A Short Proof of Moll's Minimal Conjecture

Lun Lv

School of Sciences
Hebei University of Science and Technology
Shijiazhuang 050018, P.R. China

klunlv@163.com

Submitted: Mar 1, 2017; Accepted: Sep 25, 2017; Published: Oct 6, 2017
Mathematics Subject Classifications: 05A20, 11B83, 33F99

Abstract

We give a short proof of Moll's minimal conjecture, which has been confirmed by Chen and Xia.

Keywords: Boros-Moll polynomial; Moll's minimal conjecture; spiral property

1 Introduction

The Boros-Moll polynomials, denoted by $P_m(a)$, arise in the evaluation of the following quartic integral, see [2–6, 12]. For any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = 2^{-2m}\sum_k 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k. \tag{1.1}$$

Let $d_l(m)$ be the coefficient of a^l in $P_m(a)$. Then (1.1) gives

$$d_l(m) = 2^{-2m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{l}. \tag{1.2}$$

Much progress has been made since Boros and Moll [1] proved the positivity of the coefficients of $P_m(a)$. Boros and Moll [4] have proved that the sequence $\{d_l(m)\}_{0\leqslant l\leqslant m}$ is unimodal. The log-concavity of the sequence $\{d_l(m)\}_{1\leqslant l\leqslant m-1}$ was conjectured by Moll [12], and it was proved by Kauers and Paule [11] based on recurrence relations. Chen and Xia [9] showed that the sequence $d_l(m)$ satisfies the strongly ratio monotone property which implies the log-concavity and the spiral property. Chen and Gu [7] proved the reverse ultra log-concavity of the Boros-Moll polynomials. By introducing the structure of partially 2-colored permutations, Chen, Pang and Qu [8] found a combinatorial proof of the log-concavity of the Boros-Moll polynomials. Moll also posed a conjecture that is stronger than the log-concavity of the polynomials $P_m(a)$. This conjecture was called Moll's minimum conjecture, and has been confirmed by Chen and Xia [10].

The main objective of this paper is to give a short proof of the following equivalent form of Moll's minimal conjecture, which was confirmed by Chen and Xia [10].

Theorem 1.1 (Theorem 2.1 [10]). *Given $m \geqslant 2$, for $1 \leqslant l \leqslant m$, $l(l+1)(d_l^2(m) - d_{l+1}(m)d_{l-1}(m))$ attains its minimum at $l = m$ with $m(m+1)d_m^2(m)$.*

2 The Proof of Theorem 1.1

Chen and Gu [7] proved the following theorem, which gave a lower bound of $\frac{d_l^2(m)}{d_{l+1}(m)d_{l-1}(m)}$.

Theorem 2.1 (Theorem 1.2 [7]). *For $m \geqslant 2$ and $1 \leqslant l \leqslant m-1$, we have*

$$\frac{d_l^2(m)}{d_{l+1}(m)d_{l-1}(m)} > \frac{(m-l+1)(m+l)(l+1)}{l(m-l)(m+l+1)}. \tag{2.1}$$

Multiplying both sides of (2.1) by l and then plusing $ld_l^2(m)$ to the two sides gives the following result.

Theorem 2.2. *For $m \geqslant 2$ and $1 \leqslant l \leqslant m-1$, we have*

$$l(l+1)\left(d_l^2(m) - d_{l+1}(m)d_{l-1}(m)\right) > \left(l + \frac{2l^3}{(m+l)(m-l+1)}\right) d_l^2(m). \tag{2.2}$$

On the other hand, Chen and Xia [9] have shown the spiral property of sequence $\{d_l(m)\}_{1\leqslant l\leqslant m-1}$, that is

$$d_{m-1}(m) < d_1(m) < d_{m-2}(m) < d_2(m) < \cdots < d_{[\frac{m}{2}]}(m). \tag{2.3}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f(l) = l + \frac{2l^3}{(m+l)(m-l+1)}$. Then for $1 \leqslant l \leqslant m-1$,

$$f'(l) = 1 + \frac{6l^2}{(m+l)(m-l+1)} + \frac{2l^3(2l-1)}{(m+l)^2(m-l+1)^2} > 0.$$

Restricting $l \in N^+$, we see that the sequence $\{l + \frac{2l^3}{(m+l)(m-l+1)}\}_{1\leqslant l\leqslant m-1}$ is strictly monotone increasing.

Combining (2.2) and (2.3), we get

$$\begin{aligned} l(l+1)(d_l^2(m) - d_{l+1}(m)d_{l-1}(m)) &> (l + \frac{2l^3}{(m+l)(m-l+1)})d_l^2(m) \\ &\geqslant \min\{(1+\frac{2}{(m+1)m})d_1^2(m), (m-1+\frac{(m-1)^3}{2m-1})d_{m-1}^2(m)\}. \end{aligned} \tag{2.4}$$

By direct computation we may deduce from (1.2) that

$$(1+\frac{2}{(m+1)m})d_1^2(m) \geqslant m(m+1)d_m^2(m),$$
$$(m-1+\frac{(m-1)^3}{2m-1})d_{m-1}^2(m) \geqslant m(m+1)d_m^2(m).$$

It follows by (2.4) that

$$l(l+1)(d_l^2(m) - d_{l+1}(m)d_{l-1}(m)) > m(m+1)d_m^2(m), \quad 1 \leqslant l \leqslant m-1.$$

This completes the proof. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China, the Natural Science Foundation of Hebei Province (A2014208152) and the Top Young-aged Talents Program of Hebei Province.

References

[1] T. Amdeberhan and V. Moll. A formula for a quartic integral: A survey of old proofs and some new ones. *Ramanujan J.*, 18: 91–102, 2009.

[2] G. Boros and V. Moll. An integral hidden in Gradshteyn and Ryzhik. *J. Comput. Appl. Math.*, 106: 361–368, 1999.

[3] G. Boros and V. Moll. A sequence of unimodal polynomials. *J. Math. Anal. Appl.*, 237: 272–285, 1999.

[4] G. Boros and V. Moll. A criterion for unimodality. *Electron. J. Combin.*, 6: #R10, 1999.

[5] G. Boros and V. Moll. The double square root, Jacobi polynomials and Ramanujan's Master Theorem. *J. Comput. Appl. Math.*, 130: 337–344, 2001.

[6] G. Boros and V. Moll. *Irresistible Integrals*, Cambridge University Press, Cambridge, 2004.

[7] W.Y.C. Chen and C.C.Y Gu. The reverse ultra log-concavity of the Boros-Moll polynomials. *Proc. Amer. Math. Soc.*, 137: 3991–3998, 2009.

[8] W.Y.C. Chen, S.X.M. Pang and E.X.Y. Qu. Partially 2-colored permutations and the Boros-Moll polynomials. *Ramanujan J.*, 27: 297–304, 2012.

[9] W.Y.C. Chen and E.X.W. Xia. The ratio monotonicity of the Boros-Moll polynomials. *Math. Comput.*, 78: 2269–2282, 2009.

[10] W.Y.C. Chen and E.X.W. Xia. Proof of Moll's minimum conjecture. *European J. Combin.*, 34: 787–791, 2013.

[11] M. Kauers and P. Paule. A computer proof of Moll's log-concavity conjecture. *Proc. Amer. Math. Soc.*, 135: 3847–3856, 2007.

[12] V. Moll. The evaluation of integrals: A personal story. *Notices Amer. Math. Soc.*, 49: 311–317, 2002.