A note on chromatic number and induced odd cycles

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Submitted: Sep 8, 2015; Accepted: Oct 13, 2017; Published: Nov 3, 2017
Mathematics Subject Classifications: 05C15, 05C17, 05C69

Abstract

An odd hole is an induced odd cycle of length at least 5. Scott and Seymour confirmed a conjecture of Gyárfás and proved that if a graph G has no odd holes then $\chi(G) \leqslant 2^{2^{\omega(G)+2}}$. Chudnovsky, Robertson, Seymour and Thomas showed that if G has neither K_4 nor odd holes then $\chi(G) \leqslant 4$. In this note, we show that if a graph G has neither triangles nor quadrilaterals, and has no odd holes of length at least 7, then $\chi(G) \leqslant 4$ and $\chi(G) \leqslant 3$ if G has radius at most 3, and for each vertex u of G, the set of vertices of the same distance to u induces a bipartite subgraph. This answers some questions in [17].

Keywords: chromatic number; induced odd cycles

*Research partially supported by NSFC projects 11331003 and 11571180, and by a project funded by PAPD of Jiangsu Higher Education Institutions.

1 Introduction

Let G be a graph, and let k be an integer. A *k-coloring* of G is an assignment of k colors to the vertices of G such that adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of G is the minimum integer k such that G admits a k-coloring. We use $\omega(G)$ to denote the *clique number* of G which is the largest integer l such that G contains the complete graph K_l as a subgraph. It is certain that $\chi(G) \geqslant \omega(G)$. But the difference $\chi(G) - \omega(G)$ may be arbitrarily large as there are triangle-free graphs with arbitrary large chromatic number (see [7, 23, 15]), and furthermore, Erdős [8] showed that for every positive integers k and l there exists a graph G with $\chi(G) \geqslant k$ whose shortest cycle has length at least l.

There are still quite a lot of families of graphs whose chromatic numbers are bounded by a function of their clique numbers. For instance, the *Strong Perfect Graph Theorem* [2] asserts that $\chi(G) = \omega(G)$ if G contains neither odd cycles of length at least 5 nor their complements as induced subgraphs, and Vizing's theorem [24] together with Beineke's characterization [1] shows that $\chi(G) \leqslant \omega(G) + 1$ if G contains none of the nine given graphs as induced subgraphs. As a natural question, one may ask, for a given family of graphs $\mathcal{G}$, whether there is a function f such that $\chi(G) \leqslant f(\omega(G))$ for each graph $G \in \mathcal{G}$? If such a function f does exist, then we say that the family $\mathcal{G}$ is *χ-bounded*, and call f a *binding function* of $\mathcal{G}$. In this literature, the family of perfect graphs is χ-bounded with a binding function $f(x) = x$, and the family of line graphs is χ-bounded with a binding function $f(x) = x + 1$.

For convenience, we say that a graph G induces a graph H if H is isomorphic to an induced subgraph of G. Let $\mathcal{F}$ be a family of graphs. A graph G is said to be *$\mathcal{F}$-free* if it induces no member of $\mathcal{F}$. For a finite family $\mathcal{F}$, Erdős [8] shows that if $\mathcal{F}$-free graphs are χ-bounded then $\mathcal{F}$ must contain a tree. Then, Gyárfás [10], and Sumner [22] independently, conjectured that F-free graphs are χ-bounded for every forest F. There are some partial results about this conjecture (see [4, 12, 11, 13, 14, 18]).

A *hole* of a graph is an induced cycle of length at least four. Gyárfás [12] also proposed three conjectures on the relation between chromatic number and induced cycles in graphs. Let l be an integer of at least 4.

Conjecture 1. *([12])* {odd holes}-free graphs are χ-bounded.

Conjecture 2. *([12])* {holes of length at least l}-free graphs are χ-bounded.

Conjecture 3. *([12])* {odd holes of length at least l}-free graphs are χ-bounded.

Scott [19] proved that for each $l > 0$, the family of graphs with neither odd holes nor hole of length at least l is χ-bounded. Chudnovsky, Robertson, Seymour and Thomas [3] confirmed Conjecture 1 on graphs of clique number at most 3 and showed that if G is $\{K_4, \textit{odd holes}\}$-free, then $\chi(G) \leqslant 4$ (note that if G is $\{K_3, \textit{odd holes}\}$-free, then it is bipartite).

In 2016, Scott and Seymour [20] proved that $\chi(G) \leqslant 2^{2^{\omega(G)+2}}$ for {odd holes}-free graph G and thus confirmed Conjecture 1. As to Conjectures 2 and 3, Scott and Seymour [21] first proved that for each $l > 0$, every triangle-free graph with sufficiently large chromatic number contains holes of l consecutive lengths, and thus confirmed them on triangle-free graphs. Finally, Chudnovsky, Scott and Seymour [5] confirmed Conjecture 2, and Chudnovsky, Scott, Seymour and Spirkl [6] confirmed Conjecture 3.

We use $\mathcal{G}$ to denote the family of graphs that have neither triangles nor quadrilaterals, and have no odd holes of length larger than 5.

Robertson conjectured (see [16]) that the only 3-connected, internally 4-connected graph in $\mathcal{G}$ is the Petersen graph. This conjecture is true if the requirement on internally 4-connectivity is replaced by cubic [16]. Plummer and Zha [17] presented a counterexample to Robertson's conjecture, and posed a few new questions including (1) whether every such graph has bounded chromatic number? and (2) how close to being perfect are these graphs?

In this note, we prove that, for every graph $G \in \mathcal{G}$, $\chi(G) \leqslant 4$ and $\chi(G) \leqslant 3$ if G has radius at most 3 (Theorem 6), which answers the first question of Plummer and Zha, and for each vertex u of G, the set of vertices of the same distance to u induces a bipartite subgraph (Lemma 4), which answers the second question in some sense.

We introduce some notations. Let S be a subset of $V(G)$, and let x be a vertex of G. We use $G[S]$ to denote the subgraph of G induced by S, and let $N_S(x)$ be the neighbors of x in S. Let x and y be two vertices of G. An xy-path refers to a path from x to y. We use $d_G(x, y)$ to denote the length of a shortest xy-path which is referred to as the distance between the two vertices. A cycle of length k is simply called a k-cycle.

2 Proof of the main results

Recall that $\mathcal{G}$ denotes the family of graphs that have neither C_3 nor C_4, and have no odd holes of length at least 7. First, we have the following lemma on the structure of graphs in $\mathcal{G}$.

Lemma 4. *Let G be a graph in $\mathcal{G}$, let u be an arbitrary vertex of G, and let $L_i = \{x : d_G(u, x) = i\}$ for $i = 0, 1, 2, \cdots$. Then, $G[L_i]$ is bipartite for every i.*

Proof. Since G has no C_3, $L_0 = \{u\}$, and $G[L_1]$ is an independent set. So the conclusion holds for $i = 0, 1$. Suppose that, $G[L_i]$ is bipartite for each $0 \leqslant i \leqslant h$ for some $h \geqslant 1$.

Let $H = G[L_{h+1}]$, and suppose that H is not bipartite. Then, H has an odd cycle and thus a 5-cycle, say $u_1u_2u_3u_4u_5u_1$. Let v_i be a neighbor of u_i in L_h. Since G has neither C_3 nor C_4, $v_i \neq v_j$ if $i \neq j$.

Among all paths of length h from u to v_1 or v_3, we choose P to be a uv_1-path, and P' to be a uv_3-path, such that P and P' have the most common vertices. Let $w \in L_j$ be the last common vertex of P and P', let $P_w = x_jx_{j+1} \dots x_h$ be the segment of P from w to v_1, and let $P'_w = y_jy_{j+1} \dots y_h$ be the segment of P' from w to v_3, where $x_j = y_j = w$, $x_h = v_1$ and $y_h = v_3$. Then, $j \leqslant h - 1$, and thus $C = wP_wv_1u_1u_5u_4u_3v_3P'_ww$ is an odd cycle of length $2(h - j) + 5 \geqslant 7$. Therefore, C has chords.

By the choice of P and P', each chord of C must be of the form x_iy_i for some $j+2 \leqslant i \leqslant h$. Let i_0 be the largest index such that $x_{i_0}y_{i_0} \in E(G)$. If $i_0 < h$, then,

$$u_1u_2u_3v_3y_{h-1}\dots y_{i_0}x_{i_0}x_{i_0+1}\dots x_{h-1}v_1u_1$$

is an odd hole of length at least 7. Therefore, $i_0 = h$, i.e., $v_1v_3 \in E(G)$.

With the same arguments, we can show that $\{v_2v_4, v_3v_5, v_4v_1, v_5v_2\} \subseteq E(G)$. It follows that $G[L_h]$ has a 5-cycle $v_1v_3v_5v_2v_4v_1$, a contradiction. Thus, Lemma 4 holds. □

To prove our theorem, we need the following generalization of Brook's theorem by Gallai [9]. A graph G is said to be *k-vertex-critical* if $\chi(G) = k$ and $\chi(G-v) < \chi(G)$ for each vertex v.

Theorem 5. ([9]) *Let G be a k-vertex-critical graph, and let V_1 be the set of vertices of degree $k-1$ in G. Then every 2-connected induced subgraph of $G[V_1]$ is either a complete graph or an odd hole.*

Now, we are ready to state and prove our theorem.

Theorem 6. *Let G be a graph in $\mathcal{G}$. Then, $\chi(G) \leqslant 4$, and $\chi(G) \leqslant 3$ if G has radius at most 3.*

Proof. Let u be an arbitrary vertex of G, and let $L_i = \{u : d_G(u,v) = i\}$ for an integer $i \geqslant 0$. By Lemma 4, the vertices with even distance to u induce a bipartite subgraph, and the vertices with odd distance to u induce a bipartite subgraph too. Therefore, $\chi(G) \leqslant 4$.

Next, we suppose that G has radius at most 3. Suppose that the conclusion does not hold. Let G be a counterexample in $\mathcal{G}$ with the smallest order, i.e., every proper subgraph of G is 3-colorable, and let $k = \max\{i : L_i \neq \varnothing\}$. Then,

$$k \leqslant 3, G \text{ is 4-vertex-critical, and } \delta(G) \geqslant 3.$$

If $k = 2$, then $\chi(G) \leqslant 3$ as we may color $L_2 \cup \{u\}$ with two colors by Lemma 4, and color L_1 with the third color. So, we suppose that $k = 3$.

We will show that

$$G[L_2 \cup L_3] \text{ is bipartite.}$$

Then, we may color $L_2 \cup L_3 \cup \{u\}$ with two colors, and color L_1 with the third color, and thus $\chi(G) \leqslant 3$.

Suppose to the contrary that $G[L_2 \cup L_3]$ is not bipartite. Then, $G[L_2 \cup L_3]$ has odd cycles, and thus has 5-cycles. We choose C to be a 5-cycle in $G[L_2 \cup L_3]$ that maximizes $|V(C) \cap L_2|$. Let $C = u_1u_2u_3u_4u_5u_1$, and let $m = |V(C) \cap L_2|$. By Lemma 4,

$$1 \leqslant m \leqslant 4.$$

We may assume that $V(C) \cap L_2$ does not contain two nonadjacent vertices. For otherwise, let $u_1, u_3 \in L_2$ by symmetry. Let $P_1 = uw_1u_1$ and $P_2 = uw_2u_3$ be two paths.

Since G has neither C_3 nor C_4, $w_1 \neq w_2$, $w_1w_2 \notin E(G)$, and w_1u_1 and w_2u_3 are the only two edges from $\{w_1, w_2\}$ to $\{u_1, u_3\}$. Now, $uw_1u_1u_5u_4u_3w_2u$ is an odd hole, a contradiction.

It follows that $m \leqslant 2$, and if $m = 2$ then the two vertices in $V(C) \cap L_2$ must be adjacent.

Suppose that $m = 2$, and let $u_1, u_2 \in L_2$ by symmetry. Let x be a neighbor of u_4 in L_2. Let $P_1 = uw_1u_1$, $P_2 = uw_2u_2$, and $P_3 = ux'x$. It is clear that $w_1 \neq w_2$, and so we suppose by symmetry that $x' \neq w_1$. Now, $uw_1u_1u_5u_4xx'u$ is an odd hole.

Finally, the only remaining case is that $m = 1$, i.e.,

$$\text{every 5-cycle of } G[L_2 \cup L_3] \text{ has a unique vertex in } L_2.$$

For convenience, we relabel the 5-cycle C as $v_0u_0u_1u_2u_3v_0$, and let $v_0 \in L_2$ by symmetry. Let $P = uw_0v_0$, and let $P_i = uw_iv_iu_i$ for $i \in \{0, 1, 2\}$. It is certain that

$$\{v_0, v_1, v_2\} \text{ is an independent set of size 3}$$

as G has neither C_3 nor C_4.

If $N(v_1) \cap L_1 \neq \{w_0\}$, we may choose P_1 such that $w_1 \neq w_0$, then $w_0v_1, w_1v_0 \notin E(G)$ for otherwise $G[\{u, w_0, w_1, v_0, v_1\}]$ will have a C_4, and thus $uw_0v_0u_0u_1v_1w_1u$ is an odd hole. The same contradiction occurs if $N(v_2) \cap L_1 \neq \{w_0\}$. Therefore, we have

$$(N(v_0) \cup N(v_1) \cup N(v_2)) \cap L_1 = \{w_0\}. \tag{1}$$

Recall that for a subset S of vertices and a vertex z, $N_S(z)$ denotes the neighbors of z in S.

Let B be the component of $G[L_3]$ that contains u_0. We will show that

$$B \text{ is a cycle that can be labelled as } u_0u_1 \dots u_{3q-1} \text{ for some integer } q. \tag{2}$$

and for each integer i and each integer $j \in \{0, 1, 2\}$,

$$d(u_i) = 3, \text{ and } N_{L_2}(u_i) = \{v_j\} \text{ if } i \equiv j \bmod 3. \tag{3}$$

We proceed to prove (2) and (3), and prove (3) by induction first.

If $N_{L_2}(u_3) \neq \{v_0\}$, we may choose $P_3 = uw_3v_3u_3$ such that $v_3 \neq v_0$, then $v_2 \neq v_3$, $w_3 \neq w_0$, and $\{v_2w_3, w_0v_3, v_2v_3, w_0w_3\} \cap E(G) = \varnothing$ by (1) (as G has neither C_3 nor C_4). Now, $uw_3v_3u_3u_2v_2w_0u$ is an odd hole (note that $w_2 = w_0$ by (1)), a contradiction. The same happens if $N_{L_2}(u_0) \neq \{v_0\}$. So,

$$N_{L_2}(u_0) \cup N_{L_2}(u_3) = \{v_0\}. \tag{4}$$

Since $\delta(G) \geqslant 3$ and $k = 3$, both $|N_B(u_0)| > 1$ and $|N_B(u_3)| > 1$. Choose $u_4 \in N_B(u_3) \setminus \{u_2\}$, and let $P_4 = uw_4v_4u_4$. If $N_{L_1}(v_4) \neq \{w_0\}$, we may choose P_4 such that $w_4 \neq w_0$, then it is easy to check that $uw_0v_0u_3u_4v_4w_4u$ would be an odd hole. So,

$$N_{L_1}(v_4) = \{w_0\}.$$

If $N_{L_2}(u_4) \neq \{v_1\}$, we may choose P_4 such that $v_4 \neq v_1$, then $w_0v_4u_4u_3u_2u_1v_1w_0$ would be an odd hole, a contradiction. So, we may suppose, by symmetry, that

$$N_{L_2}(x) = \{v_1\} \text{ for every vertex } x \in N_B(u_3) \setminus \{u_2\}, \tag{5}$$

and

$$N_{L_2}(y) = \{v_2\} \text{ for every vertex } y \in N_B(u_0) \setminus \{u_1\}. \tag{6}$$

Let u_{-1} be a vertex in $N_B(u_0) \setminus \{u_1\}$. If $N_B(u_0) \neq \{u_{-1}, u_1\}$, choose $z_0 \in N_B(u_0) \setminus \{u_{-1}, u_1\}$, then $z_0v_2 \in E(G)$ by (6) and thus $z_0v_2u_{-1}u_0z_0$ would be a C_4. Therefore, we may suppose, by (4) and by symmetry, that

$$N(u_0) = \{u_{-1}, u_1, v_0\} \text{ and } N(u_3) = \{u_2, u_4, v_0\}.$$

By applying the same arguments as that used on C to 5-cycles $v_1u_1u_2u_3u_4v_1$ and $v_2u_{-1}u_0u_1u_2v_2$, we see that

$$N(u_1) = \{u_0, u_2, v_1\} \text{ and } N(u_2) = \{u_1, u_3, v_2\}.$$

We have proved (3) for $i \in \{0, 1, 2, 3\}$. Suppose that (3) holds for $i \in \{0, 1, \ldots, m\}$, where $m \geqslant 3$. Let $i = m + 1$, and suppose, without loss of generality, that $m \equiv 1 \pmod 3$. Then, $v_1u_{m-3}u_{m-2}u_{m-1}u_mv_1$ is a 5-cycle, $N_{L_2}(u_{m-2}) = \{v_2\}$, and $N_{L_2}(u_{m-1}) = \{v_0\}$ by the inductive hypothesis. By applying the same argument to $v_1u_{m-3}u_{m-2}u_{m-1}u_mv_1$ as that used on C, we see that

$$N_{L_2}(u_{m+1}) = \{v_2\} \text{ and } d(u_{m+1}) = 3.$$

Therefore, (3) holds for all i.

Since G is finite, there must be an integer j such that $u_0u_j \in E(G)$. Suppose that $j \equiv r \pmod 3$. By (3), $v_0u_0u_jv_0$ would be a C_3 if $r = 0$, and $u_0u_1v_1u_ju_0$ would be a C_4 if $r = 1$. Therefore, $r = 2$, and so B is a cycle of length $j + 1$ which is a multiple of 3. This completes the proof of (2).

Let H be the subgraph of G induced by the vertices of degree 3. Now, B is a 2-connected subgraph of H, which is neither complete nor an odd hole. This contradiction to Theorem 5 shows that $G[L_2 \cup L_3]$ is bipartite. Now, by coloring $L_2 \cup L_3 \cup \{u\}$ with two colors and coloring L_1 with the third color, we can complete the proof of Theorem 6. □

Finally, we would like to mention that in [17], Plummer and Zha also conjectured that the graphs in $\mathcal{G}$ may have chromatic number at most 3. This problem is still open in general.

Acknowledgements

We thank the referees sincerely for their helpful suggestions.

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