

Disjoint Cycles of Different Lengths in Graphs and Digraphs*

Julien Bensmail[†]

Université Côte d'Azur, CNRS, Inria, I3S, France

`julien.bensmail@unice.fr`

Ararat Harutyunyan[‡]

LAMSADE, CNRS

Université Paris-Dauphine, PSL Research University
75016 Paris, France

`ararat.harutyunyan@lamsade.dauphine.fr`

Ngoc Khang Le

Laboratoire d'Informatique du Parallélisme
École Normale Supérieure de Lyon
69364 Lyon Cedex 07, France
`ngoc-khang.le@ens-lyon.fr`

Binlong Li[§]

Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi 710072, P.R. China
`libinlong@mail.nwpu.edu.cn`

Nicolas Lichiardopol

Lycée A. de Craponne
Salon, France

Submitted: Mar 25, 2017; Accepted: Nov 22, 2017; Published: Dec 8, 2017

Mathematics Subject Classifications: 05C07, 05C20, 05C38

Abstract

In this paper, we study the question of finding a set of k vertex-disjoint cycles (resp. directed cycles) of distinct lengths in a given graph (resp. digraph). In the context of undirected graphs, we prove that, for every $k \geq 1$, every graph with minimum degree at least $\frac{k^2+5k-2}{2}$ has k vertex-disjoint cycles of different lengths, where the degree bound is best possible. We also consider other cases such as when

*This paper is dedicated to the memory of Nicolas Lichiardopol.

[†]Supported by ERC Advanced Grant GRACOL, project no. 320812.

[‡]Supported by an FQRNT postdoctoral research grant and CIMI research fellowship.

[§]Supported by NSFC (11601429, 11671320) and the Natural Science Foundation of Shaanxi Province (2016JQ1002, 2014JK1353).

the graph is triangle-free, or the k cycles are required to have different lengths modulo some value r . In the context of directed graphs, we consider a conjecture of Lichiardopol concerning the least minimum out-degree required for a digraph to have k vertex-disjoint directed cycles of different lengths. We verify this conjecture for tournaments, and, by using the probabilistic method, for some regular digraphs and digraphs of small order.

Keywords: vertex-disjoint cycles; different lengths; minimum degree

1 Introduction

In this paper, we study degree conditions guaranteeing the existence in a graph (resp. digraph) of a certain number of vertex-disjoint cycles (resp. directed cycles) whose lengths satisfy particular properties. More precisely, not only we want cycles (resp. directed cycles) being vertex-disjoint, but we also require their lengths to be different somehow. Namely, we first ask for the lengths to be different only, but then also request additional properties on the lengths such as having the same remainder to some modulo.

After giving some preliminary results in Section 2, we start in Section 3 by studying how the number of vertex-disjoint cycles of different lengths in an undirected graph behaves in relation to the minimum degree of that graph. More precisely, we consider, given some $k \geq 1$, the minimum degree required for a graph to have at least k vertex-disjoint cycles of different lengths (and sometimes additional length properties). We show that this value is precisely $\frac{k^2+5k-2}{2}$ for every k (Theorem 12). Several more constrained situations are then considered, *e.g.* when the graph is triangle-free or the vertex-disjoint cycles are required to be more than just of different lengths; in these situations as well, we exhibit bounds (most of which are tight) on the least minimum degree required to guarantee the existence of the k desired vertex-disjoint cycles. We also conjecture that for every $D \geq 3$, every graph G of large enough order verifying $k+1 \leq \delta(G) \leq \Delta(G) \leq D$ has k vertex-disjoint cycles of different lengths (see Conjecture 20). To support this conjecture, we prove it for $k=2$ (Theorem 23). This in particular yields that every cubic graph of order more than 14 has two vertex-disjoint cycles of different lengths, which is tight (see Theorem 24).

We then consider, in Section 4, the same question for digraphs: What minimum out-degree is required for a digraph to have at least k vertex-disjoint directed cycles of different lengths? The existence of such a minimum out-degree was conjectured by Lichiardopol in [10], who verified it for $k=2$. We here give further support to Lichiardopol's Conjecture by showing it to hold for tournaments (see Corollary 30), and, using the probabilistic method, for some regular digraphs (Theorem 34) and digraphs of small order (Theorem 35).

2 Preliminaries

Let G be a graph and X a subset of $V(G)$. We use $G[X]$ to denote the subgraph of G induced by X , and $G - X$ to denote the subgraph of G induced by $V(G) \setminus X$. For two disjoint subsets X, Y of $V(G)$, we denote by $(X, Y)_G$ the bipartite subgraph of G with all edges between X and Y . For a subgraph H of G , we set $G - H = G - V(H)$.

We now recall results, some of which are folklore, that guarantee the existence of cycles under particular circumstances. These results will be used to obtain our main results. First, by considering a longest path of a graph, the following result is easily observed.

Proposition 1. *For every $k \geq 1$, every graph of minimum degree at least $k + 1$ contains k cycles of different lengths.*

Now, by considering a maximum cut of a graph, the following is also clear.

Proposition 2. *Let G be a graph of minimum degree at least $2k - 1$, where $k \geq 1$. Then $V(G)$ can be partitioned into sets S and T such that the bipartite subgraph $(S, T)_G$ has minimum degree at least k .*

An immediate consequence is:

Theorem 3. *For every $k \geq 1$, every graph of minimum degree at least $2k + 1$ contains k even cycles of different lengths.*

Proof. By Proposition 2, we can partition $V(G)$ into sets S and T such that the bipartite subgraph $G' = (S, T)_G$ has minimum degree at least $k + 1$. By Proposition 1, graph G' contains k cycles of different lengths. Since G' is bipartite, all these cycles have even length. \square

Proposition 2 shows the existence, in any graph, of a cut such that every vertex has ‘many’ neighbors in the different partite set. In a different flavour, the following three theorems concern cuts of graphs such that every vertex has ‘many’ neighbors in the same partite set.

Theorem 4 (Stiebitz [14]). *If s and t are non-negative integers, and G is a graph with minimum degree at least $s + t + 1$, then the vertex set of G can be partitioned into two sets which induce subgraphs of minimum degree at least s and t , respectively.*

Theorem 5 (Kaneko [9]). *If s and t are non-negative integers, and G is a triangle-free graph with minimum degree at least $s + t$, then the vertex set of G can be partitioned into two sets which induce subgraphs of minimum degree at least s and t , respectively.*

Theorem 6 (Diwan [5]). *If $s, t \geq 2$ are two integers, and G is a graph with minimum degree at least $s + t - 1$ and girth at least 5, then the vertex set of G can be partitioned into two sets which induce subgraphs of minimum degree at least s and t , respectively.*

Diwan proved in [6] the following result:

Theorem 7 (Diwan [6]). *For every $r \geq 2$, and for any natural number m , every graph of minimum degree at least $2r - 1$ contains a cycle of length $2m$ modulo r .*

Theorem 7 yields that when r is odd, every graph of minimum degree at least $2r - 1$ contains a cycle of length m modulo r , for any natural number m .

Corollary 8. *For every $k \geq 1$ and $r \geq 2$, every graph G with*

$$\delta(G) \geq \begin{cases} 2(k+1)r - 1, & \text{if } k \text{ is even and } r \text{ is odd;} \\ 2kr - 1, & \text{otherwise,} \end{cases}$$

has k cycles of different lengths all divisible by r .

Proof. If r is even, then $\delta(G) \geq 2kr - 1$. By Theorem 7, there are k cycles C_0, C_1, \dots, C_{k-1} such that for each i , cycle C_i has length $ir \pmod{kr}$. Clearly the k cycles have different lengths all divisible by r .

If both k and r are odd, then $\delta(G) \geq 2kr - 1$. By Theorem 7, there exist k cycles C_0, C_1, \dots, C_{k-1} such that for each i , cycle C_i has length $2ir \pmod{kr}$. Since kr is odd, the k cycles have different lengths all divisible by r .

If k is even and r is odd, then $\delta(G) \geq 2(k+1)r - 1$. Note that $k+1$ is odd. By the analysis above, graph G contains $k+1$ cycles of different lengths all divisible by r . \square

Let us also mention the following result due to Dirac, about cycles in 2-connected graphs with given minimum degree.

Theorem 9 (Dirac [7]). *Every 2-connected graph with order n and minimum degree at least δ contains a cycle of length at least $\min\{2\delta, n\}$.*

When it comes to oriented graphs, a result of Camion implies that every strongly-connected tournament is Hamiltonian.

Theorem 10 (Camion [4]). *Every strongly-connected tournament contains a Hamiltonian cycle.*

Still in the oriented context, the following result of Moon asserts that every strongly-connected tournament is pancyclic, *i.e.* has directed cycles of every possible length.

Theorem 11 (Moon [12]). *Every strongly-connected tournament with order n has a directed cycle of length i , for every $i = 3, \dots, n$.*

3 Disjoint cycles of different lengths in undirected graphs

Throughout this section, we exhibit conditions, notably in terms of minimum degree, implying the existence of a certain number of vertex-disjoint cycles of different lengths in a graph.

3.1 Distinct lengths and relation to girth

The central question of this work is the following: What is the smallest minimum degree $f(k)$ required so that a graph with minimum degree $f(k)$ has at least k vertex-disjoint cycles of different lengths? We answer this question in the following result.

Theorem 12.

a) For every $k \geq 1$, there exists a minimum integer $f(k)$ such that every graph of minimum degree at least $f(k)$ contains k vertex-disjoint cycles of different lengths.

b) We have $f(k) = \frac{k^2 + 5k - 2}{2}$.

Proof. a) We proceed by induction on k . Clearly $f(1)$ exists (and we have $f(1) = 2$). Suppose that the assertion is true for $k - 1$ (where $k \geq 2$), and let us study it for k . Let G be an arbitrary graph of minimum degree at least $f(k - 1) + k + 2$. By Theorem 4, there exists a partition (V_1, V_2) of $V(G)$ such that $G[V_1]$ is of minimum degree at least $f(k - 1)$ and $G[V_2]$ is of minimum degree at least $k + 1$. Then $G[V_1]$ contains $k - 1$ disjoint cycles C_1, \dots, C_{k-1} of different lengths. By Proposition 1, subgraph $G[V_2]$ contains k cycles of different lengths. Then one of these cycles has length distinct from those of the cycles C_1, \dots, C_{k-1} . We get then a collection of k vertex-disjoint cycles of different lengths. So the assertion is true for k , and the result is proved. Furthermore, we have $f(k) \leq f(k - 1) + k + 2$.

b) We have $f(i) \leq f(i - 1) + i + 2$ for $2 \leq i \leq k$. By addition and simplification we get $f(k) \leq f(1) + 4 + \dots + k + 2$, hence $f(k) \leq 2 + \frac{1}{2}(k + 2)(k + 3) - 6$, which yields

$$f(k) \leq \frac{k^2 + 5k - 2}{2}. \quad (1)$$

On the other hand since the complete graph on $f(k) + 1$ vertices is of minimum degree $f(k)$, it contains k vertex-disjoint cycles of different lengths. It follows that $f(k) + 1 \geq 3 + \dots + (k + 2)$, hence $f(k) + 1 \geq \frac{1}{2}(k + 2)(k + 3) - 3$, which yields

$$f(k) \geq \frac{k^2 + 5k - 2}{2}. \quad (2)$$

From Inequalities (1) and (2), we get $f(k) = \frac{k^2 + 5k - 2}{2}$. □

One may naturally wonder how the girth parameter acts on the task of finding vertex-disjoint cycles of distinct lengths. Towards this question, we show that, for triangle-free graphs, the function $f(k)$ from Theorem 12 can be improved to the function $g(k)$ below, which is again best possible.

Theorem 13.

- a) *For every $k \geq 1$, there exists a minimum integer $g(k)$ such that every triangle-free graph of minimum degree at least $g(k)$ contains k vertex-disjoint cycles of different lengths.*
- b) *We have $g(k) = \frac{k^2 + 3k}{2}$.*

Proof. Using Theorem 5 in place of Theorem 4, mimicking the proof of Theorem 12 results in a straightforward proof of the first part of the statement. To see that the degree bound is indeed best possible, consider the complete bipartite graph $K_{g(k),g(k)}$, which is triangle-free and has minimum degree $g(k)$, and has k vertex-disjoint cycles of distinct lengths. \square

Quite similarly as in the proofs of Theorems 12 and 13, using Theorem 6 we can refine the function $f(k)$ for graphs with girth at least 5:

Theorem 14.

- a) *For every $k \geq 1$ and $g \geq 5$, there exists a minimum integer $g_g(k)$ such that every graph of girth g and minimum degree at least $g_g(k)$ contains k vertex-disjoint cycles of different lengths.*
- b) *We have $g_g(k) \leq \frac{k^2 + k + 2}{2}$.*

We have no clue, however, whether the degree bound is tight here.

3.2 Distinct lengths modulo r

We now investigate the impact on the function $f(k)$ in Theorem 12 if we additionally require the vertex-disjoint cycles to be of specific lengths modulo some integer r . We start by determining the best degree bound $h(k)$ for even cycles (*i.e.* $r = 2$).

Theorem 15.

- a) *For every $k \geq 1$, there exists a minimum integer $h(k)$ such that every graph of minimum degree at least $h(k)$ contains k vertex-disjoint even cycles of different lengths.*
- b) *We have $h(k) = k^2 + 3k - 1$.*

Proof. **a)** We proceed by induction on k . Clearly $h(1)$ exists (and, by Theorem 3, we have $h(1) = 3$). Suppose that the assertion is true for $k - 1$, for some $k \geq 2$, and let us study it for k . Let G be an arbitrary graph of minimum degree at least $h(k - 1) + 2k + 2$. By Theorem 4, there exists a partition (V_1, V_2) of $V(G)$ such that $G[V_1]$ is of minimum degree at least $h(k - 1)$ and $G[V_2]$ is of minimum degree at least $2k + 1$. Then $G[V_1]$ contains $k - 1$ vertex-disjoint even cycles C_1, \dots, C_{k-1} being of different lengths. According to Theorem 3, subgraph $G[V_2]$ contains k even cycles of different lengths. Then one of these

cycles has length distinct from those of the cycles C_1, \dots, C_{k-1} . We get then a collection of k vertex-disjoint even cycles of different lengths. So, the assertion is true for k , and the result is proved. We also deduce $h(k) \leq h(k-1) + 2k + 2$.

b) We have $h(i) \leq h(i-1) + 2i + 2$ for $2 \leq i \leq k$. By addition and simplification, we get $h(k) \leq h(1) + 2(3 + \dots + k + 1)$, hence $h(k) \leq 3 + 2 \left(\frac{1}{2}(k+1)(k+2) - 3 \right)$, which yields

$$h(k) \leq k^2 + 3k - 1. \quad (3)$$

On the other hand since the complete graph on $h(k) + 1$ vertices is of minimum degree $h(k)$, it contains k vertex-disjoint even cycles of different lengths. It follows that $h(k) + 1 \geq 2 \cdot 2 + \dots + 2(k+1)$, hence $h(k) + 1 \geq 2 \left(\frac{1}{2}(k+1)(k+2) - 1 \right)$, which yields

$$h(k) \geq k^2 + 3k - 1. \quad (4)$$

We thus deduce $h(k) = k^2 + 3k - 1$ from Inequalities (3) and (4). \square

We extend Theorems 12 and 15 to cycles having lengths multiple to a same value r . Recall that the exact value of $f_2(k)$ is given in Theorem 15.

Theorem 16.

a) Let $r \geq 3$ be an integer. For every $k \geq 1$, there exists a minimum integer $f_r(k)$ such that every graph of minimum degree at least $f_r(k)$ contains k vertex-disjoint cycles of different lengths all divisible by r .

$$\text{b) We have } f_r(k) \leq \begin{cases} k(k+1)r - 1, & \text{if } r \text{ is even;} \\ 2(k^2 + k - 1)r - 1, & \text{if both } k \text{ and } r \text{ are odd;} \\ 2k^2r - 1, & \text{if } k \text{ is even and } r \text{ is odd,} \end{cases}$$

$$\text{and } f_r(k) \geq \begin{cases} \frac{1}{2}k(k+1)r - 1, & \text{if } r \text{ is even;} \\ \frac{1}{2}k(k+1)r, & \text{if } r \text{ is odd.} \end{cases}$$

Proof. **a)** We proceed by induction on k . Clearly $f_r(1)$ exists (and, by Theorem 7, we have $f_r(1) \leq 2r - 1$). Suppose that the assertion is true for $k - 1$, where $k \geq 2$, and let us study it for k . Let G be an arbitrary graph of minimum degree at least

$$f_r(k-1) + \begin{cases} 2(k+1)r, & \text{if } k \text{ is even and } r \text{ is odd;} \\ 2kr & \text{otherwise.} \end{cases}$$

According to Theorem 4, there exists a partition (V_1, V_2) of $V(G)$ such that $G[V_1]$ is of minimum degree at least $f_r(k-1)$ and $G[V_2]$ is of minimum degree at least $2(k+1)r - 1$ for k even and r odd, and of minimum degree at least $2kr - 1$ otherwise. Then $G[V_1]$ contains $k - 1$ disjoint cycles C_1, \dots, C_{k-1} of different lengths all divisible by r . By Corollary 8, subgraph $G[V_2]$ contains k cycles of different lengths all divisible by r . Then one of these cycles has length distinct from those of the cycles C_1, \dots, C_{k-1} . We get then a collection

of k disjoint cycles of different lengths all divisible by r . So the assertion is true for k , and the existence is proved.

b) If r is even, then we have

$$\begin{aligned} f_r(k) &\leq f_r(k-1) + 2kr \\ &\leq f_r(k-2) + 2(k-1)r + 2kr \\ &\leq f_r(1) + \sum_{i=2}^k 2ir \\ &= k(k+1)r - 1. \end{aligned}$$

If both k and r are odd, then we have

$$\begin{aligned} f_r(k) &\leq f_r(k-1) + 2kr \\ &\leq f_r(k-2) + 2kr + 2kr \\ &\leq f_r(k-3) + 2(k-2)r + 2kr + 2kr \\ &\leq f_r(1) + \sum_{i=1}^{(k-1)/2} 4(2i+1)r \\ &= 2(k^2 + k - 1)r - 1. \end{aligned}$$

If k is even and r is odd, then

$$\begin{aligned} f_r(k) &\leq f_r(k-1) + 2(k+1)r \\ &\leq 2((k-1)^2 + k - 2)r - 1 + 2(k+1)r \\ &= 2k^2r - 1. \end{aligned}$$

So the first assertion of **b)** is proved.

We now focus on the second assertion of **b)**. For even $r \geq 4$ and $k \geq 1$, the complete graph $K_{f_r(k)+1}$ contains k vertex-disjoint cycles of different lengths, all divisible by r . It follows that $f_r(k)+1 \geq r+2r+\dots+kr$, which yields $f_r(k) \geq \frac{1}{2}k(k+1)r-1$. For odd $r \geq 3$ and $k \geq 1$, the complete bipartite graph $K_{f_r(k), f_r(k)}$ (of order $2f_r(k)$) contains k disjoint cycles of different lengths, all divisible by r . It follows that $2f_r(k) \geq 2r+4r+\dots+2kr$, which yields $f_r(k) \geq \frac{1}{2}k(k+1)r$. This concludes the proof. \square

We conclude this section by considering the opposite direction, namely cycles having distinct lengths modulo some value r .

Theorem 17.

a) Let $r \geq 3$ be an odd integer. There exists a minimum integer $\phi(r)$ such that every graph of minimum degree at least $\phi(r)$ contains r vertex-disjoint cycles C_0, \dots, C_{r-1} with $v(C_i) \equiv i \pmod{r}$ for $0 \leq i \leq r-1$.

b) We have $\frac{r^2 + 5r - 2}{2} \leq \phi(r) \leq 2r^2 - 1$.

Proof. **a)** Let G be a graph with minimum degree at least $2r^2 - 1$. By repeated applications of Theorem 4, we get a partition $(V_0, V_1, \dots, V_{r-1})$ of $V(G)$ such that for every i , subgraph $G[V_i]$ has minimum degree at least $2r - 1$. By Theorem 7, subgraph $G[V_i]$ has a cycle C_i of length $i \pmod r$. Thus the collection of r cycles C_0, C_1, \dots, C_{r-1} is as required. So $\phi(r)$ exists, and $\phi(r) \leq 2r^2 - 1$.

b) The complete graph $K_{\phi(r)+1}$ contains r disjoint cycles with the required conditions. It follows that $\phi(r) + 1 \geq 3 + \dots + r + 2$, hence $\phi(r) + 1 \geq \frac{1}{2}(r+2)(r+3) - 3$, which implies $\frac{1}{2}(r^2 + 5r - 2) \leq \phi(r)$, as claimed. \square

3.3 Large graphs

Recall that Theorem 12 implies that every graph of minimum degree at least $f(k) = \frac{1}{2}(k^2 + 5k - 2)$ has k vertex-disjoint cycles of different lengths. Furthermore, the complete graph $K_{f(k)+1}$ shows that the bound on the minimum degree is best possible. However, if we allow finite exceptions, then this bound can be improved. That is, the degree bound can be reduced further for graphs with sufficiently large order.

In the proof of the main result of this section, Theorem 19, we will make use of the following notion. Consider an integer $k \geq 2$ and a graph G of minimum degree $\delta(G) \geq k + 1$. By a k -path-vertex schema of G we mean a pair $\mathcal{S} = (P, x)$ consisting of a path P and a vertex x not in P having exactly $k + 1$ neighbors in P . Using a longest path argument, it can be observed that every graph contains a path-vertex schema. Observe also that the subgraph induced by $V(P) \cup \{x\}$ contains k cycles of different length (all containing x). For convenience, we sometimes consider \mathcal{S} as a subgraph of G . So $V(\mathcal{S}) = V(P) \cup \{x\}$, and $G - \mathcal{S} = G - (V(P) \cup \{x\})$. The *cardinality* of \mathcal{S} is $|V(\mathcal{S})|$. It is obvious that this cardinality is at least $k + 2$. An *optimal* k -path-vertex schema of G is a schema of G of minimum cardinality. Observe that in this case the extremities of P are neighbors of x .

We will make use of the following result concerning optimal path-vertex schemas.

Lemma 18. *Let G be a graph with $\delta(G) \geq k + 1$, and $\mathcal{S} = (P, x)$ be an optimal k -path-vertex schema of G . Then every vertex y of $G - \mathcal{S}$ has at most $k + 2$ neighbors in $V(\mathcal{S})$. Moreover, if there exists a vertex y in $G - \mathcal{S}$ having exactly $k + 2$ neighbors in $V(\mathcal{S})$, then x is a neighbor of y , all the vertices of P are neighbors of x , and $|V(\mathcal{S})| = k + 2$.*

Proof. Suppose $P = (u_1, u_2, \dots, u_p)$. If y has at least $k + 2$ neighbors in $V(P)$, then there exists i (with $2 \leq i \leq p$) such that y has exactly $k + 1$ neighbors in $P' = (u_i, \dots, u_p)$. Thus P' and y form a k -path-vertex schema of cardinality less than \mathcal{S} , a contradiction. This implies that y has at most $k + 2$ neighbors in $V(\mathcal{S})$.

Suppose that y has exactly $k + 2$ neighbors in $V(\mathcal{S})$. By the analysis above, we can see that y has exactly $k + 1$ neighbors on P and $xy \in E(G)$. Thus (P, y) is an optimal k -path-vertex schema, and then u_1 and u_p are neighbors of y . Suppose for the sake of a contradiction that there exists a vertex of P which is not a neighbor of x . Clearly u_1 and u_p are neighbors of x . Let i be the maximum index such that $xu_i \notin E(G)$. Thus $2 \leq i \leq p - 1$. If $i \leq p - 2$, then let $P' = (u_{i+2}, \dots, u_p, y, u_1, \dots, u_{i-1})$; otherwise, if

$i = p - 1$, let $P' = (y, u_1, \dots, u_{p-2})$. Then P' and x form a k -path-vertex schema of cardinality less than \mathcal{S} , a contradiction. So, all vertices of P are neighbors of x , and clearly we have $|V(\mathcal{S})| = k + 2$. \square

We are now ready to prove a refinement of Theorem 12 for sufficiently large graphs.

Theorem 19.

- a) For every $k \geq 2$, every graph G of order $n \geq 7 \cdot \lfloor k^2/4 \rfloor$ and $\delta(G) \geq \frac{1}{2}(k^2 + 3k)$ has k vertex-disjoint cycles of different lengths.
- b) The bound on $\delta(G)$ is best possible.

Proof. a) We prove the claim by induction on k , starting from the row $k = 2$. In other words, we prove that if G is a graph of order $n \geq 7$ and $\delta(G) \geq 5$, then G contains two vertex-disjoint cycles of different lengths.

If G is triangle-free, then we are done by Theorem 13. Suppose now that G contains a triangle $C = (x_1, x_2, x_3, x_1)$. We claim that there exists a vertex x_4 of $G - C$ having three neighbors in C . Indeed, if it was not the case, then $G - C$ would be of minimum degree at least 3 and then it would contain two cycles of different lengths. One of these cycles and C would then form a pair of vertex-disjoint cycles of different lengths, and we would be done. Clearly the subgraph H induced by x_1, x_2, x_3 and x_4 is complete. Suppose that there exists at most one vertex of $G - H$ having four neighbors in H . Then the graph $G - H$ has at most one vertex of degree at most 1. Then $G - H$ contains a cycle (since any longest path in $G - H$ has one of its end-vertices having at least two neighbors on the path) and since H contains a triangle and a 4-cycle, we get two vertex-disjoint cycles of different lengths and we are done. Suppose thus that there exist two vertices x_5 and x_6 both having x_1, x_2, x_3 and x_4 as neighbors.

Let H' be the subgraph induced by x_1, \dots, x_6 . If every vertex in $G - H'$ has at most three neighbors in H' , then $\delta(G - H') \geq 2$ and $G - H'$ contains a cycle. Since $H \subset H'$ contains a triangle and a 4-cycle, we get two vertex-disjoint cycles of different lengths and we are done. So assume now that there exists a vertex x_7 having at least four neighbors in $\{x_1, \dots, x_6\}$. Then x_7 has at least two neighbors in $\{x_1, \dots, x_4\}$, say $x_7x_1, x_7x_2 \in E(G)$. Then (x_1, x_2, x_7, x_1) and $(x_3, x_5, x_4, x_6, x_3)$ are two vertex-disjoint cycles of different lengths. This achieves the proof of the case $k = 2$.

Suppose now that the assertion is true up to row $k - 1$, where $k \geq 3$, and let us study it for k . So G is a graph of order $n \geq 7 \cdot \lfloor k^2/4 \rfloor$ and of minimum degree $\delta(G) \geq \frac{1}{2}k(k + 3) = \frac{1}{2}(k + 1)(k + 2) - 1$. By applying Theorem 4 repeatedly, we get a partition (V_1, V_2, \dots, V_t) of $V(G)$ where $t = \lfloor \frac{1}{2}(k + 1) \rfloor$, such that each V_i induces a graph of minimum degree at least $k + 1$. Hence each $G[V_i]$ has a k -path-vertex schema. This implies that G has a k -path-vertex schema of order at most $\frac{n}{\lfloor (k + 1)/2 \rfloor}$.

Suppose first that there exists an optimal k -path-vertex schema $\mathcal{S} = (P, x)$ such that every vertex y of $G - \mathcal{S}$ has at most $k + 1$ neighbors in \mathcal{S} . Recall that $|V(\mathcal{S})| \leq \frac{n}{\lfloor (k + 1)/2 \rfloor}$.

Let $G' = G - \mathcal{S}$. Then the order of G' is

$$n' = n - |V(\mathcal{S})| \geq \frac{\lfloor (k-1)/2 \rfloor}{\lfloor (k+1)/2 \rfloor} n \geq \begin{cases} 7k(k-2)/4, & k \text{ is even;} \\ 7(k-1)^2/4, & k \text{ is odd,} \end{cases}$$

i.e. $n' \geq 7 \cdot \left\lfloor \frac{(k-1)^2}{4} \right\rfloor$. Note that every vertex y of G' has at least

$$\frac{k^2 + 3k}{2} - k - 1 = \frac{(k-1)^2 + 3(k-1)}{2}$$

neighbors in G' , so $\delta(G') \geq \frac{1}{2}((k-1)^2 + 3(k-1))$. By the induction hypothesis G' contains $k-1$ vertex-disjoint cycles C_1, \dots, C_{k-1} of different lengths. Plus, the induced subgraph $G[V(\mathcal{S})]$ contains k cycles of different lengths. Then one of these cycles and the cycles C_1, \dots, C_{k-1} form a collection of k vertex-disjoint cycles of different lengths. So in this case the assertion is proved for k .

Suppose now that there does not exist an optimal k -path-vertex schema \mathcal{S} such that every vertex y of $G - \mathcal{S}$ has at most $k+1$ neighbors in $V(\mathcal{S})$. We take an arbitrary optimal k -path-vertex schema $\mathcal{S} = (P, x)$, and let $y \in V(G - \mathcal{S})$ be a vertex having $k+2$ neighbors in \mathcal{S} . By Lemma 18, vertices x and y are adjacent, all the vertices of P are neighbors of x (and also of y), and $|V(\mathcal{S})| = k+2$. We put $P = (u_1, u_2, \dots, u_{k+1})$ and $\Omega = \{u_1, u_2, \dots, u_{k-1}, x, y\}$. Clearly $G[\Omega]$ contains $k-1$ cycles of different lengths (ranging from 3 to $k+1$).

Let $G' = G - \Omega$. Note that G' has minimum degree at least $\frac{1}{2}((k-1)^2 + 3(k-1))$ and has order at least $7 \cdot \lfloor (k-1)^2/4 \rfloor$. By the induction hypothesis G' contains $k-1$ disjoint cycles C_1, C_2, \dots, C_{k-1} of different lengths (where we label the C_i 's so that their lengths are non-decreasing). Since $G[\Omega]$ contains $k-1$ cycles of all lengths from 3 to $k+1$, we have $|V(C_i)| = i+2$ for $1 \leq i \leq k-1$. In particular, cycle C_{k-1} is of order $k+1$ and C_{k-2} is of order k .

Let $G'' = G - \left(\Omega \cup \bigcup_{i=1}^{k-1} V(C_i) \right)$, and let z be a vertex of G'' . We claim that z has at most one neighbor in Ω . Indeed, we would get that z and the vertices of Ω form a cycle of order $k+2$; so, together with C_1, \dots, C_{k-2} we would then get k disjoint cycles of different lengths. We also claim that z has at most $\frac{1}{2}(k+1)$ neighbors in C_{k-1} . Indeed, otherwise the vertices of C_{k-1} and z would form a cycle C'_{k-1} of order at least $k+2$; so $C_1, \dots, C_{k-2}, C'_{k-1}$ and an appropriate cycle of $G[\Omega]$ of order $k+1$ would form a collection of k disjoint cycles of different lengths. It follows that z has at most $1 + 3 + \dots + k + \frac{k+1}{2} = \frac{k^2 + 2k - 3}{2}$ neighbors in $\Omega \cup \bigcup_{i=1}^{k-1} V(C_i)$, and z has at least $\frac{1}{2}(k+3) \geq 3$ neighbors in G'' . Let $P' = (v_1, v_2, \dots, v_s)$ be a longest path in G'' . Since $\delta(G'') \geq 3$, we have $s \geq 4$.

Note that all the neighbors of v_1 in G'' are in P' . It follows that v_1 has at most k neighbors in G'' (for otherwise we would have a cycle of length at least $k+2$ and, together with the cycles C_1, \dots, C_{k-1} , we would have the desired cycles). We have seen that v_1 has at most one neighbor in Ω , and that v_1 has at most $\frac{1}{2}(k+1)$ neighbors in C_{k-1} . We claim

that v_1 has more than $k/2$ neighbors in C_{k-2} . Indeed, suppose the opposite. Then v_1 has at most

$$1 + 3 + \cdots + k - 1 + \frac{k}{2} + \frac{k + 1}{2} = \frac{k^2 + k - 3}{2}$$

neighbors in $\Omega \cup \bigcup_{i=1}^{k-1} V(C_i)$. This means that v_1 has at most $\frac{1}{2}(k^2 + k - 4)$ neighbors in $\Omega \cup \bigcup_{i=1}^{k-1} V(C_i)$ and that v_1 has at least $\frac{k^2 + 3k}{2} - \frac{k^2 + k - 4}{2} = k + 2$ neighbors in G'' , a contradiction. So as we claimed, vertex v_1 has more than $k/2$ neighbors in C_{k-2} , and similarly v_s has more than $k/2$ neighbors in C_{k-2} . But now it can be observed that the vertices of C_{k-2} and the vertices of P' form a cycle of length at least $k + 2$, and, together with the cycles C_i (where $1 \leq i \leq k - 1$ and $i \neq k - 2$) and an appropriate cycle of length k of $G[\Omega]$, we get k disjoint cycles of different lengths. This concludes this part of the proof.

b) Let G be a complete bipartite graph with two partite sets of size $\frac{1}{2}k(k + 3) - 1$ and $n - \frac{1}{2}k(k + 3) + 1$, respectively, where n is large enough. Then $\delta(G) = \frac{1}{2}k(k + 3) - 1$. Note that G does not have k vertex-disjoint cycles of different lengths. So the bound on $\delta(G)$ in our statement is sharp. \square

While the bound on δ in Theorem 19 is best possible, we have no clue about how far from optimal the bound on n is. We believe it would be interesting to investigate this aspect further.

3.4 Large graphs with bounded maximum degree

We conclude this section by pointing out that if we restrict ourselves to large graphs with bounded maximum degree, then we may further refine Theorem 19. In particular, we believe the following conjecture should be true.

Conjecture 20. *For every two integers k and D , there is an integer n_0 such that every graph G of order at least n_0 with $k + 1 \leq \delta(G) \leq \Delta(G) \leq D$ has k vertex-disjoint cycles of different lengths.*

We here make a first step towards Conjecture 20 by proving it for $k = 2$, see Theorem 23.

Before presenting the proof, we need to introduce a few preliminary results concerning graphs having only one type of cycles. In the upcoming results, we use $n(G)$ to denote the order of some graph G . For any integer i , we denote by $N_i(G)$ the set of vertices of G with degree i , and set $n_i(G) = |N_i(G)|$.

Lemma 21. *Let G be a graph with $\delta(G) \geq 2$. If all cycles of G have length 3, then $n_2(G) \geq \frac{n(G)}{3} + 2$.*

Proof. We first claim that every block (maximal 2-connected subgraph) of G is either a K_2 or a triangle. Let B be an arbitrary block of G . If $B \neq K_2$, then B is 2-connected. If

B has at least four vertices, then, by Theorem 9, it contains a cycle of order at least 4, which is not in our assumption. So we conclude that B has exactly three vertices. Thus B is a triangle.

Now we prove the lemma by induction on the number of blocks of G . If G has only one block, then G is a triangle and the assertion is trivial. If G is disconnected, then we can complete the proof by applying the induction hypothesis to each component of G . Now we assume that G is connected and separable.

Let B be an end-block of G , and x_0 be the cut-vertex of G contained in B . Recall that B is a triangle. If $d_{G-B}(x_0) \geq 2$, then let $G' = G - (B - x_0)$. By the induction hypothesis, we have $n_2(G') \geq \frac{n(G')}{3} + 2$. Note that $n(G) = n(G') + 2$ and $n_2(G) \geq n_2(G') + 1$. We then have

$$n_2(G) \geq n_2(G') + 1 \geq \frac{n(G')}{3} + 2 + 1 = \frac{n(G) - 2}{3} + 3 \geq \frac{n(G)}{3} + 2.$$

Now we assume that $d_{G-B}(x_0) = 1$. Let $P = (x_0, x_1, \dots, x_t)$ be a path of G such that $d(x_i) = 2$ for $1 \leq i \leq t-1$ and $d(x_t) \geq 3$. Let $G' = G - B - (P - x_t)$. By the induction hypothesis, we have $n_2(G') \geq \frac{n(G')}{3} + 2$. Note that $n(G) = n(G') + t + 2$ and $n_2(G) \geq n_2(G') + t$. Since $t \geq 1$, we now have

$$n_2(G) \geq n_2(G') + t \geq \frac{n(G')}{3} + 2 + t = \frac{n(G) - t - 2}{3} + t + 2 \geq \frac{n(G)}{3} + 2,$$

which concludes the proof. \square

Lemma 22. *Let G be a graph with $\delta(G) \geq 2$. If all cycles of G have length 4, then $n_2(G) \geq \frac{n(G)}{5} + 2$.*

Proof. We first claim that every block of G is either a K_2 or a complete bipartite graph $K_{2,s}$ with $s \geq 2$. Let B be an arbitrary block of G . If $B \neq K_2$, then B is 2-connected. Thus B contains a cycle, which is a C_4 by our assumption. If B is not bipartite, then B has an odd cycle, which is not a C_4 , a contradiction. This implies that B is bipartite. Let X, Y be the two partite sets of B and let $C = (x_1, y_1, x_2, y_2, x_1)$ be a C_4 of G , where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Suppose that X has a third vertex x_3 . Since B is 2-connected, there is a path P with two end-vertices in C such that $x_3 \in V(P)$ and all internal vertices of P are not in C . If P has length more than 2, then B contains a cycle longer than C , which is not a C_4 , a contradiction. Thus P has length 2 and $x_3y_1, x_3y_2 \in E(B)$. If Y has a third vertex y_3 , then by a similar analysis as above, we can see that $x_1y_3, x_2y_3 \in E(G)$. But in this case $C' = (x_1, y_3, x_2, y_2, x_3, y_1, x_1)$ is a cycle being not a C_4 , a contradiction. This implies that either $|X| = 2$ or $|Y| = 2$. We suppose without loss of generality that $|X| = 2$. Since B is 2-connected, vertices x_1, x_2 are adjacent to every vertex in Y . Thus B is a complete bipartite graph $K_{2,s}$ with $s \geq 2$.

Now we prove the lemma by induction on the block number of G . If G has only one block, then $G = K_{2,s}$ with $s \geq 2$ and the assertion is trivial. If G is disconnected, then

we can complete the proof by applying the induction hypothesis to each component of G . Now we assume that G is connected and separable.

Let B be an end-block of G , and x_0 be the cut-vertex of G contained in B . Note that $B = K_{2,s}$ with $s \geq 2$. If $d_{G-B}(x) \geq 2$, then let $G' = G - (B - x)$. By the induction hypothesis, $n_2(G') \geq \frac{n(G')}{5} + 2$. Note that $n(G) = n(G') + s + 1$ and

$$n_2(G) \geq \begin{cases} n_2(G') + 2, & \text{if } s = 2; \\ n_2(G') + s - 2, & \text{if } s \geq 3. \end{cases}$$

One can compute that $n_2(G) \geq \frac{n(G)}{5} + 2$. Now we assume that $d_{G-B}(x) = 1$. Let $P = (x_0, x_1, \dots, x_t)$ be a path of $G - (B - x_0)$ such that $d(x_i) = 2$ for $1 \leq i \leq t - 1$ and $d(x_t) \geq 3$. Let $G' = G - B - (P - x_t)$. By the induction hypothesis, we have $n_2(G') \geq \frac{n(G')}{5} + 2$. Note that $n(G) = n(G') + s + t + 1$ and

$$n_2(G) \geq \begin{cases} n_2(G') + t + 1, & \text{if } s = 2; \\ n_2(G') + s + t - 3, & \text{if } s \geq 3, \end{cases}$$

where $t \geq 1$. This yields $n_2(G) \geq \frac{n(G)}{5} + 2$. □

We are now ready to prove Conjecture 20 for $k = 2$.

Theorem 23. *For every $D \geq 3$, every graph G of order more than $20D - 46$ with $3 \leq \delta(G) \leq \Delta(G) \leq D$ has two vertex-disjoint cycles of different lengths.*

Proof. Suppose that G does not contain two vertex-disjoint cycles of different lengths. We will show that $n(G)$ is bounded above by $20D - 46$. Let g denote the girth of G . We distinguish two cases.

Case 1. $g = 3$ or $g = 4$.

Let C be a shortest cycle of G . Then C is either a C_3 or a C_4 . Let H be an induced subgraph of G defined by a sequence of vertex sets U_0, U_1, \dots, U_s such that:

1. $U_0 = V(C)$ and $U_s = V(H)$;
2. for every i , where $0 \leq i \leq s - 1$, there is a vertex $x_i \in V(G) \setminus U_i$ such that $d_{U_i}(x_i) > \frac{d(x_i)}{2}$ and $U_{i+1} = U_i \cup \{x_i\}$;
3. for every vertex $x \in V(G) \setminus U_s$, we have $d_{U_s}(x) \leq \frac{d(x)}{2}$.

Note that $n(H) = n(C) + s$. We first show that s is bounded above by a constant (depending on D). Note that $e(U_0, V(G) \setminus U_0) = \sum_{v \in U_0} (d(v) - 2)$ is bounded above by $4D - 8$. Observe that $e(U_{i+1}, V(G) \setminus U_{i+1}) \leq e(U_i, V(G) \setminus U_i) - 1$. This implies that $e(U_s, V(G) \setminus U_s) \leq 4D - 8 - s$ and $s \leq 4D - 8$.

Let $F = G - H$. Recall that for every vertex $x \in V(F)$, we have $d_F(x) \geq \left\lceil \frac{d(x)}{2} \right\rceil \geq 2$. Thus $\delta(F) \geq 2$. If F contains a cycle of length different from that of C , then we are done. So we assume that either every cycle of F is a triangle, or every cycle of F is a C_4 . Note that every vertex in $N_2(F)$ has a neighbor in H . Furthermore, we have $n_2(F) \leq e(V(H), V(F)) \leq 4D - 8 - s$. Now, applying Lemmas 21 and 22, we get

$$n(F) \leq 5(n_2(F) - 2) \leq 5(4D - 10 - s),$$

and, thus,

$$n(G) = n(H) + n(F) \leq 4 + s + 5(4D - 10 - s) \leq 20D - 46.$$

Case 2. $g \geq 5$.

Let $C = (x_1, \dots, x_g, x_1)$ be a cycle of G of order g . We may assume that any vertex of $V(G) \setminus V(C)$ has at most one neighbor in C (for otherwise we would have a cycle of length less than g). So, the graph $G' = G - C$ is of minimum degree at least 2 and since G' cannot be of minimum degree at least 3 (for otherwise G' would contain two cycles of different lengths, and then one of these cycles and C would be disjoint and of different lengths), actually G' is of minimum degree exactly 2. Let $P = (y_1, \dots, y_t)$ be a longest path in G' . By maximality of t , all neighbors of y_1 in G' are in P . If y_1 has three neighbors in G' , clearly we are done. It follows that y_1 has exactly two neighbors in P and then, necessarily, the neighbor of y_1 in P distinct from y_2 is y_g . So $C' = (y_1, \dots, y_g, y_1)$ is a cycle of G' of length g . We claim that $d_{G'}(y_2) = 2$. Indeed suppose that y_2 has a neighbor in P distinct from y_1 and y_3 . Necessarily we have $t \geq g + 1$ and the neighbor of y_2 is y_{g+1} . But then $(y_1, y_g, y_{g+1}, y_2, y_1)$ is a 4-cycle, which is impossible. Suppose now that y_2 has a neighbor y in $V(G') \setminus V(P)$. Then (y, y_2, \dots, y_t) is a longest path in G' . By similar arguments as earlier, y_g is a neighbor of y . But then (y_1, y_2, y, y_g, y_1) is a 4-cycle, which is again impossible. So, each of the vertices y_1 and y_2 has one neighbor in C . When $g \geq 7$, it can be noted that y_1, y_2 and a shortest path between their neighbors in C form a cycle of length less than g , which is impossible. So we are done when $g \geq 7$.

Assume now that $g = 5$. Recall that each of the vertices y_1, y_2 has one neighbor in C and clearly the neighbors of y_1, y_2 in C are distinct and not adjacent (for otherwise we would have a cycle of length less than 5). So vertices y_1, y_2 and a path between their neighbors in C of length 3 form a cycle C_1 of length 6. Let us consider the graph $G_1 = G - (V(C) \cup \{y_1, y_2\})$. Clearly every vertex of G_1 distinct from y_3 and y_5 is of degree at least 2 in G_1 . If one of the vertices y_3 and y_5 is of degree at least 2 in G_1 , then, by considering a longest path in G_1 , we deduce that G_1 contains a cycle, and then this cycle and one of the cycles C and C_1 would form two vertex-disjoint cycles of different lengths. Suppose now that y_3 and y_5 are both of degree 1 in G_1 . Since $(y_4, y_3, y_2, y_1, y_5, \dots, y_t)$ is a path of length t , by the previous arguments y_4 is of degree 2 in G' . Since $n \geq 15$, we have $V(G) \setminus (V(C) \cup V(C')) \neq \emptyset$, and the graph $G_2 = G - V(C) \cup V(C')$ is of minimum degree at least 2. It follows that G_2 contains a cycle, and then, as previously, we are done. This concludes the case $g = 5$.

Suppose now that $g = 6$. As $P' = (y_5, \dots, y_1, y_6, \dots, y_t)$ is also longest path of G' , by the previous reasoning, vertices y_5 and y_4 are of degree 2 in G' . It follows that each of the vertices y_1 , y_2 , and y_4 has one neighbor in C . Then the subgraph G_1 of G induced by $V(C) \cup \{y_1, y_2, y_3, y_4\}$ contains two cycles of different lengths. It is clear that $C'' = (y_t, y_{t-1}, \dots, y_{t-5}, y_t)$ is a 6-cycle of G . If $t \geq 10$, clearly $V(C'')$ is vertex-disjoint with $V(G_1)$ and we are done. It cannot be the case that $t = 9$ or $t = 7$ as y_4 and y_2 are of degree 2 in G' . Suppose now that $t = 6$. Since $n \geq 15$, we have $V(G) \setminus (V(C) \cup V(C'')) \neq \emptyset$ and we may assume that the graph $G - V(C) \cup V(P)$ does not contain cycles (for otherwise we are done). So there is a vertex z in $V(G) \setminus (V(C) \cup V(C''))$ that has a neighbor y_i (where $i = 3$ or $i = 6$). But then the vertices of C and z would form a path of order 7, a contradiction to the maximality of t .

Suppose now that $t = 8$. Clearly, vertices y_7 and y_8 are of degree 2 in G' . Since $n \geq 15$, subset $V(G) \setminus (V(C) \cup V(P))$ is non-empty, and the graph $G - V(C) \cup V(P)$ is of minimum degree at most 1. Then there exists a vertex z of $G - V(C) \cup V(P)$ having a neighbor in $V(P)$. Since y_1, y_2, y_4, y_5, y_7 and y_8 are of degree 2 in G' , this neighbor is either y_3 or y_6 . Without loss of generality, we may suppose that this neighbor is y_3 . Suppose that z has another neighbor in P . Necessarily this other neighbor is y_6 – but then $(z, y_3, y_4, y_5, y_6, z)$ would be a 5-cycle, which is impossible. So the only neighbor of z in P is y_3 . Then z has a neighbor u in $G - V(C) \cup V(P)$. Suppose that u has a neighbor v in $G - V(C) \cup V(P)$. Then $(v, u, z, y_3, \dots, y_8)$ would be a path of G' of length 9, which by maximality of $t = 8$ is impossible. It follows that u has a neighbor in P , and necessarily this neighbor is y_6 (because G' does not contain triangles). Note that z and u are of degree 2 in G' . So the eight vertices of $\Omega = \{y_1, y_2, y_4, y_5, y_7, y_8, z, u\}$ are of degree 2 in G' and then each vertex of Ω has exactly one neighbor in C (which has order 6). It follows that there exists a vertex w of C having two neighbors a and b in Ω . Since it can be checked that $d_{G'}(a, b) \leq 3$, we eventually get a cycle of length at most 5, which is not possible. So the case $g = 6$ is settled and the result is proved for $g \geq 5$. This concludes the whole proof. \square

Taking $D = 3$ in Theorem 23, we get that every cubic graph of order more than $20D - 46 = 14$ has two vertex-disjoint cycles of different lengths.

Theorem 24. *Every cubic graph of order at least 15 contains two vertex-disjoint cycles of different lengths.*

We remark that the Heawood graph $((3, 6)$ -cage, see Figure 1) of order 14 does not contain two vertex-disjoint cycles of different lengths. Thus the bound on the order in Theorem 24 is sharp.

4 Disjoint cycles of different lengths in digraphs

The results from this section are motivated by the following conjecture raised by Bermond and Thomassen in 1981 [3].

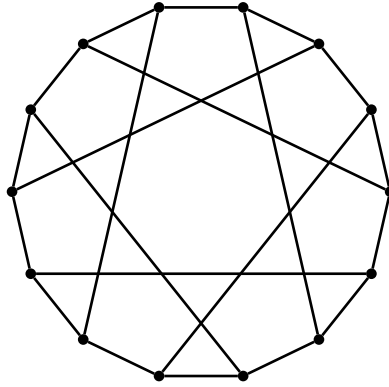


Figure 1: The Heawood graph.

Conjecture 25 (Bermond and Thomassen [3]). *For every $k \geq 2$, every digraph of minimum out-degree at least $2k - 1$ contains at least k vertex-disjoint directed cycles.*

If true, Conjecture 25 would be best possible because of the bidirected complete graph on $2k$ vertices. Towards Conjecture 25, one could more generally wonder whether for every k there is a smallest finite function $f(k)$ such that every digraph of minimum out-degree at least $f(k)$ contains at least k vertex-disjoint directed cycles (so $f(k)$ should be equal to $2k - 1$ according to Conjecture 25). Thomassen first proved in [15] that $f(k)$ exists for every $k \geq 1$. Later, Alon improved the value of $f(k)$ to $64k$ [1]. For $k = 2$, Thomassen proved in [15] that $f(k) = 3$, which confirms Conjecture 25 for $k = 2$. Later on, Lichiardopol, Pór and Sereni proved that for $k = 3$ the best value for $f(k)$ is 5, again confirming Conjecture 25 for $k = 3$ [11]. This apart, Conjecture 25 is still open, though some more partial results may be found in literature (see *e.g.* [2]).

Motivated by Conjecture 25 and in the same flavour as in Section 3, one can wonder about the existence of a (smallest) finite function $g(k)$ such that every digraph with minimum out-degree at least $g(k)$ contains k (vertex-) disjoint (directed) cycles of distinct lengths. In this context, the following was conjectured by Lichiardopol [10]:

Conjecture 26 (Lichiardopol [10]). *For every $k \geq 2$, there exists an integer $g(k)$ such that every digraph of minimum out-degree at least $g(k)$ contains k vertex-disjoint directed cycles of distinct lengths.*

It is worth pointing out that a similar function $h(k)$ for the existence of k disjoint cycles of the same length does not exist, as Alon proved that there exist digraphs of arbitrarily large minimum out-degree having no two (not necessarily disjoint) cycles of the same length [1]. Lichiardopol proved Conjecture 26 for $k = 2$ in [10], solving a question of Henning and Yeo [8]. We note that $g(k)$ should in general be a quadratic function of k ; for an illustration of this statement, consider a complete bidirected digraph D on $g(k) + 1$ vertices. Since $\delta^+(D) = g(k)$, there exist k disjoint cycles of different lengths in D . It follows that

$$g(k) + 1 \geq 2 + \cdots + k + 1,$$

hence $g(k) + 1 \geq \frac{1}{2}(k+1)(k+2) - 1$, which yields $g(k) \geq \frac{1}{2}(k^2 + 3k - 2)$ in general.

In this section, we verify Conjecture 26 in several contexts. We first verify it for tournaments in Section 4.1. Using the probabilistic method, we then verify it, in Sections 4.2 and 4.3, for some regular digraphs, and digraphs where the order is a polynomial function of the minimum out-degree. Some concluding remarks are gathered in Section 4.4.

4.1 Tournaments

We here verify Conjecture 26 for tournaments. More precisely, for every $k \geq 1$, we study the smallest finite function $g_t(k)$ such that every tournament of minimum out-degree at least $g_t(k)$ has k vertex-disjoint directed cycles of different lengths. We exhibit both an upper bound and a lower bound on $g_t(k)$ for every k .

In order to prove our upper bound, we need to introduce the following result first.

Lemma 27. *Every tournament of minimum out-degree $\delta \geq 1$ contains a directed cycle of order at least $2\delta + 1$.*

Proof. We proceed by induction on the order $n \geq 2\delta + 1$ of a tournament T of minimum out-degree δ . We claim that the assertion is true for $n = 2\delta + 1$. Indeed, in this case T is a regular tournament of degree δ . So T is strongly-connected, and then, by Theorem 10, we get that T is Hamiltonian. So the vertices of T form a directed cycle of order $2\delta + 1$, and we are done. Suppose now that the assertion is true up to the row $n - 1$, where $n \geq 2\delta + 2$, and let us study it for n . So T is a tournament of minimum out-degree δ and of order $n \geq 2\delta + 2$. If T is strongly-connected, then T is Hamiltonian and again we are done. Suppose thus that T is not strongly-connected. In that case, we consider a strongly-connected component B of T that is terminal (in the decomposition into strongly-connected components of T), *i.e.* with no arcs out-going to the rest of T . Then B has minimum out-degree at least δ . So, by the induction hypothesis, we deduce that B (and therefore T) contains a directed cycle of order at least $2\delta + 1$. So the assertion is true for n , which concludes the proof. \square

We are now ready to exhibit an upper bound on $g_t(k)$, and hence to confirm Conjecture 25 for tournaments.

Theorem 28. *For every $k \geq 1$, we have $g_t(k) \leq \frac{1}{2}(k^2 + 4k - 3)$.*

Proof. We proceed by induction on k . The assertion is clearly true for $k = 1$. So suppose that the assertion is true up to row $k - 1$ (where $k \geq 2$), and let us study it for k . Let T be a tournament of minimum out-degree at least $\frac{1}{2}(k^2 + 4k - 3)$. By the induction hypothesis T contains $k - 1$ disjoint cycles C_1, \dots, C_{k-1} of different lengths. From Theorem 11 (applied on the C_i 's), we now deduce that T has a collection C'_1, \dots, C'_{k-1} of disjoint cycles with $|V(C'_i)| = i + 2$, and therefore of different lengths.

We have

$$|V(C'_1) \cup \dots \cup V(C'_{k-1})| = 3 + \dots + k + 1 = \frac{k^2 + 3k - 4}{2} < \frac{k^2 + 4k - 3}{2}.$$

It thus follows that the tournament $T' = T - V(C'_1) \cup \dots \cup V(C'_{k-1})$ is of positive order and of minimum out-degree at least $\frac{k^2+4k-3}{2} - \frac{k^2+3k-4}{2} = \frac{k+1}{2}$. According to Lemma 27, we know that T' contains a cycle C'_k of length at least $k+2$ and, together with the directed cycles C'_1, \dots, C'_{k-1} , we get then k disjoint cycles of different lengths. Consequently, the assertion is true for k , and the result is proved. \square

We now deduce a lower bound on $g_t(k)$.

Observation 29. *For every $k \geq 1$, we have $g_t(k) \geq \frac{k^2+5k-2}{4}$.*

Proof. Let T be a regular tournament of degree $g_t(k)$. Then T contains k disjoint cycles of different lengths. Since the order of T is $2g_t(k) + 1$, it follows that

$$2g_t(k) + 1 \geq 3 + \dots + k + 2,$$

hence $2g_t(k) + 1 \geq \frac{(k+2)(k+3)}{2} - 3$, which yields $g_t(k) \geq \frac{k^2+5k-2}{4}$. \square

So, in the context of tournaments, according to Theorem 28 and Observation 29 we get the following.

Corollary 30. *For every $k \geq 1$, we have*

$$\frac{k^2 + 5k - 2}{4} \leq g_t(k) \leq \frac{k^2 + 4k - 3}{2}.$$

4.2 Regular digraphs

We now use the probabilistic method to prove, in the current section and upcoming Section 4.3, Conjecture 26 in two new contexts. To this aim, we first need to introduce a few tools and lemmas. The first two are classic tools of the probabilistic method, namely Chernoff's Inequality and the Lovász Local Lemma (see *e.g.* [13]).

Proposition 31 (Chernoff's Inequality). *Let X be a binomial random variable $BIN(n, p)$. Then, for any $0 \leq t \leq np$, we have $\Pr[X - np < -t] \leq e^{-t^2/3np}$.*

Proposition 32 (Lovász Local Lemma – Symmetric version). *Let A_1, \dots, A_n be a finite set of events in some probability space Ω such that each A_i occurs with probability at most p , where each A_i is mutually independent of all but at most d other events. If $4pd \leq 1$, then $\Pr[\cap_{i=1}^n \overline{A_i}] > 0$.*

We will also be needing the following fact on the existence of k (not necessarily disjoint) cycles with distinct lengths in a digraph with minimum out-degree at least k .

Proposition 33. *For every $k \geq 1$, every digraph D with minimum out-degree at least k contains k directed cycles of distinct lengths.*

Proof. Consider the out-neighbours of the last vertex of a longest directed path in D . \square

We are now ready to prove the main result of this section. By a r -regular digraph, we refer to a digraph whose all vertices have in- and out-degree r .

Theorem 34. *Let $k \geq 500$ and $r \geq \frac{k^2}{2}(1+7(\frac{\ln k}{k})^{\frac{1}{3}})$. Then every r -regular digraph contains at least k vertex-disjoint directed cycles of distinct lengths.*

Proof. Let D be a simple r -regular digraph, and assume $r \geq \frac{k^2}{2}(1+7(\frac{\ln k}{k})^{\frac{1}{3}})$. The main idea is to prove, using several probabilistic tools, that we can partition the vertex set of D into k parts V_1, \dots, V_k such that each V_i induces a digraph of minimum out-degree at least i . With such a partition in hand, one can then get the k desired disjoint cycles by just considering each of the V_i 's successively, and picking, in each of the digraphs they induce, one cycle whose length is different from the lengths of the previously picked cycles. This is possible according to Proposition 33 due to the out-degree property of the partition V_1, \dots, V_k .

We first introduce some notation and assumptions we use throughout (and further) to deal with our computations. Every parameter has to be thought of as a function of k . By writing $o(1)$, we refer to a term tending to 0 as k tends to infinity. Given two terms a and b , we write $a \sim b$ if a/b tends to 1, and $a \ll b$ if a/b tends to 0 as k tends to infinity. Let $k' := k + \lfloor k^{2/3}(\ln k)^{1/3} \rfloor$ and s be the sum of the first k' integers, that is $s := \frac{k'(k'+1)}{2}$. It is assumed throughout that $r \sim s \sim \frac{k^2}{2}$; so r and s will sometimes freely be replaced by $\frac{k^2}{2}$ to simplify some computation (assuming this has no impact on the computation).

We now show that, under our assumptions, the desired partition V_1, \dots, V_k of $V(D)$ exists with non-zero probability. For this purpose, let us just randomly k -color the vertices of D , where assigning color j to some vertex means that we put it into V_j . All colors are not assigned uniformly, but in such a way that, for every color $j \in \{1, \dots, k\}$, the probability p_j that some vertex is colored j is:

$$p_j := \frac{j + \lfloor k^{2/3}(\ln k)^{1/3} \rfloor}{s}.$$

Note that $\sum_{j=1}^k p_j < 1$, therefore a vertex gets no color with probability $1 - \sum_{j=1}^k p_j$.

Let X_v^j denote the number of out-neighbors of v colored j by the random process. Clearly $X_v^j \sim \text{BIN}(r, p_j)$. So that we can later apply the Lovász Local Lemma, let us define our set of bad events. Let A_v be the event

$$A_v := \bigcup_{j=1}^k (v \text{ is colored } j \text{ and } X_v^j < j).$$

Second, let B be the event that at least one of the k colors does not appear among a fixed subset $\{u_1, u_2, \dots, u_r\}$ of r vertices of D . It should be clear that any two events A_v and A_u are dependent if $(u \cup N^+(u)) \cap (v \cup N^+(v)) \neq \emptyset$. So, since D is r -regular, each A_v depends on at most r^2 other bad events A_u . Since the event B only depends on the colors of r fixed vertices, similarly B depends on at most r^2 other bad events. To apply the Lovász Local Lemma, every bad event $A \in (\cup_v A_v) \cup B$ must hence fulfill $4r^2 \Pr(A) \leq 1$.

Concerning B , we have:

$$\begin{aligned}\Pr(B) &\leq \sum_{j=1}^k (1 - p_j)^r \leq k \left(1 - \frac{\lfloor k^{2/3}(\ln k)^{1/3} \rfloor + 1}{s} \right)^r \\ &\leq k e^{-\frac{k^{2/3}(\ln k)^{1/3} r}{s}} \\ &\leq k^{-\left(\frac{k}{\ln k}\right)^{2/3} \cdot \frac{r}{k^2} + 1},\end{aligned}$$

where we used the fact that $s \leq k^2$ for $k \geq 500$. Therefore, we have that:

$$4r^2 \Pr(B) \leq 4r^2 k^{-\left(\frac{k}{\ln k}\right)^{2/3} \cdot \frac{r}{k^2} + 1} \leq 1,$$

since $r \geq k^2/2$ and $k \geq 500$.

Now consider the A_v 's. Since

$$s = \frac{(k')^2}{2} \left(1 + \frac{1}{k'} \right) \leq \frac{k^2}{2} \left(1 + \left(\frac{\ln k}{k} \right)^{\frac{1}{3}} \right)^2 \left(1 + \frac{2}{k} \right),$$

then

$$\left(1 - 4 \left(\frac{\ln k}{k} \right)^{\frac{1}{3}} \right) r p_j \geq \left(1 - 4 \left(\frac{\ln k}{k} \right)^{\frac{1}{3}} \right) \frac{r j}{s} \geq j$$

since $k \geq 500$. Applying Chernoff's Inequality, we get:

$$\begin{aligned}\Pr(X_v^j < j) &\leq \Pr \left(X_v^j < \left(1 - 4 \left(\frac{\ln k}{k} \right)^{\frac{1}{3}} \right) r p_j \right) \\ &\leq e^{-16 \left(\frac{\ln k}{k} \right)^{\frac{2}{3}} \frac{r p_j}{3}}.\end{aligned}$$

Then:

$$\Pr(A_v) \leq \sum_{j=1}^k p_j e^{-16 \left(\frac{\ln k}{k} \right)^{\frac{2}{3}} \frac{r p_j}{3}}.$$

We hence want every term in the sum to be smaller than $\frac{1}{4r^2 k}$. Thus, it suffices to show that, for each j , we have

$$\frac{\lfloor k^{2/3}(\ln k)^{1/3} \rfloor + j}{s} \exp \left(-16 \left(\frac{\ln k}{k} \right)^{\frac{2}{3}} \frac{r}{3} \frac{\lfloor k^{2/3}(\ln k)^{1/3} \rfloor + j}{s} \right) < \frac{1}{4r^2 k}.$$

Since $k^2/2 \leq s \leq k^2$ for $k \geq 500$ and $r \geq s$, this reduces to showing that we have

$$\left(\lfloor k^{2/3}(\ln k)^{1/3} \rfloor + j \right) k^{-\frac{16r}{3s}} \leq \frac{k}{8r^2}.$$

Since k is sufficiently large and $j \leq k$, this, in turn, amounts to showing that $k^{-\frac{16r}{3s}} \leq \frac{1}{16r^2}$ or equivalently, that $\frac{k^{\frac{16r}{3s}}}{16r^2} \geq 1$. Since k is sufficiently large, $r \geq s$ and $r \geq k^2/2$, the function on the left is increasing; hence it suffices to show that the inequality holds for $r = k^2/2$. Since $r \geq s$, we only need to show that $k^{16/3} \geq 4k^4$, which is clearly true when $k \geq 500$.

Under all these conditions, the requirements for applying the Lovász Local Lemma are thus met; we can hence deduce the desired partition V_1, \dots, V_k , thus the claimed k disjoint cycles of distinct lengths. \square

4.3 Small digraphs

We now prove Conjecture 26 for digraphs whose order can be expressed as some particular function of the minimum out-degree.

Theorem 35. *Let $k \geq 1$. Then every simple digraph of order at most c^{r^d} (where c and d are two constants satisfying $c > 1$ and $0 < d < 1$) and minimum out-degree $r \geq c_0 \max\{k^{\frac{1}{1-d}}, k^2\}$ contains at least k vertex-disjoint directed cycles of distinct lengths, where $c_0 = \max\{2, (24 \ln c)^{\frac{1}{1-d}}\}$.*

Proof. Let D be such a digraph with order n where we assume the out-degree of every vertex is exactly r . We herein reuse the terminology introduced in the proof of Theorem 34.

The proof is similar to that of Theorem 34, except that the random k -coloring of the vertices of D is this time performed uniformly (all k colors have the same probability to be assigned). Given any vertex v of D , let X_v denote the number of out-neighbors of v being assigned the same color by the random coloring. Clearly $X_v \sim \text{BIN}(r, 1/k)$. Our two kinds of bad events are the following. First A_v is, for every vertex v , the event that $X_v < k$. Second, let B be the event that at least one of the k colors does not appear at all.

We first assume the following:

$$\frac{r}{2k} \geq k \Leftrightarrow r \geq 2k^2. \quad (5)$$

Therefore, applying Chernoff's Inequality, we have:

$$\Pr(A_v) = \Pr(X_v < k) \leq \Pr\left(X_v < \frac{r}{2k}\right) \leq e^{-\frac{r}{12k}}.$$

By the Union Bound, we deduce:

$$\begin{aligned} \Pr\left(\bigcup_v A_v \cup B\right) &\leq \sum_v \Pr(A_v) + \Pr(B) \leq ne^{-\frac{r}{12k}} + k\left(1 - \frac{1}{k}\right)^n \\ &\leq ne^{-\frac{r}{12k}} + k\left(1 - \frac{1}{k}\right)^r \\ &\leq ne^{-\frac{r}{12k}} + ke^{-\frac{r}{k}}. \end{aligned} \quad (6)$$

Since we want this obtained sum to be smaller than 1, we would like each of its two terms to be strictly smaller than $1/2$. Then, concerning the first term:

$$ne^{-\frac{r}{12k}} < \frac{1}{2} \Leftrightarrow \frac{r}{12k} > \ln(2n).$$

Since $n \leq c^{r^d}$, we have $\ln(2n) \leq 2r^d \ln c$. So we need a stronger inequality:

$$\frac{r}{12k} \geq 2r^d \ln c \Leftrightarrow r \geq (24k \ln c)^{\frac{1}{1-d}}. \quad (7)$$

Concerning the second term of Inequality (6), we want:

$$ke^{-\frac{r}{k}} < \frac{1}{2} \Leftrightarrow r > k \ln(2k). \quad (8)$$

Suppose now that Inequalities (7) and (8) hold. Then there exists a partition V_1, \dots, V_k of $V(D)$ such that the out-degree of each vertex in the part containing it is at least k . We are now able to successfully pick a cycle from each of these parts in such a way that all picked cycles have distinct lengths (by applying Proposition 33).

So that all conditions of Inequalities (5), (7) and (8) are met, we then just need $r \geq c_0 \max\{k^{\frac{1}{1-d}}, k^2\}$, where $c_0 = \max\{2, (24 \ln c)^{\frac{1}{1-d}}\}$, as claimed. \square

4.4 Concluding remarks

In Sections 4.2 and 4.3, we have proved that Conjecture 26 holds for some regular digraphs (Theorem 34) and digraphs with bounded order (Theorem 35). About these two results, let us mention the following:

1. In the proofs of Theorems 34 and 35, if we require the parts of the partition V_1, \dots, V_k to include more vertices, then we can deduce k distinct cycles whose lengths are ‘more than just distinct’. Possible such additional properties include the cycles to be of even or odd lengths only, or to have lengths divisible by some fixed integer.
2. Let us mention that the bound on r in the statement of Theorem 34 is asymptotically best possible. This is the main reason why the random process in the proof is not uniform, and thus why the result requires k to be large enough. As a consequence, the proof could probably be adapted so that it works for all values of k , but, on the other hand, this would require to consider r -regular digraphs with r being a much bigger function of k .
3. In the statement of Theorem 35, it is worth mentioning that we can lower the requirement on the out-degree if we assume a smaller upper bound on n . For example, if n is bounded above by a polynomial function of r ($n < r^d$ for some constant $d > 0$), then we can require the digraph to have minimum out-degree at least $k^2(\ln k)^3$ only (which sticks closer to what is stated in Conjecture 26).
4. In case Conjecture 26 is false and some counterexamples exist, Theorem 35 gives a lower bound on the order of these counterexamples. These should be of large order.

References

- [1] N. Alon. Disjoint directed cycles. *Journal of Combinatorial Theory, Series B*, 68(2):167-178, 1996.
- [2] S. Bessy, N. Lichiardopol and J.-S. Sereni. Two proofs of the Bermond-Thomassen conjecture for tournaments with bounded minimum in-degree. *Discrete Mathematics*, 310(3):557-560, 2009.
- [3] J.-C. Bermond and C. Thomassen. Cycles in digraphs – a survey. *Journal of Graph Theory*, 5(1):1-43, 1981.
- [4] P. Camion. Chemins et circuits hamiltoniens des graphes complets. *Comptes Rendus de l'Académie des Sciences de Paris*, 249:2151-2152, 1959.
- [5] A.A. Diwan. Decomposing graphs with girth at least five under degree constraints. *Journal of Graph Theory*, 33:237-239, 2000.
- [6] A.A. Diwan. Cycles of even lengths modulo k . *Journal of Graph Theory*, 65(3):246-253, 2010.
- [7] G.A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, S3-2(1):69-81, 1952.
- [8] M.A. Henning and A. Yeo. Vertex Disjoint Cycles of Different Length in Digraphs. *SIAM Journal on Discrete Mathematics*, 26:687-694, 2012.
- [9] A. Kaneko. On decomposition of triangle-free graphs under degree constraints. *Journal of Graph Theory*, 27:7-9, 1998.
- [10] N. Lichiardopol. Proof of a Conjecture of Henning and Yeo on Vertex-Disjoint Directed Cycles. *SIAM Journal on Discrete Mathematics*, 28(3):1618-1627, 2014.
- [11] N. Lichiardopol, A. Pór and J.-S. Sereni. A Step toward the Bermond-Thomassen Conjecture about Disjoint Cycles in Digraphs. *SIAM Journal on Discrete Mathematics*, 23(2):979-992, 2009.
- [12] J.W. Moon. *Topics on Tournaments*. Holt, Rinehart and Winston, New York, 1968.
- [13] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method*. Springer, 2002.
- [14] M. Stiebitz. Decomposing graphs under degree constraints. *Journal of Graph Theory*, 23(3):321-324, 1996.
- [15] C. Thomassen. Disjoint cycles in digraphs. *Combinatorica*, 3(3-4):393-396, 1983.