Schur-Concavity for Avoidance of Increasing Subsequences in Block-Ascending Permutations

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Abstract

For integers $a_1, \dots, a_n \geqslant 0$ and $k \geqslant 1$, let $\mathcal{L}_{k+2}(a_1, \dots, a_n)$ denote the set of permutations of $\{1, \dots, a_1 + \cdots + a_n\}$ whose descent set is contained in $\{a_1, a_1 + a_2, \dots, a_1 + \cdots + a_{n-1}\}$, and which avoids the pattern $12 \dots (k+2)$. We exhibit some bijections between such sets, most notably showing that $\#\mathcal{L}_{k+2}(a_1, \dots, a_n)$ is symmetric in the a_i and is in fact Schur-concave. This generalizes a set of equivalences observed by Mei and Wang.

Keywords: pattern avoidance, Young tableaux

1 Introduction

1.1 Synopsis

For nonnegative integers $a_1, \dots, a_n$ an $(a_1, \dots, a_n)$-*ascending permutation* is a permutation on $\{1, 2, \dots, a_1 + \cdots + a_n\}$ whose descent set is contained in $\{a_1, a_1 + a_2, \dots, a_1 + \cdots + a_{n-1}\}$. In other words the permutation ascends in blocks of length $a_1, a_2, \dots, a_n$, and thus has the form

$$\pi = \pi_{11} \dots \pi_{1a_1} \mid \pi_{21} \dots \pi_{2a_2} \mid \cdots \mid \pi_{n1} \dots \pi_{na_n}$$

for which $\pi_{i1} < \pi_{i2} < \cdots < \pi_{ia_i}$ for all i. (The $\mid$ separators are added between blocks for readability.) These permutations were introduced at least as early as 1993, when Gessel and Reutenauer [2] exhibited a bijection between such permutations and so-called *ornaments*, preserving the cycle structure of π. Their work was then extended by others [1, 3, 6].

In this paper we study such permutations, but focusing on pattern avoidance rather than cycle structure.

Definition 1.1. Let $a_1, \ldots, a_n$ be nonnegative integers.

- Let $\mathcal{L}_{k+2}(a_1, \ldots, a_n)$ denote the set of $(a_1, \ldots, a_n)$-ascending permutations that avoid the pattern $12\ldots(k+2)$. In particular, $\mathcal{L}_{k+2}(a_1, \ldots, a_n) = \varnothing$ if $\max\{a_1, \ldots, a_n\} \geqslant k+2$. (The use of $k+2$ here is for consistency with [4] and [5].)

- Let $\mathcal{D}_h(a_1, \ldots, a_n)$ denote the set of $(a_1, \ldots, a_n)$-ascending permutations which avoid $12\ldots(h+1)$ but not $12\ldots h$, that is, the longest increasing subsequence should have length exactly equal to h. In other words,

$$\mathcal{D}_h(a_1, \ldots, a_n) = \mathcal{L}_{h+1}(a_1, \ldots, a_n) \setminus \mathcal{L}_h(a_1, \ldots, a_n).$$

Many special cases of $\mathcal{L}_{k+2}(a_1, \ldots, a_n)$ are well-studied. For example,

- $\mathcal{L}_3(1, \ldots, 1)$ is the set of 123-avoiding permutations on $\{1, \ldots, n\}$, and

- $\mathcal{L}_3(2, \ldots, 2)$ is the set of alternating or "zig-zag" permutations on $\{1, \ldots, 2n\}$ which avoid 123.

Both have cardinality equal to the nth Catalan number.

In 2011, Lewis [4, Proposition 3.1, Theorem 4.1] generalized these results to give two bijections:

- $\mathcal{L}_{k+2}(k, \ldots, k)$ to standard Young tableaux of shape $\langle (k+1)^n \rangle$, and

- $\mathcal{L}_{k+2}(k+1, \ldots, k+1)$ to standard Young tableaux of shape $\langle (k+1)^n \rangle$.

His proof uses a modified version of the Robinson-Schensted-Knuth correspondence; the hook-length formula then lets us compute the cardinalities.

In 2017, Mei and Wang [5] generalized Lewis's bijections to the 2^n sets of the form

$$\mathcal{L}_{k+2}(a_1, \ldots, a_n) \qquad \text{where} \qquad a_i \in \{k, k+1\}. \tag{1}$$

Thus this cardinality of such sets does not depend on the choice of which a_i are equal to k or $k+1$ [5, Theorem 2.3]. Mei and Wang then proposed the problem of finding a direct bijection between these sets of permutations, without appealing to the RSK correspondence [5, Problem 4.2].[1]

1.2 Statement of Results

The two major results we will prove are:

Theorem 1.2. *For each h, the cardinality of $\mathcal{D}_h(a_1, \ldots, a_n)$ does not depend on the order of the a_i's, and there is an explicit bijection between the sets.*

[1]Actually, in the statement of Problem 4.2 in E-JC 24(1) 2017, there is a benign typo: $S_{nk}(123)$ should be replaced by just $\mathcal{L}(n; k; \emptyset)$ (which corresponds to $\mathcal{L}_{k+2}(k, \ldots, k)$ in our notation). In any case, our approach does not treat $\mathcal{L}_{k+2}(k, \ldots, k)$ specially.

Theorem 1.3. *Fix h, and suppose the sequence $a_1 \geqslant \cdots \geqslant a_n$ majorizes the sequence $b_1 \geqslant \ldots \geqslant b_n$. Then there is an explicit injection*

$$\#\mathcal{D}_h\left(a_1, a_2, \ldots, a_n\right) \hookrightarrow \#\mathcal{D}_h\left(b_1, b_2, \ldots, b_n\right).$$

(Recall that a sequence $a_1 \geqslant \cdots \geqslant a_n$ *majorizes* a sequence $b_1 \geqslant \cdots \geqslant b_n$ if $a_1 + \cdots + a_i \geqslant b_1 + \cdots + b_i$ for all i and $a_1 + \cdots + a_n = b_1 + \cdots + b_n$.) In other words, $\#\mathcal{D}_h$ is Schur-concave.

Because $\mathcal{L}_{k+2}(a_1, \ldots, a_n) = \bigcup_{h \leqslant k+1} \mathcal{D}_h(a_1, \ldots, a_n)$ the Schur-concavity holds for $\#\mathcal{L}_{k+2}$ as well:

Corollary 1.4. *For each k, the cardinality of $\mathcal{L}_{k+2}(a_1, \ldots, a_n)$ does not depend on the order of the a_i's, and there is an explicit bijection between the sets.*

Corollary 1.5. *Fix k, and suppose the sequence $a_1 \geqslant \cdots \geqslant a_n$ majorizes the sequence $b_1 \geqslant \cdots \geqslant b_n$. Then there is an explicit injection*

$$\#\mathcal{L}_{k+2}\left(a_1, a_2, \ldots, a_n\right) \hookrightarrow \#\mathcal{L}_{k+2}\left(b_1, b_2, \ldots, b_n\right).$$

We will also make the following simple observation:

Lemma 1.6. *For all k, a_2, $\ldots$, a_n,*

$$\#\mathcal{L}_{k+2}(k+1, a_2, \ldots, a_n) = \#\mathcal{L}_{k+2}(k, a_2, \ldots, a_n)$$

and there is an explicit bijection between these sets.

The proofs of Theorem 1.2 (hence Corollary 1.4) and Lemma 1.6 are explicit bijections, not relying on the RSK correspondence. Hence these two results resolve Mei and Wang's problem [5, Problem 4.2], because by composing them appropriately we may obtain a direct bijection between any two sets of the form described in (1).

1.3 Outline

The rest of the paper is structured as follows. First in Section 2 we quickly prove Lemma 1.6. Then, in Section 3 we describe two maps **W** and **V** in the special situation $n = 2$, which will form the core of the proof. In Section 4 we show how to extend the maps **W** and **V** in order to obtain the desired bijection. Finally in Section 5 we compute some specific values of $\#\mathcal{L}_{k+2}(a_1, \ldots, a_n)$.

2 Proof of Lemma 1.6

First, we make the following observation.

Lemma 2.1. *If $\pi \in \mathcal{L}_{k+2}(k+1, a_2, \ldots, a_n)$ then $\pi_{1,k+1}$ is the largest element of π, that is, $\pi_{1,k+1} = (k+1) + a_2 + \cdots + a_n$.*

Proof. By definition $\pi_{1,1} < \cdots < \pi_{1,k+1}$. Moreover if $i \geqslant 2$ and $\pi_{i,j} > \pi_{1,k+1}$ then $\pi_{1,1} < \cdots < \pi_{1,k+1} < \pi_{i,j}$ would be a $12\ldots(k+2)$ pattern. $\square$

This gives us the map

$$\mathcal{L}_{k+2}(k+1, a_2, \ldots, a_n) \to \mathcal{L}_{k+2}(k, a_2, \ldots, a_n)$$

defined by

$$\pi_{1,1}\ldots\pi_{1,k}\pi_{1,k+1} \mid \pi_{2,1}\ldots \mapsto \pi_{1,1}\ldots\pi_{1,k} \mid \pi_{2,1}\ldots$$

where we simply delete the maximal element from the $(k+1)$st position. This map obviously admits an inverse, since inserting a maximal element in the $(k+1)$st position cannot introduce a $1\ldots(k+2)$ pattern. This produces the claimed bijection.

3 The Bijections W and V

In this section we define two maps $\mathbf{W}$ and $\mathbf{V}$ between sets of the form $\mathcal{D}_h(p,q)$ for a fixed h. These maps form the heart of the proof of Theorem 1.2.

First, we introduce some notation for permutations of $\mathcal{D}_h(p,q)$, where $0 \leqslant p, q \leqslant h$. Consider a permutation

$$\pi = x_1x_2\ldots x_p \mid y_qy_{q-1}\ldots y_1 \in \mathcal{D}_h(p,q).$$

As the maximal increasing subsequence of π has length h, there should be an index j such that

$$x_1 < \cdots < x_j < y_{h-j} < \cdots < y_1. \tag{2}$$

However, this j may not be unique; for example, $1368 \mid 2457$ has two maximal increasing subsequences, namely 12457 and 13457. Nonetheless, we are interested in the *largest* and *smallest* indices with this property.

Definition 3.1. For $\pi \in \mathcal{D}_h(p,q)$, we denote by $\nu_h(\pi)$ and $\omega_h(\pi)$ the smallest and largest index j, respectively, which satisfies (2).

With this definition we may define the map $\mathbf{W}$.

Definition 3.2. Suppose $p \in \{0,\ldots,h-1\}$ and $q \in \{1,\ldots,h\}$. We define the map

$$\mathcal{D}_h(p,q) \xrightarrow{\mathbf{W}} \mathcal{D}_h(p+1,q-1)$$

by

$$\begin{aligned} \pi &= x_1\ldots x_p \mid y_q\ldots y_1 \in \mathcal{D}_h(p,q) \\ \mapsto \mathbf{W}(\pi) &= x_1\ldots x_jy_{h-j}x_{j+1}\ldots x_p \mid y_q\ldots y_{h-j+1}y_{h-j-1}\ldots y_1 \end{aligned}$$

where $j = \omega_h(\pi)$.

In other words (in the notation of Definition 3.2),

$$x_1 < \cdots < x_j < y_{h-j} < \cdots < y_1 \tag{3}$$

is an increasing subsequence of maximal length. Observe that this requires $y_{h-j} = x_j + 1$ (or $y_{h-j} = 1$ if $j = 0$), since otherwise $y_{h-j} - 1$ could be inserted into (3).

Example 3.3. For $(p, q) = (3, 5)$, $h = 6$, we have an example

$$\begin{gathered} \mathcal{D}_6(3,5) \xrightarrow{\mathbf{W}} \mathcal{D}_6(4,4) \\ 236 \mid 14578 \mapsto 2346 \mid 1578. \end{gathered}$$

Proposition 3.4. *This map is well-defined; that is, the longest increasing subsequence of* $\mathbf{W}(\pi)$ *has length* h.

Proof. Assume not, and that moving y_{h-j} introduces some increasing subsequence with length $h + 1$. Then there must be some index k such that

$$x_1 < \cdots < x_j < y_{h-j} < x_{j+1} < \cdots < x_k < y_{h-k} < y_{h-k-1} < \cdots < y_1.$$

But then $x_1 < \cdots < x_k < y_{h-k} < \cdots < y_1$ is an increasing subsequence of length h in π, contradicting the choice of $j = \omega_h(\pi)$ being maximal. $\square$

The map $\mathbf{V}$ is defined in an analogous way, in the reverse direction.

Definition 3.5. Suppose $p \in \{1, \ldots, h\}$ and $q \in \{0, \ldots, h-1\}$. We define the map

$$\mathcal{D}_h(p-1, q+1) \xleftarrow{\mathbf{V}} \mathcal{D}_h(p, q)$$

by

$$\begin{aligned} \pi &= x_1 \ldots x_p \mid y_q \ldots y_1 \in \mathcal{D}_h(p, q) \\ \mapsto \mathbf{V}(\pi) &= x_1 \ldots x_{j-1} x_{j+1} \ldots x_p \mid y_q \ldots y_{h-j+1} x_j y_{h-j} y_{h-j-1} \ldots y_1 \end{aligned}$$

where $j = \nu_h(\pi)$.

In exactly the same way as before we have the following.

Proposition 3.6. *This map is well-defined; that is, the longest increasing subsequence of* $\mathbf{V}(\pi)$ *has length* h.

Proposition 3.7. *The maps* $\mathbf{W}$ *and* $\mathbf{V}$ *are inverses, and hence bijections.*

Proof. We will check that $\mathbf{V}(\mathbf{W}(\pi)) = \pi$, with the other direction being analogous. Write

$$\begin{aligned} \pi &= x_1 \ldots x_p \mid y_q \ldots y_1 \in \mathcal{D}_h(p, q) \\ \mapsto \mathbf{W}(\pi) &= x_1 \ldots x_j y_{h-j} x_{j+1} \ldots x_p \mid y_q \ldots y_{h-j+1} y_{h-j-1} \ldots y_1 \end{aligned}$$

where $j = \omega(\pi)$. Now, observe that $\mathbf{W}(\pi)$ still has a subsequence

$$x_1 < \cdots < x_j < y_{h-j} < y_{h-j+1} < \cdots < y_1$$

and consequently, we have $\nu(\mathbf{W}(\pi)) \leqslant j + 1$.

We now contend that $\nu(\mathbf{W}(\pi)) = j + 1$. (Informally, this is because all length h subsequences of smaller index in the original sequence relied on y_{h-j}, and hence are killed by the application of $\mathbf{W}$.) Assume for contradiction that $\nu(\mathbf{W}(\pi)) < j + 1$, so there is a $\ell \leqslant j$ such that

$$x_1 < \cdots < x_\ell < y_{h-\ell+1} < \cdots < y_{h-j+1} < y_{h-j-1} < \cdots < y_1$$

is an increasing subsequence in $\mathbf{W}(\pi)$. But this would imply that

$$x_1 < \cdots < x_\ell < y_{h-\ell+1} < \cdots < y_{h-j+1} < y_{h-j} < y_{h-j-1} < \cdots < y_1$$

is an increasing subsequence of length $h + 1$ in π, which is a contradiction. $\square$

By composing the bijection $\mathbf{V}$, we deduce the following corollaries.

Corollary 3.8. *Let $h \geqslant 1$ and $p, q, p', q' \in \{1, \ldots, h\}$ such that $p + q = p' + q'$. Then*

$$\#\mathcal{D}_h(p, q) = \#\mathcal{D}_h(p', q').$$

Observe that this already implies Theorem 1.2 (and hence Corollary 1.4) in the case $n = 2$; that is, composition of $\mathbf{W}$ induces a map

$$\mathcal{L}_{k+2}(p, q) \xrightarrow{\mathbf{W}} \mathcal{L}_{k+2}(q, p) \tag{4}$$

whenever $p < q$.

4 Proofs of Theorem 1.2 and Theorem 1.3

4.1 Structure Preservation Lemma

First, we will make the following useful observation about the map $\mathbf{W}$.

Lemma 4.1. *Let $\mathcal{D}_h(p, q) \xrightarrow{\mathbf{W}} \mathcal{D}_h(p + 1, q - 1)$, and $\pi \in \mathcal{D}_h(p, q)$. For $1 \leqslant a < b \leqslant p + q$, the following are equivalent:*

- *There is an increasing subsequence of length r in π consisting of only elements in the interval $[a, b]$.*
- *There is an increasing subsequence of length r in $\mathbf{W}(\pi)$ consisting of only elements in the interval $[a, b]$.*

Proof. We will check only the forward direction, the reverse direction being analogous using $\mathbf{V}$ in place of $\mathbf{W}$. As always, let $j = \omega(\pi)$ and write

$$\begin{aligned} \pi &= x_1 \dots x_p \mid y_q \dots y_1 \in \mathcal{D}_h(p,q) \\ \mapsto \mathbf{W}(\pi) &= x_1 \dots x_j y_{h-j} x_{j+1} \dots x_p \mid y_q \dots y_{h-j+1} y_{h-j-1} \dots y_1 \end{aligned}$$

Clearly it suffices to consider subsequences which involve y_{h-j}, since any other subsequence remains intact under $\mathbf{W}$.

We claim that $y_{h-j+\ell} < x_{j-\ell+1}$ for $1 \leqslant \ell \leqslant j$. Indeed, if this was not the case, then we could construct a sequence of length greater than h in π by taking

$$x_1 < \cdots < x_{j-\ell+1} < y_{h-j+\ell} < \cdots < y_{h-j} < \cdots < y_1.$$

Thus, given any subsequence, if there are any y terms less than y_{h-j} then we may replace them with corresponding x terms instead. Explicitly, if our subsequence of length r in π is

$$a \leqslant x_{i_1} < \cdots < x_{i_2} < y_{h-j+\ell} < \cdots < y_{h-j} < \cdots < y_{i_3} \leqslant b$$

then in $\mathbf{W}(\pi)$ we have

$$a \leqslant x_{i_1} < \cdots < x_{i_2} < x_{j-(\ell-1)} < \cdots < x_j < y_{h-j} < \cdots < y_{i_3} \leqslant b.$$

This proves the lemma. □

4.2 Proof of Theorem 1.2

We are now ready to prove the following result, which implies Theorem 1.2 directly.

Theorem 4.2. *For any index ℓ, if $a_\ell \leqslant a_{\ell+1}$ then we have a bijection*

$$\mathcal{D}_h\left(a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_n\right) \to \mathcal{D}_h\left(a_1, \dots, a_{\ell+1}, a_\ell, \dots, a_n\right) \tag{5}$$

obtained by applying $\mathbf{W}$ in (4) on only the ℓth and $(\ell+1)$st blocks, viewed as a permutation on $\{1, \dots, a_\ell + a_{\ell+1}\}$. The inverse map is given by applying $\mathbf{V}$ in the same way.

In other words, we may swap two adjacent a_i's.

Example 4.3. For an example with $\mathcal{D}_5(1,2,4,1) \to \mathcal{D}_5(1,4,2,1)$ we have

$$\begin{aligned} 1 \mid 37 \mid 2458 \mid 6 &\mapsto 1 \mid 347 \mid 258 \mid 6 \\ &\mapsto 1 \mid 3478 \mid 25 \mid 6. \end{aligned}$$

Proof of Theorem 4.2. Each permutation in $\mathcal{D}_h(a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_n)$ naturally induces a permutation of $\mathcal{D}_r(a_\ell, a_{\ell+1})$ for some $r \geqslant a_{\ell+1}$, by looking at the relative ordering of the $a_\ell + a_{\ell+1}$ elements in these two blocks. (To be exact, r is the length of the longest increasing subsequence among $\pi_{\ell 1} \dots \pi_{\ell a_\ell} \mid \pi_{(\ell+1)1} \dots \pi_{(\ell+1)\pi_{\ell+1}}$.) In this way, we obtain a partition

$$\mathcal{D}_h(a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_n) = \bigcup_{r \geqslant a_{\ell+1}} X_r$$

where X_r is the set of permutations in $\mathcal{D}_h(a_1,\ldots,a_n)$ whose longest increasing subsequence among the ℓth and $(\ell+1)$st block has length exactly r.

Similarly, each permutation in $\mathcal{D}_h(a_1,\ldots,a_{\ell+1},a_\ell,\ldots,a_n)$ naturally induces a permutation of $\mathcal{D}_r(a_{\ell+1},a_\ell)$ for some $r \geqslant a_{\ell+1}$. So in exactly the same fashion we partition the left-hand side as

$$\mathcal{D}_h(a_1,\ldots,a_{\ell+1},a_\ell,\ldots,a_n) = \bigcup_{r\geqslant a_{\ell+1}} Y_r$$

with Y_r denoting those permutations in the right-hand side whose longest increasing subsequence among the ℓth and $(\ell+1)$st block has length exactly r.

We claim that applying $\mathbf{W}$ as described in Theorem 4.2 yields a bijection $X_r \to Y_r$. This follows from Lemma 4.1: the lemma then ensures that at each application of $\mathbf{W}$, no $1\ldots(r+1)$ patterns are created, nor are any $1\ldots r$ patterns destroyed. So the image of this map on X_r really does lie in Y_r, as claimed.

In the same way we may use $\mathbf{V}$ to define a map in the reverse direction. Since $\mathbf{W}$ and $\mathbf{V}$ are inverses, we have produced a bijection $X_r \to Y_r$. Putting these together for all $r \geqslant a_{\ell+1}$ gives the desired result. $\square$

4.3 Proof of Theorem 1.3

In analogy to before, we will prove the following result, which implies Theorem 1.3.

Theorem 4.4. *For any index ℓ, if $a_{\ell+1} \geqslant a_\ell + 2$ then we have an injective map*

$$\mathcal{D}_h\left(a_1,\ldots,a_\ell,a_{\ell+1},\ldots,a_n\right) \hookrightarrow \mathcal{D}_h\left(a_1,\ldots,a_\ell+1,a_{\ell+1}-1,\ldots,a_n\right) \tag{6}$$

obtained by applying $\mathbf{W}$ in (4) on only the ℓth and $(\ell+1)$st blocks, viewed as a permutation on $\{1,\ldots,a_\ell+a_{\ell+1}\}$.

Proof. This is really an observation made within the proof of Theorem 4.2. Retaining the notation in our earlier proof, we decompose

$$\begin{aligned}\mathcal{D}_h(a_1,\ldots,a_\ell,a_{\ell+1},\ldots,a_n) &= \bigcup_{r\geqslant a_{\ell+1}} X_r \\ \mathcal{D}_h(a_1,\ldots,a_\ell+1,a_{\ell+1}-1,\ldots,a_n) &= \bigcup_{r\geqslant a_{\ell+1}-1} Y_r.\end{aligned}$$

As in the proof of Theorem 1.2, we obtain bijections $X_r \to Y_r$ for $r \geqslant a_{\ell+0}$ which collate to give a bijection

$$\bigcup_{r\geqslant a_{\ell+1}} X_r \to \bigcup_{r\geqslant a_{\ell+1}} Y_r.$$

The change from the previous proof is that we now have a set $Y_{a_{\ell+1}-1}$ on the right-hand side which is not in the image of our map. Nonetheless we may still conclude our map is injective, which proves Theorem 4.4. $\square$

5 Enumeration

Now that we have a symmetry result, we turn our attention to actually computing $\#\mathcal{L}_{k+2}(a_1, \ldots, a_n)$ in certain situations. By the main result of this paper, it suffices to assume

$$1 \leqslant a_1 \leqslant \cdots \leqslant a_n \leqslant k.$$

The general problem of computing the value seems difficult, since the special case $a_1 = \cdots = a_n = 1$ is equivalent to computing the number of $12\ldots(k+2)$ avoiding permutations; no closed formula is known for $k \geqslant 3$. Nonetheless, even computing the cardinality for special cases other than those for which $a_i \in \{k, k+1\}$ would be interesting. We give some examples here.

5.1 The $n = 2$ Case

We show that $\#\mathcal{D}_h(p, q)$ is given by the entries of Catalan's triangle.

Proposition 5.1. *As usual, let*

$$C(n, k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!} = \binom{n+k}{k} - \binom{n+k}{k-1}$$

denote the (n, k)th entry of Catalan's triangle. Then for any $1 \leqslant p \leqslant q \leqslant h$, we have

$$\#\mathcal{D}_h\,(p, q) = \begin{cases} C(h, p+q-h) & p+q \geqslant h \\ 0 & p+q < h. \end{cases}$$

Proof. Assume $p + q \geqslant h$, and let $m = p + q - h \geqslant 0$ for brevity. Thus by Corollary 3.8, we have

$$\#\mathcal{D}_h(p, q) = \#\mathcal{D}_h(h, m).$$

If $m = 0$ the result is clear so assume $m > 0$. We now prove the result by induction on $h + m$. From Lemma 1.6 and the definition of $\mathcal{L}_{h+1}$,

$$\begin{aligned} \#\mathcal{L}_{h+1}(h, m) &= \#\mathcal{L}_{h+1}(h-1, m). \\ \#\mathcal{D}_h(h, m) &= \#\mathcal{D}_h(h-1, m) + \#\mathcal{D}_{h-1}(h-1, m) \\ &= C(h, m-1) + C(h-1, m) = C(h, m), \end{aligned}$$

which completes the inductive step. (The term $\mathcal{D}_{h-1}(h-1, m)$ is omitted when $m = h$.) □

5.2 Young Tableaux

We now give some examples of possible generalizations of the cardinality computed in [5].

Proposition 5.2. *For $p \leqslant k$, the cardinality*

$$\#\mathcal{L}_{k+2}(p, \underbrace{k, k, \ldots, k}_{n-1})$$

is equal to the number of standard Young tableaux of shape

$$\left\langle (k+1)^{n-1}, p \right\rangle .$$

Of course, this cardinality may be computed using the hook-length formula.

Proof. This is essentially identical to [4, Proposition 3.1]. By our results, it suffices to consider the cardinality of

$$\mathcal{L}_{k+2}(p, \underbrace{k+1, k+1, \ldots, k+1}_{n-1}).$$

Given a permutation $\pi = \pi_{11}\pi_{12}\ldots\pi_{1p} \mid \cdots \mid \pi_{n1}\ldots\pi_{n(k+1)}$, we construct a tableau as follows:

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
\pi_{n,1} & \pi_{n,2} & \pi_{n,3} & \cdots & \pi_{n,p} & \cdots & \pi_{n,k+1} \\
\hline
\pi_{n-1,1} & \pi_{n-1,2} & \pi_{n-1,3} & \cdots & \pi_{n-1,p} & \cdots & \pi_{n-1,k+1} \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\hline
\pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots & \pi_{2,p} & \cdots & \pi_{2,k+1} \\
\hline
\pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots & \pi_{1,p} & & \\
\hline
\end{array}$$

Obviously each row is increasing; then, one observes that π has no $12\ldots(k+2)$ pattern exactly if the tableau T is a standard Young tableau (the columns are increasing as well). $\square$

5.3 Skew Young Tableau

It is possible to generalize both the results above using the concept of skew Young tableaux.

Proposition 5.3. *For $p \leqslant q \leqslant k$, the cardinality*

$$\#\mathcal{L}_{k+2}(p, q, \underbrace{k, k, \ldots, k}_{n-2})$$

is equal to the number of standard skew Young tableaux of shape

$$\left\langle (k+1)^{n-1}, p \right\rangle / \left\langle k+1-q \right\rangle .$$

Proof. By our results, it suffices to consider the cardinality of

$$\mathcal{L}_{k+2}(p, \underbrace{k+1, k+1, \ldots, k+1}_{n-2}, q).$$

Given a permutation $\pi = \pi_{11}\pi_{12}\ldots\pi_{1p} \mid \cdots \mid \pi_{n1}\ldots\pi_{nq}$, we write it in an array as follows:

		$\pi_{n,1}$	$\cdots$	$\pi_{n,k+1-q}$	$\cdots$	$\pi_{n,q}$
$\pi_{n-1,1}$	$\cdots$	$\pi_{n-1,k+2-q}$	$\cdots$	$\pi_{n-1,p}$	$\cdots$	$\pi_{n-1,k+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\pi_{2,1}$	$\cdots$	$\pi_{2,k+1-q}$	$\cdots$	$\pi_{2,p}$	$\cdots$	$\pi_{2,k+1}$
$\pi_{1,1}$	$\cdots$	$\pi_{1,k+2-q}$	$\cdots$	$\pi_{1,p}$		

In the same way as before, one observes that π has no $12\ldots(k+2)$ pattern exactly if this tableau has increasing columns. $\square$

Example 5.4. To compute $\#\mathcal{L}_8(4,5,6,6,6)$, we biject it to $\mathcal{L}_8(4,7,7,7,5)$ and arrange the permutations of the latter in the following fashion:

		π_{51}	π_{52}	π_{53}	π_{54}	π_{55}
π_{41}	π_{42}	π_{43}	π_{44}	π_{45}	π_{46}	π_{47}
π_{31}	π_{32}	π_{33}	π_{34}	π_{35}	π_{36}	π_{37}
π_{21}	π_{22}	π_{23}	π_{24}	π_{25}	π_{26}	π_{27}
π_{11}	π_{12}	π_{13}	π_{14}			

Thus, $\#\mathcal{L}_8(4,5,6,6,6)$ is equal to the number of standard Young tableaux of shape $\langle 7,7,7,7,4\rangle \,/\, \langle 2\rangle$.

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