On r-uniform linear hypergraphs with no Berge-$K_{2,t}$

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Abstract

Let $\mathcal{F}$ be an r-uniform hypergraph and G be a multigraph. The hypergraph $\mathcal{F}$ is a *Berge-G* if there is a bijection $f : E(G) \to E(\mathcal{F})$ such that $e \subseteq f(e)$ for each $e \in E(G)$. Given a family of multigraphs $\mathcal{G}$, a hypergraph $\mathcal{H}$ is said to be *$\mathcal{G}$-free* if for each $G \in \mathcal{G}$, $\mathcal{H}$ does not contain a subhypergraph that is isomorphic to a Berge-G. We prove bounds on the maximum number of edges in an r-uniform linear hypergraph that is $K_{2,t}$-free. We also determine an asymptotic formula for the maximum number of edges in a linear 3-uniform 3-partite hypergraph that is $\{C_3, K_{2,3}\}$-free.

Keywords: hypergraph Turán problem; Sidon sets; Berge-$K_{2,t}$

1 Introduction

Let G be a multigraph and $\mathcal{F}$ be a hypergraph. Following Gerbner and Palmer [5], we say that $\mathcal{F}$ is a *Berge-G* if there is a bijection $f : E(G) \to E(\mathcal{F})$ with the property that $e \subseteq f(e)$ for all $e \in E(G)$. This definition generalizes both Berge-cycles and Berge-paths in hypergraphs. Recall that for an integer $k \geqslant 2$, a Berge k-cycle is an alternating sequence $v_1 e_1 v_2 e_2 \cdots v_k e_k v_1$ of distinct vertices and edges such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \leqslant i \leqslant k-1$, and $\{v_k, v_1\} \subseteq e_k$. A Berge k-path is defined in a similar way (omit e_k and v_1 from the sequence). Given a family of multigraphs $\mathcal{G}$, the hypergraph $\mathcal{H}$ is *$\mathcal{G}$-free* if for every $G \in \mathcal{G}$, the hypergraph $\mathcal{H}$ does not contain a subhypergraph that is isomorphic to a Berge-G. Observe that Berge-G is a family of hypergraphs. For example, $\{\{a,b,c\},\{c,d,e\}\}$ and $\{\{a,b,c\},\{b,c,d\}\}$ are non-isomorphic hypergraphs, but both are Berge-G's where G is the path whose edges are $\{b,c\}$ and $\{c,d\}$.

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Write $\mathrm{ex}_r(n, \mathcal{G})$ for the maximum number of edges in an n-vertex r-uniform hypergraph that is $\mathcal{G}$-free. The function $\mathrm{ex}_r(n, \mathcal{G})$ is the *Turán number* or *extremal number* of $\mathcal{G}$. When $r = 2$ and $\mathcal{G}$ consists of simple graphs, $\mathrm{ex}_2(n, \mathcal{G})$ coincides with the usual definition of Turán numbers. When $\mathcal{G} = \{G\}$, we write $\mathrm{ex}_r(n, G)$ instead of $\mathrm{ex}_r(n, \{G\})$.

One of the most important results in graph theory is the so-called Erdős-Stone-Simonovits Theorem which is a statement about Turán numbers of graphs.

Theorem 1 (Erdős, Stone, Simonovits). *If G is a graph with chromatic number $k \geqslant 2$, then*

$$\mathrm{ex}_2(n, G) = \left(1 - \frac{1}{k-1}\right)\binom{n}{2} + o(n^2).$$

Theorem 1 provides an asymptotic formula for the Turán number of any non-bipartite graph. No such result is known for $r \geqslant 3$ and in general, hypergraph Turán problems are considerably harder than graph Turán problems. Despite this, there has been some success in estimating $\mathrm{ex}_r(n, \mathcal{G})$ when $\mathcal{G}$ contains short cycles. For instance, Bollobás and Győri [3] proved that

$$\frac{1}{3\sqrt{3}} n^{3/2} - o(n^{3/2}) \leqslant \mathrm{ex}_3(n, C_5) \leqslant \sqrt{2} n^{3/2} + 4.5n.$$

In other words, the maximum number of triples in an n-vertex 3-uniform hypergraph with no Berge 5-cycle is $\Theta(n^{3/2})$. One of the motivations behind estimating $\mathrm{ex}_3(n, C_5)$ is the problem of finding the maximum number of triangles in a graph with no 5-cycle. We refer the reader to [3] and the papers of Győri, Li [11], and Alon and Shikhelman [2] for more on the intriguing problem of finding the maximum number of copies of a graph F in an H-free graph G.

Lazebnik and Verstraëte [13] proved several results concerning r-uniform hypergraphs that are $\{C_2, C_3, C_4\}$-free. Here C_2 is the multigraph consisting of two parallel edges. Recall that a hypergraph $\mathcal{F}$ is *linear* if any two distinct edges of $\mathcal{F}$ intersect in at most one vertex. It is easy to check that

a hypergraph is linear if and only if it is C_2-free.

Lazebnik and Verstraëte showed that

$$\mathrm{ex}_3(n, \{C_2, C_3, C_4\}) = \frac{1}{6} n^{3/2} + o(n^{3/2}). \tag{1}$$

A consequence of this result is the asymptotic formula $T_3(n, 8, 4) = \frac{1}{6} n^{3/2} + o(n^{3/2})$ for the generalized Turán number $T_r(n, k, l)$. This is defined to be the maximum number of edges in an n-vertex r-uniform hypergraph with the property that no k vertices span l or more edges. Provided cycles are defined in the Berge sense as above, one may say that a $\{C_2, C_3, C_4\}$-free hypergraph is a hypergraph of girth 5, and this is the terminology that is used in [13]. The interest in $\mathrm{ex}_3(n, \{C_2, C_3, C_4\})$ has its origins in determining the maximum number of edges in a graph with girth 5 which is a well-known, unsolved problem of Erdős (see (2) below).

For related results, including results for paths, cycles, and some general bounds, see [10], [7], and [5], respectively. The case of cycles has received considerable attention. Collier-Cartaino, Graber, and Jiang [4] investigated so-called linear cycles in linear hypergraphs. Their paper has a particularly nice introduction that discusses several results in this area. Lastly, the papers of Győri and Lemons [8, 9, 10], in which bounds on the number of edges in a hypergraph with no Berge k-cycle are obtained, are also important contributions.

In this paper we consider what happens in (1) when C_4 is replaced by $K_{2,3}$. Our main result is given in the following theorem.

Theorem 2. *For any integer $r \geqslant 3$,*

$$\frac{1}{r^{3/2}}n^{3/2} - o(n^{3/2}) \leqslant \mathrm{ex}_r(n, \{C_2, C_3, K_{2,2r-3}\}) \leqslant \frac{\sqrt{2r-4}}{r(r-1)}n^{3/2} + \frac{n}{r}.$$

Since $\frac{1}{3^{3/2}} > \frac{1}{6}$, Theorem 2 implies that there are 3-uniform hypergraphs that are $\{C_2, C_3, K_{2,3}\}$-free and have more edges than any $\{C_2, C_3, C_4\}$-free 3-uniform hypergraph. For graphs, the best known bounds on the Turán number of $\{C_3, C_4\}$ are

$$\frac{1}{2\sqrt{2}}n^{3/2} - o(n^{3/2}) \leqslant \mathrm{ex}_2(n, \{C_3, C_4\}) \leqslant \frac{1}{2}n^{3/2} + o(n^{3/2}). \tag{2}$$

In [1] it is shown that $\mathrm{ex}_2(n, \{C_3, K_{2,3}\}) \geqslant \frac{1}{\sqrt{3}}n^{3/2} - o(n^{3/2})$. Putting all of these results together, we see that in both the graph case and the 3-uniform hypergraph case, forbidding $K_{2,3}$ instead of C_4 allows one to have significantly more edges. It is not known if this is also true for $r \geqslant 4$. On an interesting related note, Erdős has conjectured that the lower bound in (2) is correct while in [1] it is conjectured that the lower bound in (2) can be improved.

Our construction that establishes the lower bound in Theorem 2 is r-partite. In this case, the upper bound of Theorem 2 can be improved by adapting the counting argument of [13] to the $K_{2,3}$-free case.

Theorem 3. *If $\mathcal{F}$ is a $\{C_2, C_3, K_{2,3}\}$-free 3-uniform 3-partite hypergraph with n vertices in each part, then*

$$|E(\mathcal{F})| \leqslant \sqrt{\frac{2}{r-1}}n^{3/2} + n.$$

Furthermore, for any q that is a power of an odd prime, there is a 3-uniform 3-partite $\{C_2, C_3, K_{2,3}\}$-free hypergraph with q^2 vertices in each part and $q^2(q-1)$ edges.

A similar result for 3-uniform 3-partite $\{C_2, C_3, C_4\}$-free graphs was proved in [13]. Let us write $z_r(n, \mathcal{G})$ for the maximum number of edges in a $\mathcal{G}$-free r-uniform r-partite hypergraph with n vertices in each part. Using this notation, we can state Theorem 2.6 of [13] as $z_3(n, \{C_2, C_3, C_4\}) \leqslant \frac{1}{\sqrt{2}}n^{3/2} + n$ for all $n \geqslant 3$, and $z_3(n, \{C_2, C_3, C_4\}) \geqslant \frac{1}{2}n^{3/2} - 3n$ for infinitely many n. Theorem 3 gives the asymptotic formula

$$z_3(n, \{C_2, C_3, K_{2,3}\}) = n^{3/2} + o(n^{3/2}).$$

One drawback to Theorem 2 is that the size of the forbidden graph $K_{2,2r-3}$ depends on r. There are two natural directions to pursue. On one hand, we can fix r and attempt to construct $K_{2,t}$-free hypergraphs where t tends to infinity and at the same time, the number of edges increases with t. Our next theorem shows that this can be done at the cost of allowing C_3.

Theorem 4. *Let $r \geqslant 3$ be an integer and l be any integer with $2l+1 \geqslant r$. If $q \geqslant 2lr^3$ is a power of an odd prime and $n = rq^2$, then*

$$\mathrm{ex}_r(n, \{C_2, K_{2,t+1}\}) \geqslant \frac{l}{r^{3/2}} n^{3/2} - \frac{l}{r} n$$

where $t = (r-1)(2l^2 - l)$.

The other direction is to fix t and let r become large. This is a much more difficult problem as suggested by the results and discussion in [13]. We were unable to answer the following slight variation of a question posed to us by Verstraëte [16].

Question 5. Is there a bipartite graph F that contains a cycle for which the following holds: there is a positive integer $r(F)$ such that for all $r \geqslant r(F)$, we have

$$\mathrm{ex}_r(n, \{C_2, F\}) = o(\mathrm{ex}_2(n, F)). \tag{3}$$

Using the graph removal lemma, one can show that (3) holds whenever F is a non-bipartite graph provided $r \geqslant |V(F)|$. When $F = C_4$, the formula (1) implies that $\mathrm{ex}_3(n, \{C_2, C_4\}) = \Omega(\mathrm{ex}_2(n, C_4))$, but it is not known if the same lower bound holds for larger r. Using blow ups of extremal graphs, Gerbner and Palmer [5] (see also [8, 10] for cycles) proved that $\mathrm{ex}_r(n, K_{s,t}) = \Omega(\mathrm{ex}_2(n, K_{s,t}))$ whenever $2 \leqslant r \leqslant s+t$, but the hypergraphs constructed using this method are not C_2-free. Improving the lower bound on $\mathrm{ex}_3(n, \{C_2, C_{2k}\})$ that comes from random constructions is a problem that was mentioned explicitly by Füredi and Özkahya in [7].

In the next section we prove the upper bounds stated in Theorems 2 and 3. Both of these upper bounds use the counting arguments of [13]. We include their proofs for completeness, but we do want to make it clear that proving our upper bounds using the methods of [13] is straightforward. The lower bounds of Theorems 2, 3, and 4 are our main contribution. Section 3.1 contains algebraic lemmas which are required for our construction. Section 3.2 gives the construction which is a generalization of the one found in [15] and is based on a construction Allen, Keevash, Sudakov, and Verstraëte (see Theorem 1.6 [1]).

2 Upper bounds

2.1 The upper bound of Theorem 2

Using the counting argument of [13] we can prove an upper bound on the number of edges in a $\{C_2, C_3, K_{2,t+1}\}$-free r-uniform hypergraph. Given a set S, write $S^{(2)}$ for the set of pairs of elements of S. In this section we prove the following which implies the upper bound given in Theorem 2.

Theorem 6. *If $r \geqslant 3$ and $t \geqslant 1$ are integers, then*

$$\mathrm{ex}_r(n, \{C_2, C_3, K_{2,t+1}\}) \leqslant \frac{\sqrt{t}}{r(r-1)} n^{3/2} + \frac{n}{r}.$$

Proof. Let $\mathcal{F}$ be a $\{C_2, C_3, K_{2,t+1}\}$-free r-uniform hypergraph with n vertices. Let V be the vertex set of $\mathcal{F}$. For $v \in V$, let $e_1^v, \dots, e_{d(v)}^v$ be the edges in $\mathcal{F}$ that contain v where $d(v)$ is the degree of v in $\mathcal{F}$. For $1 \leqslant i < j \leqslant d(v)$, let

$$P(e_i^v, e_j^v) = \left\{\{x, y\} \in V^{(2)} : x \in e_i^v \backslash \{v\} \text{ and } y \in e_j^v \backslash \{v\}\right\}.$$

Since $\mathcal{F}$ is linear, the sets $e_1^v \backslash \{v\}, e_2^v \backslash \{v\}, \dots, e_{d(v)}^v \backslash \{v\}$ are pairwise disjoint so we have $|P(e_i^v, e_j^v)| = (r-1)^2$. For any fixed vertex v,

$$\sum_{1 \leqslant i < j \leqslant d(v)} |P(e_i^v, e_j^v)| = (r-1)^2 \binom{d(v)}{2} \tag{4}$$

and the sum in (4) never counts a pair $\{x, y\} \in V^{(2)}$ more than once.

Now consider the sum

$$\sum_{v \in V} \sum_{1 \leqslant i < j \leqslant d(v)} |P(e_i^v, e_j^v)|. \tag{5}$$

Suppose a pair $\{x, y\} \in V^{(2)}$ is counted more than t times in this sum. Let $v_1, \dots, v_{t+1}$ be distinct vertices such that there are edges $e_i \neq f_i \in E(\mathcal{F})$, both of which contain v_i, and $\{x, y\} \in P(e_i, f_i)$ for $1 \leqslant i \leqslant t+1$. Assume $x \in e_i$ and $y \in f_i$. By definition of $P(e, f)$, $\{x, y\} \cap \{v_1, \dots, v_{t+1}\} = \emptyset$ so $x, y, v_1, \dots, v_{t+1}$ are all distinct. If $e_1, \dots, e_{t+1}$, $f_1, \dots, f_{t+1}$ are all distinct, then $\mathcal{F}$ contains a $K_{2,t+1}$ so these $2t+2$ edges cannot all be distinct. We will show that this leads to a contradiction.

If $e_i = e_j$ for some $1 \leqslant i < j \leqslant t+1$, then $v_j \in e_i$ and $\{f_i, f_j, e_i\}$ is a C_3 since $v_i \in e_i \cap f_i$, $y \in f_i \cap f_j$, and $v_j \in f_j \cap e_i$. Note that $f_i \neq f_j$ otherwise $\{v_i, v_j\} \subseteq f_i \cap e_i$ contradicting the linearity of $\mathcal{F}$. We conclude that $e_i \neq e_j$ for $1 \leqslant i < j \leqslant t+1$. A similar argument shows that $f_i \neq f_j$ for $1 < i < j \leqslant t+1$. The only remaining possibility is that $e_i = f_j$ for some $1 \leqslant i \neq j \leqslant t+1$. If this is the case, then $y \in e_i$ so $\{v_i, y\} \subseteq e_i \cap f_i$ which, by linearity, implies $e_i = f_i$ which is a contradiction.

We conclude that the sum (5) counts any pair $\{x, y\} \in V^{(2)}$ at most t times. Let m be the number of edges of $\mathcal{F}$. By (4) and Jensen's Inequality applied to the convex function

$$f(x) = \begin{cases} \binom{x}{2} & \text{if } x \geqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$t\binom{n}{2} \geqslant \sum_{v \in V} \sum_{1 \leqslant i < j \leqslant d(v)} |P(e_i^v, e_j^v)| = (r-1)^2 \sum_{v \in V} \binom{d(v)}{2} \geqslant n(r-1)^2 \binom{rm/n}{2}.$$

This is a quadratic inequality in m and implies that

$$m \leqslant \left(\frac{tn^3}{r^2(r-1)^2} + \frac{n^2}{4r^2}\right)^{1/2} + \frac{n}{2r} \leqslant \frac{\sqrt{t}}{r(r-1)} n^{3/2} + \frac{n}{r}. \qquad \square$$

2.2 The upper bound of Theorem 3

The upper bound of Theorem 3 essentially follows from Theorem 2.3 in [13] with some modifications to the proof. We include the proof for completeness.

Theorem 7. *Let $r \geqslant 3$. If $\mathcal{F}$ is a $\{C_2, C_3, K_{2,3}\}$-free r-uniform r-partite hypergraph with n vertices in each part, then*

$$|E(\mathcal{F})| \leqslant \sqrt{\frac{2}{r-1}} n^{3/2} + n.$$

Proof. Let $\mathcal{F}$ be an r-partite r-uniform hypergraph with n vertices in each part. Let $V_1, \dots, V_r$ be the parts of $\mathcal{F}$ and assume that $\mathcal{F}$ is $\{C_2, C_3, K_{2,3}\}$-free. Let S be the set of all pairs of the form $(v, \{x, y\})$ where $v \in V(\mathcal{F})$, $\{x, y\}$ is a pair of vertices in the same part with $x \neq v$, $y \neq v$, and there are distinct edges e and f with $\{v, x\} \subset e$ and $\{v, y\} \subset f$. We will count the cardinality of S in two ways. Given a vertex $v \in \mathcal{F}$, we again write $d(v)$ for the number of edges that contain v.

If we first choose the vertex v, there are $\binom{d(v)}{2}(r-1)$ ways to choose a pair $\{x, y\}$ for which $(v, \{x, y\})$ belongs to S. Here we are using the fact that $\mathcal{F}$ is linear and so every edge of $\mathcal{F}$ contains exactly one vertex in each part. Therefore,

$$|S| = \sum_{v \in V(\mathcal{F})} \binom{d(v)}{2}(r-1) = (r-1) \sum_{i=1}^{r} \sum_{v \in V_i} \binom{d(v)}{2}. \tag{6}$$

Next we show that

$$|S| \leqslant 2 \sum_{i=1}^{r} \binom{|V_i|}{2}. \tag{7}$$

We first pick a pair $\{x, y\}$ that are in the same part, say $\{x, y\} \subset V_i$. We now claim that there are at most two distinct v's for which $(v, \{x, y\})$ belongs to S. Aiming for a contradiction, suppose that $(v, \{x, y\})$, $(v', \{x, y\})$, and $(v'', \{x, y\})$ all belong to S, where v, v', and v'' are all distinct. Let e, e', and e'' be the edges through x that contain v, v', and v'', respectively. Let f, f', and f'' be the edges through y that contain v, v', and v'', respectively. We will show that since $\mathcal{F}$ is $\{C_2, C_3\}$-free, all of the edges e, e', e'', f, f', and f'' are distinct and so form a $K_{2,3}$, which provides the needed contradiction. If $e \in \{f, f', f''\}$, then e contains both x and y which is impossible since x and y are in the same part. The same argument shows $e' \notin \{f, f', f''\}$ and $e'' \notin \{f, f', f''\}$, so that $\{e, e', e''\} \cap \{f, f', f''\} = \emptyset$. Now suppose $e = e'$. Then $\{v, v'\} \subset e$, and now $v \in e \cap f$, $y \in f \cap f'$, and $v' \in f' \cap e$. The edges e, f, and f' cannot form a C_3 since $\mathcal{F}$ is C_3-free. Therefore, $f = f'$ so $v' \in f$. Since $\{v, v'\} \subset f$, $\{v, v'\} \subset e$, and $\mathcal{F}$ is C_2-free, the edges e and f must be the same, but we have shown already that this cannot occur. By symmetry, $e \neq e''$ and $e' \neq e''$. We conclude that the edges e, e', and e'' are all distinct. A similar argument shows that f, f', and f'' are all distinct. This gives a $K_{2,3}$ in $\mathcal{F}$ which is a contradiction. Therefore, there are at most two distinct vertices v and v' for which the pairs $(v, \{x, y\})$ and $(v', \{x, y\})$ belong to S.

Combining (6) and (7) and using the fact that $|V_i| = n$ for every i, we have

$$2r\binom{n}{2} = 2\sum_{i=1}^{r}\binom{|V_i|}{2} \geqslant |S| = (r-1)\sum_{i=1}^{r}\sum_{v\in V_i}\binom{d(v)}{2}.$$

By Jensen's Inequality, $\sum_{v\in V_i}\binom{d(v)}{2} \geqslant n\binom{m/n}{2}$ where m is the number of edges of $\mathcal{F}$. Together, these two estimates give $2r\binom{n}{2} \geqslant (r-1)rn\binom{m/n}{2}$ so

$$rn(n-1) \geqslant (r-1)rn\frac{(m/n)(m/n-1)}{2}.$$

It follows that

$$m \leqslant \sqrt{\frac{2}{r-1}}n^{3/2} + n. \qquad \square$$

3 Lower bounds

In this section we prove the lower bounds of Theorems 2, 3, and 4.

3.1 Algebraic Lemmas

In this subsection we prove some lemmas that are needed to prove our lower bounds. We write $\mathbb{F}_q$ for the finite field with q elements and $\mathbb{F}_q^*$ for the group $\mathbb{F}_q\backslash\{0\}$ under multiplication.

The first lemma is due to Ruzsa [14] and was key to the construction in [15]. A proof can be found in [15].

Lemma 8. *Suppose α, β, γ, and δ are nonzero elements of $\mathbb{F}_q$ with $\alpha + \beta = \gamma + \delta$. If $a_1, a_2, a_3, a_4 \in \mathbb{F}_q^*$, $\alpha a_1 + \beta a_2 = \gamma a_3 + \delta a_4$, and $\alpha a_1^2 + \beta a_2^2 = \gamma a_3^2 + \delta a_4^2$, then*

$$\alpha\beta(a_1 - a_2)^2 = \gamma\delta(a_3 - a_4)^2.$$

The next lemma is known. It is merely asserting the well-known fact that $\{(a, a^2) : a \in \mathbb{F}_q^*\}$ is a Sidon set in the group $\mathbb{F}_q \times \mathbb{F}_q$ where the group operation is componentwise addition.

Lemma 9. *If $a_1, a_2, a_3, a_4 \in \mathbb{F}_q^*$, $a_1 + a_2 = a_3 + a_4$, and $a_1^2 + a_2^2 = a_3^2 + a_4^2$, then $\{a_1, a_2\} = \{a_3, a_4\}$.*

The next two lemmas will be used to control the appearance of small graphs in our construction. The idea is that a copy of some small graph in our construction corresponds to a nontrivial solution to some system of equations over $\mathbb{F}_q$. Variations of these lemmas have appeared in [15].

Lemma 10. *Let α, β, and γ be distinct elements of $\mathbb{F}_q$. If $a_1, a_2, a_3 \in \mathbb{F}_q^*$,*

$$0 = \alpha(a_2 - a_1) + \beta(a_3 - a_2) + \gamma(a_1 - a_3), \tag{8}$$

and

$$0 = \alpha(a_2^2 - a_1^2) + \beta(a_3^2 - a_2^2) + \gamma(a_1^2 - a_3^2),$$

then $a_1 = a_2 = a_3$.

Proof. Adding βa_1 to both sides of (8) and rearranging gives

$$(\gamma - \beta)(a_3 - a_1) = (\alpha - \beta)(a_2 - a_1). \tag{9}$$

A similar manipulation yields $(\gamma - \beta)(a_3^2 - a_1^2) = (\alpha - \beta)(a_2^2 - a_1^2)$ which is equivalent to

$$(\gamma - \beta)(a_3 - a_1)(a_3 + a_1) = (\alpha - \beta)(a_2 - a_1)(a_2 + a_1). \tag{10}$$

Note that $\gamma - \beta \neq 0$ and $\alpha - \beta \neq 0$ since α, β, and γ are all different. If $a_3 = a_1$, then (9) implies that $a_2 = a_1$ and we are done. Otherwise, we divide (10) by (9) to get $a_3 + a_1 = a_2 + a_1$ which gives $a_3 = a_2$. This equality, together with (8), implies $0 = \alpha(a_2 - a_1) + \gamma(a_1 - a_2)$ so

$$\gamma(a_2 - a_1) = \alpha(a_2 - a_1).$$

If $a_2 - a_1 = 0$, then with $a_3 = a_2$ we get $a_1 = a_2 = a_3$ and we are done. Otherwise, we may cancel $a_2 - a_1$ to get $\gamma = \alpha$ which contradicts the fact that $\gamma \neq \alpha$. □

Lemma 11. *Let $\alpha, \beta \in \mathbb{F}_q^*$ with $\alpha + \beta \neq 0$. If $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}_q^*$,*

$$\alpha a_1 + \beta b_1 = \alpha a_2 + \beta b_2 = \alpha a_3 + \beta b_3, \tag{11}$$

and

$$\alpha a_1^2 + \beta b_1^2 = \alpha a_2^2 + \beta b_2^2 = \alpha a_3^2 + \beta b_3^2,$$

then there is a pair $\{i, j\} \subset \{1, 2, 3\}$ with $a_i = b_i$ and $a_j = b_j$.

Proof. By Lemma 8,

$$\alpha\beta(a_1 - b_1)^2 = \alpha\beta(a_2 - b_2)^2. \tag{12}$$

Since $\alpha\beta \neq 0$, (12) implies that $(a_1 - b_1)^2 = (a_2 - b_2)^2$ so either $a_1 - b_1 = a_2 - b_2$, or $a_1 - b_1 = b_2 - a_2$.

Suppose $a_1 - b_1 = a_2 - b_2$. We multiply this equation through by α and subtract the resulting equation from the first equation in (11) to get

$$(\alpha + \beta)b_1 = (\alpha + \beta)b_2.$$

As $\alpha + \beta \neq 0$, it must be the case that $b_1 = b_2$ which, with (11), gives $a_1 = a_2$ and we are done.

Now suppose that $a_1 - b_1 = b_2 - a_2$. By symmetry, we may then assume that $a_1 - b_1 = b_3 - a_3$. We then have $a_2 - b_2 = a_3 - b_3$ and the argument from the previous paragraph gives $a_2 = a_3$ and $b_2 = b_3$. □

3.2 The Construction

Let $r \geqslant 2$ and $l \geqslant 1$ be integers. Let q be a power of an odd prime. Let $\alpha_1, \ldots, \alpha_r$ be distinct elements of $\mathbb{F}_q$. We choose q large enough so that there are distinct elements $m_1, \ldots, m_l \in \mathbb{F}_q^*$ that satisfy the condition

$$m_s(\alpha_k - \alpha_i) \neq m_t(\alpha_k - \alpha_j) \tag{13}$$

whenever $1 \leqslant s, t \leqslant l$ and i, j, and k are distinct integers with $1 \leqslant i, j, k \leqslant r$.

For $1 \leqslant i \leqslant r$, let $V_i = \mathbb{F}_q \times \mathbb{F}_q \times \{i\}$. The union $V_1 \cup V_2 \cup \cdots \cup V_r$ will be the vertex set of our hypergraph. We now define the edges. Each edge will contain exactly one element from each V_i. Given $x, y \in \mathbb{F}_q$, $a \in \mathbb{F}_q^*$, and an integer $s \in \{1, 2, \ldots, l\}$, let

$$\begin{aligned} e(x, y, a, m_s) &= \{(x + \alpha_1(m_s a), y + \alpha_1(m_s a^2), 1), (x + \alpha_2(m_s a), y + \alpha_2(m_s a^2), 2), \\ &\quad \ldots, (x + \alpha_r(m_s a), y + \alpha_r(m_s a^2), r)\}. \end{aligned}$$

We define $\mathcal{H}$ to be the r-uniform hypergraph with vertex set

$$V(\mathcal{H}) = \{(x, y, i) : x, y \in \mathbb{F}_q, 1 \leqslant i \leqslant r\}$$

and edge set

$$E(\mathcal{H}) = \{e(x, y, a, m_s) : x, y \in \mathbb{F}_q, a \in \mathbb{F}_q^*, s \in \{1, \ldots l\}\}.$$

The vertex set of $\mathcal{H}$ can be written as $V(\mathcal{H}) = V_1 \cup \cdots \cup V_r$ so $\mathcal{H}$ is r-partite.

Lemma 12. *The hypergraph $\mathcal{H}$ is linear.*

Proof. Suppose $e(x_1, y_1, a_1, m_s)$ and $e(x_2, y_2, a_2, m_t)$ are edges of $\mathcal{H}$ that share at least two vertices, say (u_i, v_i, i) in V_i and (u_j, v_j, j) in V_j, where $1 \leqslant i < j \leqslant r$. We have

$$u_i = x_1 + \alpha_i(m_s a_1) = x_2 + \alpha_i(m_t a_2), \quad v_i = y_1 + \alpha_i(m_s a_1^2) = y_2 + \alpha_i(m_t a_2^2),$$
$$u_j = x_1 + \alpha_j(m_s a_1) = x_2 + \alpha_j(m_t a_2), \quad v_j = y_1 + \alpha_j(m_s a_1^2) = y_2 + \alpha_j(m_t a_2^2).$$

Taking differences yields

$$u_i - u_j = m_s a_1(\alpha_i - \alpha_j) = m_t a_2(\alpha_i - \alpha_j)$$

and

$$v_i - v_j = m_s a_1^2(\alpha_i - \alpha_j) = m_t a_2^2(\alpha_i - \alpha_j).$$

Since α_i and α_j are distinct, we may cancel $\alpha_i - \alpha_j$ to obtain $m_s a_1 = m_t a_2$ and $m_s a_1^2 = m_t a_2^2$. All of the elements m_s, m_t, a_1, and a_2 are not zero so that this pair of equations implies that $a_1 = a_2$ and $m_s = m_t$. It then follows from $x_1 + \alpha_i(m_s a_1) = x_2 + \alpha_i(m_t a_2)$ that $x_1 = x_2$ and similarly, $y_1 = y_2$. We conclude that $e(x_1, y_1, a_1, m_s) = e(x_2, y_2, a_2, m_t)$ and so $\mathcal{H}$ is linear. ☐

From Lemma 12 we see that $\mathcal{H}$ has $lq^2(q-1)$ edges and it is clear that $\mathcal{H}$ has rq^2 vertices. When $r=2$, $\mathcal{H}$ is a graph.

Example Let $r=2$, $l=1$, $q \geqslant 3$ be any power of an odd prime, $\alpha_1=0$, $\alpha_2=1$, and $m_1=1$. In this case, $\mathcal{H}$ is a $(q-1)$-regular bipartite graph with q^2 vertices in each part. It can be shown that $\mathcal{H}$ is isomorphic to a subgraph of the incidence graph of the projective plane $PG(2,q)$. In particular, $\mathcal{H}$ is C_4-free.

In the terminology of forbidden subgraphs, Lemma 12 tells us that $\mathcal{H}$ is C_2-free.

Lemma 13. *If $l=1$, then the hypergraph $\mathcal{H}$ is C_3-free.*

Proof. This is certainly true if $r=2$ as in this case $\mathcal{H}$ is a bipartite graph. Assume that $r \geqslant 3$ and suppose $\mathcal{H}$ contains a C_3. By Lemma 12, there are three distinct edges $e(x_1,y_1,a_1,m_1)$, $e(x_2,y_2,a_2,m_1)$, and $e(x_3,y_3,a_3,m_1)$ and integers $1 \leqslant i < j < k \leqslant r$ such that

$$(x_1+\alpha_i(m_1a_1), y_1+\alpha_i(m_1a_1^2), i) = (x_2+\alpha_i(m_1a_2), y_2+\alpha_i(m_1a_2^2), i),$$
$$(x_2+\alpha_j(m_1a_2), y_2+\alpha_j(m_1a_2^2), j) = (x_3+\alpha_j(m_1a_3), y_3+\alpha_j(m_1a_3^2), j),$$
$$(x_3+\alpha_k(m_1a_3), y_3+\alpha_k(m_1a_3^2), k) = (x_1+\alpha_k(m_1a_1), y_1+\alpha_k(m_1a_1^2), k).$$

The first equation represents the vertex in V_i that is the unique vertex in the intersection of the edges $e(x_1,y_1,a_1,m_1)$ and $e(x_2,y_2,a_2,m_1)$.

By considering the equations coming from the first components, we get

$$\begin{aligned} 0 &= (x_1-x_2)+(x_2-x_3)+(x_3-x_1) \\ &= m_1\alpha_i(a_2-a_1)+m_1\alpha_j(a_3-a_2)+m_1\alpha_k(a_1-a_3). \end{aligned}$$

Similarly, the equations from the second components give

$$0 = m_1\alpha_i(a_2^2-a_1^2)+m_1\alpha_j(a_3^2-a_2^2)+m_1\alpha_k(a_1^2-a_3^2).$$

By Lemma 10 with $\alpha=m_1\alpha_i$, $\beta=m_1\alpha_j$, and $\gamma=m_1\alpha_k$, we have $a_1=a_2=a_3$. Since

$$(x_1+\alpha_i(m_1a_1), y_1+\alpha_i(m_1a_1^2), i) = (x_2+\alpha_i(m_1a_2), y_2+\alpha_i(m_1a_2^2), i),$$

we obtain $x_1=x_2$ and $y_1=y_2$ which gives $e(x_1,y_1,a_1,m_1)=e(x_2,y_2,a_2,m_1)$, a contradiction. □

For the next sequence of lemmas we will require some additional notation and terminology. For $1 \leqslant i \neq j \leqslant r$, let $\mathcal{H}(V_i,V_j)$ be the bipartite graph with parts V_i and V_j where $(u,v,i) \in V_i$ is adjacent to $(u',v',j) \in V_j$ if and only if there is an edge $e \in E(\mathcal{H})$ such that

$$\{(u,v,i),(u',v',j)\} \subseteq e. \tag{14}$$

An equivalent way of defining adjacencies in $\mathcal{H}(V_i, V_j)$ is to say that (u, v, i) is adjacent to (u', v', j) if and only if there are elements $x, y \in \mathbb{F}_q$, $a \in \mathbb{F}_q^*$, and an $s \in \{1, 2, \ldots, l\}$ such that

$$u' = u + m_s(\alpha_j - \alpha_i)a \quad \text{and} \quad v' = v + m_s(\alpha_j - \alpha_i)a^2. \tag{15}$$

This is because if (14) holds with $e = e(x, y, a, m_s)$, then

$$u = x + \alpha_i(m_s a),\ v = y + \alpha_i(m_s a^2),\ u' = x + \alpha_j(m_s a), \text{ and } v' = y + \alpha_j(m_s a^2).$$

For three distinct integers i, j, and k with $1 \leqslant i, j, k \leqslant r$, let $\mathcal{H}(V_i, V_j, V_k)$ be the union of the graphs $\mathcal{H}(V_i, V_j)$, $\mathcal{H}(V_j, V_k)$, and $\mathcal{H}(V_k, V_i)$.

For any $x, y \in \mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the edge $e(x, y, a, m_s)$ in $\mathcal{H}$ is said to have *color* m_s. An edge f in the graph $\mathcal{H}(V_i, V_j)$ or $\mathcal{H}(V_i, V_j, V_k)$ is said to have *color* m_s if the unique edge e in $\mathcal{H}$ with $f \subseteq e$ has color m_s. The edge e is unique by Lemma 12.

Lemma 14. *For any $1 \leqslant i \neq j \leqslant r$ and $1 \leqslant s \leqslant l$, the edges of color m_s in the graph $\mathcal{H}(V_i, V_j)$ induce a $K_{2,2}$-free graph.*

Proof. Suppose $\{(u_1, v_1, i), (u_2, v_2, j), (u_3, v_3, i), (u_4, v_4, j)\}$ forms a $K_{2,2}$ in $\mathcal{H}(V_i, V_j)$ where each of the edges of this $K_{2,2}$ have color m_s. Using (15) as our condition for adjacency in $\mathcal{H}(V_i, V_j)$, we have

$$\begin{aligned} u_2 &= u_1 + m_s(\alpha_j - \alpha_i)a_1 = u_3 + m_s(\alpha_j - \alpha_i)a_2, \\ v_2 &= v_1 + m_s(\alpha_j - \alpha_i)a_1^2 = v_3 + m_s(\alpha_j - \alpha_i)a_2^2, \\ u_4 &= u_1 + m_s(\alpha_j - \alpha_i)a_3 = u_3 + m_s(\alpha_j - \alpha_i)a_4, \\ v_4 &= v_1 + m_s(\alpha_j - \alpha_i)a_3^2 = v_3 + m_s(\alpha_j - \alpha_i)a_4^2 \end{aligned}$$

for some $a_1, a_2, a_3, a_4 \in \mathbb{F}_q^*$. By the first and third set of equations,

$$m_s^{-1}(\alpha_j - \alpha_i)^{-1}(u_1 - u_3) = a_2 - a_1 = a_4 - a_3.$$

Similarly, by the second and fourth set of equations, $a_2^2 - a_1^2 = a_4^2 - a_3^2$. By Lemma 9, either $(a_1, a_4) = (a_2, a_3)$ or $(a_1, a_4) = (a_3, a_2)$.

If $a_1 = a_2$, then $u_1 = u_3$ by the first set of equations and $v_1 = v_3$ by the second set of equations. This implies (u_1, v_1, i) and (u_3, v_3, i) are the same vertex which is a contradiction.

If $a_1 = a_3$, then by taking differences of the first and third set of equations we get $u_2 = u_4$. By taking differences of the second and fourth set of equations we get $v_2 = v_4$. This implies that the vertices (u_2, v_2, j) and (u_4, v_4, j) are the same which is another contradiction. □

Lemma 15. *If $1 \leqslant i \neq j \leqslant r$, then for any $l \geqslant 1$, the graph $\mathcal{H}(V_i, V_j)$ is $K_{2,2l^2-l+1}$-free.*

Proof. If $l = 1$, then we are done by Lemma 14 as all of the edges in $\mathcal{H}(V_i, V_j)$ will have the same color, namely m_1.

Assume that $l \geqslant 2$ and suppose $u, v, w_1, \ldots, w_{2l^2-l+1}$ are the vertices of of $K_{2,2l^2-l+1}$ in $\mathcal{H}(V_i, V_j)$ with $u, v \in V_i$ and $w_1, \ldots, w_{2l^2-l+1} \in V_j$. Since $\frac{2l^2-l+1}{l} > 2l - 1$, there are at least $2l$ edges of the form $\{u, w_z\}$ that have the same color. Without loss of generality, assume that for $1 \leqslant z \leqslant 2l$, the edges $\{u, w_z\}$ have color m_1. Let $W = \{w_1, \dots, w_{2l}\}$. By Lemma 14, there cannot be two distinct edges, both with color m_1, that are incident with v and a vertex in W. Thus, at least $2l - 1$ of the edges between W and v have a color other than m_1. As $\frac{2l-1}{l-1} > 2$, there must be three edges between W and v that all have the same color. Without loss of generality, assume that $\{v, w_1\}$, $\{v, w_2\}$, and $\{v, w_3\}$ all have color m_2. Let $v = (x_v, y_v, i)$, $u = (x_u, y_u, i)$, and $w_z = (x_{w_z}, y_{w_z}, j)$ for $z \in \{1, 2, 3\}$. For each $z \in \{1, 2, 3\}$, there are elements $a_z, b_z \in \mathbb{F}_q^*$ with

$$x_{w_z} = x_u + m_1(\alpha_j - \alpha_i)a_z = x_v + m_2(\alpha_j - \alpha_i)b_z$$

and

$$y_{w_z} = y_u + m_1(\alpha_j - \alpha_i)a_z^2 = y_v + m_2(\alpha_j - \alpha_i)b_z^2.$$

From these equations we obtain

$$\begin{aligned} x_v - x_u = m_1(\alpha_j - \alpha_i)a_1 + m_2(\alpha_i - \alpha_j)b_1 &= m_1(\alpha_j - \alpha_i)a_2 + m_2(\alpha_i - \alpha_j)b_2 \\ &= m_1(\alpha_j - \alpha_i)a_3 + m_2(\alpha_i - \alpha_j)b_3 \end{aligned}$$

and

$$\begin{aligned} y_v - y_u = m_1(\alpha_j - \alpha_i)a_1^2 + m_2(\alpha_i - \alpha_j)b_1^2 &= m_1(\alpha_j - \alpha_i)a_2^2 + m_2(\alpha_i - \alpha_j)b_2^2 \\ &= m_1(\alpha_j - \alpha_i)a_3^2 + m_2(\alpha_i - \alpha_j)b_3^2. \end{aligned}$$

We want to apply Lemma 11 with $\alpha = m_1(\alpha_j - \alpha_i)$ and $\beta = m_2(\alpha_i - \alpha_j)$ but before doing so, we verify that we have satisfied the hypothesis of Lemma 11. Since $m_i \neq 0$, and $\alpha_i - \alpha_j \neq 0$, both α and β are not zero. If $\alpha + \beta = 0$, then

$$0 = m_1(\alpha_j - \alpha_i) + m_2(\alpha_i - \alpha_j) = \alpha_i(m_2 - m_1) - \alpha_j(m_2 - m_1)$$

so $\alpha_i(m_2 - m_1) = \alpha_j(m_2 - m_1)$. As m_1 and m_2 are distinct, $m_2 - m_1 \neq 0$ so $\alpha_i = \alpha_j$ which contradicts the fact that α_i and α_j are distinct. We conclude that $\alpha + \beta \neq 0$ and Lemma 11 applies so we may assume that $a_1 = b_1$ and $a_2 = b_2$. These two equalities together with

$$m_1(\alpha_j - \alpha_i)a_1 + m_2(\alpha_i - \alpha_j)b_1 = m_1(\alpha_j - \alpha_i)a_2 + m_2(\alpha_i - \alpha_j)b_2$$

give

$$(m_1 - m_2)(\alpha_j - \alpha_i)a_1 = (m_1 - m_2)(\alpha_j - \alpha_i)a_2.$$

Therefore, $a_1 = a_2$.

From the equations

$$x_{w_1} = x_u + m_1(\alpha_j - \alpha_i)a_1 \quad \text{and} \quad x_{w_2} = x_u + m_1(\alpha_j - \alpha_i)a_2$$

we get $x_{w_1} = x_{w_2}$. A similar argument gives $y_{w_1} = y_{w_2}$, thus

$$w_1 = (x_{w_1}, y_{w_1}, j) = (x_{w_2}, y_{w_2}, j) = w_2$$

which provides the needed contradiction. We conclude that $\mathcal{H}(V_i, V_j)$ is $K_{2,2l^2-l+1}$-free. $\square$

Lemma 16. *Let i, j, and k be distinct integers with $1 \leqslant i, j, k \leqslant r$. For any $l \geqslant 1$, the graph $\mathcal{H}(V_i, V_j, V_k)$ does not contain a $K_{2,2l^2+1}$ with one vertex in V_i, one vertex in V_j, and $2l^2 + 1$ vertices in V_k.*

Proof. We proceed as in the proof of Lemma 15. Suppose $\{u, v\}$ and $\{w_1, \ldots, w_{2l^2+1}\}$ are the parts of the $K_{2,2l^2+1}$ with $u \in V_i$, $v \in V_j$, and $w_1, \ldots, w_{2l^2+1} \in V_k$. As $\frac{2l^2+1}{l} > 2l$, we can assume that the edges $\{u, w_1\}, \ldots, \{u, w_{2l+1}\}$ all have the same color, say m_1. Since $\frac{2l+1}{l} > 2$, we can assume that at least three of the edges $\{v, w_1\}, \ldots, \{v, w_{2l+1}\}$ have the same color. Let $\{v, w_1\}, \{v, w_2\}$, and $\{v, w_3\}$ have color m_s. As in the proof of Lemma 15, we have elements $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}_q^*$ such that

$$\begin{aligned} m_1(\alpha_k - \alpha_i)a_1 + m_s(\alpha_j - \alpha_k)b_1 &= m_1(\alpha_k - \alpha_i)a_2 + m_s(\alpha_j - \alpha_k)b_2 \\ &= m_1(\alpha_k - \alpha_i)a_3 + m_s(\alpha_j - \alpha_k)b_3, \end{aligned}$$

and

$$\begin{aligned} m_1(\alpha_k - \alpha_i)a_1^2 + m_s(\alpha_j - \alpha_k)b_1^2 &= m_1(\alpha_k - \alpha_i)a_2^2 + m_s(\alpha_j - \alpha_k)b_2^2 \\ &= m_1(\alpha_k - \alpha_i)a_3^2 + m_s(\alpha_j - \alpha_k)b_3^2. \end{aligned}$$

If $s = 1$ (so $m_s = m_1$), then we apply Lemma 11 with $\alpha = m_1(\alpha_k - \alpha_i)$ and $\beta = m_1(\alpha_j - \alpha_k)$ noting that $\alpha + \beta = m_1(\alpha_j - \alpha_i) \neq 0$. If $s \neq 1$, then without loss of generality, assume that $s = 2$. We apply Lemma 11 with

$$\alpha = m_1(\alpha_k - \alpha_i) \text{ and } \beta = m_2(\alpha_j - \alpha_k).$$

Here we recall that by (13), the m_t's have been chosen so that $m_1(\alpha_k - \alpha_i) \neq m_2(\alpha_k - \alpha_j)$ so $\alpha + \beta \neq 0$. In both cases, we can apply Lemma 11 to get $a_1 = b_1$ and $a_2 = b_2$. The remainder of the proof is then identical to that of Lemma 15. □

Proof of the lower bound in Theorem 2 and Theorem 3. Let $r \geqslant 3$ be an integer and $l = 1$. Let $q \geqslant r$ be a power of an odd prime and $\alpha_1, \ldots, \alpha_r$ be distinct elements of $\mathbb{F}_q$. Let $m_1 = 1 \in \mathbb{F}_q$ and note that (13) holds for $\alpha_1, \ldots, \alpha_r$ and m_1 since in this case, (13) is equivalent to the statement that $\alpha_1, \ldots, \alpha_r$ are all different. Let $\mathcal{H}$ be the corresponding hypergraph defined at the beginning of Section 3.2. By Lemmas 12 and 13, $\mathcal{H}$ is $\{C_2, C_3\}$-free. Now we show that $\mathcal{H}$ is $K_{2,2r-3}$-free.

Suppose $\{u, v\}$ and $W = \{w_1, \ldots, w_{2r-3}\}$ are the parts of a $K_{2,2r-3}$ in $\mathcal{H}$. If $\{u, v\} \subset V_i$ for some $i \in \{1, 2, \ldots, r\}$, then by Lemma 14, $|V_j \cap W| \leqslant 1$ for each $j \in \{1, 2, \ldots, r\} \backslash \{i\}$. This is impossible since $2r - 3 > r - 1$ as $r > 2$. Now suppose $u \in V_i$ and $v \in V_j$ where $1 \leqslant i < j \leqslant r$. By Lemma 16, $|V_k \cap W| \leqslant 2$ for each $k \in \{1, 2, \ldots, r\} \backslash \{i, j\}$. Once again this is impossible since $2r - 3 > 2(r - 2)$. This shows that $\mathcal{H}$ is $K_{2,2r-3}$-free. The proof is completed by observing that $\mathcal{H}$ has q^2 vertices in each part $V_1, \ldots, V_r$ and $\mathcal{H}$ has $q^2(q-1)$ edges. □

Proof of Theorem 4. Let $r \geqslant 3$ and let l be any integer with $2l + 1 \geqslant r$. This assumption on l implies that

$$(r-2)(2l^2) \leqslant (r-1)(2l^2 - l). \tag{16}$$

Let q be a power of an odd prime chosen large enough so that there are r distinct elements $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q$ and l distinct elements $m_1, \ldots, m_l \in \mathbb{F}_q^*$ that satisfy condition (13). We claim that choosing $q \geqslant 2lr^3$ is sufficient for such elements to exist. Indeed, we first choose $\alpha_1, \ldots, \alpha_r$ so that these elements are all distinct. We then choose the m_z's. If we have chosen $m_1, \ldots, m_t$ so that (13) holds for $\alpha_1, \ldots, \alpha_r$ and $m_1, \ldots, m_t$, then as long as we choose m_{t+1} so that $m_{t+1} \neq m_z(\alpha_k - \alpha_j)(\alpha_k - \alpha_i)^{-1}$, then (13) holds for $\alpha_1, \ldots, \alpha_r$ and $m_1, \ldots, m_t, m_{t+1}$. There are at most tr^3 products of the form $m_z(\alpha_k - \alpha_j)(\alpha_k - \alpha_i)^{-1}$ with $z \in \{1, \ldots, t\}$ and $1 \leqslant i, j, k \leqslant r$ so $q \geqslant 2lr^3$ is enough to choose m_{t+1}.

Having chosen $\alpha_1, \ldots, \alpha_r$ and $m_1, \ldots, m_l$, let $\mathcal{H}$ be the corresponding hypergraph. By Lemma 12, $\mathcal{H}$ is C_2-free. Now we show that $\mathcal{H}$ is $K_{2,(r-1)(2l^2-l)+1}$-free.

Suppose $\{u, v\}$ and $W = \{w_1, \ldots, w_t\}$ are the parts of a $K_{2,t}$ in $\mathcal{H}$. If $\{u, v\} \subset V_i$ for some i, then by Lemma 15, $|V_j \cap W| \leqslant 2l^2 - l$ for each $j \in \{1, 2, \ldots, r\} \backslash \{i\}$ so $t \leqslant (r-1)(2l^2 - l)$. If $u \in V_i$ and $v \in V_j$ for some $1 \leqslant i < j \leqslant r$, then by Lemma 16, $|V_k \cap W| \leqslant 2l^2$ for each $k \in \{1, 2, \ldots, r\} \backslash \{i, j\}$ so $t \leqslant (r-2)(2l^2)$ thus by (16), $t \leqslant (r-1)(2l^2 - l)$. We conclude that $\mathcal{H}$ is $K_{2,(r-1)(2l^2-l)+1}$-free. The proof of Theorem 4 is completed by observing that $\mathcal{H}$ has rq^2 vertices and $lq^2(q-1)$ edges. □

4 Concluding Remarks and Acknowledgments

It was pointed out to the author by Cory Palmer that the argument used to prove Theorem 6 can be used to show that

$$\mathrm{ex}_r(n, \{C_2, K_{2,t+1}\}) \leqslant \frac{\sqrt{2(t+1)}}{r} n^{3/2} + \frac{n}{r}$$

for all $r \geqslant 3$ and $t \geqslant 1$. This shows that the lower bound in Theorem 4 gives the correct order of magnitude but determining the correct constant could be difficult. It is known that in the case of graphs, $\mathrm{ex}_2(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t} n^{3/2} + o(n^{3/2})$ (see Füredi [6]).

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