Note on a Ramsey theorem for posets with linear extensions

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Abstract

In this note we consider a Ramsey type result for partially ordered sets. In particular, we give an alternative short proof of a theorem for a posets with multiple linear extensions recently obtained by Solecki and Zhao (2017).

Keywords: Ramsey theorem; posets

1 Preliminary definitions

A poset is a pair (X, P^X), where X is a set and P^X is a partial order on X. We consider partial orders that are strict, i.e. not reflexive.

We say that a partial order L^X on X *extends* a partial order P^X on X if for all $x, y \in X$

$$xP^X y \Rightarrow xL^X y.$$

If (X, P^X) is a poset and $U \subset X$ we denote by $P^X|_U$ the *restriction* of P^X onto U.

Below, we consider collections $\mathcal{L}_k^X = (L_1^X, L_2^X, \ldots, L_k^X)$, where each of L_i^X is a linear order on X.

Definition 1. We denote by $PL^{(k)}$ the set consisting of all triplets $(X, P^X, \mathcal{L}_k^X)$, where (X, P^X) is a poset and each L_i^X for $i \in [k]$ is a linear order that extends P^X.

Definition 2. Let $\mathcal{X}, \mathcal{Y} \in PL^{(k)}$, where $\mathcal{X} = (X, P^X, \mathcal{L}_k^X)$ and $\mathcal{Y} = (Y, P^Y, \mathcal{L}_k^Y)$. We write $\mathcal{X} \subseteq \mathcal{Y}$ if

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- $X \subseteq Y$ and $P^Y|_X$ extends P^X.
- $L_i^Y|_X = L_i^X$ for all $i \in [k]$.

Definition 3. Let $\mathcal{X}, \mathcal{Y} \in PL^{(k)}$, where $\mathcal{X} = (X, P^X, \mathcal{L}_k^X)$ and $\mathcal{Y} = (Y, P^Y, \mathcal{L}_k^Y)$. We say that a mapping $\pi : X \to Y$ is order preserving for $\mathcal{X}$ and $\mathcal{Y}$ if for any $i \in [k]$ and any $x, y \in X$ we have

$$xL_i^X y \Leftrightarrow \pi(x)L_i^Y \pi(y) \quad \text{and} \quad xP^X y \Leftrightarrow \pi(x)P^Y \pi(y).$$

Definition 4. We say that π is an isomorphism between $\mathcal{X} \in PL^{(k)}$ and $\tilde{\mathcal{X}} \in PL^{(k)}$ if it is order preserving bijection. We say that $\mathcal{X} \in PL^{(k)}$ is isomorphic to $\tilde{\mathcal{X}} \in PL^{(k)}$ if there is an isomorphism between $\mathcal{X}$ and $\tilde{\mathcal{X}}$.

Definition 5. Let $k > 0$ and $\mathcal{X}, \mathcal{Y} \in PL^{(k)}$. We say that $\tilde{\mathcal{X}} \in PL^{(k)}$ is a copy of $\mathcal{X}$ in $\mathcal{Y}$ if $\tilde{\mathcal{X}} \subseteq \mathcal{Y}$ and $\tilde{\mathcal{X}}$ is isomorphic to $\mathcal{X}$. For $\mathcal{X}, \mathcal{Y} \in PL^{(k)}$ denote by $\binom{\mathcal{Y}}{\mathcal{X}}$ the set of all copies of $\mathcal{X}$ in $\mathcal{Y}$.

For any $\tilde{\mathcal{X}} \in \binom{\mathcal{Y}}{\mathcal{X}}$ there is unique order preserving mapping $\pi : X \to \tilde{X}$. On other hand, any order preserving mapping $\pi : X \to Y$ induces a copy $\tilde{\mathcal{X}} = \pi(\mathcal{X}) \in \binom{\mathcal{Y}}{\mathcal{X}}$. We identify each $\tilde{\mathcal{X}} \in \binom{\mathcal{Y}}{\mathcal{X}}$ with corresponding order preserving mapping π and will say that π is a copy of $\mathcal{X}$ in $\mathcal{Y}$ instead of saying that $\tilde{\mathcal{X}}$ is a copy of $\mathcal{X}$ in $\mathcal{Y}$ with corresponding order preserving mapping π.

The following theorem follows from the result of [3] (see [2] and [4]). Different proof of Theorem 6 was also given by Sokić [9] (using results of [5] and [1]).

Theorem 6. *For any integer r and any $\mathcal{X}, \mathcal{Y} \in PL^{(1)}$ there is $\mathcal{Z} \in PL^{(1)}$, such that for any r-colouring of set $\binom{\mathcal{Z}}{\mathcal{X}}$ there is $\tilde{\mathcal{Y}}$, a copy of $\mathcal{Y}$ in $\mathcal{Z}$, such that $\binom{\tilde{\mathcal{Y}}}{\mathcal{X}}$ is monochromatic.*

Next theorem is a product version of the Theorem 6, that we are going to use in Section 3. Proof of this theorem is based on a standard folkloristic product argument. For similar results of this type see e.g. [6].

Theorem 7. *For any $\mathcal{X}_i, \mathcal{Y}_i \in PL^{(1)}$ with $i \in [k]$ there are $\mathcal{Z}_i \in PL^{(1)}$ with $i \in [k]$, such that for any 2-colouring of set $\binom{\mathcal{Z}_1}{\mathcal{X}_1} \times \cdots \times \binom{\mathcal{Z}_k}{\mathcal{X}_k}$ there are $\tilde{\mathcal{Y}}_i$, a copies of $\mathcal{Y}_i$ in $\mathcal{Z}_i$ for $i \in [k]$, such that $\binom{\tilde{\mathcal{Y}}_1}{\mathcal{X}_1} \times \cdots \times \binom{\tilde{\mathcal{Y}}_k}{\mathcal{X}_k}$ is monochromatic.*

Based on Theorem 7, in Section 3 we provide a proof of the following result, first obtained in [7].

Theorem 8. *For any integer k any $\mathcal{A}, \mathcal{B} \in PL^{(k)}$ there is $\mathcal{C} \in PL^{(k)}$, such that for any colouring 2-colouring of set $\binom{\mathcal{C}}{\mathcal{A}}$ there is $\tilde{\mathcal{B}}$, a copy of $\mathcal{B}$ in $\mathcal{C}$, such that $\binom{\tilde{\mathcal{B}}}{\mathcal{A}}$ is monochromatic.*

To distinguish between the objects of $PL^{(1)}$, which play a special role in our proof, and $PL^{(k)}$ for $k \geqslant 2$, from now on, we use letters $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ for elements of $PL^{(1)}$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ for elements of $PL^{(k)}$.

For the ease of notation we will give a proof of Theorem 8 for case $k = 2$. The proof of the general case follows the same lines (and is accessible on arxiv.org).

2 Properties of join and canonical copies

First, we define the join of two elements of $PL^{(1)}$.

Definition 9. Let $\mathcal{Z}_i = (Z_i, P^{Z_i}, L^{Z_i}) \in PL^{(1)}$ for $i = 1, 2$. Define $\mathcal{C} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2$ by

$$\mathcal{C} = (Z_1 \times Z_2, <_{\mathcal{C}}, <_{lx_1}, <_{lx_2}),$$

where $Z_1 \times Z_2$ is Cartesian product of sets Z_1 and Z_2, $<_{\mathcal{C}}$ is a partial order and $<_{lx_1}, <_{lx_2}$ are linear orders on $Z_1 \times Z_2$ defined by:

$$(x_1, x_2) <_{\mathcal{C}} (y_1, y_2) \Leftrightarrow x_1 P^{Z_1} y_1 \text{ and } x_2 P^{Z_2} y_2 \ ,$$

$$(x_1, x_2) <_{lx_1} (y_1, y_2) \Leftrightarrow x_1 L^{Z_1} y_1 \text{ or } \{x_1 = y_1 \text{ and } x_2 L^{Z_2} y_2 \ \},$$
$$(x_1, x_2) <_{lx_2} (y_1, y_2) \Leftrightarrow x_2 L^{Z_2} y_2 \text{ or } \{x_2 = y_2 \text{ and } x_1 L^{Z_1} y_1 \ \}.$$

We say that $\mathcal{Z}_1 \sqcup \mathcal{Z}_2$ is the join of $\mathcal{Z}_1$ and $\mathcal{Z}_2$.

Note, that for $\mathcal{Z}_1, \mathcal{Z}_2 \in PL^{(1)}$ we have that $\mathcal{Z}_1 \sqcup \mathcal{Z}_2 \in PL^{(2)}$. Indeed, since both L^{Z_i} extend P^{Z_i} we infer that both $<_{lx_i}$ also extend $<_{\mathcal{C}}$ for $i = 1, 2$.

Claim 10. *Let $\mathcal{Z}_i = (Z_i, P^{Z_i}, L^{Z_i}) \in PL^{(1)}$ for $i = 1, 2$ and let $\mathcal{B} = (Y, P^Y, L_1^Y, L_2^Y) \in PL^{(2)}$. Set $\mathcal{C} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2$ and let $\pi_i : Y \to Z_i$ be a copy of $\mathcal{Y}_i = (Y, P^Y, L_i^Y)$ in $\mathcal{Z}_i$ for $i = 1, 2$. Then the image of the mapping $\pi : Y \to Z_1 \times Z_2$, defined by*

$$\pi(y) = (\pi_1(y), \pi_2(y))$$

for each $y \in Y$, is a copy of $\mathcal{B}$ in $\binom{\mathcal{C}}{\mathcal{B}}$.

Remark 11.

- We say that the image of the mapping π from Claim 10, is a *canonical* copy of $\mathcal{B}$ in $\mathcal{C} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2$.
- By $\binom{\mathcal{C}}{\mathcal{B}}_{can} \subseteq \binom{\mathcal{C}}{\mathcal{B}}$ we denote a set of all canonical copies of $\mathcal{B}$ in $\mathcal{C}$.

Proof. We need to verify that $\pi : Y \to Z_1 \times Z_2$ is order preserving for $\mathcal{A}$ and $\mathcal{C}$. Indeed, we observe that if $x, y \in Y$, then fact that $\pi_i : Y \to Z_i$ preserves P^Y for $i = 1, 2$ combined with definition of $\mathcal{Z}_1 \sqcup \mathcal{Z}_2$ yields

$$xP^Y y \Leftrightarrow \begin{array}{c} \pi_1(x) P^{Z_1} \pi_1(y), \\ \pi_2(x) P^{Z_2} \pi_2(y) \end{array} \Leftrightarrow \pi(x) <_{\mathcal{C}} \pi(y).$$

Since π_i preserves L_i^Y for $i = 1, 2$, we have

$$x L_i^Y y \Leftrightarrow \pi_i(x) L_i^{Z_i} \pi_i(y) \Leftrightarrow \pi(x) <_{lx_i} \pi(y)$$

for $i = 1, 2$. Hence, π preserves P^Y and L_i^Y for $i = 1, 2$. $\square$

For the rest of this section we assume that $\mathcal{C} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2 = (Z_1 \times Z_2, <_{\mathcal{C}}, <_{lx_1}, <_{lx_2})$, $\mathcal{A} = (X, P^X, L_1^X, L_2^X)$ and $\mathcal{B} = (Y, P^Y, L_1^Y, L_2^Y)$.

Fact 12. *By construction,* $\binom{\mathcal{C}}{\mathcal{A}}_{can}$ *is in 1-1 correspondence with the set* $\binom{\mathcal{Z}_1}{\mathcal{X}_1} \times \binom{\mathcal{Z}_2}{\mathcal{X}_2}$ *and the function* $\lambda : (\pi_1(X), \pi_2(X)) \mapsto \pi(X)$ *is the bijection between sets* $\binom{\mathcal{Z}_1}{\mathcal{X}_1} \times \binom{\mathcal{Z}_2}{\mathcal{X}_2}$ *and* $\binom{\mathcal{C}}{\mathcal{A}}_{can}$*.*

The following Claim states that if π is a canonical copy of $\mathcal{B}$ in $\mathcal{C}$ and $\tilde{\mathcal{A}}$ is a copy of $\mathcal{A}$ in $\mathcal{B}$, then $\pi(\tilde{\mathcal{A}})$ is a canonical copy of $\mathcal{A}$ in $\mathcal{C}$.

Claim 13. *If* $\pi \in \binom{\mathcal{C}}{\mathcal{B}}_{can}$ *and* $\tau \in \binom{\mathcal{B}}{\mathcal{A}}$*, then* $\sigma = \pi \circ \tau \in \binom{\mathcal{C}}{\mathcal{A}}_{can}$*.*

Proof. Since $\pi : Y \to Z_1 \times Z_2$ is a canonical copy, we have that $\pi = (\pi_1, \pi_2)$, where $\pi_i : Y \to Z_i$ are copies of Y in Z_i for $i = 1, 2$. Define $\sigma_i = \pi_i \circ \tau$ for $i = 1, 2$. It is sufficient to prove that for $i = 1, 2$ σ_i is order preserving for $\mathcal{X}$ and $\mathcal{Z}_i$.

Indeed, since τ preserves P^X, L_1^X, L_2^X and that π_i preserves P^Y, L_i^Y for $i = 1, 2$, we have for any $x, y \in X$ and for $i = 1, 2$

$$xP^X y \Leftrightarrow \tau(x)P^Y\tau(y) \Leftrightarrow \pi_i(\tau(x))P^{Z_i}\pi_i(\tau(y)) \Leftrightarrow \sigma_i(x)P^{Z_i}\sigma_i(y),$$

$$xL_i^X y \Leftrightarrow \tau(x)L_i^Y\tau(y) \Leftrightarrow \pi_i(\tau(x))L^{Z_i}\pi_i(\tau(y)) \Leftrightarrow \sigma_i(x)L^{Z_i}\sigma_i(y).$$

Consequently, for $i = 1, 2$, σ_i is order preserving for $\mathcal{X}$ and $\mathcal{Z}_i$, and $\sigma = (\sigma_1, \sigma_2)$ is a canonical copy of $\mathcal{A}$ in $\mathcal{C}$. $\square$

Our final Claim states that if $\tilde{\mathcal{B}}$ is a canonical copy of $\mathcal{B}$ in $\mathcal{C}$, and $\tilde{\mathcal{A}}$ is a copy of $\mathcal{A}$ in $\tilde{\mathcal{B}}$, then $\tilde{\mathcal{A}}$ is a canonical copy of $\mathcal{A}$ in $\mathcal{C}$.

Claim 14. *If* $\pi \in \binom{\mathcal{C}}{\mathcal{B}}_{can}$ *and* $\sigma \in \binom{\pi(\mathcal{B})}{\mathcal{A}}$*, then* $\sigma \in \binom{\mathcal{C}}{\mathcal{A}}_{can}$*.*

Proof. Since π is an isomorphism between $\mathcal{B}$ and $\pi(\mathcal{B})$, the mapping π^{-1} exists and is order preserving for $\pi(\mathcal{B})$ and $\mathcal{B}$. Since $\sigma : \mathcal{A} \to \sigma(\mathcal{A}) \subseteq \pi(\mathcal{B})$ and $\pi^{-1} : \pi(\mathcal{B}) \to \mathcal{B}$, the mapping $\tau = \pi^{-1} \circ \sigma : \mathcal{A} \to \mathcal{B}$ is well defined. Moreover, σ and π^{-1} are order preserving, so is also τ. Finally, Claim 13 applied for π and τ gives that $\pi \circ \tau = \sigma$ is a canonical copy of $\mathcal{A}$. $\square$

3 Proof of Theorem 8

Let $\mathcal{A} = (X, P^X, L_1^X, L_2^X)$ and $\mathcal{B} = (Y, P^Y, L_1^Y, L_2^Y)$ be given. Applying Theorem 7 with $\mathcal{X}_i = (X, P^X, L_i^X)$ for $i = 1, 2$ and $\mathcal{Y}_i = (Y, P^Y, L_i^Y)$ for $i = 1, 2$ we obtain $\mathcal{Z}_i = (Z_i, P^{Z_i}, L_i^{Z_i})$ for $i = 1, 2$.

Set $\mathcal{C} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2$. Let $\chi : \binom{\mathcal{C}}{\mathcal{A}} \to \{red, blue\}$ be a colouring. Since $\binom{\mathcal{C}}{\mathcal{A}}_{can} \subseteq \binom{\mathcal{C}}{\mathcal{A}}$, colouring χ induces $\{red, blue\}$ colouring of $\binom{\mathcal{C}}{\mathcal{A}}_{can}$. By Fact 12, $\binom{\mathcal{Z}_1}{\mathcal{X}_1} \times \binom{\mathcal{Z}_2}{\mathcal{X}_2}$ and $\binom{\mathcal{C}}{\mathcal{A}}_{can}$ are in 1-1 correspondence and thus χ induces a colouring of $\binom{\mathcal{Z}_1}{\mathcal{X}_1} \times \binom{\mathcal{Z}_2}{\mathcal{X}_2}$. By a choice of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (recall that $\mathcal{Z}_i \in PL^{(1)}$, $i = 1, 2$) there are $\tilde{Y}_i \in \binom{\mathcal{Z}_i}{\mathcal{Y}_i}$ for $i = 1, 2$, such that

$\binom{\tilde{\mathcal{Y}}_1}{\mathcal{X}_1} \times \binom{\tilde{\mathcal{Y}}_2}{\mathcal{X}_2}$ is monochromatic and w.l.o.g we assume that all elements of $\binom{\tilde{\mathcal{Y}}_1}{\mathcal{X}_1} \times \binom{\tilde{\mathcal{Y}}_2}{\mathcal{X}_2}$ are red.

Let $\pi_i : \mathcal{Y}_i \to \mathcal{Z}_i$ be a copy of $\mathcal{Y}_i$ in $\mathcal{Z}_i$, such that $\pi_i(\mathcal{Y}_i) = \tilde{\mathcal{Y}}_i$ for $i = 1, 2$. Then, by Claim 10, the mapping $\pi : Y \to Z_1 \times Z_2$ defined by $\pi(y) = (\pi_1(y), \pi_2(y))$ is a canonical copy of $\mathcal{B}$ in $\mathcal{C}$ (see Remark 11) i.e. $\pi \in \binom{\mathcal{C}}{\mathcal{B}}_{can}$. Let $\sigma \in \binom{\pi(\mathcal{B})}{\mathcal{A}}$, then, by Claim 14, $\sigma \in \binom{\pi(\mathcal{B})}{\mathcal{A}}_{can}$.

Therefore, σ is of the form $\sigma(x) = (\sigma_1(x), \sigma_2(x))$, where $\sigma_1 \in \binom{\tilde{\mathcal{Y}}_1}{\mathcal{X}_1}$ and $\sigma_2 \in \binom{\tilde{\mathcal{Y}}_2}{\mathcal{X}_2}$. Since all elements of $\binom{\tilde{\mathcal{Y}}_1}{\mathcal{X}_1} \times \binom{\tilde{\mathcal{Y}}_2}{\mathcal{X}_2}$ are red, we get that the pair (σ_1, σ_2) and σ itself is red. Consequently, every element of $\binom{\pi(\mathcal{B})}{\mathcal{A}}$ is colored red.

Concluding remarks

We chose to present the argument for $k = 2$ for its notational ease. With the concept of join of two posets replaced with join of k posets, as in definition below, the proof follows the line of the argument presented in this note.

Definition 15. Let $\mathcal{Z}_i = (Z_i, P^{Z_i}, L^{Z_i}) \in PL^{(1)}$ for $i \in [k]$ and set $C = \Pi_{i=1}^k Z_i$.

Define partial order $<_C$ on set C by $\overline{x} <_C \overline{y}$ if $x_i P^{Z_i} y_i$ for all $i \in [k]$.

For all $i \in [k]$ define shifted lexicographic orders $<_{lx_i}$ on set $\Pi_{i=1}^k Z_i$, by

$$\overline{x} <_{lx_i} \overline{y} \Leftrightarrow x_{i+\delta} L^{Z_{i+\delta}} y_{i+\delta},$$

where δ is the smallest non-negative number j, for which $x_{i+j} \neq y_{i+j}$ (with addition mod k). Let $\mathcal{L}_k^C = (<_{lx_1}, <_{lx_2}, \ldots, <_{lx_k})$. Then the join of $\mathcal{Z}_1, \ldots, \mathcal{Z}_k$ is

$$\mathcal{C} = (C, <_C, \mathcal{L}_k^C).$$

During preparation of this paper it was brought to our attention that Theorem 8 also follows from the results of Sokić [8]. Alternative proof of Theorem 8 can be deduced from Theorem 1 in [8] and follows the same steps as the proof presented in this note.

The original version of this note is available on arxiv.org.

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