A Euclidean Ramsey result in the plane

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Abstract

An old question in Euclidean Ramsey theory asks, if the points in the plane are red-blue coloured, does there always exist a red pair of points at unit distance or five blue points in line separated by unit distances? An elementary proof answers this question in the affirmative.

1 Introduction

Many problems in Euclidean Ramsey theory ask, for some $d \in \mathbb{Z}^+$, if the d-dimensional Euclidean space $\mathbb{E}^d$ is coloured with $r \geqslant 2$ colours, does there exist a colour class containing some desired geometric structure? Research in Euclidean Ramsey theory was surveyed in [4–6] by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus; for a more recent survey, see Graham [7].

Say that two geometric configurations are congruent iff there exists an isometry (distance preserving bijection) between them. For $d \in \mathbb{Z}^+$, and geometric configurations F_1, F_2, let the notation $\mathbb{E}^d \to (F_1, F_2)$ mean that for any red-blue coloring of $\mathbb{E}^d$, either the red points contain a congruent copy of F_1, or the blue points contain a congruent copy of F_2.

For a positive integer i, denote by ℓ_i the configuration of i collinear points with distance 1 between consecutive points. One of the results in [5] states that

$$\mathbb{E}^2 \to (\ell_2, \ell_4). \tag{1}$$

In the same paper, it was asked if $\mathbb{E}^2 \to (\ell_2, \ell_5)$, or perhaps a weaker result holds: $\mathbb{E}^3 \to (\ell_2, \ell_5)$.

The result (1) was generalised by Juhász [10], who proved that if T_4 is any configuration of 4 points, then $\mathbb{E}^2 \to (\ell_2, T_4)$. Juhász [9] informed me that Iván's thesis [8] contains a proof that for any configuration T_5 of 5 points, $\mathbb{E}^3 \to (\ell_2, T_5)$ (which implies that $\mathbb{E}^3 \to (\ell_2, \ell_5)$). Arman and Tsaturian [1] proved that $\mathbb{E}^3 \to (\ell_2, \ell_6)$.

In this paper, it is proved that $\mathbb{E}^2 \to (\ell_2, \ell_5)$:

Theorem 1. *Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Then there exist five blue points that form an ℓ_5.*

The existence of a k, such that $\mathbb{E}^2 \not\to (\ell_2, \ell_k)$, was first noted by Erdős and Graham [3], who mention the upper bound of "10000000, more or less". A more precise bound for $k = 10^{10}$ follows from a recent result of Conlon and Fox [2], who showed that for all $n \geqslant 2$, $\mathbb{E}^n \not\to (\ell_2, \ell_{10^{5n}})$.

2 Proof of Theorem 1

The proof is by contradiction; it is assumed that there are no five blue points forming an ℓ_5. The following lemmas are needed.

Lemma 2. *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. If there is no blue ℓ_5, then there are no three blue points forming an equilateral triangle with side length 3 and with a red centre.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose that blue points A, B and C form an equilateral triangle with side length 3 and with red centre O. Consider the part of the unit triangular lattice shown in Figure 1(a). The points D, E, F, G are blue, since they are distance 1 apart from O. The point X is red; otherwise $XADEB$ is a red ℓ_5. Similarly, Y is red (to prevent red $YAFGC$). Then X and Y are two red points distance 1 apart, which contradicts the assumption. □

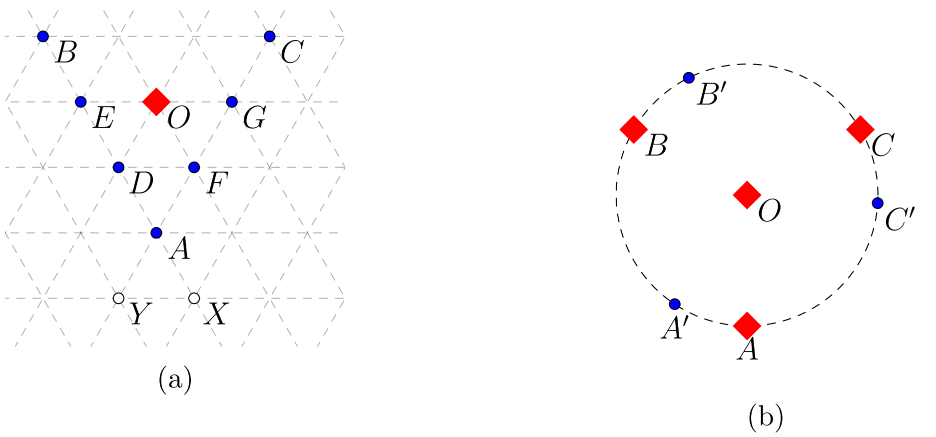

Figure 1: Red points are denoted by diamonds, blue points are denoted by discs.

Lemma 3. *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. If there is no blue ℓ_5, then there are no three red points forming an equilateral triangle with side length 3 and with a red centre.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose that blue points A, B and C form an equilateral triangle with side length 3 and with red centre O. Let A', B', C' be the images of A, B and C, respectively, under a rotation about O so that $AA' = BB' = CC' = 1$ (see Figure 1(b)). Then A', B', C' are blue and form an equilateral triangle with side length 3 and red center O, which contradicts the result of Lemma 2. □

Define $\mathfrak{T}_3$, $\mathfrak{T}_4$, $\mathfrak{T}_5$, $\mathfrak{T}_6$, $\mathfrak{T}_7$ to be the configurations of three, four, five, six and seven points (respectively), depicted in Figure 2 (all the smallest distances between the points are equal to $\sqrt{3}$).

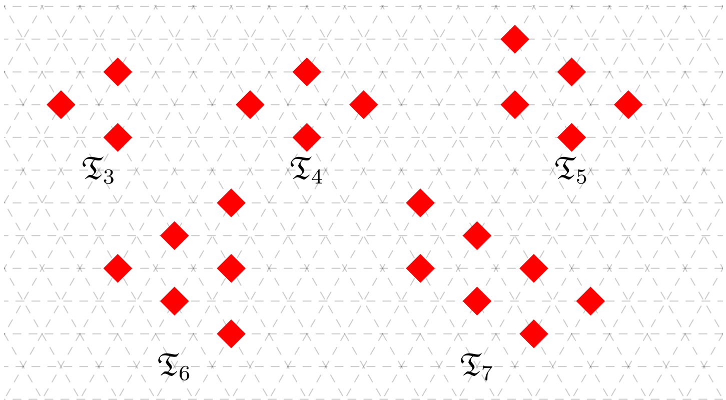

Figure 2

Lemma 4. *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. If there is no blue ℓ_5, then there are no seven red points forming a $\mathfrak{T}_7$.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose that A, B, C, D, E, F and G are red points forming a $\mathfrak{T}_7$ (as in Figure 3). Let X be the reflection of F in BC. Let X', A', F' be the images of X, A, F, respectively, under the clockwise rotation about B such that $XX' = AA' = FF' = 1$. Since A and F are red, A' and F' are blue. If X' is blue, then $X'A'F'$ is a blue equilateral triangle with side length 3 and red center B, which contradicts the result of Lemma 2. Therefore, X' is red.

Let X'', D'', F'' be the images of X, D, F, respectively, under the clockwise rotation about C such that $XX'' = DD'' = FF'' = 1$. Since D and F are red, D'' and F'' are blue. If X'' is blue, then $X''D''F''$ is a blue equilateral triangle with side length 3 and red center C, which contradicts the result of Lemma 2. Therefore, X'' is red. Consider the clockwise rotation through 60° about X. This rotation sends C to B, and so every point on the circle with radius $\sqrt{3}$ centered at C is sent to the corresponding point on the circle with radius $\sqrt{3}$ centered at B; in particular, X' can be viewed as the image of X''. Therefore $XX'X''$ is a unit equilateral triangle, hence $X'X''$ is a red ℓ_2, which contradicts the assumption of the lemma. □

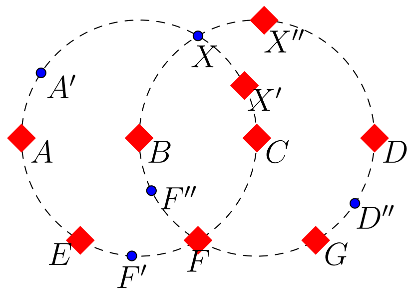

Figure 3

Lemma 5. *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. Let A, B, C be three red points forming a $\mathfrak{T}_3$. If there is no blue ℓ_5, then there exists a red $\mathfrak{T}_6$ that contains $\{A, B, C\}$ as a subset.*

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Let A, B, C be three red points forming a $\mathfrak{T}_3$. Consider the unit triangular lattice depicted in Figure 4.

Suppose that there is no red point D such that A, B, C, D form a $\mathfrak{T}_4$. Then points X, Y, Z are blue. Points E, F, G, H, I, J are blue, since each of them is distance 1 apart from a red point. If the point K is red, then the points L and M are blue and $LMYGH$ is a blue ℓ_5. Therefore, K is blue. Then N is red (otherwise $KJIZN$ is a blue ℓ_5), hence P and Q are blue, which leads to a blue ℓ_5 $PQFEX$. A contradiction is obtained, therefore there exists a red point D such that A, B, C, D form a $\mathfrak{T}_4$.

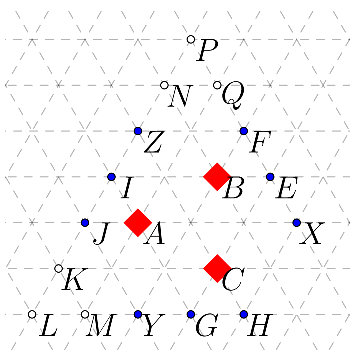

Figure 4

Let A, B, C, D form a red $\mathfrak{T}_4$. Consider the part of the unit triangular lattice depicted in Figure 5. Suppose that there is no red point E such that A, B, C, D, E form a $\mathfrak{T}_5$. Then the points X, F and G are blue. Points H, I, K, L, M, N are blue, since each of them is distance 1 apart from a red point. Point P is red (otherwise $FHIGP$ is a blue ℓ_5), therefore Q and R are blue. Then X, N, M, Q, R form a blue ℓ_5, which gives a contradiction. Hence, there exists a red point E such that A, B, C, D, E form a $\mathfrak{T}_5$.

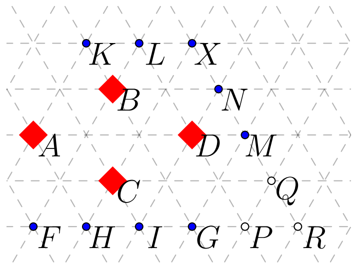

Figure 5

Let A, B, C, D, E form a $\mathfrak{T}_5$ (Figure 6). Suppose that F is blue. By Lemma 3, points X and Y are blue (otherwise X, E, C (Y, A, D) form a red triangle with side length 3 and red center B). Points G, H, I, J, K, L, M, N are blue, since each one of them is at distance 1 from a red point. If point P is blue, then Q is red (otherwise $QPKLF$ is a blue ℓ_5), U and T are blue and form a blue ℓ_5 with points G, H and X. Therefore, P is red. Similarly, R is red (otherwise S is red and $VWJIY$ is a blue ℓ_5). Then A, B, C, D, E, P and R form a red $\mathfrak{T}_7$, which is not possible by Lemma 4. Therefore, F is red and A, B, C, D, E, F form a red $\mathfrak{T}_6$. □

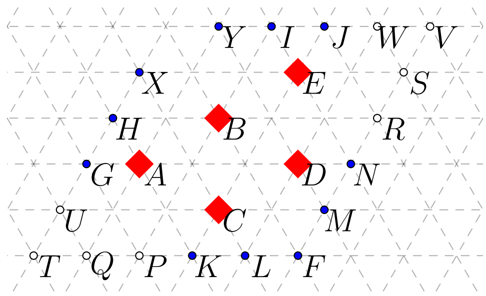

Figure 6

Lemma 6. *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. Let $\mathfrak{L}$ be a unit triangular lattice that contains three red points forming a $\mathfrak{T}_3$. If there is no blue ℓ_5 in $\mathbb{E}^2$, then the colouring of $\mathfrak{L}$ is unique (up to translation or rotation by a multiple of $60°$), and is depicted in Figure 7.*

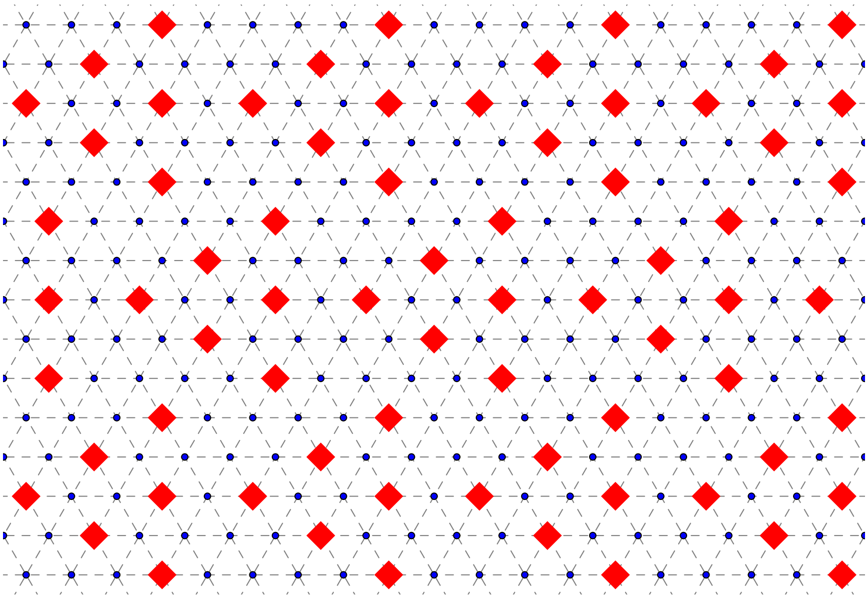

Figure 7

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5. Suppose there exist three red points of $\mathfrak{L}$ that form a $\mathfrak{T}_3$. By Lemma 5, it may be assumed that there is a red $\mathfrak{T}_6$. Denote its points by A, B, C, D, E, F (see Figure 8). It will be proved that the translate $A'B'C'D'E'F'$ of $ABCDEF$ by the vector of length 5 collinear to $\overrightarrow{AD}$ is red.

Consider the points shown in Figure 8. Since A, D and F are red, by Lemma 3, I is blue. Since C, F and D are red, by Lemma 3, J is blue. Points K, L, M, N are blue, since each one is distance 1 apart from a red point. If R is red, then both P and Q are blue and form a blue ℓ_5 with K, L and I. Therefore R is blue. Then the point A' is red (otherwise $A'JNMR$ is a blue ℓ_5).

Since S_1, S_2, S_3, S_4 are blue (as distance 1 apart from red points D and A'), B' is red. Similarly, F' is red. Points V and W are blue as they are distance 1 apart from C. Points U is blue by Lemma 3 (since A, D and B are red). If X is red, then X_1 and X_2 are blue and a blue ℓ_5 $UVWX_1X_2$ is formed. Therefore, X is blue. Similarly, Y is blue. By Lemma 5, $A'B'F'$ must be contained in a red $\mathfrak{T}_6$, and since X and Y are blue, the only possible such $\mathfrak{T}_6$ is $A'B'C'D'E'F'$. Hence, A', B', C', D', E', F' are blue.

Similarly, the translates of $ABCDEF$ by vectors of length 5 collinear to $\overrightarrow{EB}$ and $\overrightarrow{CF}$ are red. By repeatedly applying the same argument to the new red translates, it can be seen that all the translates of $ABCDEF$ by a multiple of 5 in $\mathfrak{L}$ are red. All the other points are blue, as each one is distance 1 apart from a red point. Hence, the colouring as in Figure 7 is obtained. □

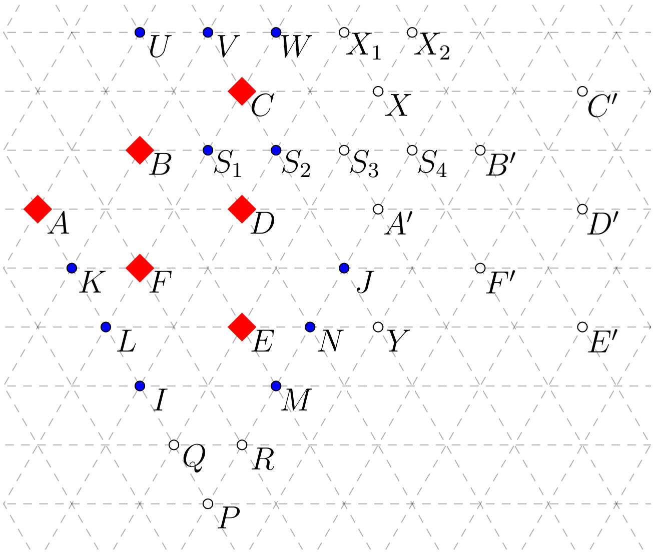

Figure 8

Lemma 7. *Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red ℓ_2. Let $\mathfrak{L}$ be a unit triangular lattice that does not contain three red points forming a $\mathfrak{T}_3$. If there is no blue ℓ_5 in $\mathbb{E}^2$, then the colouring of $\mathfrak{L}$ is unique (up to translation or rotation by a multiple of* 60°*), and is depicted in Figure 9.*

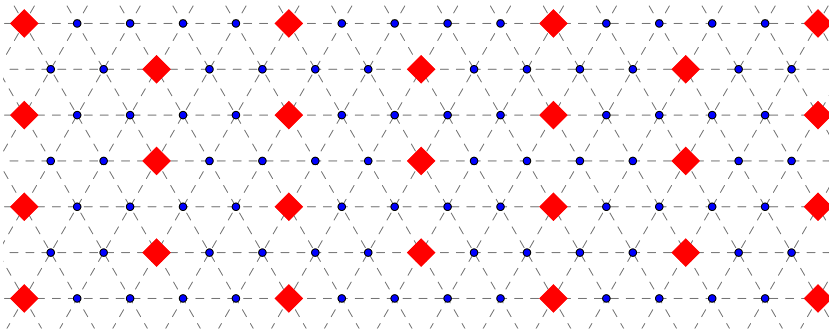

Figure 9

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red ℓ_2 and no blue ℓ_5.

If $\mathfrak{L}$ does not contain a red point, then any ℓ_5 is blue, therefore $\mathfrak{L}$ contains a red point A. By Lemma 2, one of the points of $\mathfrak{L}$ at distance $\sqrt{3}$ to A is red (otherwise the three such points form a blue triangle with side length 3 and red centre A). Denote this point by B (Figure 10). Since $\mathfrak{L}$ does not contain a red $\mathfrak{T}_3$, the points D and G are blue. Points E, F, I, H, K, J are blue, since they are distance 1 apart from B. Then the point B'

is red (otherwise blue ℓ_5 $DEFGB'$ is formed). Point N is 1 apart from B', hence blue. Then C and A' are red (otherwise a blue ℓ_5 is formed).

By repeating the same argument for points B and C, B and A (instead of A and B), and so on, it can be shown that any node of $\mathfrak{L}$ on the line AB is red. Similarly, since A' and B' are both red, any node of $\mathfrak{L}$ on the line $A'B'$ is red. By the same argument, A'', B'' and any node on the line containing them is red; A''', B''' and any node on the line containing them is red, and so on. By colouring all point distance 1 apart form red points blue, the colouring in Figure 9 is obtained. □

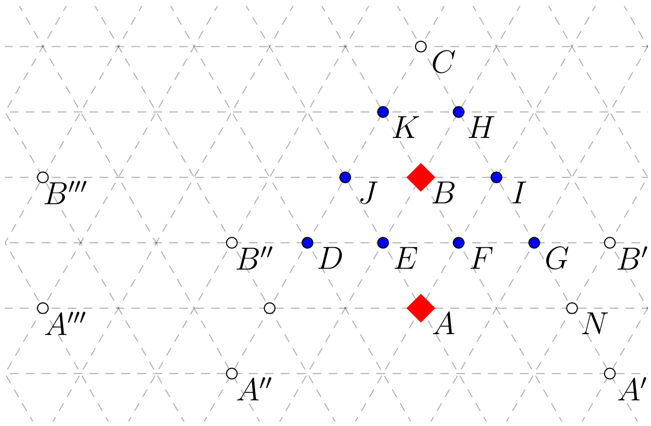

Figure 10

Proof of Theorem 1. Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Suppose that there are no five blue points that form an ℓ_5. Then there is a red point A. Consider two points B and C, both distance 5 apart from A, such that $|BC| = 1$. At least one of the points B and C (say, B) is blue. Consider the unit triangular lattice $\mathfrak{L}$ that contains A and B. By Lemma 6 and Lemma 7, $\mathfrak{L}$ is coloured either as in Figure 7 or as in Figure 9. But neither one of the colourings contains two points of different colour distance 5 apart, which gives a contradiction. Therefore, there exist five blue points that form an ℓ_5. □

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