Antimagic orientation of biregular bipartite graphs

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Abstract

An antimagic labeling of a directed graph D with n vertices and m arcs is a bijection from the set of arcs of D to the integers $\{1, \dots, m\}$ such that all n oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. An undirected graph G is said to have an antimagic orientation if G has an orientation which admits an antimagic labeling. Hefetz, Mütze, and Schwartz conjectured that every connected undirected graph admits an antimagic orientation. In this paper, we support this conjecture by proving that every biregular bipartite graph admits an antimagic orientation.

Keywords: Labeling; Antimagic labeling; Antimagic orientation

1 Introduction

Unless otherwise stated explicitly, all graphs considered are simple and finite. A *labeling* of a graph G with m edges is a bijection from $E(G)$ to a set S of m integers, and the *vertex sum* at a vertex $v \in V(G)$ is the sum of labels on the edges incident to v. If there are two vertices have same vertex sums in G, then we call them *conflict*. A labeling of $E(G)$ with no conflicting vertex is called a *vertex distinguishable labeling*. A labeling is *antimagic* if it is vertex distinguishable and $S = \{1, 2, \dots, m\}$. A graph is *antimagic* if it has an antimagic labeling.

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Hartsfield and Ringel [8] introduced antimagic labelings in 1990 and conjectured that every connected graph other than K_2 is antimagic. There have been significant progresses toward this conjecture. Let G be a graph with n vertices other than K_2. In 2004, Alon, Kaplan, Lev, Roditty, and Yuster [1] showed that there exists a constant c such that if G has minimum degree at least $c \cdot \log n$, then G is antimagic. They also proved that G is antimagic when the maximum degree of G is at least $n-2$, and they proved that all complete multipartite graphs (other than K_2) are antimagic. The latter result of Alon et al. was improved by Yilma [15] in 2013.

Apart from the above results on dense graphs, the antimagic labeling conjecture has been also verified for regular graphs. Started with Cranston [5] showing that every bipartite regular graph is antimagic, regular graphs of odd degree [6], and finally all regular graphs [2, 3] were shown to be antimatic sequentially. For more results on the antimagic labeling conjecture for other classes of graphs, see [7, 9, 11, 12].

Hefetz, Mütze, and Schwartz [10] introduced the variation of antimagic labelings, i.e., antimagic labelings on directed graphs. An *antimagic* labeling of a directed graph with m arcs is a bijection from the set of arcs to the integers $\{1, ..., m\}$ such that any two oriented vertex sums are pairwise distinct, where an *oriented vertex sum* is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. A digraph is called *antimagic* if it admits an antimagic labeling. For an undirected graph G, if it has an orientation such that the orientation is antimagic, then we say G admits an *antimagic orientation*. Hefetz et al. in the same paper posted the following problems.

Question 1 ([10])**.** Is every connected directed graph with at least 4 vertices antimagic?

Conjecture 2 ([10])**.** Every connected graph admits an antimagic orientation.

Parallel to the results the on antimagic labelling conjecture, Hefetz, Mütze, and Schwartz [10] showed that every orientation of a dense graph is antimagic and almost all regular graphs have an antimagic orientation. Particulary, they showed that every orientation of stars (other than $K_{1,2}$), wheels, and complete graphs (other than K_3) is antimagic. Recently, Li et al. [13] showed that every connected even regular graph has an antimagic orientation. Observe that if a bipartite graph is antimagic, then it has an antimagic orientation obtained by directing all edges from one partite set to the other. Thus by the result of Cranston [5], regular bipartite graphs have an antimagic orientation. A bipartite graph is *biregular* if vertices in each of the same partite set have the same degree. In this paper, by supporting Conjecture 2, we obtain the result below.

Theorem 3. *Every biregular bipartite graph admits an antimagic orientation.*

2 Notation and Lemmas

Let G be a graph. If G is bipartite with partite sets X and Y, we denote G by $G[X, Y]$. Given an orientation of G and a labeling on $E(G)$, for a vertex $v \in V(G)$ and a subgraph H of G, we use $\omega_H(v)$ to denote the *oriented sum at v* in H, which is the sum of labels of all arcs entering v minus the sum of labels of all arcs leaving it in the graph H. If v is of degree 2 in G, we say the labels at edges incident to v the *label at* v and write it as a pair in $\{(a,b),(-a,b),(a,-b),(-a,-b)\}$, where a,b are the labels on the two edges incident to v, and $-a$ is used if the edge with label a is leaving v and a is used otherwise; similar situation for the value $-b$ or b.

A *trail* is an alternating sequence of vertices and edges $v_0, e_1, v_1, \dots, e_t, v_t$ such that v_{i-1} and v_i are the endvertices of e_i, for each i with $1 \leqslant i \leqslant t$, and the edges are all distinct (but there might be repetitions among the vertices). A trail is *open* if $v_0 \neq v_t$. The *length* of a trail is the number of edges in it. Occasionally, a trail T is also treated as a graph whose vertex set is the set of distinct vertices in T and edge set is the set of edges in T. We use the terminology "trail" without distinguishing if it is a sequence or a graph, but the meaning will be clear from the context. For two integers a,b with $a<b$, let $[a,b] := \{a, a+1, \dots, b\}$.

We need the result below which guarantees a matching in a bipartite graph. It is an exercise to prove it by applying Hall's Matching Theorem.

Lemma 4. *Let H be a bipartite graph with partite sets X and Y. If there is no isolated vertex in X and $d_H(x) \geqslant d_H(y)$ holds for every edge xy with $x \in X$ and $y \in Y$, then H has a matching which saturates X.*

For even regular graphs, Petersen proved that a 2-factor always exists.

Lemma 5 ([14])**.** *Every regular (multi)graph with positive even degree has a 2-factor.*

Also we need the following result on decomposing edges in a graph into trails.

Lemma 6 ([2])**.** *Given a connected graph G, and let $T = \{v \in V : d_G(v)$ is odd$\}$. If $T \neq \emptyset$, then $E(G)$ can be partitioned into $\frac{|T|}{2}$ open trails.*

Lemma 7. *Every simple 2-regular graph G admits a vertex distinguishable labeling with labels in $[a,b]$, where a,b are two positive integers with $b-a = |E(G)|-1$. Moreover, the vertex sums belong to $[2a+1, 2b-1]$.*

Proof. Note that G is antimagic by Corollary 3 in [5]. Assume that $\phi : E(G) \to [1, |E(G)|]$ is an antimagic labeling of G. Define another labeling $\varphi : E(G) \to [a,b]$ based on ϕ as follows.

$$\varphi(e) = \phi(e) + a - 1, \quad \forall\, e \in E(G).$$

Since G is regular and ϕ is antimagic, it is clear that φ is a vertex distinguishable labeling of G. Furthermore, the sums fall into the interval $[2a+1, 2b-1]$. $\square$

Lemma 8. *Let $T[X,Y]$ be an open trail with all vertices in Y having degree 2 except precisely two having degree 1. Suppose T has $2m$ edges. Let y_1 and y_{m+1} be the two degree 1 vertices in Y such that T starts at y_1 and ends at y_{m+1}. Let a,b be two integers with $a \leqslant b-2m+1$. Then there exists a bijection from $E(T)$ to $[a, a+m-1] \cup [b-m+1, b]$ such that each of the following holds.*

(i) $\omega_T(x) = \frac{d_T(x)(a+b)}{2}$ *for any* $x \in X$*; and* $\omega_T(y) \neq \omega_T(z)$ *for any distinct* $y, z \in Y - \{y_1, y_{m+1}\}$.

(ii) If $m \equiv 0 \pmod 2$*, then* $\omega_T(y_1) = b$*,* $\omega_T(y_{m+1}) = b - m + 1$*, and* $\omega_T(y)$ *is an odd number in* $[2a+1, 2a+2m-3] \cup [2b-2m+5, 2b-3]$ *for any* $y \in Y - \{y_1, y_{m+1}\}$.

(iii) If $m \equiv 1 \pmod 2$*, then* $\omega_T(y_1) = a$*,* $\omega_T(y_{m+1}) = b - m + 1$*, and* $\omega_T(y)$ *is an odd number in* $[2a+3, 2a+2m-3] \cup [2b-2m+5, 2b-1]$ *for any* $y \in Y - \{y_1, y_{m+1}\}$.

Proof. Since $|E(T)| = 2m$, and except precisely two degree 1 vertices, all other vertices in Y have degree 2, we conclude that $|Y| = m + 1$. Let $Y = \{y_1, y_2, \ldots, y_{m+1}\}$. Then there are precisely m edges of T incident to vertices in Y with even indices, and m edges of T incident to vertices in Y with odd indices. We treat T as an alternating sequence of vertices and edges starting at y_1 and ending at y_{m+1}.

If $m \equiv 0 \pmod 2$, following the order of the appearances of edges in T, assign edges incident to vertices in Y of even indices with labels

$$a, a+1, \ldots, a+m-1,$$

and assign edges incident to vertices in Y of odd indices with labels

$$b, b-1, \ldots, b-m+1.$$

That is, the label at y_i is $(a+i-2, a+i-1)$ if i is even; and $(b-i+2, b-i+1)$ if i is odd and not equal to 1 or $m+1$.

If $m \equiv 1 \pmod 2$, following the order of the appearances of edges in T, assign edges incident to vertices in Y of odd indices with labels

$$a, a+1, \ldots, a+m-1,$$

and assign edges incident to vertices in Y of even indices with labels

$$b, b-1, \ldots, b-m+1.$$

That is, the label at y_i is $(a+i-2, a+i-1)$ if i is odd and not equal to 1; and $(b-i+2, b-i+1)$ if i is even and not equal to $m+1$.

If $m \equiv 0 \pmod 2$, for each $y_i \in Y$ with $1 \leqslant i \leqslant m+1$, by the assignment of labels, we have that

$$\omega_T(y_i) = \begin{cases} b, & \text{if } i = 1; \\ b-m+1, & \text{if } i = m+1; \\ 2a+2i-3, & \text{if } i \text{ is even and } 2 \leqslant i \leqslant m; \\ 2b-2i+3, & \text{if } i \text{ is odd and } 3 \leqslant i \leqslant m-1. \end{cases}$$

If $m \equiv 1 \pmod 2$, for each $y_i \in Y$ with $1 \leqslant i \leqslant m+1$, by the assignment of labels, we have that

$$\omega_T(y_i) = \begin{cases} a, & \text{if } i = 1; \\ b-m+1, & \text{if } i = m+1; \\ 2a+2i-3, & \text{if } i \text{ is odd and } 3 \leqslant i \leqslant m; \\ 2b-2i+3, & \text{if } i \text{ is even and } 2 \leqslant i \leqslant m-1. \end{cases}$$

The sum on each vertex y_i with $y_i \in Y - \{y_1, y_{m+1}\}$ is expressed as either $2a + 2i - 3$ or $2b - 2i + 3$, which is an odd number. Furthermore, the sums on $y_2, y_4, \dots, y_m$, starting at $2a+1$, strictly increase to $2a+2m-3$ if $m \equiv 0 \pmod 2$, and the sums on $y_3, y_5, \dots, y_{m-1}$, starting at $2a + 3$, strictly increase to $2a + 2m - 3$ if $m \equiv 1 \pmod 2$. The sums on $y_3, y_5, \dots, y_{m-1}$, starting at $2b - 3$, strictly decrease to $2b - 2m + 5$ if $m \equiv 0 \pmod 2$, and the sums on $y_2, y_4, \dots, y_m$, starting at $2b - 1$, strictly decrease to $2b - 2m + 5$ if $m \equiv 1 \pmod 2$. So these sums are all distinct. Since $a = b - 2m + 1$, it holds that $2a+2m-3 < 2b-2m+5$. Thus all $\omega_T(y)$ are distinct for $y \in Y$ with $y \in Y - \{y_1, y_{m+1}\}$.

Let x be a vertex in X. Suppose that one appearance of x is adjacent to y_i and y_{i+1} in the sequence T. If $m \equiv 0 \pmod 2$, for even i with $2 \leqslant i \leqslant m$, the labels on the two edges xy_i and xy_{i+1} contribute a value of $(a + i - 1) + (b - (i + 1) + 2) = a + b$ to $\omega_T(x)$; for odd i with $1 \leqslant i \leqslant m-1$, the labels on the two edges xy_i and xy_{i+1} contribute a value of $(b - i + 1) + (a + (i + 1) - 2) = a + b$ to $\omega_T(x)$. Since x appears $d_T(x)/2$ times in T, $\omega_T(x) = \frac{d_T(x)(a+b)}{2}$. If $m \equiv 1 \pmod 2$, for even i with $2 \leqslant i \leqslant m$, the labels on the two edges xy_i and xy_{i+1} contribute a value of $(b - i + 1) + (a + (i + 1) - 2) = a + b$ to $\omega_T(x)$; for odd i with $1 \leqslant i \leqslant m-1$, the labels on the two edges xy_i and xy_{i+1} contribute a value of $(a + i - 1) + (b - (i + 1) + 2) = a + b$ to $\omega_T(x)$. Since x appears $d_T(x)/2$ times in T, $\omega_T(x) = \frac{d_T(x)(a+b)}{2}$. □

Lemma 9. *Let $C[X, Y]$ be a cycle of length $2m$ with $m \equiv 0 \pmod 2$, and let a, b be two integers with $a \leqslant b - 2m + 1$. Then there exists a bijection from $E(C)$ to $[a, a + m - 1] \cup [b - m + 1, b]$ such that each of the following holds.*

(i) $\omega_C(x) = a + b$ *for any* $x \in X$.

(ii) $\omega_C(y) \neq \omega_C(z)$ *for any distinct* $y, z \in Y$.

(iii) $\omega_C(y) \in [2a + 1, 2a + 2m - 3] \cup [2b - 2m + 5, 2b - 2]$ *for all* $y \in Y$*, and the sums in* $[2a + 1, 2a + 2m - 3]$ *are odd.*

Proof. Denote by $C = x_1y_1x_2y_2 \cdots x_my_mx_1$ with $x_i \in X$ and $y_i \in Y$. Following the order of the appearances of edges in C, assign edges incident to vertices in $\{y_1, y_3, \dots, y_{m-1}\}$ with labels

$$a + 1, a, a + 2, a + 3, \dots, a + m - 2, a + m - 1.$$

Note that the labels are increasing consecutive integers after exchanging the positions of the first two; assign edges incident to vertices in $\{y_2, y_4, \dots, y_m\}$ with labels

$$b, b - 2, b - 3, \dots, b - m + 2, b - m + 1, b - 1.$$

Note that the labels are decreasing consecutive integers after inserting the last number between the first two labels.

For each $y_i \in Y$ with $1 \leqslant i \leqslant m$, following the appearances of the edges in the sequence of C, we denote the labels on the edges incident to y_i by an ordered pair. Particularly, by the assignment of the labels, we have that

$$\text{label at } y_i = \begin{cases} (a+1, a), & \text{if } i = 1; \\ (b, b-2), & \text{if } i = 2; \\ (b-m+1, b-1), & \text{if } i = m; \\ (a+i-1, a+i), & \text{if } i \text{ is odd with } 3 \leqslant i \leqslant m-1; \\ (b-i+1, b-i), & \text{if } i \text{ is even with } 4 \leqslant i \leqslant m-2. \end{cases}$$

The sums on $y_1, y_3, \ldots, y_{m-1}$ starting at $2a+1$ strictly increase to $2a+2m-3$ and all of them are odd; and the sums on $y_4, y_6, \ldots, y_{m-2}$ starting at $2b-7$ strictly decrease to $2b-2m+5$ and all of them are odd; the sums on y_2, y_m are even numbers $2b-2$ and $2b-m$, respectively. Since $a = b-2m+1$, $2a+2m-3 < 2b-2m+5$. Hence all $\omega_C(y)$ are distinct for $y \in Y$. This shows both (ii) and (iii).

Let x_i be a vertex in X for $1 \leqslant i \leqslant m$. If $i = 1$, the labels on the two edges x_1y_1 and x_1y_m are $a+1$ and $b-1$, respectively; if $i = 2$, the labels on the two edges x_2y_1 and x_2y_2 are a and b, respectively. Thus $\omega_C(x_i) = a+b$ if $i = 1, 2$. Suppose $i \geqslant 3$. If i is even, then the labels on the two edges x_iy_{i-1} and x_iy_i are $a+(i-1)$ and $b-i+1$, respectively; if i is odd, the labels on the two edges xy_{i-1} and xy_i are $b-(i-1)$ and $a+i-1$, respectively. Thus $\omega_C(x_i) = a+b$. This proves (i). □

Lemma 10. *Let G be a bipartite graph, and $H[X,Y]$ a subgraph of G. Suppose that $E(H)$ can be decomposed into edge-disjoint $p+q$ open trails $T_1, \ldots, T_p$, $T_{p+1}, \ldots, T_{p+q}$, and ℓ cycles $C_{p+q+1}, \ldots, C_{p+q+\ell}$. Suppose further that these $p+q+\ell$ subgraphs have no common vertex in Y, and for each of the trails, its vertices in Y are all distinct and its endvertices are contained in Y. Let $2m := |E(H)|$, and c, d be two integers with $c = d-2m+1$. If the length of $T_1, \ldots, T_p$ are congruent to 2 modulo 4, and the length of each of the remaining trails and cycles is congruent to 0 modulo 4, then there exists a bijection from $E(H)$ to $[c, c+m-1] \cup [d-m+1, d] = [c, d]$ such that each of the following holds.*

(i) $\omega_H(x) = \frac{d_H(x)(c+d)}{2}$ *for any* $x \in X$.

(ii) For each i with $1 \leqslant i \leqslant p+q$, let T_i start at y_{1i} and end at $y_{(m_i+1)i}$, where $y_{1i}, y_{(m_i+1)i} \in Y$. Suppose that

$$\begin{cases} \omega_G(y_{1i}) = \omega_H(y_{1i}) + (c-2p+i-1), & \text{if } 1 \leqslant i \leqslant p, \\ \omega_G(y_{(m_i+1)i}) = \omega_H(y_{(m_i+1)i}) + (c-i), & \text{if } 1 \leqslant i \leqslant p, \\ \omega_G(y_{1i}) = \omega_H(y_{1i}) + (d+2q-2(i-p-1)), & \text{if } p+1 \leqslant i \leqslant p+q, \\ \omega_G(y_{(m_i+1)i}) = \omega_H(y_{(m_i+1)i}) + (d+2q-2(i-p-1)-1), & \text{if } p+1 \leqslant i \leqslant p+q, \\ \omega_G(y) = \omega_H(y), & \text{if } y \neq y_{1i}, y_{(m_i+1)i}. \end{cases}$$

Then $\omega_G(y) \neq \omega_G(z)$ for any distinct $y, z \in Y$, and $\omega_G(y) \in [2c-2p, \max\{2d+2q-\sum_{i=1}^{p} m_i, 2d\}]$ for all $y \in Y$.

Proof. Since $T_1, \ldots, T_{p+q}$ and $C_{p+q+1}, \ldots, C_{p+q+\ell}$ are edge-disjoint and pairwise have no common vertex in Y, and for each of the trail, its vertices in Y are all distinct and its endvertices are contained in Y, we conclude that in the graph H, all vertices in X have even degree, all the endvertices of T_i are precisely the degree 1 vertices in Y, and all other vertices in Y have degree 2. For each i with $1 \leqslant i \leqslant p+q$ and each j with $p+q+1 \leqslant j \leqslant p+q+\ell$, let

$$|E(T_i)| = 2m_i, \quad |E(C_j)| = 2m_j.$$

Since $V(T_i) \cap Y$ contains two degree 1 vertices and $|V(T_i) \cap Y| - 2$ degree 2 vertices in Y, we conclude that $|V(T_i) \cap Y| = m_i + 1$.

Apply Lemma 8 on each T_i, $1 \leqslant i \leqslant p+q$, with

$$a := a_i := c + \sum_{j=1}^{i-1} m_j, \quad b := b_i := d - \sum_{j=1}^{i-1} m_j;$$

and apply Lemma 9 on each C_i, $p+q+1 \leqslant i \leqslant p+q+\ell$, with

$$a := a_i := c + \sum_{j=1}^{i-1} m_j, \quad b := b_i := d - \sum_{j=1}^{i-1} m_j.$$

Note that $a_i + b_i = c + d$, $1 \leqslant i \leqslant p+q+\ell$. By Lemma 8 and Lemma 9, we get that

$$\omega_H(x) = \frac{d_H(x)(c+d)}{2} \quad \text{for any } x \in X.$$

By Lemma 8, the sums at $y_{1i}, y_{m_i i}$, respectively, are

$$\begin{cases} \omega_H(y_{1i}) = c + \sum_{j=1}^{i-1} m_j, \quad \omega_H(y_{(m_i+1)i}) = d - \sum_{j=1}^{i} m_j + 1, & \text{if } 1 \leqslant i \leqslant p; \quad (1) \\ \omega_H(y_{1i}) = d - \sum_{j=1}^{i-1} m_j, \quad \omega_H(y_{(m_i+1)i}) = d - \sum_{j=1}^{i} m_j + 1, & \text{if } p+1 \leqslant i \leqslant p+q; \quad (2) \end{cases}$$

and for each i with $1 \leqslant i \leqslant p+q$, the sums at vertices in $V(T_i) \cap Y - \{y_{1i}, y_{(m_i+1)i}\}$ fall into the intervals

$$\begin{cases} [2a_i+3, 2a_i+2m_i-3] \cup [2b_i-2m_i+5, 2b_i-1], & \text{if } 1 \leqslant i \leqslant p, \\ [2a_i+1, 2a_i+2m_i-3] \cup [2b_i-2m_i+5, 2b_i-3], & \text{if } p+1 \leqslant i \leqslant p+q, \end{cases}$$

and all these sums are distinct and odd.

By Lemma 9, for each i with $p+q+1 \leqslant i \leqslant p+q+\ell$, the sums at vertices in $V(C_i) \cap Y$ are all distinct and fall into the intervals

$$[2a_i+1, 2a_i+2m_i-3] \cup [2b_i-2m_i+5, 2b_i-2],$$

and all the sums in $[2a_i+1, 2a_i+2m_i-3]$ are odd.

Since for each i, j with $1 \leqslant i < j \leqslant p+q+\ell$,

$$\begin{cases} 2a_j+1 > 2a_i+2m_i-3; \\ 2b_i-2m_i+5 > 2b_j-1, \end{cases}$$

we see that $2a_{p+q+\ell}+2m_{p+q+\ell}-3$ is the largest value in the set

$$\left(\bigcup_{i=1}^{p}[2a_i+3, 2a_i+2m_i-3]\right) \cup \left(\bigcup_{i=p+1}^{p+q+\ell}[2a_i+1, 2a_i+2m_i-3]\right),$$

and $2b_{p+q+\ell}-2m_{p+q+\ell}+5$ is the smallest value in the set

$$\begin{aligned} &\left(\bigcup_{i=1}^{p}[2b_i-2m_i+5, 2b_i-1]\right) \cup \left(\bigcup_{i=p+1}^{p+q}[2b_i-2m_i+5, 2b_i-3]\right) \\ &\qquad \cup \left(\bigcup_{i=p+q+1}^{p+q+\ell}[2b_i-2m_i+5, 2b_i-2]\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} & 2b_{p+q+\ell}-2m_{p+q+\ell}+5-(2a_{p+q+\ell}+2m_{p+q+\ell}-3) \\ = \; & 2d-\sum_{j=1}^{p+q+\ell}2m_i+5-(2c+\sum_{j=1}^{p+q+\ell}2m_i-3) \\ = \; & 2(c+2m-1)-2m+5-(2c+2m-3) \quad (d=c+2m-1 \text{ and } \sum_{j=1}^{p+q+\ell}2m_i=2m) \\ = \; & 6. \end{aligned}$$

Hence, all the vertex sums at $Y-\{y_{1i}, y_{(m_i+1)i} \mid 1 \leqslant i \leqslant p+q\}$ are pairwise distinct. Let

$$k_1 := \sum_{i=1}^{p} m_i, \quad \text{and} \quad k_2 := \sum_{i=p+1}^{p+q} m_i.$$

By Equalities (1) and (2), and the assumptions on the parity of each m_i, if $1 \leqslant i \leqslant p$,

$$
\begin{cases}
\omega_G(y_{1i}) = \omega_H(y_{1i}) + (c-2p+i-1), \\
\qquad\qquad = c + \sum_{j=1}^{i-1} m_j + c - 2p + i - 1 \\
\qquad\qquad = 2c - 2p + \sum_{j=1}^{i-1}(m_j+1) \\
\qquad\qquad \equiv 0 \,(\mathrm{mod}\, 2) & \text{since each } m_j \text{ is odd} \\
\qquad\qquad \in [2c-2p, 2c-p-1+k_1-m_p], & (3) \\
\omega_G(y_{(m_i+1)i}) = \omega_H(y_{(m_i+1)i}) + (c-i) \\
\qquad\qquad = d - \sum_{j=1}^{i} m_j + 1 + c - i \\
\qquad\qquad = c + 2m - 1 - \sum_{j=1}^{i} m_j + 1 + c - i \\
\qquad\qquad = 2c + 2m - \sum_{j=1}^{i}(m_j - 1) \equiv 0 \,(\mathrm{mod}\, 2) & \text{since each } m_j \text{ is odd} \\
\qquad\qquad \in [2c+2m-k_1-p, 2c+2m-m_1-1], & (4)
\end{cases}
$$

and if $p+1 \leqslant i \leqslant p+q$,

$$
\begin{cases}
\omega_G(y_{1i}) = \omega_H(y_{1i}) + (d+2q-2(i-p-1)) & (5) \\
\qquad\qquad = d - \sum_{j=1}^{i-1} m_j + d + 2q - 2(i-p-1) \equiv 0 \,(\mathrm{mod}\, 2), & \text{since each } m_j \text{ is even} \\
\omega_G(y_{(m_i+1)i}) = \omega_H(y_{(m_i+1)i}) + (d+2q-2(i-p-1)-1) & (6) \\
\qquad\qquad = d - \sum_{j=1}^{i} m_j + 1 + d + 2q - 2(i-p-1) - 1 \\
\qquad\qquad \equiv 0 \,(\mathrm{mod}\, 2), & \text{since each } m_j \text{ is even}
\end{cases}
$$

By the above analysis, for each i with $1 \leqslant i \leqslant p+q$, both $\omega_G(y_{1i})$ and $\omega_G(y_{(m_i+1)i})$ are even. As all the sums at vertices in $\bigcup_{i=1}^{p+q} V(T_i) \cap Y - \{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ are odd by Lemma 8, and by Lemma 9 all the sums on vertices in $\bigcup_{i=p+q+1}^{p+q+\ell} V(C_i) \cap Y$ which fall into the set $\bigcup_{i=p+q+1}^{p+q+\ell} [2a_i+1, 2a_i+2m_i-3]$ are odd, all of them are distinct from these $2(p+q)$ ω_G sums on vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$. Hence, to show that all the vertex sums at vertices in Y are distinct, we are left to check that all these $2(p+q)$ ω_G sums are distinct with the sums on vertices in $\bigcup_{i=p+q+1}^{p+q+\ell} V(C_i) \cap Y$ which fall into the set $\bigcup_{i=p+q+1}^{p+q+\ell} [2b_i - 2m_i + 5, 2b_i - 2]$, and all these $2(p+q)$ ω_G sums at vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ are pairwise distinct.

If $1 \leqslant i < j \leqslant p$, by (3) and (4), $\omega_G(y_{1i}) < \omega_G(y_{1j})$ and $\omega_G(y_{(m_i+1)i}) > \omega_G(y_{(m_i+1)j})$. Thus, all the sums at vertices either in $\{y_{1i} \,|\, 1 \leqslant i \leqslant p\}$ or in $\{y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p\}$ are all distinct, and

$$\omega_G(y_{1p}) = \max\{\omega_G(y_{1i}) \,|\, 1 \leqslant i \leqslant p\}, \quad \omega_G(y_{(m_p+1)p}) = \min\{\omega_G(y_{(m_i+1)i}) \,|\, 1 \leqslant i \leqslant p\}.$$

Furthermore, by (3) and (4),

$$\begin{aligned} & \omega_G(y_{(m_p+1)p}) - \omega_G(y_{1p}) \\ = \; & 2c + 2m - k_1 - p - (2c - p - 1 + k_1 - m_p) \\ = \; & 2m - 2k_1 + m_p + 1 \geqslant 2 \quad (2m \geqslant 2k_1, m_p \geqslant 1). \end{aligned}$$

Thus, the ω_G sums on vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p\}$ are all distinct.

By (5) and (6), $\omega_G(y_{1i}) > \omega_G(y_{(m_i+1)i})$ for all i with $p+1 \leqslant i \leqslant p+q$, and $\omega_G(y_{(m_i+1)i}) > \omega_G(y_{1j})$ if $p+1 \leqslant i < j \leqslant p+q$. Thus, all the ω_G sums on vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, p+1 \leqslant i \leqslant p+q\}$ are all distinct, and

$$\omega_G(y_{(m_{p+q}+1)(p+q)}) = \min\{\omega_G(y_{1i}), \omega_G(y_{(m_i+1)i}) \,|\, p+1 \leqslant i \leqslant p+q\}.$$

Furthermore,

$$\omega_G(y_{(m_1+1)1}) = \max\{\omega_G(y_{1i}), \omega_G(y_{(m_i+1)i}) \,|\, 1 \leqslant i \leqslant p\},$$

and by (6) and (4),

$$\begin{aligned} & \omega_G(y_{(m_{p+q}+1)(p+q)}) - \omega_G(y_{(m_1+1)1}) \\ = \; & 2d - k_1 - k_2 + 2 - (2c + 2m - m_1 - 1) \\ = \; & 2m - k_1 - k_2 + m_1 + 1 \geqslant 2 \quad (d = c + 2m - 1, 2m \geqslant k_1 + k_2, m_1 \geqslant 1). \end{aligned}$$

Thus, all the ω_G sums on vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ are all distinct. We may assume that $\ell \geqslant 1$. Otherwise, we are done.

By the definition of the parameters b_i and easy calculations,

$$\bigcup_{i=p+q+1}^{p+q+\ell} [2b_i - 2m_i + 5, 2b_i - 2] \subseteq [2b_{p+q+\ell} - 2m_{p+q+\ell} + 5, 2b_{p+q+1} - 2].$$

By (3), (4),(5), and (6), the ω_G sums at vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ fall into the intervals

$$[2c - 2p, 2c + 2m - m_1 + 1] \cup [2d - k_1 - k_2 + 2, 2d + 2q - k_1].$$

Since

$$\begin{aligned} & (2b_{p+q+\ell} - 2m_{p+q+\ell} + 5) - (2c + 2m - m_1 - 1) \\ = \; & 2d - 2k_1 - 2k_2 + 5 - (2(d - 2m + 1) + 2m - m_1 - 1) \\ = \; & 2m - 2k_1 - 2k_2 + m_1 + 4 \\ \geqslant \; & 5 \quad (2m \geqslant 2k_1 + 2k_2 \text{ and } m_1 \geqslant 1), \end{aligned}$$

and

$$\begin{aligned} & (2d-k_1-k_2+2)-(2b_{p+q+1}-2) \\ = \;& 2d-2k_1-k_2+2-(2d-2k_1-2k_2-2) \\ = \;& k_1+k_2+4 \geqslant 4, \end{aligned}$$

we then conclude that these $2(p+q)$ ω_G sums at vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ are all distinct with the ω_G sums at vertices in $\bigcup_{i=p+q+1}^{p+q+\ell} V(C_i) \cap Y$ which fall into the set $\bigcup_{i=p+q+1}^{p+q+\ell}[2b_i-2m_i+5, 2b_i-2]$.

By all the arguments above, we have shown that $\omega_G(y) \neq \omega_G(z)$ for any distinct $y, z \in Y$. Since the sums on vertices in $Y - \{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ fall into the interval $[2c, 2d]$ and the sums on vertices in $\{y_{1i}, y_{(m_i+1)i} \,|\, 1 \leqslant i \leqslant p+q\}$ fall into the interval $[2c-2p, 2d+2q-k_1]$, where the value $2d+2q-k_1$ is attained at $\omega_G(y_{1(p+1)})$, we have that $\omega_G(y) \in [2c-2p, \max\{2d+2q-k_1, 2d\}]$ for all $y \in Y$. $\square$

3 Proof of Theorem 3

Let $G = [X, Y]$ be a biregular bipartite graph. Assume that $|X| = m$, $|Y| = n$, $d_G(x) = s \geqslant d_G(y) = t$, where $x \in X, y \in Y$. Consequently $m \leqslant n$ and $|E(G)| = ms = nt$. Given an orientation of G, we will denote the orientation by $\overrightarrow{G}$.

If $t = 1$, then G is the union of vertex-disjoint stars with centers in X. Denote

$$X = \{x_1, x_2, \ldots, x_m\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_n\}.$$

For each x_i, $1 \leqslant i \leqslant m$, we assign arbitrarily edges incident to x_i with labels

$$s(i-1)+1, s(i-1)+2, \ldots, s(i-1)+s.$$

Orient edges of G from X to Y. Thus, the oriented vertex sums for vertices in X are negative, and the oriented vertex sums for vertices in Y are positive. Hence, no two vertices x and y conflict if $x \in X$ and $y \in Y$. Also, it is routine to check that no two vertices in X conflicting and no two vertices in Y conflicting. Hence the labeling of $\overrightarrow{G}$ is antimagic. Thus we assume $t \geqslant 2$. We distinguish three cases for finishing the proof.

Case 1: $t \geqslant 3$

Orient edges of G from X to Y, and denote the orientation by $\overrightarrow{G}$. By the orientation of G, the sums of vertices in X are negative while the sums at vertices in Y are positive. Hence in the following, we just need to find a labeling of $\overrightarrow{G}$ using labels in $[1, sm]$, which guarantees that the sums at vertices in X are all distinct and the sums at vertices in Y are all distinct. By Lemma 4, G has a matching M saturating X. Assume, w.l.o.g, that $M = \{x_1y_1, x_2y_2, \ldots, x_my_m\}$. Let $H = G - M$. Note that $d_H(y_i) = t-1$ for $1 \leqslant i \leqslant m$ and $d_H(y_i) = t$ for $m+1 \leqslant i \leqslant n$.

Subcase 1.1: $t \geqslant 3$ and t is odd

Reserve labels in $[1, m]$ for edges in M, and use labels in $[m+1, tn = sm]$ for edges in H. For each y_i with $1 \leqslant i \leqslant m$, assign arbitrarily the edges incident to y_i with labels

$$m+i, 3m-i+1, 3m+i, 5m-i+1, 5m+i, \ldots, (t-2)m-i+1, (t-2)m+i, tm;$$

and for each y_i with $m+1 \leqslant i \leqslant n$, assign arbitrarily the edges incident to y_i with labels

$$t(i-m-1)+tm+1, t(i-m-1)+tm+2, \ldots, t(i-m-1)+tm+t.$$

Under the above assignment of labels, we have that for any $y_i, y_j \in \{y_1, y_2, \ldots, y_m\}$, $\omega_H(y_i) = \omega_H(y_j)$. Thus any assignment of distinct labels on the edges in M results in a labeling of G such that the sums of labels at vertices in Y are all distinct. Hence, we can choose an assignment of distinct labels on the edges in M just based on the ordering of the values in $\{\omega_H(x_i) \mid 1 \leqslant i \leqslant m\}$. Therefore, up to a reordering of edges in M, we may assume that $\omega_H(x_1) \leqslant \omega_H(x_2) \leqslant \cdots \leqslant \omega_H(x_m)$. Now for each edge $x_iy_i \in M$, $1 \leqslant i \leqslant m$, assign the edge x_iy_i with the label i.

We verify now that the labeling of $\overrightarrow{G}$ given above is antimagic. For each $x_i, x_j \in X$ with $i < j$, since $\omega_H(x_i) \leqslant \omega_H(x_j)$, it holds that $\omega_G(x_i) = \omega_H(x_i) + i < \omega_H(x_j) + j = \omega_G(x_j)$.

Next for each $y_i, y_j \in \{y_1, y_2, \ldots, y_m\}$ with $i < j$, since $\omega_H(y_i) = \omega_H(y_j) = \frac{t-1}{2}((t+1)m+1)$, we have that $\omega_G(y_i) = \omega_H(y_i) + i < \omega_H(y_j) + j = \omega_G(y_j)$. By the assignment of labels on edges incident to y_i with $m+1 \leqslant i \leqslant n$, the sums at y_i are pairwise distinct. The smallest vertex sum among these values is $\omega_G(y_{m+1}) = t^2m + \sum_{i=1}^{t} i$. The largest vertex sum among values in $\{\omega_G(y_1), \ldots, \omega_G(y_m)\}$ is $\omega_G(y_m) = \frac{t-1}{2}((t+1)m+1) + m$. It is easy to check that $\omega_G(y_m) < \omega_G(y_{m+1})$. Hence, all the sums at vertices in Y are distinct.

Subcase 1.2: $t \geqslant 3$ and t is even

Reserve labels in $\{2, 4, \ldots, 2m\}$ for edges in M, and use the labels in $\{1, 3, \ldots, 2m-1\} \cup \{2m+1, \ldots, tn = sm\}$ for edges in H. For each y_i with $1 \leqslant i \leqslant m$, assign arbitrarily the edges incident to y_i with labels

$$2i-1, 3m-i+1, 4m-i+1, 4m+i, 6m-i+1, \ldots, (t-2)m+i, tm-i+1;$$

and for each y_i with $m+1 \leqslant i \leqslant n$, assign arbitrarily the edges incident to y_i with labels

$$t(i-m-1)+tm+1, t(i-m-1)+tm+2, \ldots, t(i-m-1)+tm+t.$$

Assume, w.l.o.g., that under the above assignment of labels, $\omega_H(x_1) \leqslant \omega_H(x_2) \leqslant \cdots \leqslant \omega_H(x_m)$. Now for each edge $x_iy_i \in M$, $1 \leqslant i \leqslant m$, assign the edge x_iy_i with the label $2i$.

We verify now that the labeling of $\overrightarrow{G}$ given above is antimagic. Obviously, for each $x_i, x_j \in X$ with $1 \leqslant i < j \leqslant m$, because $\omega_H(x_i) \leqslant \omega_H(x_j)$, it holds that $\omega_G(x_i) = \omega_H(x_i) + 2i < \omega_H(x_j) + 2j = \omega_G(x_j)$.

Next for each y_i, y_j with $1 \leqslant i < j \leqslant m$, since $\omega_H(y_i) = \omega_H(y_j) = (t^2-9)m + t - 3$, we have that $\omega_G(y_i) = \omega_H(y_i) + 2i < \omega_H(y_j) + 2j = \omega_G(y_j)$. By the assignment of labels on edges incident to y_i with $m+1 \leqslant i \leqslant n$, the sums at y_i are pairwise distinct. The smallest sum among these values is $\omega_G(y_{m+1}) = t^2m+\sum_{i=1}^{t} i$. The largest sum among values in $\{\omega_G(y_1), \ldots, \omega_G(y_m)\}$ is $\omega_G(y_m) = (t^2-9)m+t-3+2m$. It is easy to check that $\omega_G(y_m) < \omega_G(y_{m+1})$. Hence, all the sums at vertices in Y are distinct.

Case 2: $t = 2$ and s is odd

By Lemma 4, there exists a matching M saturating vertices in X. In each component of $G-M$, the vertices contained in X are all of even degree $s-1$, and all vertices contained in Y are of degree 2 or 1. Thus, the number of vertices with degree 1 in the component is even. Since there are in total m vertices of degree 1 in Y, by Lemma 6, we can decompose $E(G-M)$ into $m/2$ open trials with endvertices in Y. Denote the trails by $T_1, \ldots, T_{m/2}$. Since the endvertices of each T_i are in Y, all the vertices in $V(T_i) \cap X$ have even degree. Consequently, T_i has even length. For each i with $1 \leqslant i \leqslant m/2$, let

$$2m_i := |E(T_i)| \quad \text{and} \quad T_i = y_{1i}x_{1i} \cdots x_{m_i i}y_{(m_i+1)i},$$

where $x_{1i}, x_{2i}, \ldots, x_{m_i i} \in X$ and $y_{1i}, y_{2i}, \ldots, y_{m_i+1i} \in Y$. Note that $x_{1i}, x_{2i}, \ldots, x_{m_i i}$ may not be distinct vertices in X, but $y_{1i}, y_{2i}, \ldots, y_{m_i+1i}$ are distinct vertices in Y because all vertices in Y have degree 2 in G. Assume further, w.l.o.g., that there are p trails $T_1, \ldots, T_p$ of length congruent to 2 modulo 4, and q trails $T_{p+1}, \ldots, T_{p+q}$ of length congruent to 0 modulo 4. Set

$$c := 2p+1, \quad d := sm-2q, \quad \text{and} \quad H = \bigcup_{i=1}^{p+q} T_i.$$

The endvertices of the $m/2$ open trails are exactly the set of Y-endvertices of the m matching edges. Thus, for each i with $1 \leqslant i \leqslant m/2$, and for each edge $e \in M$, e is incident to either y_{1i} or $y_{(m_i+1)i}$. We assign labels in $[1, 2p] \cup [sm-2q+1, sm]$ on e as below. If $1 \leqslant i \leqslant p$,

$$\text{label on } e = \begin{cases} i, & \text{if } e \text{ is incident to } y_{1i}; \quad (7) \\ 2p-i+1, & \text{if } e \text{ is incident to } y_{(m_i+1)i}; \quad (8) \end{cases}$$

if $p+1 \leqslant i \leqslant p+q$,

$$\text{label on } e = \begin{cases} sm-2(i-p-1), & \text{if } e \text{ is incident to } y_{1i}; \quad (9) \\ sm-2(i-p-1)-1, & \text{if } e \text{ is incident to } y_{(m_i+1)i}. \quad (10) \end{cases}$$

Thus,

$$\begin{cases}
\omega_G(y_{1i}) = \omega_H(y_{1i}) + i, & \text{if } 1 \leqslant i \leqslant p,\\
\qquad\quad = \omega_H(y_{1i}) + (c - 2p + i - 1), & \\
\omega_G(y_{(m_i+1)i}) = \omega_H(y_{(m_i+1)i}) + (2p - i + 1), & \text{if } 1 \leqslant i \leqslant p,\\
\qquad\qquad\quad = \omega_H(y_{(m_i+1)i}) + (c - i), & \\
\omega_G(y_{1i}) = \omega_H(y_{1i}) + (sm - 2(i - p - 1)), & \text{if } p+1 \leqslant i \leqslant p+q,\\
\qquad\quad = \omega_H(y_{1i}) + (d + 2q - 2(i - p - 1)), & \\
\omega_G(y_{(m_i+1)i}) = \omega_H(y_{(m_i+1)i}) + (sm - 2(i - p - 1) - 1), & \text{if } p+1 \leqslant i \leqslant p+q,\\
\qquad\qquad\quad = \omega_H(y_{(m_i+1)i}) + (d + 2q - 2(i - p - 1) - 1), & \\
\omega_G(y) = \omega_H(y), & \text{if } y \in Y, y \neq y_{1i}, y_{(m_i+1)i}.
\end{cases}$$

Applying Lemma 10 on H with $c = 2p + 1$ and $d = sm - 2q$ defined as above, we get an assignment of labels on $E(H)$ such that

(i) For any $x \in V(H) \cap X$, $\omega_H(x) = \frac{d_H(x)(c+d)}{2} = \frac{(s-1)(sm-2q+2p+1)}{2}$; and

(ii) For any distinct $y, z \in V(H) \cap Y$, $\omega_G(y) \neq \omega_G(z)$.

Orient all the edges of G from X to Y, and denote the orientation by $\overrightarrow{G}$.

Claim 1: The labeling of $\overrightarrow{G}$ given above is antimagic.

Proof. We first show that all the set of labels used is the set $[1, sm]$. The set of labels used on edges in M is the set $[1, 2p] \cup [sm-2q+1, sm]$, and the set of labels used on edges in H is the set $[c, d] = [2p + 1, sm - 2q]$. The union of these two sets is the set $[1, sm]$.

We then show that all the oriented sums on vertices in $\overrightarrow{G}$ are pairwise distinct. We first examine the sums on vertices in Y. For any $y \in Y$, since $Y = V(H) \cap Y$, we know that all the sums on vertices in Y are pairwise distinct by (ii) preceding Claim 1.

Finally, we show that for any $x \in X$ and $y \in Y$ the oriented sums at x and y are distinct in G. This is clear since all the oriented sums at vertices in Y are positive while that at vertices in X are negative. ☐

Case 3: $t = 2$ and s is even

We may assume that $s \geqslant 4$. Otherwise G is 2-regular and $|E(G)| = 2m$. By Lemma 7, G has an antimatic labeling by taking $a := 1$ and $b := 2m$, and the labeling is also an antimagic labeling of $\overrightarrow{G}$ obtained by orienting all edges from X to Y.

Claim 2: The graph $G[X, Y]$ contains a subgraph F such that

(1) F is a set of vertex disjoint cycles; and

(2) $V(F) \cap X = X$.

Proof. Suppressing all degree 2 vertices in Y (removing the vertex and adding an edge joining the two neighbors of the removed vertex), we obtain an s-regular (multi)graph G'. Since s is even, by applying Lemma 5, we find a 2-factor of G'. Subdivide each edge in the 2-factor of G', we get the desired graph F. $\square$

Now $G - E(F)$ is a graph with all vertices having even degree. So $G - E(F)$ can be decomposed into edge-disjoint cycles. Assume that there are in total ℓ edge-disjoint cycles in $G - E(F)$ such that each of them has length congruent to 0 modulo 4, and there are in total h edge-disjoint cycles in $G - E(F)$ such that each of them has length congruent to 2 modulo 4. For each i with $1 \leqslant i \leqslant h$, denote by

$$C_i = x_{1i}y_{1i}\cdots x_{m_i i}y_{m_i i}x_{1i}, \quad \text{where } x_{1i}, x_{2i}, \ldots, x_{m_i i} \in X, \quad y_{1i}, y_{2i}, \ldots, y_{m_i i} \in X.$$

the i-th cycle of length congruent to 2 modulo 4.

We pre-label edges in $\{x_{1i}y_{1i}, x_{1i}y_{m_i i}, y_{1i}x_{2i}, y_{m_i i}x_{m_i i}, x_{2i}y_{2i}, x_{m_i i}y_{(m_i-1)i} \mid 1 \leqslant i \leqslant h\}$. In doing so, we distinguish if $s = 4$ or $s \geqslant 6$.

If $s = 4$, for each i with $1 \leqslant i \leqslant h$, use the labels in $[1, 3h] \cup [2m - 3h + 1, 2m]$ to label each edge e indicated below.

$$\text{label on } e = \begin{cases} i, & \text{if } e = x_{1i}y_{1i}; & (11)\\ 2m - (i-1), & \text{if } e = x_{1i}y_{m_i i}; & (12)\\ h + 2i - 1, & \text{if } e = y_{1i}x_{2i}; & (13)\\ h + 2i, & \text{if } e = y_{m_i i}x_{m_i i}; & (14)\\ 2m + 1 - (h + 2i - 1), & \text{if } e = x_{2i}y_{2i}; & (15)\\ 2m + 1 - (h + 2i), & \text{if } e = x_{m_i i}y_{(m_i-1)i}. & (16) \end{cases}$$

If $s \geqslant 6$, for each i with $1 \leqslant i \leqslant h$, use the labels in $[1, 4h] \cup [(s-2)m - 2h + 1, (s-2)m]$ to label each edge e indicated below.

$$\text{label on } e = \begin{cases} i, & \text{if } e = x_{1i}y_{1i}; & (17)\\ h + i, & \text{if } e = x_{1i}y_{m_i i}; & (18)\\ 2h + 2i - 1, & \text{if } e = y_{1i}x_{2i}; & (19)\\ 2h + 2i, & \text{if } e = y_{m_i i}x_{m_i i}; & (20)\\ (s-2)m + 2h + 1 - (2h + 2i - 1), & \text{if } e = x_{2i}y_{2i}; & (21)\\ (s-2)m + 2h + 1 - (2h + 2i), & \text{if } e = x_{m_i i}y_{(m_i-1)i}. & (22) \end{cases}$$

Assume there are q paths with positive length after deleting the vertices $x_{1i}, x_{2i}, x_{m_i i}$, $y_{1i}, y_{m_i i}$ in each C_i for $1 \leqslant i \leqslant h$. Assume, w.l.o.g., that these paths are obtained from $C_{h-q+1}, \ldots, C_h$. For each $1 \leqslant i \leqslant q$, denote these paths by

$$P_i = C_{h-q+i} - \{x_{1(h-q+i)}, x_{2(h-q+i)}, x_{m_{(h-q+i)}(h-q+i)}, y_{1(h-q+i)}, y_{m_{(h-q+i)}(h-q+i)}\},$$

and assume that P_i starts at $y_{2(h-q+i)}$ and ends at $y_{(m_{h-q+i}-1)(h-q+i)}$. Then each P_i has length $2(m_{h-q+i} - 3) \equiv 0 (\mathrm{mod}\, 4)$. Under this assumption, we know that $C_1, C_2, \ldots, C_{h-q}$ are 6-cycles.

Denote the ℓ edge-disjoint cycles in $G - E(F)$ such that each of them has length congruent to 0 modulo 4 by $D_{q+1}, D_{q+2}, \ldots, D_{q+\ell}$. Let

$$H = \left(\bigcup_{i=1}^{q} P_i \right) \bigcup \left(\bigcup_{i=q+1}^{q+\ell} D_i \right).$$

If $s = 4$, then let

$$c := 3h + 1 \quad \text{and} \quad d := 2m - 3h.$$

It is clear that $d - c = 2m - 6h - 1 = |E(H)| - 1$. For $h - q + 1 \leqslant j \leqslant h$, let $i = j - h + q$. Then by Equations (15) and (16), the labels on the other edges not in H incident to y_{2j} and $y_{(m_j-1)j}$, respectively, are

$$\begin{aligned} 2m + 1 - (h + 2j - 1) &= (s-2)m - h - 2(i + h - q) + 2 \\ &= (s-2)m - 3h + 2q - 2i + 2 = d + 2q - 2(i-1), \\ 2m + 1 - (h + 2j) &= (s-2)m - h - 2(i + h - k) + 1 \\ &= (s-2)m - 3h + 2q - 2i + 1 = d + 2q - 2(i-1) - 1. \end{aligned}$$

If $s \geqslant 6$, then let

$$c := 4h + 1 \quad \text{and} \quad d := (s-2)m - 2h.$$

Again $d - c = (s-2)m - 6h - 1 = |E(H)| - 1$. For $h - q + 1 \leqslant j \leqslant p$, let $i = j - h + q$. Then by Equations (21) and (22), the labels on the other edges not in H incident to y_{2j} and $y_{(m_j-1)j}$, respectively, are

$$\begin{aligned} (s-2)m + 2h + 1 - (2h + 2j - 1) &= (s-2)m - 2(i + h - q) + 2 \\ &= (s-2)m - 2h + 2q - 2i + 2 = d + 2q - 2(i-1), \\ (s-2)m + 2h + 1 - (2h + 2j) &= (s-2)m - 2(i + h - q) + 1 \\ &= (s-2)m - 2h + 2q - 2i + 1 \\ &= d + 2q - 2(i-1) - 1. \end{aligned}$$

Thus,

$$\begin{aligned} \omega_G(y_{2j}) &= \omega_H(y_{2j}) + (d + 2q - 2(i - p - 1)), \\ \omega_G(y_{(m_j-1)j}) &= \omega_H(y_{(m_j-1)j}) + (d + 2q - 2(i - p - 1) - 1). \end{aligned}$$

Apply Lemma 10 on H with c and d defined above (according to if $s = 4$ or $s \geqslant 6$) and with $p = 0$, we get an assignment of labels on $E(H)$ such that

(i) For any $x \in V(H) \cap X$, $\omega_H(x) = \frac{d_H(x)(c+d)}{2}$; and

(ii) For any distinct $y, z \in V(H) \cap Y$, $\omega_G(y) \neq \omega_G(z)$, and $\omega_G(y) \in [2c, 2d + 2q]$.

Apply Lemma 7 on F with

$$a := (s-2)m + 1 \quad \text{and} \quad b := sm,$$

we get an antimagic labeling on F.

If $s = 4$, orient the edges in $\{x_{1i}y_{1i}, x_{1i}y_{m_i i} \mid 1 \leqslant i \leqslant h\}$ from Y to X, and orient all the remaining edges from X to Y. If $s \geqslant 6$, orient the edges in $\{x_{1i}y_{1i} | 1 \leqslant i \leqslant h\}$ from Y to X, and orient all the remaining edges from X to Y. Denote the orientation of G by $\overrightarrow{G}$.

Claim 3: The labeling of $\overrightarrow{G}$ given above is antimagic.

Proof. We first show that the set of labels used is the set $[1, sm]$. The labels used on edges in F are exactly numbers in the set $[(s-2)m+1, sm]$. If $s = 4$, then the set of labels used on edges in $\{x_{1i}y_{1i}, x_{1i}y_{m_i i}, x_{2i}y_{1i}, x_{m_i i}y_{m_i i}, x_{2i}y_{2i}, x_{m_i i}y_{(m_i-1)i} \mid 1 \leqslant i \leqslant h\}$ is $[1, 3h] \cup [2m-3h+1, 2m]$, and the set of labels used on $E(H)$ is $[3h+1, 2m-3h]$. If $s \geqslant 6$, then the set of labels used on edges in $\{x_{1i}y_{1i}, x_{1i}y_{m_i i}, x_{2i}y_{1i}, x_{m_i i}y_{m_i i}, x_{2i}y_{2i}, x_{m_i i}y_{(m_i-1)i} \mid 1 \leqslant i \leqslant h\}$ is $[1, 4h] \cup [(s-2)m-2h+1, (s-2)m]$, and the set of labels used on $E(H)$ is $[4h+1, (s-2)m-2h]$. The union of these sets is the set $[1, sm]$.

We then show that the oriented sums on vertices in $\overrightarrow{G}$ are all distinct. We separate the proof according to if $s = 4$ or $s \geqslant 6$.

Case $s = 4$: For each i with $1 \leqslant i \leqslant h$, by (11)-(14) and the orientation of G, the labels at y_{1i}, $y_{m_i i}$, respectively, are

$$(-i, h+2i-1) \quad \text{and} \quad (-2m+(i-1), h+2i).$$

Thus,

$$\begin{aligned}
\omega_G(y_{1i}) &= h + i - 1 \in [h, 2h-1], \\
\omega_G(y_{m_i i}) &= -2m + h + 3i - 1 \in [-2m+h+2, -2m+4h-1].
\end{aligned}$$

All these $2h$ values are pairwise distinct and fall into the interval $[-2m+h+2, 2h-1]$.

For each i with $1 \leqslant i \leqslant h-q$, by (15) and (16), the label at y_{2i} is,

$$(2m+1-(h+2i-1), 2m+1-(h+2i)).$$

Thus,

$$\omega_G(y_{2i}) = 4m + 3 - 2h - 4i \in [4m+3-6h+4q, 4m-1-2h].$$

All these $h-q$ values are pairwise distinct.

The sums at vertices in $V(H) \cap Y$ are all distinct and fall into the interval $[2c, 2d+2q] = [6h+2, 4m-6h+2q] \subseteq [6h+2, 4m]$ ($q \leqslant h$) by Lemma 10. The sums at vertices in $V(F) \cap Y$ are all distinct and fall into the interval $[4m+3, 8m-1]$ by Lemma 7. Since these sets $[-2m+h+2, 2h-1], [6h+2, 4m-6h+2q], [4m+3-6h+4q, 4m-1-2h]$, and $[4m+3, 8m-1]$ are pairwise disjoint, we see that the oriented sums on vertices in Y are all distinct.

Next we show that in $\overrightarrow{G}$, the oriented sums on vertices in X are all distinct. For each i with $1 \leqslant i \leqslant h$,

$$\begin{cases} \omega_{G-E(F)}(x_{1i}) = 2m+1 & \text{by (11) and (12) ,} \\ \omega_{G-E(F)}(x_{2i}) = \omega_{G-E(F)}(x_{m_i i}) = -2m-1 & \text{by (13) (15), and (14) (16),} \\ \omega_{G-E(F)}(x) = -\frac{d_H(x)(c+d)}{2} = -(c+d) = -2m-1 & \text{if } x \in V(H) \cap X. \end{cases}$$

Hence,

$$|\omega_{G-E(F)}(u) - \omega_{G-E(F)}(v)| = 0 \text{ or } 4m+2, \text{for any } u, v \in X. \tag{23}$$

For the graph F, by Lemma 7, the sums on vertices in $V(F) \cap X$ are pairwise distinct. Since the set of labels used on $E(F)$ is $[2m+1, 4m]$ and F is 2-regular, it follows that in F, $\omega_F(x) \in [-8m+1, -4m-3]$ and any two of the sums at vertices in $V(F) \cap X$ differ an absolute value of at most $4m-4$. Because of $\omega_G(x) = \omega_{G-E(F)}(x) + \omega_F(x)$ for $x \in X$ and the fact in (23), we conclude that the total oriented vertex sums at vertices in X are all distinct.

Finally, we show that for any $x \in X$ and $y \in Y$ the oriented vertex sums at x and y are distinct in $\overrightarrow{G}$. By the analysis above, $\omega_G(y) \in [-2m+h+2, 8m-1]$ for any $y \in Y$. And $\omega_G(x) \in [-10m, -2m-2]$ for any $x \in X$, which follows by the facts that $\omega_{G-E(F)}(x) = -2m-1$ or $2m+1$, $\omega_F(x) \in [-8m+1, -4m-3]$, and $\omega_G(x) = \omega_{G-E(F)}(x) + \omega_F(x)$. Thus, $\omega_G(x) \neq \omega_G(y)$.

Case $s \geqslant 6$: For each i with $1 \leqslant i \leqslant h$, by (17)-(20) and the orientation of G, the labels at y_{1i}, $y_{m_i i}$, respectively, are

$$(-i, 2h+2i+1) \quad \text{and} \quad (h+i, 2h+2i).$$

Thus,

$$\omega_G(y_{1i}) = 2h+i+1 \in [2h+2, 3h+1] \quad \text{and} \quad \omega_G(y_{m_i i}) = 3h+3i \in [3h+3, 6h].$$

All these $2h$ values are pairwise distinct and fall into the interval $[2h+2, 6h]$.

For each i with $1 \leqslant i \leqslant h-q$, by (21) and (22), the label at y_{2i} is,

$$((s-2)m+2h+1-(h+2i-1), (s-2)m+2h+1-(h+2i).$$

Thus,

$$\omega_G(y_{2i}) = 2(s-2)m+3+2h-4i \in [2(s-2)m+3-2h+4q, 2(s-2)m+2h-1].$$

All these $h-q$ values are pairwise distinct.

The sums at vertices in $V(H) \cap Y$ are all distinct and fall into the interval $[2c, 2d+2q] = [8h+2, 2(s-2)m-4h+2q]$ by Lemma 10. The sums at vertices in $V(F) \cap Y$ are all distinct and fall into the interval $[2(s-2)m+3, 2sm-1]$ by Lemma 7. Since these sets $[2h+2, 6h]$, $[8h+2, 2(s-2)m-4h+2q]$, $[2(s-2)m+3-2h+4q, 2(s-2)m+2h-1]$, and $[2(s-2)m+3, 2sm-1]$ are pairwise disjoint, we see that the oriented vertex sums on vertices in Y are all distinct.

Next we show that in $\overrightarrow{G}$, the oriented sums at vertices in X are all distinct. Assume that for each $x \in X$, x appears α_x times in $\{x_{1i} \mid 1 \leqslant i \leqslant h\}$, and β_x times in $\{x_{2i}, x_{m_i i} \mid 1 \leqslant i \leqslant h\}$. Since each of x_{1i}, x_{2i}, and $x_{m_i i}$ has two distinct neighbors in Y, x has degree $s-2-2\alpha_x-2\beta_x$ in H. In addtion, each appearance of x in $\{x_{1i} \mid 1 \leqslant i \leqslant h\}$ contributes a value of $-h$ to the oriented sum at x by (17) and (18), and each appearance of x in $\{x_{2i}, x_{m_i i} \mid 1 \leqslant i \leqslant h\}$ contributes a value of $-(c+d)$ to the oriented sum at x by (19) (21), and (20) (22). By (i) preceeding Claim 3, $\omega_H(x) = -\frac{d_H(x)(c+d)}{2} = -\frac{(s-2-2\alpha_x-2\beta_x)(c+d)}{2}$. Hence,

$$\begin{aligned}\omega_{G-E(F)}(x) &= -\frac{(s-2-2\alpha_x-2\beta_x)(c+d)}{2} - \alpha_x h - \beta_x(c+d)\\ &= -\frac{(s-2)(c+d)}{2} + \alpha_x(c+d-h).\end{aligned}$$

Thus, for any $u, v \in X$,

$$\begin{aligned}|\omega_{G-E(F)}(u) - \omega_{G-E(F)}(v)| &= |\alpha_u - \alpha_v|(c+d-h)\\ &= |\alpha_u - \alpha_v|((s-2)m+h+1) = 0 \text{ or } > 4m. \quad (24)\end{aligned}$$

For the graph F, by Lemma 7, the sums on vertices in $V(F) \cap X$ are pairwise distinct, any two of the sums at vertices in $V(F) \cap X$ differ an absolute value of at most $4m-4$. Because of $\omega_G(x) = \omega_{G-E(F)}(x) + \omega_F(x)$ for $x \in X$ and the fact in (24), we conclude that the total oriented sums at vertices in X are all distinct.

Finally, we show that for any $x \in X$ and $y \in Y$ the oriented sums at x and y are distinct in $\overrightarrow{G}$. By the analysis above, for any $y \in Y$, $\omega_G(y) \in [2h+2, 2sm-1]$ is a positive integer. For any $x \in X$, $\omega_{G-F}(x)$ is negative and $\omega_F(x)$ is negative, so $\omega_G(x) = \omega_{G-E(F)}(x) + \omega_F(x)$ is negative. Hence $\omega_G(x) \neq \omega_G(y)$. □

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