ b_1 & b_2 & \dots & b_k & & & \end{array}$$

were a_1 and b_1 are the first element *after* the basement and $a_1 \leq b_1$. As before, F' is the filling obtained from the reverse filling rule. We now proceed by casework on how the filling appears after the filling rule is used.

Case 1: There exists $\ell \geq 3$ such that $a_\ell \leq b_\ell$, $a_i > b_i$ for $2 \leq i \leq \ell - 1$ and a_ℓ appears on top on b_ℓ in F' . By inductive hypothesis, the major index is preserved after the ℓ^{th} column.⁴

Subcase 1a: Suppose that $b_{\ell-1} \geq b_\ell$. It follows that $a_{\ell-1} > b_{\ell-1} \geq b_\ell \geq a_\ell$. Since F is inversion-free, it follows that $b_{i+1} \leq a_i < a_{i+1}$ for $2 \leq i \leq \ell - 2$ and since $a_i > b_i$ over the same indices it follows that $b_i < a_i < a_{i+1}$. Therefore, using the filling rule it follows that F' is

$$\begin{array}{cccc} \sigma(a_1, b_1) & b_2 & \dots & b_{\ell-1} \geq a_\ell \dots \\ \sigma(a_1, b_1) & a_2 & \dots & a_{\ell-1} \geq b_\ell \dots & c_1 & c_2 & \dots & c_m \end{array}$$

and the major index is clearly preserved from the second column onwards. To prove that the major index is preserved between the first and second column, there are two cases. If $b_1 < a_2$, then $b_2 \leq a_1 \leq b_1 < a_2$ as F has no inversions. Thus

$$F = \begin{array}{cccc} a_1 < a_2 & \dots & & \\ b_1 \geq b_2 & \dots & & \end{array} \quad \text{and} \quad F' = \begin{array}{cccc} a_1 \geq b_2 & \dots & & \\ b_1 < a_2 & \dots & & \end{array}$$

Since the descent is in the longest row in both fillings, it is clear that in both cases it has the same contribution to major index.

In the second subcase, $a_2 \leq b_1$ and since F has no inversions it follows that $b_2 \leq a_1 < a_2 \leq b_1$. In this case it follows that we are in the situation

$$F = \begin{array}{cccc} a_1 < a_2 & \dots & & \\ b_1 \geq b_2 & \dots & & \end{array} \quad \text{and} \quad F' = \begin{array}{cccc} b_1 \geq b_2 & \dots & & \\ a_1 < a_2 & \dots & & \end{array}$$

and major index is preserved in this case as well.

Subcase 1b: Otherwise $b_{\ell-1} < b_\ell$.

Subcase 1b.i: If $a_{\ell-1} < b_\ell$ then it follows $b_{\ell-1} < a_{\ell-1} < b_\ell$. Furthermore since F has no inversions it follows that $b_{\ell-1} < a_{\ell-1} < a_\ell \leq b_\ell$. This property also implies that $b_i < a_i < a_{i+1}$ for $2 \leq i \leq \ell - 2$ and thus F' has the form

$$\begin{array}{cccc} \dots & b_{\ell-1} < a_\ell & \dots & \\ \dots & a_{\ell-1} < b_\ell & \dots & c_1 & c_2 & \dots & c_m. \end{array}$$

⁴This can be done as taking the original basement and appending the filling as it appears at the ℓ^{th} column and after gives an inversion free filling. This logic is used whenever the inductive hypothesis is invoked.

Using the same treatment in Subcase 1a, it follows that major index is preserved.

Subcase 1b.ii: Otherwise $a_\ell \leq b_\ell \leq a_{\ell-1}$ and $b_{\ell-1} < b_\ell$.

- If $b_{\ell-1} \leq b_{\ell-2}$ then it follows that $b_{\ell-1} \leq b_{\ell-2} < a_{\ell-2} < a_{\ell-1}$ as there are no inversions in F . Then F and F' are of the forms

$$F = \begin{array}{ccccccc} \dots & a_{\ell-2} & < & a_{\ell-1} & & \geq & a_\ell \dots \\ \dots & b_{\ell-2} & & \geq & b_{\ell-1} & < & b_\ell \dots \end{array}$$

and

$$F' = \begin{array}{ccccccc} \dots & b_{\ell-2} & < & a_{\ell-1} & & \geq & a_\ell \dots \\ \dots & a_{\ell-2} & & \geq & b_{\ell-1} & < & b_\ell \dots \end{array}$$

and the major index is preserved up to the $(\ell - 2)^{nd}$ column by the reasoning in Subcase 1a. Therefore the major index is preserved on the entire filling and statement follows.

- Otherwise, $b_{\ell-1} > b_{\ell-2}$. Since F is inversion-free it follows that $b_{\ell-1} \leq a_{\ell-2} < a_{\ell-1}$. In this case, both F and F' have two rows which are of the form

$$\begin{array}{ccccccc} \dots & a_{\ell-2} & < & a_{\ell-1} & & \geq & a_\ell \dots \\ \dots & b_{\ell-2} & < & b_{\ell-1} & < & b_\ell \dots \end{array}$$

Continuing left, we either reach an index such that $b_{j+1} \leq b_j$ for $2 \leq j \leq \ell - 2$ or there is no such index. If $b_{j+1} \leq b_j$ is the greatest such index it follows using identical reasoning as above that

$$F = \begin{array}{ccccccccccc} \dots & a_j & < & a_{j+1} & \dots & < & a_{\ell-1} & & \geq & a_\ell & \dots & c_1 & \dots & c_m \\ \dots & b_j & & \geq & b_{j+1} & \dots & < & b_{\ell-1} & < & b_\ell & \dots & & & & \end{array}$$

while F' has the form

$$F' = \begin{array}{ccccccccccc} \dots & b_j & < & a_{j+1} & \dots & < & a_{\ell-1} & & \geq & a_\ell & \dots & & & \\ \dots & a_j & & \geq & b_{j+1} & \dots & < & b_{\ell-1} & < & b_\ell & \dots & c_1 & \dots & c_m \end{array}$$

and the major index is preserved between from column j and column ℓ . The remaining columns have major index preserved by the logic of Subcase 1a, and thus the major index is preserved overall. If there is no such j , note that $a_2 > b_2$ and $b_1 \geq a_1$ and since F is inversion-free it follows that $b_2 \leq a_1 < a_2$. Therefore F is of the form

$$F = \begin{array}{ccccccccccc} a_1 & < & a_2 & \dots & < & a_{\ell-1} & & \geq & a_\ell & \dots & c_1 & \dots & c_m \\ b_1 & & \geq & b_2 & \dots & \geq & b_{\ell-1} & < & b_\ell & \dots & & & & \end{array}$$

while F' has the form

$$F' = \begin{array}{ccccccccccc} a_1 & < & a_2 & \dots & < & a_{\ell-1} & & \geq & a_\ell & \dots & & & \\ b_1 & & \geq & b_2 & \dots & \geq & b_{\ell-1} & < & b_\ell & \dots & c_1 & \dots & c_m \end{array}$$

and major index is preserved as the sum of leg lengths are the same.

Case 2: There exist $3 \leq \ell \leq k$ such that $a_\ell \leq b_\ell$, $a_i > b_i$ for $2 \leq i \leq \ell - 1$ and a_ℓ is below b_ℓ in F' .

Subcase 2a: If $a_{\ell-1} < a_\ell$ then it follows that $b_{\ell-1} < a_{\ell-1} < a_\ell \leq b_\ell$. Thus F' has the form

$$\begin{array}{ccccccc} \dots & b_{\ell-1} & b_\ell & \dots & a_k & & \\ \dots & a_{\ell-1} & a_\ell & \dots & b_k & c_1 & c_2 \dots c_m \end{array}$$

and the major index is preserved using Subcase 1a.

Subcase 2b: Otherwise $a_\ell \leq a_{\ell-1}$.

Subcase 2b.i: Suppose $a_\ell \leq b_{\ell-1}$. Since $a_{\ell-1} > b_{\ell-1}$ it follows that $a_{\ell-1} > b_{\ell-1} \geq a_\ell$. Therefore, it follows that

$$F = \begin{array}{ccccccc} \dots & a_{\ell-1} & a_\ell & \dots & c_1 \dots & & \\ \dots & b_{\ell-1} & b_\ell & \dots & & & \end{array} \quad \text{and} \quad F' = \begin{array}{ccccccc} \dots & b_{\ell-1} & b_\ell & \dots & & & \\ \dots & a_{\ell-1} & a_\ell & \dots & c_1 \dots & & \end{array}$$

and the major index is preserved using the reasoning in Subcase 1a.

Subcase 2b.ii: Otherwise $b_{\ell-1} < a_\ell \leq b_\ell$, $a_{\ell-1} \geq a_\ell$. Furthermore $a_{\ell-1} \geq b_\ell \geq a_\ell$ as F has no inversions and it follows that the F and F' have the form

$$F = \begin{array}{ccccccc} \dots & a_{\ell-1} & \geq & a_\ell \dots a_k & c_1 \dots & & \\ \dots & b_{\ell-1} & < & b_\ell \dots b_k & & & \end{array} \quad \text{and} \quad F' = \begin{array}{ccccccc} \dots & a_{\ell-1} & \geq & b_\ell \dots a_k & & & \\ \dots & b_{\ell-1} & < & a_\ell \dots b_k & c_1 \dots & & \end{array}$$

The major index is preserved using the reasoning in Subcase 1b.ii

Case 3: Case $\ell = 2$. In this case $a_2 \leq b_2$ and by inductive hypothesis, the major index is preserved from the second column onward. We need therefore simply to verify that the major index is preserved among descent between the first and second column.

Subcase 3a: Suppose that $b_2 \leq a_1$ then it follows that $a_2 \leq b_2 \leq a_1 \leq b_1$. In this case, we have

$$F = \begin{array}{ccccccc} a_1 & \geq & a_2 \dots a_k & c_1 \dots c_m & & & \\ b_1 & \geq & b_2 \dots b_k & & & & \end{array}$$

while F' is in one of the forms

$$\begin{array}{ccccccc} a_1 & \geq & a_2 \dots a_k & & & & \\ b_1 & \geq & b_2 \dots b_k & c_1 \dots c_m & & & \end{array} \quad \text{or} \quad \begin{array}{ccccccc} a_1 & \geq & b_2 \dots a_k & & & & \\ b_1 & \geq & a_2 \dots b_k & c_1 \dots c_m & & & \end{array}$$

The major index is preserved between the first and second columns since descents are not present in either possible position.

Subcase 3b: Otherwise $b_2 > a_1$ and then there are two possible cases.

Subcase 3b.i: a_2 appears below b_2 in F' :

- Case $b_1 < a_2$, where it follows that $a_1 \leq b_1 < a_2 \leq b_2$ so

$$F = \begin{array}{ccccccc} a_1 & < & a_2 \dots & c_1 \dots c_m & & & \\ b_1 & < & b_2 \dots & & & & \end{array}$$

and

$$F' = \begin{array}{l} a_1 < b_2 \dots \\ b_1 < a_2 \dots \quad c_1 \dots c_m \end{array}$$

and major index is preserved.

- Case $b_1 \geq a_2$. With F being inversion-free and $b_1 \geq a_2$, $a_2 \leq b_2$, $a_1 < b_2$ it follows that $a_1 < a_2 \leq b_2$. Therefore,

$$F = \begin{array}{l} a_1 \quad a_2 \dots \quad c_1 \dots c_m \\ b_1 \quad b_2 \dots \end{array} \quad \text{and} \quad F' = \begin{array}{l} b_1 \quad b_2 \dots \\ a_1 \quad a_2 \dots \quad c_1 \dots c_m \end{array}$$

and major index is preserved.

Subcase 3b.ii: a_2 appears above b_2 in F' :

- Case $b_1 \geq b_2$. It follows that $b_1 \geq b_2 > a_1$ and F has no inversions it follows that $a_1 < a_2 \leq b_2 \leq b_1$. Thus, the situation is

$$F = \begin{array}{l} a_1 < a_2 \dots \quad c_1 \dots c_m \\ b_1 \geq b_2 \dots \end{array} \quad \text{and} \quad F' = \begin{array}{l} b_1 \geq a_2 \dots \\ a_1 < b_2 \dots \quad c_1 \dots c_m \end{array}$$

and major index is preserved.

- Case $b_1 < b_2$. Since $b_2 > a_1$ and F has no inversions it follows that $a_1 < a_2 \leq b_2$. Furthermore $a_1 \leq b_1$ and we have

$$F = \begin{array}{l} a_1 < a_2 \dots \quad c_1 \dots c_m \\ b_1 < b_2 \dots \end{array} \quad \text{and} \quad F' = \begin{array}{l} a_1 < a_2 \dots \\ b_1 < b_2 \dots \quad c_1 \dots c_m \end{array}$$

and major index is preserved.

Case 4: The final case occurs if $a_1 \leq b_1$ and $a_i > b_i$ for $2 \leq i \leq k$.

Subcase 4a: If $c_1 > a_k > b_k$ or $a_k > b_k \geq c_1$ then F' is

$$\begin{array}{l} \dots \quad b_2 \quad \dots \quad b_k \\ \dots \quad a_2 \quad \dots \quad a_k \quad c_1 \quad c_2 \quad \dots \quad c_m \end{array}$$

using the logic of Subcase 1a and the major index is preserved similarly to Subcase 1a.

Subcase 4b: Otherwise $a_\ell \geq c_1 > b_\ell$. We have

$$F = \begin{array}{l} \dots \quad a_\ell \geq c_1 \dots c_m \\ \dots \quad b_\ell \end{array} \quad \text{and} \quad F' = \begin{array}{l} \dots \quad a_\ell \\ \dots \quad b_\ell > c_1 \dots c_m \end{array}$$

Using the reasoning from Subcase 1b.ii it follows that there exists $j \geq 1$ such that

$$F = \begin{array}{l} \dots a_j < a_{j+1} \dots < a_\ell \geq c_1 \dots c_m \\ \dots b_j \geq b_{j+1} \dots < b_\ell \end{array}$$

and

$$F' = \begin{array}{l} \dots b_j < a_{j+1} \dots < a_\ell \\ \dots a_j \geq b_{j+1} \dots < b_\ell < c_1 \dots c_m \end{array}$$

The marked descents in F and F' have leg lengths that sum to the same value so the major index is preserved after column j while the major index is preserved from column 1 to column j using Subcase 1a.

We have now verified all possible cases and the result that the filling rule is major index preserving follows via strong induction on the length of the shorter row.

To demonstrate that is is a bijection, we refer to Theorem⁵ 5.1.1 in [5], with $t = 0$. It follows that the two sets of fillings φ map between are equinumerous. This together with the fact that φ is injective implies that φ is also bijective. \square

Analogous to Proposition 6, it remains to show that φ is compatible with a larger filling. We proceed as before, and reduce this to a finite set of verifications. The approach is similar to the case when we treated coinversion-free fillings, and the main difference is the details in Claim 14.

Lemma 13. *The map φ applied to two adjacent rows in a larger filling does not introduce any inversions.*

Proof. As before, it suffices to consider the three-row case, where φ is applied to two adjacent rows and the third row is fixed. Again, we consider a 2×3 -grid with entries in $1, \dots, 6$, and all three entries in the first column are present. There are again exactly three things that can occur locally in the grid. We verify this using the computer.

The diagram is degenerate. One of the rows being swapped has an element missing in the grid — the entries in F' are uniquely determined. By checking all such local cases, we see that all corresponding grids are inversion-free.

Both possibilities are valid.

The non-fixed entries in the second column can either be swapped or not by the filling rule, and *both* these possibilities yield an inversion-free grid using the filling rule. This is verified by computer. As an example of this situation, we might have

$$F = \begin{array}{l} 3 : 3 \ 4 \\ 1 : 5 \ 6 \\ 2 : 1 \ 2 \end{array} \quad F' = \begin{array}{l} 1 : 3 \ 4 \\ 3 : 5 \ 6 \\ 2 : 1 \ 2 \end{array} \quad F'' = \begin{array}{l} 1 : 5 \ 6 \\ 3 : 3 \ 4 \\ 2 : 1 \ 2 \end{array}$$

and in either case, the filling rule produces valid (inversion-free) grid.

Only one of the grids is inversion-free.

This situation requires a more careful analysis, and we need to do a non-local analysis to prove that F' is indeed of the form that produce a inversion-free filling.

Computer check verifies that the event that only one of the two grids are valid occurs only under the conditions in the following claim, which then determines that we are in the case that produces a valid grid:

⁵Theorem 5.1.1 only treats unaugmented fillings. However, since we fix a basement on both sides that cannot introduce inversions, the result follows.

Claim 14. Suppose we swap the longest and shortest row in the 2×3 grid, as in

$$\text{swap} \begin{cases} 3: & a & d \\ 1: & b & e \\ 2: & c & f \end{cases} \quad \text{or} \quad \text{swap} \begin{cases} 2: & a & d \\ 3: & b & e \\ 1: & c & f \end{cases} \quad (18)$$

Furthermore, suppose that one of $e < d < f$ or $d < f < e$ or $f < e < d$ or $d = f$ and $e \neq f$ hold (up-increasing condition). Then the corresponding grid in F' must be of the respective forms

$$\begin{array}{lcl} 1: & * & e \\ 3: & * & d \\ 2: & c & f \end{array} \quad \text{and} \quad \begin{array}{lcl} 2: & a & d \\ 1: & * & f \\ 3: & * & e \end{array} \quad (19)$$

Proof. Suppose that the entries in the second column of F' are not as in Eq. (19), that is, we assume they did not “flip”.

If the adjacent column to the right of the second column in F also has all three entries present, it follows (via computer verification) that these entries also have the up-increasing condition. This third column in F then also appears identically in F' . The up-increasing condition is therefore an invariant, present in all further columns to the right, via induction.

Eventually, we reach the end one of the shortest row, where last complete column satisfies the up-increasing condition and is identical in both F and F' . Finally, an exhaustive search on the computer shows that it is impossible for F and F' to be of these specified forms and simultaneously be inversion-free.

Hence, the entries in F' must be arranged as in Eq. (19). Computer verification on the local situation in Eq. (19) verifies that the filling rule produces no inversions. \square

Thus, φ is an injection and together with Theorem 5.1.1 in [5], it is also a bijection. It is also possible to prove that this is a bijection by a computer-style method similar to what is done above. \square

As a final comment on this section, note that the set $\text{InvFree}(\alpha, \sigma)$ (with big basement) is empty if there are $i < j$ such that $\alpha_i < \alpha_j$ and $\sigma_i < \sigma_j$. This is due to the fact that the first entry in row j , and the two basement entries σ_i and σ_j form an inversion triple.

5 Applications

Despite the relatively lengthy proofs given, the existence of these column-set preserving maps allow for a multitude of short corollaries.

We begin with a technical result that has been partially proved in a variety of special cases.

Proposition 15. *Given fixed column-sets, there exists at most one coinversion-free filling with shape α and decreasing basement.*

Proof. To prove that there is at most one coinversion free filling with shape α and decreasing basement, simply apply the column-set preserving operator ϕ until the shape becomes a partition.

In the partition case, any list of column-sets with sizes compatible with α admits a unique coinversion-free filling. This unique filling can be constructed from the column-sets via the following iterative process going column by column, and in each column, top to bottom: Given an entry e in column i , the adjacent entry in column $i + 1$ is found by taking the largest unused entry in column-set $i + 1$ less or equal to e , and if there is no such element, simply take the largest entry. It is easy to verify that this gives a coinversion-free filling and given $E_\lambda(\mathbf{x}; 1, 0) = e_{\lambda'}(\mathbf{x})$ the result follows. \square

Given a coinversion-free filling F , we construct the *biword* of F as follows: Let the top row be the non-basement entries of F listed in increasing order and the bottom row be the corresponding columns the entries belong to, listed in decreasing order in case of a tie in the first row, see Eq. (20) for an example.

$$\begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 3 & 2 & 1 & 3 \\ \hline 3 & 2 & 2 & 1 & & \\ \hline 2 & 1 & 4 & & & \\ \hline 1 & & & & & \\ \hline \end{array} \quad \longrightarrow \quad \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 4 & 3 & 1 & 3 & 2 & 1 & 5 & 2 & 1 & 2 \end{pmatrix} \quad (20)$$

We let $\text{cw}(F)$ denote the lower row in this biword, the *charge word* of F . Finally, define the charge of F as $\text{charge}(\text{cw}(F))$, where $\text{charge}(\cdot)$ is defined as in *e.g.*, [6], by decomposing $\text{cw}(F)$ into *standard subwords*, computing $\text{charge}(w) := \text{maj}(\text{rev}(w^{-1}))$ of each such subword w (interpreted as a permutation), followed by adding the results. The standard subwords are extracted iteratively by finding the rightmost occurrence of the smallest element, then scanning right to left for the next smallest element, looping around it necessary, and then repeating this process until one has found an occurrence of largest element to the word. These letters form the first subword and this process is repeated on the remaining letters until there are no more letters left.

For example, the word 1322133241214 has the subword decomposition $w_1 = 3214$, $w_2 = 3241$, $w_3 = 321$, $w_4 = 12$, extracted as

$$\begin{array}{ll} 132213\bar{3}\bar{2}41\bar{2}\bar{1}\bar{4} & \rightarrow 3214 \quad (1) \\ 13221 \quad \bar{3}\bar{2}\bar{4}\bar{1} & \rightarrow 3241 \quad (1) \\ \bar{1}\bar{3}\bar{2}\bar{2}\bar{1} & \rightarrow 321 \quad (0) \\ \bar{1} \quad \bar{2} & \rightarrow 12 \quad (1) \end{array}$$

The charge for the permutations is displayed to the right, so the total charge of the word is 3.

Theorem 16. *Let F be a coinversion-free filling of shape α with the key basement w_0 . Then $\text{charge}(\text{cw}(F)) = \text{maj}(F)$.*

Proof. Note that $\text{cw}(F)$ is uniquely determined by the column-sets of F by construction. Therefore, after applying ϕ repeatedly, it suffices to prove the theorem when F is of partition shape, and from hereon, F is assumed to be of partition shape. Under this assumption, F can be constructed from the column-sets via the following iterative process given in Proposition 15.

Claim: The first column (the column adjacent to the basement) of F is strictly decreasing. Suppose the largest entry in the basement (and thus in the filling) is n . Note that the i^{th} largest entry in the first column is at most $n + 1 - i$, since all entries in a fixed column are distinct. Furthermore, since the i^{th} largest element in the basement is exactly $n + 1 - i$, the previous construction rule implies that the first column is strictly decreasing.

Now note that the subwords obtained in the word decomposition of $\text{cw}(F)$ naturally correspond to rows of F : The claim ensures that the largest entry in the first column is in the topmost row. It follows that entry is also the first entry of the first subword of $\text{cw}(F)$. Now, by the iterative process on how to recover F from its column-sets, it is straightforward to see that the first subword of $\text{cw}(F)$ corresponds to the first row of F . By using the fact that the first column is decreasing, this argument can be repeated for the remaining rows to prove that subword j of $\text{cw}(F)$ corresponds to row j of F .

Finally, since the inverse of the j^{th} subword w_j can be seen to have the same relative order as the entries in row j , it follows that $\text{maj}(\text{rev}(w_j^{-1}))$ is equal to the contribution of row j to $\text{maj}(F)$. Hence, $\text{maj}(F)$ is equal to $\text{charge}(\text{cw}(F))$. \square

Note that Theorem 16 critically relies on the presence of a decreasing basement, and the equality does not hold for non-decreasing basements.

We are now prepared to give an analog of the famous cocharge formula for partition case of the non-symmetric Macdonald polynomials. Note that the following identity can also be proven using properties of LLT polynomials proved in [5] along with the symmetric cocharge formula, however this proof is far more in the style of the proof of the original cocharge formula given in [5]. Furthermore, this particular version seems to generalize to the non-symmetric setting, see Conjecture 18 below.

Theorem 17. *Let λ be a partition. Then*

$$E_\lambda(\mathbf{x}; q, 0) = \sum_{\mu} s_{\mu}(\mathbf{x}) \sum_{P \in \text{SSYT}(\mu', \lambda')} q^{\text{charge}(P)}.$$

Proof. The partition case of Proposition 15 implies that every possible second row (with strictly decreasing blocks in the second row corresponding to runs of the first row), in a biword can be obtained from some coinversion-free filling of shape λ .

By performing RSK on such biwords, every possible pair (P, Q) of SSYT's with the insertion tableau P having shape μ and the recording tableau Q having shape μ' appears. The proof of this is identical to the one given for the traditional RSK in [19]. Since charge is Knuth-invariant, two coinversion-free fillings F_1 and F_2 with identical insertion tableau P

also have the same charge, and Theorem 16 implies that $\text{maj}(F_1) = \text{maj}(F_2) = \text{charge}(P)$. It now follows that

$$\begin{aligned} E_\lambda(\mathbf{x}; q, 0) &= \sum_{F \in \text{CoInvFree}(\lambda, w_0)} q^{\text{maj}(F)} \mathbf{x}^F = \sum_{F \in \text{CoInvFree}(\lambda, w_0)} q^{\text{charge}(c(F))} \mathbf{x}^F \\ &\stackrel{RSK}{=} \sum_{P \in \text{SSYT}(\mu', \lambda')} q^{\text{charge}(P)} \sum_{Q \in \text{SSYT}(\mu)} \mathbf{x}^Q \\ &= \sum_{\mu} s_{\mu}(\mathbf{x}) \sum_{P \in \text{SSYT}(\mu', \lambda')} q^{\text{charge}(P)} \end{aligned}$$

as desired. □

We believe that the above theorem carries over in a more general setting:

Conjecture 18. *Let α be a composition. Then*

$$E_\alpha(\mathbf{x}; q, 0) = \sum_{F \in \text{CoInvFree}(\alpha, w_0)} q^{\text{maj } F} \mathbf{x}^F = \sum_{P \in \text{SSYT}(\mu', \lambda')} q^{\text{charge}(P)} \mathcal{K}_{\gamma(P, \alpha)}(\mathbf{x})$$

where $\lambda = \text{sort}(\alpha)$ and $\gamma(P, \alpha)$ is a composition⁶ determined by α and P . In particular, if S_Q is the subset of fillings in $\text{CoInvFree}(\alpha, w_0)$ with recording tableau $Q \in \text{SSYT}(\mu', \lambda')$, then $\sum_{F \in S_Q} \mathbf{x}^F$ is a key polynomial.

Note that a recent combinatorial proof of a positive key expansion of $E_\alpha(\mathbf{x}; q, 0)$ using *weak dual equivalence* was given in [2], which partially resolves the above conjecture.

The following example illustrates Conjecture 18.

Example 19. We have the following expansions in the key basis:

$$\begin{aligned} E_{320}(\mathbf{x}; q, 0) &= \mathcal{K}_{320} + q\mathcal{K}_{311} + q\mathcal{K}_{221} + q^2\mathcal{K}_{221} \\ E_{203}(\mathbf{x}; q, 0) &= \mathcal{K}_{203} + q\mathcal{K}_{113} + q\mathcal{K}_{212} + q^2\mathcal{K}_{221} \\ E_{023}(\mathbf{x}; q, 0) &= \mathcal{K}_{023} + q\mathcal{K}_{113} + q\mathcal{K}_{122} + q^2\mathcal{K}_{221} \end{aligned}$$

The corresponding recording tableaux are

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline & & \\ \hline \end{array}.$$

Observe that all coefficients are identical, and only the composition indexing the key polynomials differ by some permutation. Furthermore, several different permutations of the same partition might be present, as with \mathcal{K}_{212} and \mathcal{K}_{221} above. This refines the Kostka–Foulkes polynomials, as different terms in the Kostka–Foulkes polynomials might be associated to different key polynomials in the non-symmetric setting.

The question of finding an appropriate generalization of charge that explains this phenomena was raised by Lascoux in [11, p. 267–268], and thus Theorem 16 answers this question.

⁶That rearranges to the conjugate shape of P .

5.1 Inversion-free fillings and Hall–Littlewood polynomials

Similar to Proposition 15, we have the following result:

Proposition 20. *Given fixed column-sets there exists at most one inversion-free filling with shape α and big basement σ .*

Proof. Proof is similar to the one in Proposition 15. □

A number of analogous results are also possible for the modified Hall–Littlewood polynomials. The following identity is given in [7, Thm. 5.1.1]: Let τ be a permutation and λ be a partition. Then

$$\tilde{H}_{\tau\lambda}^{\tau w_0}(\mathbf{x}; q, t) = \tilde{H}_{\lambda}(\mathbf{x}; q, t). \quad (21)$$

The proof of this is rather indirect, and J. Haglund asked for a bijective proof of this identity. With our bijection φ , we can prove the $t = 0$ case:

Theorem 21. *For every $\tau \in S_n$ and partition λ , there is a column-set preserving bijection that establish the identity*

$$\tilde{H}_{\tau\lambda}^{\tau w_0}(\mathbf{x}; q, 0) = \tilde{H}_{\lambda}(\mathbf{x}; q, 0). \quad (22)$$

Proof. Consider the fillings contributing to $\tilde{H}_{\tau\lambda}^{\tau w_0}(\mathbf{x}; q, 0)$. We may repeatedly apply φ until the resulting fillings are the ones contributing to $\tilde{H}_{\lambda}(\mathbf{x}; q, 0)$. □

Although it might be tedious to carry out the bijections, remember that the resulting bijection is uniquely defined by the column-set preserving property.

We have not been able to prove the more general Eq. (21), but computer experiments suggests the following refinement of the equality:

Conjecture 22. *Let $\text{FIL}(\lambda, \tau, C)$ be all fillings with shape λ , big basement τ and column-sets C . Then*

$$\sum_{F \in \text{FIL}(\lambda, w_0, C)} q^{\text{maj}(F)} t^{\text{inv}(F)} = \sum_{F \in \text{FIL}(\tau\lambda, \tau w_0, C)} q^{\text{maj}(F)} t^{\text{inv}(F)}.$$

This indicates that one should be able to find a *column-set preserving bijection* proving Haglund’s identity.

Remark 23. One can modify Lemma 7 to show that for every τ , we have

$$[t^{\text{top}}] \tilde{H}_{\tau\lambda}^{\tau w_0}(\mathbf{x}; q, t) = [t^{\text{top}}] \tilde{H}_{\lambda}(\mathbf{x}; q, t),$$

where $[t^{\text{top}}]$ indicate the coefficient of the maximal power of q , and the bijection proving this is given by ϕ . To give an outline of the proof, note first that fillings contributing to both sides of the above identity are exactly coinversion-free fillings. Furthermore, given the big basement, the first set in the right hand side of Proposition 6 is empty.

We conclude this paper by proving an analogue of Theorem 16, in the case of inversion-free fillings. Let F be an inversion-free filling with a basement. Define the following biword of F (different from the above) as follows: Let the top row be the non-basement entries of F listed in decreasing order and the bottom row be the corresponding columns the entries belong to, listed in *increasing* order in case of a tie in the first row. Let the cocharge word of F , denoted $\text{ccw}(F)$, be the bottom row in this biword. As an example, the tableau

$$\begin{array}{|c|c|} \hline 7 & 1 \\ \hline 9 & 1 & 4 & 2 & 3 \\ \hline 8 & 2 & 1 & 3 \\ \hline 6 & & & & \\ \hline \end{array} \quad \longrightarrow \quad \begin{pmatrix} 9 & 8 & 7 & 6 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 4 & 5 & 2 & 4 & 2 & 2 & 3 \end{pmatrix}$$

has subword decomposition 13542, 1423, 12, 1, and cocharge value $4 + 1 + 0 + 0$.

Theorem 24. *Let F be an inversion-free filling of composition shape $\tau\lambda$ where λ is partition and basement τw_0 . Then*

$$\text{cocharge}(\text{ccw}(F)) = \text{maj}(F).$$

where $\text{cocharge}(\cdot)$ is defined as in [5].

Proof. The case where α is partition is given in [5].⁷ The result then follows by noting that the $\text{ccw}(F)$ is only dependent on the columns sets of F , so by applying the column-set and major-index preserving map φ until the partition shape is reached, the statement follows. \square

This answers a conjecture given in [15].⁸

Acknowledgements

The authors would like to thank Jim Haglund for helpful discussions, and the hospitality of University of Pennsylvania where this project was carried out. The second author would also like to thank the first author for his extensive mentorship throughout the project. PA is funded by the *Knut and Alice Wallenberg Foundation* (2013.03.07).

References

- [1] Per Alexandersson. Non-symmetric Macdonald polynomials and Demazure–Lusztig operators. pages 1–19, 2015. [arXiv:1602.05153](https://arxiv.org/abs/1602.05153).
- [2] Sami Assaf. Nonsymmetric Macdonald polynomials and a refinement of Kostka–Foulkes polynomials. 2017. [arXiv:1703.02466](https://arxiv.org/abs/1703.02466) (to appear in *Trans. Amer. Math. Soc.*).

⁷The proposition in [5] does not include a basement explicitly, but adding ω_0 as basement in the partition case leaves the analysis in [5] unchanged.

⁸In [15], only fillings of shape (v_1, v_2, \dots, v_k) and an index ℓ such that $v_1 \geq v_2 \geq \dots \geq v_\ell$ and $v_1 < v_{\ell+1} < v_{\ell+2} \dots < v_k$ are considered and the (implicit) basement is of the same form as in Theorem 24.

- [3] Evgeny Feigin, Ievgen Makedonskyi, and Daniel Orr. Generalized Weyl modules and nonsymmetric q -Whittaker functions, 2016.
- [4] Jeffrey Paul Ferreira. *Row-strict Quasisymmetric Schur Functions, Characterizations of Demazure Atoms, and Permuted Basement Nonsymmetric Macdonald Polynomials*. PhD thesis, University of California Davis, 2011.
- [5] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18(03):735–762, jul 2005.
- [6] James Haglund. *The q, t -Catalan Numbers and the Space of Diagonal Harmonics (University Lecture Series)*. American Mathematical Society, 2007.
- [7] James Haglund, Mark Haiman, and Nick Loehr. A Combinatorial Formula for Nonsymmetric Macdonald Polynomials. *Amer. J. Math.*, 130(2):359–383, 2008.
- [8] James Haglund, Kurt W. Luoto, Sarah Mason, and Stephanie van Willigenburg. Refinements of the Littlewood–Richardson Rule. *Trans. Amer. Math. Soc.*, 363:1665–1686, 2011.
- [9] James Haglund, Sarah Mason, and Jeffrey Remmel. Properties of the nonsymmetric Robinson–Schensted–Knuth algorithm. *J. Algebr. Comb.*, 38(2):285–327, oct 2012.
- [10] Daniel Kurland. An exploration of key polynomials. B.S thesis, University of Pennsylvania, 2016.
- [11] Alain Lascoux. *Polynomials*. Web: <http://phalanstere.univ-mlv.fr/~al/>, September 2013.
- [12] Alain Lascoux and Marcel-Paul Schützenberger. Keys & standard bases. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 125–144. Springer, New York, 1990.
- [13] I. G. Macdonald. Affine Hecke algebras and orthogonal polynomials. *Séminaire Bourbaki*, 37:189–207, 1994-1995.
- [14] Sarah Mason. An explicit construction of type A Demazure atoms. *Journal of Algebraic Combinatorics*, 29(3):295–313, 2009.
- [15] Kendra Nelsen. A New Interpretation of the Cocharge Statistic. M.s thesis, University of Pennsylvania, 2005.
- [16] Anna Pun. *On Decomposition of the Product of Demazure Atoms and Demazure Characters*. PhD thesis, University of Pennsylvania, 2016.
- [17] Arun Ram and Martha Yip. A combinatorial formula for Macdonald polynomials. *Advances in Mathematics*, 226(1):309–331, 2011.
- [18] Victor Reiner and Mark Shimozono. Key polynomials and a flagged Littlewood–Richardson rule. *J. Combin. Theory Ser. A*, 70(1):107–143, 1995.
- [19] Richard P. Stanley. *Enumerative Combinatorics: Volume 2*. Cambridge University Press, 1st edition, 2001.