On the connectivity of graphs in association schemes

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Abstract

Let $(X, \mathcal{R})$ be a commutative association scheme and let $\Gamma = (X, R \cup R^\top)$ be a connected undirected graph where $R \in \mathcal{R}$. Godsil (resp., Brouwer) conjectured that the edge connectivity (resp., vertex connectivity) of Γ is equal to its valency. In this paper, we prove that the deletion of the neighborhood of any vertex leaves behind at most one non-singleton component. Two distinct vertices $a, b \in X$ are called "twins" in Γ if they have identical neighborhoods: $\Gamma(a) = \Gamma(b)$. We characterize twins in polynomial association schemes and show that, in the absence of twins, the deletion of any vertex and its neighbors in Γ results in a connected graph. Using this and other tools, we prove lower bounds on the connectivity of Γ, especially in the case where Γ has diameter two. Among the applications of these results, we prove that the only connected relations in symmetric association schemes which admit a disconnecting set of size two are those which are ordinary polygons.

Keywords: Association scheme; Connectivity

1 Overview

Let X be a finite set of size v and let $\mathcal{R} = \{R_0, \ldots, R_d\}$ be a partition of $X \times X$ into binary relations such that R_0 is the identity relation on X and for each $i \in \{1, \ldots, d\}$ there exists $i' \in \{1, \ldots, d\}$ such that $R_i^\top = R_{i'}$ where $R^\top = \{(b, a) \mid (a, b) \in R\}$. We say $(X, \mathcal{R})$ is an *association scheme* (with *d classes*) if there exist integers p_{ij}^k $(0 \leqslant i, j, k \leqslant d)$ such that

$$|\{c \in X \mid (a, c) \in R_i \ \wedge \ (c, b) \in R_j\}| = p_{ij}^k$$

whenever $(a, b) \in R_k$. Throughout this paper, all association schemes are *commutative*: we require $p_{ij}^k = p_{ji}^k$ for all i, j, k. The problems addressed here immediately reduce to the *symmetric* case where $i' = i$ for all i; i.e., we will work with symmetric relations only.

Association schemes arise in group theory, graph theory, design theory, coding theory and more. For example, if X is a finite group with conjugacy classes $\mathcal{C}[g] = \{hgh^{-1} : h \in X\}$ $(g \in X)$, then the conjugacy class relations $R_g = \{(a, b) \mid ab^{-1} \in \mathcal{C}[g]\}$ yield a commutative association scheme on the vertex set X. The orbits on $X \times X$ of any permutation group G acting generously transitively on a set X give a symmetric association scheme. Some of the most well-studied association schemes are distance-regular graphs, including Moore graphs, distance-transitive graphs, strongly regular graphs, generalized polygons, etc. One studies q-ary error-correcting codes of length n as vertex subsets of the Hamming association scheme $H(n,q)$ [4, Sec. 9.2] and one studies t-(v,k,λ) designs as vertex subsets of the Johnson association scheme $J(v,k)$ [4, Sec. 9.1]. For an introduction to the extensive literature on the subject, the reader may consult [13, 2, 4, 17], the survey [22], or the more recent book of Bailey [1] which focuses on connections to the statistical design of experiments.

Let $(X,\mathcal{R})$ be a commutative d-class association scheme with *basis relations* $\mathcal{R} = \{R_0, \ldots, R_d\}$. For $1 \leqslant i \leqslant d$, we have a (possibly directed) simple graph $\Gamma_i = (X, R_i)$ on X. For $a \in X$, the set X is partitioned into *subconstituents* $R_i(a) = \{b \in X \mid (a,b) \in R_i\}$ $(0 \leqslant i \leqslant d)$ with respect to a. The association scheme is *symmetric* if all basis relations are symmetric; each Γ_i may be considered as an undirected graph in this case as $i' = i$ for all i. The association scheme is *primitive* [4, Sec. 2.4] if Γ_i is connected for all $i = 1, \ldots, d$ and *imprimitive* otherwise. A *system of imprimitivity* for $(X,\mathcal{R})$ is any non-trivial partition of X consisting of the components of some graph (X,R) where R is a union of basis relations. (The trivial partitions $\{X\}$ and $\{\{a\} \mid a \in X\}$ are not systems of imprimitivity.) For each i, we may construct an undirected graph H_i (possibly with loops) on vertex set $\{0, 1, \ldots, d\}$, joining j to k if $p_{ij}^k + p_{ik}^j > 0$. We call this the *unweighted distribution diagram* corresponding to basis relation R_i.

With reference to a fixed undirected graph Γ with vertex set $V\Gamma$ and edge set $E\Gamma$, we say that a and b are *twins* if $a \neq b$ yet $\Gamma(a) = \Gamma(b)$, where $\Gamma(a)$ denotes the set of neighbors of a in graph Γ. Write[1] $a^\perp = \{a\} \cup \Gamma(a)$. A graph Γ is *complete multipartite* if any two non-adjacenct vertices are twins: i.e., the complement of Γ is a union of complete graphs.

The main goal of this paper is to prove the following theorem:

Theorem 1. *Let $(X,\mathcal{R})$ be a symmetric association scheme. Assume the graph $\Gamma = (X, R_i)$ is connected and not complete multipartite. Let $H = H_i$ be the corresponding unweighted distribution diagram on $\{0, 1, \ldots, d\}$. The following are equivalent:*

(1) there exists $a \in X$ for which the subgraph $\Gamma \setminus a^\perp$ is connected;

(2) for all $a \in X$, the subgraph $\Gamma \setminus a^\perp$ is connected;

(3) the subgraph $H \setminus \{0, i\}$ is connected;

(4) Γ contains no twins.

We obtain the following corollaries.

Corollary 2. *Let $(X,\mathcal{R})$ be a commutative association scheme. Assume the undirected graph $\Gamma = (X, R_i \cup R_{i'})$ is connected and $a \in X$. Then $\Gamma \setminus \Gamma(a)$ contains at most one non-singleton component.*

[1] Note that some authors assign another meaning to $\perp$; here, we follow [4, p. 440].

Corollary 3. *Let $(X, \mathcal{R})$ be a commutative association scheme. Assume the undirected graph $\Gamma = (X, R_i \cup R_{i'})$ is connected and $a \in X$. Then, for any $T \subseteq a^\perp$ with $\Gamma(a) \not\subseteq T$, the graph $\Gamma \setminus T$ is connected.*

Corollary 4. *Let $(X, \mathcal{R})$ be a commutative association scheme. Assume the undirected graph $\Gamma = (X, R_i \cup R_{i'})$ is connected and $C \subseteq X$ is the vertex set of a clique in Γ. Then $\Gamma \setminus C$ is connected.*

The graphs considered in these theorems are all undirected graphs, either a symmetric basis relation in our association scheme or the symmetrization $(X, R_i \dot{\cup} R_{i'})$ of some directed basis relation. In both cases, the edge set of Γ is a basis relation of the *symmetrization* $(X, \mathcal{R}')$ of $(X, \mathcal{R})$ where

$$\mathcal{R}' = \left\{ R \cup R^\top \mid R \in \mathcal{R} \right\}.$$

In this way, the main theorem, while dealing only with the symmetric case, extends immediately to give these corollaries.

We should remark that these last two results extend naturally to the case where Γ is not connected in that the deletion of vertices does not increase the number of components. One verifies this by applying the respective corollary to the subscheme induced by vertices in a particular component of Γ.

2 Connectivity results for highly regular graphs

Before we provide proofs of these results and explore various consequences, we now survey earlier work on the connectivity of graphs in certain association schemes.

Brouwer and Mesner [3] showed in 1985 that the vertex connectivity of a strongly regular graph Γ is equal to its valency and that the only disconnecting sets of minimum size are the neighborhoods $\Gamma(a)$ of its vertices. (Brouwer [5] mentions that the corresponding result for edge connectivity was established by Plesník in 1975.) This result on vertex connectivity was extended by Brouwer and Koolen [6] in 2009 to show that a distance-regular graph of valency at least three has vertex connectivity equal to its valency and that the only disconnecting sets of minimum size are again the neighborhoods $\Gamma(a)$. Meanwhile a conjecture of Brouwer on the size and nature of the "second smallest" disconnecting sets in a strongly regular graph has inspired both new results and interesting examples by Cioabă, et al. [8, 9, 10, 11, 12].

Godsil [16] conjectured in 1981 that the edge connectivity of a connected basis relation in any symmetric association scheme is equal to the valency of that graph. Brouwer [5] claimed in 1996 that the same should hold for the vertex connectivity. In [16], Godsil proves that if $\Gamma = (X, R_1)$ is regular of valency v_1, then the edge connectivity of Γ is at least $\frac{v_1}{2} \frac{|X|}{|X|-1}$. In 2006, Evdokimov and Ponomarenko proved Brouwer's conjecture for $\Gamma = (X, R_1)$ in the case when $(X, \mathcal{R})$ is equal to the projection onto X of the v_1-fold tensor product $\bigotimes_{h=1}^{v_1} (X, \mathcal{R})$. See [15] for definitions and details.

Much more is known about the connectivity of vertex- and edge-transitive graphs. (See [18, Sec. 3.3-4].) Mader [19] and Watkins [26] independently obtained the following two results in 1970. The vertex connectivity of an edge-transitive graph is equal to the smallest valency. A vertex transitive graph of valency k has vertex connectivity at least

$\frac{2}{3}(k+1)$. Further, in 1971, Mader [20] proved that any vertex transitive graph has edge connectivity equal to its valency.

3 Preliminary results

Throughout this section, $(X, \mathcal{R})$ denotes a commutative association scheme.

In preparation for the proof of our main result, we now prove a few lemmas. We utilize basic terminology and notation regarding symmetric association schemes. We refer the reader to Section 2.2 of [4] for basic facts about the Bose-Mesner algebra and Section 2.4 of [4] for information on imprimitivity.

Let A_i denote the 01-matrix with rows and columns indexed by X and (a,b)-entry equal to one if $(a,b) \in R_i$ and equal to zero otherwise. Then the *Bose-Mesner algebra* $\text{span}(A_0, \ldots, A_d)$ is a complex vector space closed under both ordinary and entrywise multiplication. So it admits a basis $E_0 = \frac{1}{|X|}J, \ldots, E_d$ of pairwise orthogonal idempotents ($E_iE_j = \delta_{i,j}E_i$) and the change of basis matrices $[P_{ij}]_{i,j=0}^d$ and $[Q_{ij}]_{i,j=0}^d$ given by

$$A_j = \sum_{i=0}^{d} P_{ij}E_i \qquad \text{and} \qquad E_j = \frac{1}{|X|}\sum_{i=0}^{d} Q_{ij}A_i$$

satisfy $QP = |X|I$ (in particular, $\sum_{j=0}^d Q_{ij} = 0$ for $i \neq 0$) and $A_iE_j = P_{ji}E_j$ [4, p. 45], as well as $P_{ji} = \frac{v_i}{m_j}\bar{Q}_{ij}$ where $v_i = P_{0i}$ and $m_j = Q_{0j}$ [4, Lemma 2.2.1(iv)].

3.1 Twins

Let $\Gamma = (X, R)$ be the graph of a basis relation in $(X, \mathcal{R})$. Write $R(a) = \Gamma(a)$. Examples where twins arise (i.e., $R(a) = R(b)$ for $a \neq b$) include not only complete multipartite graphs but antipodal distance-regular graphs such as the n-cube in which case R is the distance-$\frac{n}{2}$ relation of the association scheme.

Lemma 5. *Let $(X, \mathcal{R})$ be a symmetric association scheme and let $\Gamma = (X, R_i)$ for some $i \neq 0$. If a and b are twins, then $(X, \mathcal{R})$ is imprimitive and some system of imprimitivity exists in which a and b belong to the same fibre.*

Proof. One easily checks that the following relation $\sim$ on X is an equivalence relation: $a \sim b$ if either $a = b$ or a and b are twins. To see that this is a system of imprimitivity, we verify that $\sim$ is the union of basis relations R_j for which $p_{ii}^j = p_{ii}^0$. Since $p_{ii}^i < p_{ii}^0$ and we are assuming at least one pair of twins exists, the equivalence relation is non-trivial and $(X, \mathcal{R})$ is imprimitive. □

Remark 6. We now discuss twins in polynomial association schemes. We use the well-known fact that an association scheme is imprimitive if and only if some idempotent E_j ($1 \leqslant j \leqslant d$) has repeated columns (see, e.g., [21, Theorem 2.1]). Denote by $u_j(a)$ the column of E_j indexed by $a \in X$. If $R_i(a) = R_i(b)$, then for each $0 \leqslant j \leqslant d$

$$P_{ji}u_j(a) = A_iu_j(a) = \sum_{(x,a)\in R_i} u_j(x) = \sum_{(x,b)\in R_i} u_j(x) = A_iu_j(b) = P_{ji}u_j(b)$$

so that either $P_{ji} = 0$ or $u_j(a) = u_j(b)$.

1. Assume $(X, \mathcal{R})$ is the association scheme coming from a distance-regular graph $\Gamma = (X, R_1)$ with distance-k relation R_k for $0 \leqslant k \leqslant d$ and assume $R_i(a) = R_i(b)$ for distinct vertices a and b. Suppose a and b do not belong to a common antipodal fibre in an antipodal system of imprimitivity. Then Γ must be bipartite, in which case columns a and b of E_j can be identical only for $j \in \{0, d\}$ (where $E_0, \ldots, E_d$ are ordered so that $P_{01} > P_{11} > \cdots > P_{d1} = -P_{01}$ [4, Prop. 4.4.7]). But then, except for $d = 2$, there is some $j \neq 0, d$ for which $P_{ji} \neq 0$; thus $a = b$ for $d > 2$. So bipartite systems of imprimitivity only arise for $d = 2$. Viewing complete bipartite graphs as having the antipodal property, we then have that any distinct a and b with $R_i(a) = R_i(b)$ must belong to the same antipodal fibre, d is even, and $i = d/2$.

2. Assume $(X, \mathcal{R})$ is a Q-polynomial ("cometric") association scheme [4, Section 2.7], not a polygon, and $a \neq b$ yet $R_i(a) = R_i(b)$. Then, by a theorem of Suzuki, et al. [24, 7, 25], $(X, \mathcal{R})$ is either Q-bipartite or Q-antipodal. Let $E_0, \ldots, E_d$ be a Q-polynomial ordering of the primitive idempotents and order relations such that $Q_{01} > Q_{11} > \cdots > Q_{d1}$. If a and b belong to the same fibre of a Q-bipartite imprimitivity system, then d must be even and $i = \frac{d}{2}$ by Corollary 4.2 in [21]. Otherwise, a and b must belong to the same Q-antipodal fibre and $u_j(a) = u_j(b)$ only for $j \in \{0, d\}$. So $P_{ji} = 0$ for $1 \leqslant j < d$, forcing (X, R_i) to be an imprimitive strongly regular graph (as it is regular with three eigenvalues). Since the scheme is cometric with an imprimitive strongly regular graph as a basis relation, we must have $d = 2$ and a and b are non-adjacent vertices in a complete multipartite graph.

3.2 The graph homomorphism φ_a

For $0 < i \leqslant d$, let $\Gamma_i = (X, R_i)$ and let H_i denote the unweighted distribution diagram corresponding to symmetric relation R_i.

Proposition 7. *For any $a \in X$, the map $\varphi_{a,i} : \Gamma_i \to H_i$ sending $b \in X$ to j where $(a, b) \in R_j$ is a graph homomorphism. Under this map, every walk in Γ_i projects to a walk in H_i of the same length. As a partial converse, for any $b \in X$ with $(a, b) \in R_{j_0}$ and any walk*

$$w = (j_0, j_1, \ldots, j_\ell)$$

in H_i, there is at least one walk $(b = b_0, b_1, \ldots, b_\ell)$ of length ℓ in Γ_i such that $\varphi_{a,i}(b_s) = j_s$ for each $0 \leqslant s \leqslant \ell$. □

We will call $\varphi_{a,i}$ the *projection map* and will omit the second subscript when it is clear from the context.

For vertices x and y in an undirected graph Δ, we use $d_\Delta(x, y)$ to denote the path-length distance from x to y in Δ, setting $d_\Delta(x, y) = \infty$ when no path from x to y exists in Δ.

Lemma 8. *Let $(X, \mathcal{R})$ be a symmetric association scheme and, for some $0 < i \leqslant d$, let $\Gamma = (X, R_i)$ with corresponding unweighted distribution diagram H. If Γ is connected, then for $(a, b) \in R_j$, $d_\Gamma(a, b) = d_H(0, j)$.*

Proof. A shortest path in H from j to 0 lifts via $\varphi_{a,i}^{-1}$ to a walk in Γ from b to a vertex in $R_0(a)$ — i.e., lifts to a walk from b to a — of length $d_H(j, 0)$. Conversely, each path from b to a in Γ projects to a walk of the same length from j to 0 in H. □

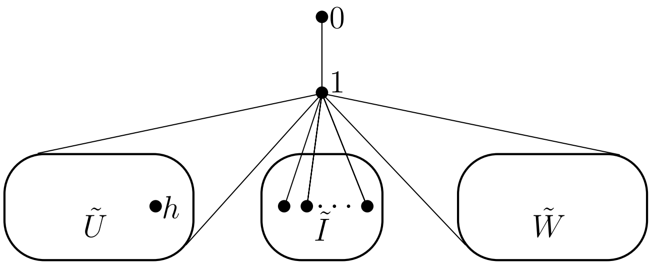

Figure 1: Graph H. Upon deletion of 0 and 1, the isolated vertices in $\tilde{I}$ are those subconstituents which contain all twins of the basepoint while $\tilde{U}$ is vertex set of a component outside $\tilde{I}$ which minimizes $\sum_{i\in\tilde{U}} v_i$.

3.3 The decomposition $\{I_a, U_a, W_a\}$ with respect to a basepoint a

For simplicity, we henceforth take $\Gamma = (X, R_1)$ with unweighted distribution diagram $H = H_1$ in the symmetric association scheme $(X, \mathcal{R})$. We assume throughout the remainder of the paper that Γ itself is a connected graph. We will compare the graphs $\Gamma_a := \Gamma \setminus a^\perp$ and $H' := H \setminus \{0, 1\}$ and show that, with known exceptions, one is connected if and only if the other is connected. One direction is straightforward.

Proposition 9. *If H' is not a connected graph, then for any $a \in X$, Γ_a is also disconnected. If i and j are in distinct components of H', then Γ_a contains no path from $R_i(a)$ to $R_j(a)$.*

Proof. Let $x \in R_i(a)$ and $y \in R_j(a)$ and suppose $x = x_0, x_1, \ldots, x_\ell = y$ is a path in Γ_a. Then $i = \varphi_a(x_0), \varphi_a(x_1), \ldots, \varphi_a(x_\ell) = j$ is a walk from i to j in H. Since H' is disconnected, $\varphi_a(x_t) \leqslant 1$ for some t which forces $x_t \in a^\perp$, a contradiction. $\square$

Proposition 10. *If x and y lie in distinct components of Γ_a, then $\Gamma(x) \cap \Gamma(y) \subseteq \Gamma(a)$.* $\square$

For $\tilde{U} \subseteq \{0, 1, \ldots, d\}$, note that $|\varphi_a^{-1}(\tilde{U})| = \sum_{i\in\tilde{U}} v_i$. We now assume that H' is disconnected and we define a decomposition of its vertex set. Let

$$\tilde{I} = \{i > 0 \mid p^i_{11} = p^0_{11}\}.$$

Now the set $\{2, \ldots, d\} \setminus \tilde{I}$ decomposes naturally into the vertex sets of the connected components of H', excluding the isolated vertices in $\tilde{I}$. Let $\tilde{U}$ be the vertex set of some component of $H' \setminus \tilde{I}$ such that $|\varphi_a^{-1}(\tilde{U})|$ is minimized. Let $\tilde{W} = \{2, \ldots, d\} \setminus \left(\tilde{I} \cup \tilde{U}\right)$ as depicted in Figure 1. For $x \in X$, set

$$I_x = \varphi_x^{-1}(\tilde{I}), \qquad U_x = \varphi_x^{-1}(\tilde{U}), \qquad W_x = \varphi_x^{-1}(\tilde{W})$$

and note that $|I_x|$, $|U_x|$, and $|W_x|$ are independent of the choice of $x \in X$. Observe that x and y are twins if and only if $y \in I_x$. While our basepoint will vary in what follows, our choice of $\tilde{U}$, $\tilde{W}$ and $\tilde{I}$ will remain fixed for this connected graph Γ.

Lemma 11. *If $\tilde{W} \neq \emptyset$, then for every $u \in U_x$, $d_\Gamma(x, u) = 2$.*

Proof. By way of contradiction, assume $u \in U_x$ with $\Gamma(x) \cap \Gamma(u) = \emptyset$. For any $w \in W_x$, we note that Γ contains an xw-path which does not pass through $u^\perp$. So x and w lie in the same connected component of Γ_u. But if $(x,u) \in R_h$ then $h \in \tilde{U}$ so $x \in U_u$ by symmetry. It follows that $W_x \cup \{x\} \subseteq U_u$. But this contradicts $|\varphi_u^{-1}(\tilde{U})| \leqslant |\varphi_x^{-1}(\tilde{W})|$. $\square$

3.4 Comparing the view from multiple basepoints

Proposition 12. *For any $a \in X$ and any $b \in U_a$, we have $W_a \cap I_b = \emptyset$.*

Proof. If x and b are twins, then x cannot be a twin of a since b is not a twin of a. So $\Gamma(x) = \Gamma(b) \subseteq U_a \cup \Gamma(a)$ gives $\Gamma(x) \cap U_a \neq \emptyset$. So $x \notin W_a$. $\square$

Now fix $a \in X$ and choose $b \in U_a$. Consider the component Δ of Γ_b containing a. Since b and a are not twins, some element of $\Gamma(a)$ is a vertex of Δ and hence Δ contains vertices in W_a unless $\tilde{W} = \emptyset$. Let $Z_a = V\Delta \cap W_a$ and let $Y_a = W_a \setminus Z_a$. This vertex decomposition is depicted in Figure 2. Since $b \in U_a$, we have $a \in U_b$ and, since Δ is connected, $Z_a \subseteq U_b$.

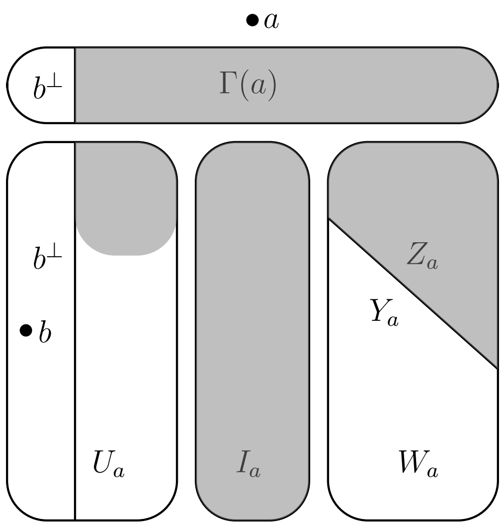

Figure 2: This diagram depicts Γ as decomposed relative to basepoint a. In Γ_b, vertex a belongs to component Δ, whose vertex set is indicated by the shaded region. The set W_a splits into Z_a and Y_a according to membership in $V\Delta$.

In the next two results, we proceed under the hypotheses stated at the beginning of Section 3.3 and assume that vertices a and $b \in U_a$ have been chosen and the sets Y_a and Z_a are defined as above relative to this pair of vertices.

Lemma 13. *Let $w = (u_0, u_1, \ldots, u_\ell)$ be a walk in Γ with $u_0 \in Y_a$ and u_ℓ lying some other component of Γ_a. Let $t \in \{1, \ldots, \ell\}$ be the smallest subscript with $u_t \notin Y_a$. Then $u_t \in \Gamma(a)$.* $\square$

Lemma 14. *For $0 \leqslant i \leqslant d$, $R_i(a) \cap Y_a \neq \emptyset$ implies $R_i(a) \cap Z_a \neq \emptyset$. So no subconstituent of Γ with respect to a is entirely contained in Y_a.*

Proof. Let $y \in Y_a$ and consider a shortest ya-path π in Γ, of length ℓ say, and label its vertices as follows: $\pi = (y = v_\ell, v_{\ell-1}, \dots, v_1, v_0 = a)$. Then, by Lemma 13, $v_s \in Y_a$ for $1 < s \leqslant \ell$. Consider $j_s = \varphi_a(v_s)$, $0 \leqslant s \leqslant \ell$, and assume $j_\ell = i$. Then we have $p^{j_s}_{1,j_{s+1}} > 0$ for $0 \leqslant s < \ell$. Note $j_0 = 0$ and $j_1 = 1$. Now we lift the walk $(j_0, \dots, j_\ell)$ in H to a different walk in Γ. Since a and b are not twins, we may choose $v_1' \in \Gamma(a) \setminus \Gamma(b)$. Since $p^1_{1j_2} > 0$, there exists $v_2' \in R_{j_2}(a)$ with v_2' adjacenct to v_1' in Γ. Continuing in this manner, we may construct a walk $\pi' = (a = v_0', v_1', \dots, v_\ell')$ in Γ with $\varphi_a(v_s') = j_s$. Since $\Gamma(b) \subseteq \Gamma(a) \cup U_a$, none of the vertices v_s' lie in $\Gamma(b)$, so the entire walk π' is contained in one component of Γ_b. By definition of Z_a, we then have $v_\ell' \in Z_a \cap R_i(a)$. □

Lemma 15. *If $\tilde{W} \neq \emptyset$, then Γ has diameter two; i.e., $p^i_{11} > 0$ for all $i > 1$.*

Proof. Let $a, x \in X$ with $x \notin a^\perp$. Choose $b \in U_a$ as above and consider, in turn, each part of the decomposition

$$V\Gamma_a \;=\; I_a \,\dot\cup\, U_a \,\dot\cup\, Z_a \,\dot\cup\, Y_a$$

relative to a and b. If $x \in I_a$, $\Gamma(x) = \Gamma(a)$; if $x \in U_a$, then $d_\Gamma(a,x) = 2$ by Lemma 11. Next consider $x \in Z_a$. Then $d(x,b) = 2$ but $\Gamma(x) \cap \Gamma(b) \subseteq \Gamma(a)$ since x and b lie in distinct components of Γ_a (Proposition 10). Finally, consider $x \in Y_a$ with $(a,x) \in R_i$. By Lemma 14, there exists $x' \in Z_a \cap R_i(a)$. Since x' has a neighbor in $\Gamma(a)$, $p^i_{11} > 0$ which then implies that some neighbor of x lies in $\Gamma(a)$ as well. □

Theorem 16. *Let $(X, \mathcal{R})$ be any symmetric association scheme and let $\Gamma = (X, R_1)$ be any connected basis relation. With reference to the above definitions, $\tilde{W} = \emptyset$.*

Proof. By way of contradiction, assume $\tilde{W} \neq \emptyset$ and define

$$\mu = \min\{p^i_{11} \mid i \in \tilde{U}\}, \qquad \omega = \min\{p^i_{11} \mid i \in \tilde{W}\}$$

and select $k \in \tilde{U}$ and $\ell \in \tilde{W}$ with $p^k_{11} = \mu$ and $p^\ell_{11} = \omega$. Note that $\mu > 0$ and $\omega > 0$ by Lemma 15. Now choose $a \in X$, and select x in $R_k(a)$. Since x is not a twin of a, we may choose $a' \in \Gamma(a) \setminus \Gamma(x)$ and since $p^1_{1\ell} > 0$, we may choose and y in $R_\ell(a)$ which is a neighbor of a'. Since Γ_x contains a path from a to y and $a \in U_x$, we have $y \in U_x$. So $|\Gamma(x) \cap \Gamma(y)| \geqslant \mu$. By Proposition 10, $\Gamma(x) \cap \Gamma(y) \subseteq \Gamma(a)$. (See Figure 3.) But $a' \in \Gamma(y) \cap \Gamma(a)$. So

$$\omega \geqslant 1 + |\Gamma(x) \cap \Gamma(y)| > \mu.$$

Now we simply reverse the roles of x and y; more precisely, we swap ℓ and k.

Select x in $R_\ell(a)$ and, choosing $a' \in \Gamma(a) \setminus \Gamma(x)$, we may find a vertex y in $R_k(a)$ which is a neighbor of a'. Since Γ_x contains a path from a to y and $a \in W_x$, we have $y \in W_x$. So $|\Gamma(x) \cap \Gamma(y)| \geqslant \omega$. By Proposition 10, $\Gamma(x) \cap \Gamma(y) \subseteq \Gamma(a)$. But $a' \in \Gamma(y) \cap \Gamma(a)$. So

$$\mu \geqslant 1 + |\Gamma(x) \cap \Gamma(y)| > \omega.$$

We have $\omega > \mu$ and $\mu > \omega$, producing the desired contradiction. □

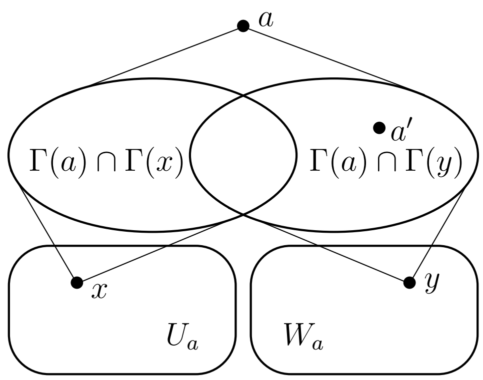

Figure 3: Since Γ has diameter two, all common neighbors of x and y are contained in $\Gamma(a)$.

4 Proofs of the main theorem and its corollaries

We are now ready to present the proof of our main theorem. The notation in this section is defined either in the statements of the results, the introductory material leading up to the statement of Theorem 1 in Section 1, with the exception of the index sets $\tilde{I}$ and $\tilde{W}$ which are fixed in Section 3.3. Recall that, for $a \in X$, $\Gamma_a := \Gamma \setminus a^\perp$ is the subgraph of $\Gamma = (X, R_i)$ obtained by deleting basepoint a and all its neighbors.

Proof of Theorem 1. As before, we assume $i = 1$ for notational convenience.

We begin by showing (3) $\Leftrightarrow$ (4). If a and b are twins in Γ with $(a, b) \in R_j$, then $j > 1$ and $p^j_{11} = v_1$ so that $j \in \tilde{I}$ and $\{j\}$ is the entire vertex set of some component of $H' = H \setminus \{0, 1\}$. So either H' is not connected or $d = j = 2$ and Γ, being imprimitive, is a complete multipartite graph. Conversely, by Theorem 16, $\tilde{W} = \emptyset$ so if $\tilde{I} = \emptyset$ we have that H' is connected.

The assertion (2) $\Rightarrow$ (1) is trivial. Proposition 9 gives us (1) $\Rightarrow$ (3). So we need only check that (3) implies (2).

Assume now that H' is connected and yet there is some $a \in X$ with Γ_a not connected. By Proposition 7, any x in Γ_a is joined by a walk in Γ_a to some vertex in $R_j(a)$ for every $j > 1$. (Simply lift a walk in H' from ℓ to j where $(a, x) \in R_\ell$.) So for every $j > 1$ every connected component of Γ_a intersects every subconstituent $R_j(a)$ non-trivially. Select $j > 1$ so as to maximize $D := d_H(0, j)$ and choose $x, y \in R_j(a)$ such that x and y lie in distinct components of Γ_a. Then every xy-path in Γ must include a vertex in $\Gamma(a)$, so $d_\Gamma(x, y) \geqslant 2(D - 1)$. Since $d_\Gamma(x, y) \leqslant D$ by Lemma 8, this forces $D \leqslant 2$. In particular, $p^\ell_{11} > 0$ for every $\ell > 1$.

Select $\ell > 1$ so as to minimize p^ℓ_{11} and select $x, y \in R_\ell(a)$ from distinct components of Γ_a. Then $(x, y) \in R_j$ for some $j > 1$ and so $|\Gamma(x) \cap \Gamma(y)| \geqslant p^\ell_{11}$. But since these two vertices lie in distinct components, Proposition 10 gives us

$$\Gamma(x) \cap \Gamma(y) \subseteq \Gamma(a) \cap \Gamma(y)$$

so $p^j_{11} = p^\ell_{11}$ and $\Gamma(x) \cap \Gamma(y) = \Gamma(a) \cap \Gamma(y)$. If $a' \in \Gamma(a)$, then a' has $p^1_{1\ell} > 0$ neighbors in $R_\ell(a)$. For any such neighbor z, we must have either $\Gamma(z) \cap \Gamma(a) = \Gamma(x) \cap \Gamma(a)$ or

$\Gamma(z) \cap \Gamma(a) = \Gamma(y) \cap \Gamma(a)$, both of which force $a' \in \Gamma(x)$. So vertices a and x must be twins. The only possibility that remains is that Γ is a complete multipartite graph. $\square$

The proofs of Corollaries 2, 3, and 4 are now rather immediate. Since each is a statement about the symmetrization of some commutative scheme, Theorem 1 applies.

Proof of Corollary 2. This is essentially Theorem 16. $\square$

Proof of Corollary 3. We apply Theorem 1 to prove this. First, if we have no twins then Γ_a is connected. Any $a' \in \Gamma(a)$ has at least one neighbor in $V\Gamma_a$. If $a \notin T$, then some $a' \in \Gamma(a)$ is also not included in T. So the graph $\Gamma \setminus T$ is connected as long as $T \neq \Gamma(a)$.

If b is a twin of a in Γ, then b is adjacent to every $x \in \Gamma(a)$. Since $\Gamma(a) \not\subseteq T$, some $a' \in \Gamma(a)$ is a vertex of $\Gamma \setminus T$. By Corollary 2, $\Gamma \setminus a^{\perp}$ has at most one non-singleton component. Let Ξ be the component of $\Gamma \setminus T$ containing this component as a connected subgraph. (If $\Gamma \setminus a^{\perp}$ consists only of singletons, choose Ξ to be any component of $\Gamma \setminus T$ containing some twin of a.) Since a' has at least one neighbor in $V\Gamma_a$, the component Ξ contains a' and every twin b of a since each of these is a neighbor of a'. Likewise, if $a \notin T$, then a belongs to Ξ since it is adjacent to a'. So in this case as well, $\Gamma \setminus T$ is connected. $\square$

Proof of Corollary 4. Let $a \in C$ and take $T = C$. Then apply Corollary 3. $\square$

We finish this section with a simple generalization arising from the proof above.

Theorem 17. *Assume $(X, \mathcal{R})$, Γ and H are defined as in Theorem 1. Let $B_{H,t}(0) = \{i \mid 0 \leqslant i \leqslant d,\ d_H(0,i) \leqslant t\}$ and $B_{\Gamma,t}(a) = \cup_{B_{H,t}(0)} R_i(a)$.*

(a) If $\Gamma' := \Gamma \setminus B_{\Gamma,t}(a)$ is disconnected and $b \in X$ with $d_\Gamma(a,b) = D$ (the diameter of Γ), then for any $x \notin B_{\Gamma,t}(a)$ not in the same component of Γ' as b, we have $d_\Gamma(a,x) \leqslant 2t$.

(b) If $H \setminus B_{H,t}(0)$ is connected and yet Γ' is disconnected, then $D \leqslant 2t$.

Proof. (a) Since x and b are in distinct components of Γ', there must exist some $y \in X$ such that $d_\Gamma(a,y) \leqslant t$ and $d_\Gamma(x,b) = d_\Gamma(x,y) + d_\Gamma(y,b)$. This gives $d_\Gamma(y,b) \geqslant D - t$ which then implies $d_\Gamma(x,a) \leqslant d_\Gamma(x,y) + d_\Gamma(y,a) \leqslant 2t$.

(b) Since $H \setminus B_{H,t}(0)$ is connected, for every $j \notin B_{H,t}(0)$, $R_j(a)$ has non-trivial intersection with every component of Γ'. So we may select x, b in distinct components of Γ' both satisfying $d_\Gamma(a,x) = d_\Gamma(a,b) = D$ and then apply part (a). $\square$

5 Further results on connectivity

In this section, we develop some machinery for the study of small disconnecting sets which are not localized. We then apply these tools to show that, with the exception of polygons, a basis relation in a symmetric association scheme has vertex connectivity at least three. We can say a bit more in the case where Γ has diameter two. For the remainder of this paper, we assume without loss of generality that $\Gamma = (X, R_1)$ in order to simplify notation.

Elementary graph theoretic techniques allow us to handle the case where Γ is in some sense locally connected. For example, if $\Gamma(y)$ induces a connected subgraph for every $y \in T$ and $d_\Gamma(y,y') \geqslant 3$ for any pair of distinct elements $y, y' \in T$, then $\Gamma \setminus T$ is connected. The proof of this claim is essentially the same as the proof of the following proposition, which applies more generally to any connected simple graph Γ.

Proposition 18. *Let $\Gamma = (X, R_1)$ be the graph associated to a connected basis relation in a symmetric association scheme $(X, \mathcal{R})$. Suppose any two vertices at distance two in Γ lie in some common cycle of length at most g and $T \subseteq V\Gamma$ satisfies $d_\Gamma(y,y') \geqslant g+1$ for all pairs y, y' of distinct vertices from T. Then $\Gamma \setminus T$ is connected.*

Proof. Set $\delta = \lfloor g/2 \rfloor$ and, for $y \in T$ set $B_\delta(y) = \{x \in X \mid d_\Gamma(x,y) \leqslant \delta\}$. The induced subgraph $\Gamma[B]$ of Γ determined by $B = B_\delta(y)$ is connected so admits a spanning tree. Moreover, since y is not a cut vertex of $\Gamma[B]$, there exists a spanning tree $\mathcal{T}_y$ for $\Gamma[B]$ in which y is a leaf vertex. For $y \in T$, let E_y denote the edge set of $\mathcal{T}_y$ with the sole edge incident to y removed.

Now consider the minor Δ of Γ obtained by contracting $B_\delta(y)$ to a single vertex for every $y \in T$. Since Δ is again a connected graph, it admits a spanning tree $\mathcal{T}$. Lift the edge set $E_\mathcal{T}$ of $\mathcal{T}$ back to $E\Gamma$ and note that $E_\mathcal{T}$ contains no edge from any of the induced subgraphs $\Gamma[B_\delta(y)]$, $y \in T$. So $E_\mathcal{T} \cup (\cup_{y \in T} E_y)$ is the edge set of a spanning tree in $\Gamma \setminus T$, which demonstrates that $\Gamma \setminus T$ is connected. $\square$

5.1 A spectral lemma

Eigenvalue techniques such as applications of eigenvalue interlacing play an important role in [3] and [6]. The following lemma is inspired by those ideas. This can be used, in conjunction with Lemma 23, to show that a graph with a small disconnecting set T whose elements are not too close together must be locally a disjoint union of cliques of size at most $|T|$.

Lemma 19. *Let $(X, \mathcal{R})$ be a symmetric association scheme and let $\Gamma = (X, R_1)$ be the graph associated to a connected basis relation. Assume that Γ contains no induced subgraph isomorphic to $K_{2,1,1}$. If $T \subseteq X$ is a disconnecting set for Γ, then $|T| > p^1_{11}$.*

Proof. The result obviously holds when Γ is complete multipartite, so assume Γ is not a complete multipartite graph. By [4, Cor. 3.5.4(ii)], we then know that the second largest eigenvalue θ of Γ is positive. Order the eigenspaces of the scheme so that $A_1 E_1 = \theta E_1$ and abbreviate $E = E_1$. For $K, L \subseteq X$, denote by $E_{K,L}$ the submatrix of E obtained by restricting to rows in K and columns in L. Let C be any clique in Γ. Then, because $v_1 > \theta > 0$, the matrix $E_{C,C} = \frac{m_1}{|X|} I + \frac{\theta m_1}{v_1 |X|}(J - I)$ is invertible.

Assume now that some disconnecting set $T \subseteq X$ has $|T| \leqslant p^1_{11}$. Let Ξ and Ξ' be two connected components of $\Gamma \setminus T$ with vertex sets B and B', respectively, and let ρ and ρ' denote the spectral radii of these two graphs. Assume, without loss, that $\rho \leqslant \rho'$. By eigenvalue interlacing, $\rho \leqslant \theta$. (see, e.g.,[4, Theorem 3.3.1].) We now show $\rho = \theta$.

Since Γ does not contain $K_{2,1,1}$ as an induced subgraph, it is locally a disjoint union of cliques and every edge of Γ lies in a clique C of size $p^1_{11} + 2$. If Ξ is edgeless, then T contains all neighbors of some vertex, which is impossible since $|T| \leqslant p^1_{11} < v_1$. So Ξ contains at least one edge and $B \cup T$ contains some clique C of size at least $p^1_{11} + 2$. It follows that the submatrix $E_{X,B\cup T}$ has rank at least p_{11}^1+2. But $|T|\leqslant p_{11}^1$. So the row space of $E_{X,B\cup T}$ contains at least two linearly independent vectors which are zero in every entry indexed by an element of T. Restricting these two vectors to coordinates in B only, we obtain two linearly independent eigenvectors for graph Ξ belonging to eigenvalue θ. It follows that $\rho=\theta$ and ρ, the spectral radius of Ξ, is not a simple eigenvalue. This contradicts the Perron-Frobenius Theorem (see, e.g., [4, Theorem 3.1.1]) since Ξ was chosen to be a connected graph. □

Remark 20. The hypotheses of the above lemma may clearly be weakened. The proof simply requires that both $B\cup T$ and $B'\cup T$ contain cliques of size $|T|+2$ or larger and that the entries E_{xy} of idempotent E are the same for all adjacent x and y in $V\Gamma$.

5.2 Intervals and metric properties of Γ

Let $(X,\mathcal{R})$ be a symmetric association scheme and $\Gamma=(X,R_1)$ with unweighted distribution diagram H. For $a,b\in X$, if $(a,b)\in R_i$, Lemma 8 tells us that the path-length distance $d_\Gamma(a,b)$ between a and b in graph Γ is equal to the path-length distance $d_H(0,i)$ between 0 and i in H. It follows that the diameter, D say, of Γ is equal to $\max_i d_H(0,i)$, which happens to be the diameter of H. We thus partition the index set $\{0,1,\ldots,d\}$ according to distance from 0 in H. For each $0\leqslant h\leqslant D$, define $I_h=\{i : d_H(0,i)=h\}$. For $0\leqslant i\leqslant d$ with $i\in I_h$, define

$$c(i)=\sum_{j\in I_{h-1}} p_{1j}^i \ .$$

Proposition 21. *With $c(i)$ defined as above*

(a) For any geodesic $0=\ell_0,1=\ell_1,\ell_2,\ldots,\ell_h$ in H,

$$1=c(\ell_1)\leqslant c(\ell_2)\leqslant\cdots\leqslant c(\ell_h).$$

(b) If $c(i)=1$, then for any $\ell\in\{1,\ldots,d\}$ which lies along a geodesic from 0 to i in H, $c(\ell)=1$ as well.

(c) If $c(i)=1$, then there is a unique shortest path in H from 0 to i and, for $(a,b)\in R_i$, there is a unique shortest path in Γ from a to b.

Proof. For part (a), observe that for $(a,b)\in R_{\ell_h}$ there exists $a'\in R_{\ell_{h-1}}(b)$ adjacent to a since $p_{1,\ell_{h-1}}^{\ell_h}>0$ so that

$$\{x\mid (x,b)\in R_1,\ d_\Gamma(x,a')=d_\Gamma(b,a')-1\}\subseteq\{x\mid (x,b)\in R_1,\ d_\Gamma(x,a)=d_\Gamma(b,a)-1\}.$$

Parts (b) and (c) follow immediately. □

For $a,b\in X$, we define the *interval* $[a,b]$ to be the union of the vertex sets of all geodesics in Γ from a to b:

$$[a,b]=\{x\in X\mid d_\Gamma(a,x)+d_\Gamma(x,b)=d_\Gamma(a,b)\}\,.$$

For the purpose of the present discussion, we introduce a piece of terminology. For $x\in X$ and $y\in T\subseteq X$, we say that x is *proximal* to y (relative to T) if $d_\Gamma(x,y)\leqslant d_\Gamma(x,y')$ for all $y'\in T$. Vertex x is then *proximal only* to $y\in T$ if $d_\Gamma(x,y)<d_\Gamma(x,y')$ for all $y'\in T$ distinct from y.

Proposition 22. *Let T be a disconnecting set for Γ and let x and z be vertices lying in different components of $\Gamma \setminus T$ with $(x, z) \in R_i$. Suppose there is some $y \in T$ such that x is proximal only to y and z is proximal to y with $(x, y) \in R_s$ and $(z, y) \in R_t$. If $c(s) = 1$ or $c(t) = 1$, then $c(i) = c(s) = c(t) = 1$.*

Proof. Every shortest path joining x to z in Γ must pass through y. Apply Proposition 21(b). □

5.3 Small disconnecting sets

We continue under the assumption that $\Gamma = (X, R_1)$ is the graph of some connected basis relation in the symmetric association scheme $(X, \mathcal{R})$. We begin by examining a simple condition which guarantees that Γ is locally a disjoint union of cliques.

Lemma 23. *Let T be a minimal disconnecting set for Γ, $y \in T$. Suppose $d_\Gamma(y, y') \geqslant 4$ for all $y' \in T$ with $y' \neq y$. Then $c(j) = 1$ for all indices j in I_2.*

Proof. Let $j \in I_2$ and let $x \in R_j(y)$. Let $z \sim y$ be some vertex lying in a different component of $\Gamma \setminus T$ from that containing x. For $(z, x) \in R_i$, we find $c(i) = 1$ by Proposition 22. So $c(j) = 1$ by Lemma 21. □

Lemma 24. *Let T be a disconnecting set for Γ, $y \in T$.*

(a) Let x and z be vertices lying in different components of $\Gamma \setminus T$. If $d_\Gamma(x, y') + d_\Gamma(y', z) > D$ for every $y' \in T$ except y, then z has a unique neighbor lying closer to x and z has a unique neighbor lying closer to y.

(b) Suppose $x \in X \setminus T$ satisfies $d_\Gamma(x, y') = D$ for every $y' \in T$ except y. If $z \in X$ lies in a component of $\Gamma \setminus T$ distinct from that containing x, then z has a unique neighbor lying closer to x and z has a unique neighbor lying closer to y.

In both cases, for $(x, z) \in R_i$, and $(y, z) \in R_j$, we have $c(i) = c(j) = 1$.

Proof. Clearly (b) follows from (a). So first verify (a) for the case $z \sim y$. Next, observe that any geodesic joining x to z passes through y. So $[x, z] = [x, y] \cup [y, z]$. Let $z' \in \Gamma(y) \cap [y, z]$. Since $[x, y] \subseteq [x, z]$ and $[x, z'] = [x, y] \cup \{z'\}$, we find $\Gamma(x) \cap [x, z] = \Gamma(x) \cap [x, z']$, a set of size one. By the same token, $[y, z] \subseteq [x, z]$ and so $\Gamma(z) \cap [y, z] \subseteq \Gamma(z) \cap [x, z]$ gives $|\Gamma(z) \cap [y, z]| = 1$. □

Lemma 25. *Let T be a minimal disconnecting set for Γ, $y \in T$, and suppose $x \in X$ satisfies $d_\Gamma(x, y') = D$ for every $y' \in T$ except y. Then*

(a) for $(x, y) \in R_i$ where $i \in I_h$, we have $\sum_{\ell \in I_h} p^i_{1\ell} = p^1_{11}$.

(b) for $z \in X \setminus T$ which is separated from x by deletion of T, if $\Gamma(z) \cap T \subseteq \{y\}$, then $\sum_{\ell \in I_k} p^j_{1\ell} = p^1_{11}$ where $(y, z) \in R_j$ with $j \in I_k$.

Proof. Let z be a neighbor of y which is separated from x by deletion of T. Since $d_\Gamma(x, z) \leqslant D$, we see that x is proximal only to y and $[x, z] = [x, y] \cup \{z\}$. The set $\Gamma(y) \cap \Gamma(z)$ has size p^1_{11} and every $z' \in \Gamma(y) \cap \Gamma(z)$ lies at distance $h + 1$ from x in Γ. Since every other neighbor of z, with the exception of y, is further away from x, we have

$\sum_{\ell \in I_{h+1}} p_{1\ell}^j = p_{11}^1$ where $(x,z) \in R_j$. Reversing roles, we see that x then has exactly p_{11}^1 neighbors which lie at distance $h+1$ from z. But, for $x' \sim x$, $d_\Gamma(x',y) = d_\Gamma(x',z) - 1$. This gives (a). To obtain (b), observe that every neighbor x' of x with $d_\Gamma(x',z) = d_\Gamma(x,z)$ must have $d_\Gamma(x',y) = d_\Gamma(x,y)$. By part (a), there are exactly p_{11}^1 such vertices. So, for $(x,z) \in R_s$, $\sum_{\ell \in I_{h+k}} p_{1\ell}^s = p_{11}^1$. Reversing roles, we see that exactly p_{11}^1 neighbors of z lie at distance $h+k$ from x. But this is precisely the set of vertices adjacent to z which lie at distance k from y. □

Theorem 26. *Let $(X, \mathcal{R})$ be a symmetric association scheme and let $\Gamma = (X, R_1)$ be the graph associated to a connected basis relation. If Γ admits a disconnecting set of size two, then Γ is isomorphic to a polygon.*

Proof. Let $T = \{y, y'\}$ be a disconnecting set of size two. Let $D = \operatorname{diam} \Gamma$ and let B be the vertex set of some connected component of $\Gamma \setminus T$. First consider the case where y' is the unique vertex at distance D from y in Γ. Then every vertex is at distance D from exactly one other vertex. On the other hand, if $x \in B \cap \Gamma(y)$, then any neighbor of y' not lying in B must be at distance D from x by the triangle inequality. It follows that y has exactly one neighbor not in B and, symmetrically, exactly one neighbor in B. So the graph has valency two in this special case. For the remainder of the proof, assume $d_\Gamma(y,y') < D$.

By Corollary 3, we have $d_\Gamma(y,y') \geqslant 3$. Let x (resp., x') denote some vertex at distance D from y' (resp., y). (Note $x \neq y$, $x' \neq y'$.) Let B and B' be the vertex sets of two connected components Ξ and Ξ', respectively, of $\Gamma \setminus T$ and assume $x \in B$. By Lemma 24(b), any $z \in B'$ has a unique neighbor lying closer to y. (Choosing $j \in I_2$ and $z \in R_j(y)$, we see that this implies Γ is $K_{2,1,1}$-free.) By Lemma 25(a), any $z \in B' \setminus \Gamma(y')$ has exactly p_{11}^1 neighbors z' satisfying $d_\Gamma(z',y) = d_\Gamma(z,y)$. Since $d_\Gamma(x,y) + d_\Gamma(y,x') > D$ and $d_\Gamma(x,y') + d_\Gamma(y',x') > D$, we must have $x' \in B$ also. So we can swap the roles of x and x', y' and y, to find that any $z \in B' \setminus \Gamma(y)$ has a unique neighbor closer to y' and exactly p_{11}^1 neighbors z' with $d_\Gamma(z',y') = d_\Gamma(z,y')$. Now select $z \in B'$ so as to maximize $d_\Gamma(z,y) + d_\Gamma(z,y')$. Since $d_\Gamma(y,y') \geqslant 3$, z is non-adjacent to at least one of y, y'; assume z is not adjacent to y'. Then z has exactly p_{11}^1 neighbors z' satisfying $d_\Gamma(z',y) = d_\Gamma(z,y)$. Since z maximizes $d_\Gamma(z,y) + d_\Gamma(z,y')$, any neighbor of z which lies farther away from y must lie closer to y'. But there is exactly one such vertex. In all, we have $|\Gamma(z)| = 1 + p_{11}^1 + 1$. But Γ is $K_{2,1,1}$-free so the neighborhood of any vertex is partitioned into cliques of size $p_{11}^1 + 1$. We find that $p_{11}^1 + 1$ divides $p_{11}^1 + 2$. This can only happen if $p_{11}^1 = 0$; i.e., Γ is triangle-free. But then z has degree two and Γ must be a polygon. □

Our final two results deal with the special case where graph Γ has diameter two.

Theorem 27. *Let $(X, \mathcal{R})$ be a symmetric association scheme and let $\Gamma = (X, R_1)$ be the graph associated to a connected basis relation. If Γ has diameter two and $|X| > v_1(t-1) + 2$, then Γ has vertex connectivity at least $t+1$ unless $t = v_1$.*

Proof. Let T be a minimal disconnecting set of size at most t. For each $y \in T$, we use the fact that any two vertices have at least one common neighbor to obtain

$$\left| \bigcup_{y \neq y' \in T} \Gamma(y') \right| \leqslant (v_1 - 1)(t-1) + 1$$

so that there is some $x \in X \setminus T$ not adjacent to any element of T except possibly y. Let B be the component of $\Gamma \setminus T$ containing x. Since Γ has diameter two, $x \sim y$ and every $z \in X \setminus (B \cup T)$ must also be adjacent to y. Swapping roles of the vertices in T, we find that, for every y in T, there is some vertex x (necessarily in B) with $\Gamma(x) \cap T = \{y\}$. But this implies that every $z \in X \setminus (B \cup T)$ is adjacent to every vertex in T, so $T = \Gamma(z)$ for every $z \notin B \cup T$. □

Remark 28. We expect very few exceptions to arise here. If $t = v_1$, then we find that $X \setminus (B \cup T) = \{z\}$ is a singleton and all but at most $v_1 - 2$ elements of B have exactly one neighbor in $T = \Gamma(z)$. With $|X| \geqslant v_1^2 - v_1 + 3$ so close to the Moore bound, does this condition force Γ to be a Moore graph?

Theorem 29. *Let $(X, \mathcal{R})$ be a symmetric association scheme and let $\Gamma = (X, R_1)$ be the graph associated to a connected basis relation. If Γ has diameter two, then either Γ has vertex connectivity at least four or Γ is isomorphic to one of the following graphs: the 4-cycle, the 5-cycle, $K_{3,3}$, the Petersen graph.*

Proof. The case where Γ admits a disconnecting set of size two is handled by Theorem 26. Let $T = \{y_1, y_2, y_3\}$ be a minimal disconnecting set of size three.
Case (i): $T \subseteq a^{\perp}$ for some $a \in X$.

By Corollary 3, we have $T = \Gamma(a)$ and Γ has valency three; i.e., Γ is isomorphic to either $K_{3,3}$ or the Petersen graph.

Case (ii): Assume T is not contained in $a^{\perp}$ for any vertex a.

In view of Theorem 27, we may assume $|X| \leqslant 2v_1+2$. (There is no cubic graph on nine vertices.) Let B and B' denote the vertex sets of two distinct connected components of $\Gamma \setminus T$ and assume, without loss of generality, that $|B| \leqslant |B'|$. Then we have $|B| \leqslant \frac{|X|-3}{2}$. So $|B| - 1 \leqslant v_1 - 2$. In view of Case (i), we may assume each $x \in B$ is adjacent to exactly two members of T and every pair of distinct vertices in B is adjacent. This forces $|B| = v_1 - 1$. Looking at $x \sim x'$ in B, we find that $p_{11}^1 \geqslant |B| - 2 + 1$ since x and x' must share a common neighbor in T. Now compare this to some $y \in T$. Since we are not in Case (i), some $y \in T$ is not adjacent to any other element of T. For this y, choose some neighbor z of y where $z \in B$ if $|\Gamma(y) \cap B| \leqslant \frac{v_1}{2}$ and $z \in B'$ if $|\Gamma(y) \cap B| > \frac{v_1}{2}$. The number of common neighbors of y and z is then at most $\frac{v_1}{2} - 1$. The inequalities $v_1 - 2 \leqslant p_{11}^1 \leqslant \frac{v_1}{2} - 1$ then imply that Γ is a polygon, which is impossible as T was chosen to be minimal. □

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