Local resilience for squares of almost spanning cycles in sparse random graphs

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Abstract

In 1962, Pósa conjectured that a graph $G = (V, E)$ contains a square of a Hamiltonian cycle if $\delta(G) \geqslant 2n/3$. Only more than thirty years later Komlós, Sárközy, and Szemerédi proved this conjecture using the so-called Blow-Up Lemma. Here we extend their result to a random graph setting. We show that for every $\epsilon > 0$ and $p = n^{-1/2+\epsilon}$ a.a.s. every subgraph of $G_{n,p}$ with minimum degree at least $(2/3+\epsilon)np$ contains the square of a cycle on $(1-o(1))n$ vertices. This is almost best possible in three ways: (1) for $p \ll n^{-1/2}$ the random graph will not contain any square of a long cycle (2) one cannot hope for a resilience version for the square of a *spanning* cycle (as deleting all edges in the neighborhood of single vertex destroys this property) and (3) for $c < 2/3$ a.a.s. $G_{n,p}$ contains a subgraph with minimum degree at least cnp which does not contain the square of a path on $(1/3+c)n$ vertices.

Keywords: random graphs, resilience, almost spanning subgraphs

1 Introduction

A classical result of Dirac [6] states that any graph G on $n \geqslant 3$ vertices with minimum degree $\delta(G) \geqslant n/2$ contains a Hamilton cycle. This result is not difficult and a proof can be found in most text books on graph theory, see e.g. [4, 21]. One also easily checks that the constant $1/2$ is best possible: the complete bipartite graph on $(n-1)/2$ and $(n+1)/2$ vertices (assuming n odd) has minimum degree $n/2 - 1/2$ but does not contain a Hamilton cycle.

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In 1962, Pósa conjectured that $G(V, E)$ contains a square of a Hamiltonian cycle if $\delta(G) \geqslant 2n/3$. A square of a cycle C is the cycle C together with all edges between vertices that have distance 2 in C. Again it is not difficult to see that the constant $2/3$ is best possible, just consider the complete tripartite graph on $(n-1)/3$, $(n-1)/3$ and $(n+2)/3$ vertices (assuming 3 divides $n-1$). However proving that minimum degree $2n/3$ actually suffices turned out to be a difficult problem. It required the development of powerful tools, most notably Szemerédi's Regularity Lemma [18, 19] and the so-called Blow-Up Lemma [10], before Pósa's conjecture was proven, at least for all sufficiently large n [12].

Theorem 1.1 (Komlós, Sárközy, Szemerédi). *There exists a natural number n_0 such that if G has order n with $n \geqslant n_0$ and*

$$\delta(G) \geqslant \frac{2}{3}n,$$

then G contains the square of a Hamiltonian cycle.

The above results can also be stated as resilience statements. For a monotone increasing graph property $\mathcal{P}$ the (local) resilience of a graph $G = (V, E)$ with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that by deleting at each vertex $v \in V$ at most an r-fraction of the edges incident to v one can obtain a graph that does not have property $\mathcal{P}$. Dirac's theorem implies that the local resilience of the property 'containing a Hamilton cycle' of the complete graph is $1/2$, while the proof of Posa's conjecture implies that the property 'containing a square of a Hamilton cycle' of the complete graph is $1/3$.

A natural extension for resilience results is to consider instead of the complete graph the random graph $G_{n,p}$ and ask for the resilience as a function of the edge probability p. It is natural to expect that there exists a threshold p_0 so that for $p \gg p_0$ the local resilience of $G_{n,p}$ of a property $\mathcal{P}$ is w.h.p. equal to the local resilience of the complete graph, while for $p \ll p_0$ the random graph $G_{n,p}$ w.h.p. does not satisfy the property $\mathcal{P}$ at all. Indeed, such a result is known, up to constant factors, for the property 'contains a Hamilton cycle'. The threshold for existence of a Hamilton cycle is $p = (\log n + \log\log n + \omega(1))/n$ [11, 3], while Lee and Sudakov [14] showed that for every positive ϵ, there exists a constant $C = C(\epsilon)$ such that for $p \geqslant C \log n/n$ w.h.p. the local resilience is $1/2 - \epsilon$.

The aim of this paper is to study the local resilience for the property 'containing a square of a Hamilton cycle'. For this problem already the threshold for existence is a hard problem. Since a square of a Hamilton cycle contains many triangles, $p \geqslant c/\sqrt{n}$ is certainly necessary. Indeed improving on a series of previous bounds ([13][15]) Benett, Dudek and Frieze have recently determined the threshold to lie at $p = 1/\sqrt{n}$ [2].

For the resilience problem one is thus tempted to speculate that at least for $p \geqslant \text{poly}(\log n)/\sqrt{n}$, for an appropriate polylog-factor, we have that the resilience of $G_{n,p}$ with respect to the property 'containing a square of a Hamilton cycle' is $1/3 - o(1)$. However, for this property it is easy to see that this is far too optimistic. By deleting all edges in the neighborhood of a vertex v we can ensure that v cannot be part of any square of a cycle. Thus for any $p = o(1)$ we have that the resilience for 'containing a square of a Hamilton cycle' is $o(1)$. In order to obtain a non-trivial result we thus need to weaken the required property. One easily checks that for constant resilience the best one can hope for is to find a square of a cycle that covers all but $\Theta(1/p^2)$ vertices. Here we show an approximate version of such a best possible result.

Theorem 1.2. *For every $\gamma, \nu > 0$ and $p = n^{-\frac{1}{2}+\gamma/2}$ a.a.s. every subgraph of $G_{n,p}$ with minimum degree at least $(2/3 + \nu)np$ contains a square of a cycle on at least $(1 - \nu)n$ vertices.*

Our result should be compared to a recent result of Peter Allen, Julia Böttcher, Julia Ehrenmüller and Anusch Taraz [1]. The authors prove a sparse version of the bandwidth theorem, which in particular implies that $G_{n,p}$ has local resilience for the property 'contains a square of a cycle on $n - C/p^2$ vertices' as long as $p \gg (\log n/n)^{1/4}$. Their proof technique as well as previous universality results hit a natural barrier around $p = n^{-1/\Delta}$ where Δ denotes the maximum degree of the embedded graph. Note that this density is required for any greedy / sequential type of embedding scheme, as for $p \ll n^{-1/\Delta}$ the typical neighbourhood of any Δ vertices is empty and one thus need to design more sophisticated look-ahead schemes. We achieve this by designing some pruning process that identifies edges that satisfy some good expansion properties. In this way we obtain the first nontrivial resilience result for almost spanning subgraphs that achieves, up to polylog factors, the optimal density.

A natural question is whether a similar result holds for higher powers of a cycle. Applying the aforementioned bandwidth theorem gives a bound of $p \gg (\log n/n)^{1/2k}$ for the k-th power of an almost spanning cycle. Similarly to the case $k = 2$ this does not match the obvious lower bound of $p \geqslant n^{-1/k}$. Our approach does not generalize easily to this setting. This is mostly due to our reliance on sparse regularity techniques which yield very strong statements about the distribution of the edges, but not on larger structures like triangles or larger complete graphs.

2 Proof

The proof makes heavy use of the sparse regularity lemma (see [9]) and related techniques. The definition of an (ϵ, p)-regular graph is briefly stated below. For a more in depth introduction to the topic see for example [8].

Definition 2.1. *A bipartite graph $B = (U \cup W, E)$ is called (ϵ, p)-regular if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geqslant \epsilon|U|$ and $|W'| \geqslant \epsilon|W|$,*

$$\left| \frac{|E(U', W')|}{|U'||W'|} - \frac{|E|}{|U||W|} \right| \leqslant \epsilon p.$$

We write (ϵ)-regular in case p equals the density $|E|/(|U||W|)$.

B is called (ϵ, p)-lower-regular if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geqslant \epsilon|U|$ and $|W'| \geqslant \epsilon|W|$,

$$\frac{|E(U', W')|}{|U'||W'|} \geqslant (1 - \epsilon)p.$$

The next two lemmas follow immediately from the above definition.

Lemma 2.2. *Let $B = (U \cup W, E)$ be an (ϵ, p)-lower-regular bipartite graph. Then for every $W' \subseteq W$ of size at least $\epsilon|W|$ all but at most $\epsilon|U|$ vertices in U have at least $(1-\epsilon)|W'|p$ neighbours in W'. If B is (ϵ)-regular with density p, then all but at most $2\epsilon|U|$ vertices in U have at least $(1-\epsilon)|W'|p$ and at most $(1+\epsilon)|W'|p$ neighbours in W'.* □

Lemma 2.3. *Let $B = (U \cup W, E)$ be an (ϵ)-regular bipartite graph for some $\epsilon < 1/3$. Then every subgraph $(U \cup W, E')$ of B such that $|E \setminus E'| \leqslant \epsilon^4|E|$ is (2ϵ)-regular.* □

The so-called sparse regularity lemma allows us to partition every subgraph of a random graph $G_{n,p}$ in (ϵ)-regular parts. To make this more precise we need one more definition.

Definition 2.4. *We say that a partition $(V_i)_0^k$ of a set V is (ϵ, k)-equitable if $|V_0| \leqslant \epsilon|V|$ and $|V_1| = \cdots = |V_k|$. We call a partition $(V_i)_0^k$ (ϵ, p)-regular if at most $\epsilon\binom{k}{2}$ pairs (V_i, V_j) with $1 \leqslant i < j \leqslant k$ are not (ϵ, p)-regular.*

The original Sparse regularity lemma [9] required the graph to fulfill a certain density condition. We use a more recent version of Scott [16] that does require such a condition.

Theorem 2.5 (Sparse regularity lemma, [16])**.** *For any $\epsilon > 0$ and $m \geqslant 1$, there exists a constant $M(\epsilon, m) \geqslant m$ such that for any $p \in [0,1]$, any graph G admits an (ϵ, p)-regular partition $(V_i)_{i=0}^k$ with exceptional class V_0 such that $m \leqslant k \leqslant M$.*

For our proof we need the slightly stronger statement that a.a.s. every subgraph of $G_{n,p}$ that satisfies some minimum degree condition contains a particularly nice regular partition. The proof follows routinely by standard arguments. We include it for convenience of the reader.

Corollary 2.6. *For every $\mu, \nu, \epsilon > 0$ and every positive integer $r_{\min}$ there exists $\alpha(\nu) > 0$ and $r_{\max}(\nu, \epsilon, r_{\min})$ such that for $p \gg 1/n$ a.a.s. every spanning subgraph $\tilde{G} \subseteq G_{n,p}$ with minimum degree at least $(\mu+\nu)np$ contains a partition of the vertices $V = V_0 \cup V_1 \cup \cdots \cup V_r$, where $r \in [r_{\min}, r_{\max}]$, such that $|V_0| \leqslant \epsilon n$, $|V_1| = \cdots = |V_r|$ and such that for every i there exist at least μr indices $j \in [r] \setminus \{i\}$ such that $\tilde{G}[V_i, V_j]$ contains a spanning (ϵ)-regular subgraph with $\lfloor |V_i||V_j|\alpha p \rfloor$ edges.*

Proof. We choose $\alpha(\nu)$, $m(r_{\min})$ and $\epsilon_0(\mu, \nu, \epsilon)$ such that inequality $(*)$ from below and the following inequalities are satisfied simultaneously:

$$\sqrt{\epsilon_0} \leqslant \frac{\nu}{2}, \qquad 2\epsilon_0/\alpha < \epsilon, \qquad (1-\sqrt{\epsilon_0})m \geqslant r_{\min}, \qquad \epsilon_0 + 2\sqrt{\epsilon_0} \leqslant \epsilon.$$

Suppose that $\tilde{G}$ is a spanning subgraph of $G_{n,p}$ with $\delta(\tilde{G}) \geqslant (\mu+\nu)np$. For every $\eta > 0$ a.a.s. every subgraph of the random graph $G_{n,p}$ is η-upper-uniform with density p and thus the sparse regularity lemma can be applied to every subgraph of $G_{n,p}$. The sparse regularity lemma (Theorem 2.5) gives us a constant $M(\epsilon_0, m)$ such that we find an (ϵ_0, p)-regular (ϵ_0, k)-equitable partition $(V_i)_0^k$ of $\tilde{G}$ for some $k \in [m, M]$.

Denote with $\tilde{n} \in [(1-\epsilon_0)n/k, n/k]$ the size of the partition classes. For $i \in [k]$ define

$$d_i := \{j \in [k] \setminus \{i\} \mid V_i,\, V_j \text{ has density at least } \alpha p \text{ in } \tilde{G}\}.$$

Note that a.a.s. in $G_{n,p}$ we have that $|E(V_i, V_j)| \leqslant (1+\epsilon_0)\tilde{n}^2 p$ for all $i, j \in [k]$ and, with room to spare, $|E(V_i, V_0 \cup V_i)| \leqslant 2(\epsilon_0 n + \tilde{n})\tilde{n}p$ for all $i \in [k]$. The minimum degree condition of $\tilde{G}$ thus implies that for all $i \in [k]$

$$\tilde{n}\,(\mu+\nu)\,np \leqslant E(V_i, V_0 \cup V_i) + \sum_{j \in [k]\setminus i} E(V_i, V_j) \leqslant 2(\epsilon_0 n + \tilde{n})\tilde{n}p + k\alpha p\tilde{n}^2 + d_i(1+\epsilon_0)\tilde{n}^2 p$$

and thus

$$d_i \geqslant \frac{(\mu+\nu)\,n - 2(\epsilon_0 n + \tilde{n}) - k\alpha\tilde{n}}{(1+\epsilon_0)\tilde{n}} \geqslant \frac{(\mu+\nu)\,k - 2(\frac{\epsilon_0}{1-\epsilon_0}k + 1) - k\alpha}{(1+\epsilon_0)} \overset{(*)}{\geqslant} \left(\mu + \frac{\nu}{2}\right) k.$$

Observe that the fact that at most $\epsilon_0\binom{k}{2}$ pairs (V_i, V_j) are not (ϵ_0, p)-regular implies that there are at most $\sqrt{\epsilon_0}k$ indices in $[k]$ for which the set

$$\{j \in [k] \setminus \{i\} \mid (V_i, V_j) \text{ is not an } (\epsilon_0, p)\text{-regular graph}\}$$

has size at least $\sqrt{\epsilon_0}k$. Every (ϵ_0, p)-regular graph with density at least αp is (ϵ_0/α)-regular and thus contains a $(2\epsilon_0/\alpha)$-regular subgraph with $\lfloor \tilde{n}^2\alpha p \rfloor$ edges (see for example Lemma 4.3 in [8]). As $\sqrt{\epsilon_0} \leqslant \frac{\nu}{2}$, we can thus find a subset $R \subseteq [k]$ of at least $(1-\sqrt{\epsilon_0})k$ indices such that for every $i \in R$ the set

$$\{j \in R \setminus \{i\} \mid V_i,\, V_j \text{ contain an } (2\epsilon_0/\alpha)\text{-regular graph with } \left\lfloor \tilde{n}^2\alpha p \right\rfloor \text{ edges}\}$$

is of size at least μk. Without loss of generality we may assume that $R = \{1, \dots, r\}$, where $r \geqslant (1-\sqrt{\epsilon_0})k \geqslant r_{\min}$. By choice of ϵ_0 we know that the cardinality of $V_0 \cup \bigcup_{i>r} V_i$ is at most $\epsilon_0 n + \sqrt{\epsilon_0}k\tilde{n} \leqslant \epsilon n$. Thus $V_0 \cup \bigcup_{i>r} V_i, V_1, \dots, V_r$ is the desired partition and the corollary thus holds for $r_{\max} = M$. $\square$

With Corollary 2.6 at hand, an alert reader will certainly be able to guess our proof strategy: apply the Komlós, Sárközy, Szemerédi Theorem to the partition guaranteed by Corollary 2.6 in order to find a square of a cycle for this partition and then find a long square of a cycle within this structure. The next definitions provide the notations to make this idea precise. First we define the notion of a regular blow-up of a graph.

Definition 2.7. *Denote with $\mathcal{G}(F, n, p, \epsilon)$ the class of graphs that consist of $|V(F)|$ pairwise disjoint vertex sets of size n. Each vertex set represents a vertex of F, and two vertex sets span an (ϵ)-regular graph with density $(1 \pm \epsilon)p$ whenever the corresponding vertices are adjacent in F.*

With P_k we denote a path of length k, with vertex set $\{1,\dots,k+1\}$ and edges $\{i,i+1\}$ for $1 \leqslant i \leqslant k$. The cycle C_k is obtained from P_k by identifying the vertices 1 and $k+1$. The square of a path is denoted with P_k^2 and the square of a cycle with C_k^2. For a graph $F \in \{P_k^2, C_k^2\}$ and its blow-up $G \in \mathcal{G}(F,n,p,\epsilon)$ we denote the vertex sets of G by $V_1,\dots,V_{|F|}$ and assume that the vertex set V_i represents the i-th vertex along the path (or cycle). We collect some additional properties of $\mathcal{G}(P_k^2,n,p,\epsilon)$ in the following definition:

Definition 2.8. *Denote with $\tilde{\mathcal{G}}(P_k^2,n,p,\epsilon) \subseteq \mathcal{G}(P_k^2,n,p,\epsilon)$ the class of graphs in which for $i \in \{1,\dots,k-1\}$*

(i) every edge spanned by V_i, V_{i+1} closes a triangle with at least $(1-\epsilon)np^2$ vertices in V_{i+2}, and

(ii) all but ϵn vertices $v \in V_{i+1}$ have neighborhoods into V_i and V_{i+2} which induce an (ϵ,p)-lower-regular subgraph.

We also need two auxiliary lemmas related to the above definition. Their proofs are deferred to Section 3 and Section 4. The first lemma states that in the random graph the restrictions imposed by Definition 2.8 are easy to satisfy:

Lemma 2.9. *For every $\alpha,\epsilon,\gamma,k_{\max} > 0$ there exists $\epsilon' > 0$ such that for every $\eta > 0$ a.a.s. in $G_{n,p}$ with $p = n^{-1/2+\gamma/2}$ every subgraph $G \subseteq G_{n,p}$, where $G \in \mathcal{G}(P_k^2,n_0,\alpha p,\epsilon')$, $n_0 \geqslant \eta n$, $k \leqslant k_{\max}$ contains a spanning subgraph from $\tilde{\mathcal{G}}(P_k^2,n_0,\alpha p,\epsilon)$.*

For $G \in \tilde{\mathcal{G}}(P_k^2,n,p,\epsilon)$ with vertex partitions $V_1,\dots,V_k$ we say that an edge $e \in E(V_1,V_2)$ *expands* to a set of edges $E' \subseteq E(V_i,V_{i+1})$ if there exists a square of a path of length i in G between e and every edge of E'. The next lemma asserts that in the random graph every edge in $E(V_1,V_2)$ expands to a majority of the edges in $E(V_k,V_{k+1})$, whenever k is sufficiently large. Property (1) of Definition 2.8 already guarantees that every edge spanned by V_1,V_2 is contained in many squares of a path ending in $E(V_k,V_{k+1})$. But since these paths may overlap this alone does not imply that every edge expands to a large portion of $E(V_k,V_{k+1})$.

Lemma 2.10. *Let $\eta,\alpha,\gamma > 0$ and $k \geqslant \frac{3}{\gamma}$ be fixed and let $p = n^{-1/2+\gamma/2}$. For ϵ small enough depending on α a.a.s. every subgraph $G \subseteq G_{n,p}$ which is from $G \in \tilde{\mathcal{G}}\left(P_{k+4}^2,n_0,\alpha p,\epsilon\right)$ for some $n_0 \geqslant \eta n$ satisfies the following: all edges in $G[V_1 \cup V_2]$ expand to at least a 0.52-fraction of the edges in $G[V_{k+4} \cup V_{k+5}]$.*

With these two lemmas at hand we can prove our main result.

Theorem 1.2. *For every $\gamma,\nu > 0$ and $p = n^{-\frac{1}{2}+\gamma/2}$ a.a.s. every subgraph of $G_{n,p}$ with minimum degree at least $(2/3+\nu)np$ contains a square of a cycle on at least $(1-\nu)n$ vertices.*

Proof. Suppose that $\tilde{G}$ is a spanning subgraph of $G_{n,p}$ with $\delta(\tilde{G}) \geqslant (\frac{2}{3}+\nu)np$. We will the fix constants $\alpha(\nu), \epsilon(\nu,\alpha), \epsilon'(\nu,\alpha,\epsilon,\gamma) > 0$ throughout the proof.

Set $k_0 = \lceil \frac{3}{\gamma} \rceil + 4$ and let $r_{\min} = \max\{3k_0, n_0\}$ where n_0 denotes the constant from Theorem 1.1. For $\alpha(\nu)$ small enough we may invoke Corollary 2.6 with $\mu \leftarrow 2/3$, $\nu \leftarrow \nu$, $\epsilon \leftarrow \epsilon'$ to obtain, for some $r \in [r_{\min}, r_{\max}(\nu, \epsilon')]$, a partition $V = V_0 \cup \cdots \cup V_r$ of the vertices of $\tilde{G}$ such that $|V_1| = \cdots = |V_r| = \tilde{n} \in [(1-\epsilon')n/r, n/r]$ and for every $i \in [r]$ the number of indices $j \in [r] \setminus \{i\}$ such that $\tilde{G}[V_i, V_j]$ contains a spanning (ϵ')-regular subgraph with $\lfloor \tilde{n}^2 \alpha p \rfloor$ edges is at least $\frac{2}{3}r$.

Since $r \geqslant n_0$ Theorem 1.1 tells us that then $\tilde{G}$ must contain a subgraph $G \in \mathcal{G}(C_r^2, \tilde{n}, \alpha p, \epsilon')$ on the partitions $V_1, \ldots, V_r$. We shall assume without loss of generality that V_i represents the i-th vertex of the cycle. Furthermore for ease of notation we identify V_{r+i} with V_i.

For integers $i \in [r]$ and $t \in [k_0, 2k_0]$ consider a collection of subsets $V_i' \subseteq V_i, \ldots, V_{i+t}' \subseteq V_{i+t}$ each of size $n' \geqslant \epsilon \tilde{n}$. By definition of (ϵ')-regularity the sets $V_i', \ldots, V_{i+t}'$ induce a subgraph $G' \in \mathcal{G}(P_t^2, n', \alpha p, 2\epsilon'/\epsilon)$ in G. By Lemma 2.9 we may pick ϵ' small enough depending on $\alpha, \epsilon, 2k_0$ such that a.a.s. G' contains a spanning subgraph $G_0 \subseteq G'$ with $G_0 \in \tilde{\mathcal{G}}(P_t^2, n', \alpha p, \epsilon)$.

We call an edge $e \in E(V_i', V_{i+1}')$ *good* (w.r.t. $V_i', \ldots, V_{i+t}'$) if it expands to at least a 0.51-fraction of the edges in $E(V_{i+t-1}', V_{i+t}')$ (through $V_i', \ldots, V_{i+t}'$ in G_0). Lemma 2.10 with $\eta \leftarrow \epsilon/2r_{\max}$, $\alpha \leftarrow \alpha$, $\gamma \leftarrow \gamma$, $k \leftarrow t-4 \geqslant k_0 - 4 \geqslant \frac{\gamma}{3}$ tells us that all edges in $G_0[V_i', V_{i+1}']$ expand to at least a 0.52-fraction of the edges in $G_0[V_{i+t-1}', V_{i+t}']$. Since $G_0 \subseteq G$ and since the density of G_0 and G differs by at most $2\epsilon\alpha p$ this implies that say a 0.99-fraction of the edges in $E(V_i', V_{i+1}')$ are good.

We can now find a long square of a path as follows: Fix an edge $e \in E(V_1, V_2)$ which is good with respect to $V_1, \ldots, V_{k_0+1}$. e expands to at least a 0.51 fraction of the edges spanned by V_{k_0}, V_{k_0+1} and a 0.99 fraction of the edges in $E(V_{k_0}, V_{k_0+1})$ are good with respect to $V_{k_0}, \ldots, V_{2k_0}$. Thus we may fix a square of a path $P \subseteq G[V_1 \cup \cdots \cup V_{k_0+1}]$ of length k_0 from e to some edge $e' \in E(V_{k_0}, V_{k_0+1})$ which is good with respect to $V_{k_0}, \ldots, V_{2k_0}$.

Now remove $V(P) \setminus e'$ from G. Observe that since $r \geqslant 3k_0$ we did not remove any vertices from $V_{k_0}, \ldots, V_{2k_0}$ and therefore e' is still good (w.r.t. $V_{k_0}, \ldots, V_{2k_0}$). Thus we may extend P to end in an edge $e'' \in E(V_{2k_0-1}, V_{2k_0})$ which is good w.r.t. to $V_{2k_0-1}, \ldots, V_{3k_0-1}$. This procedure can be continued for as long as $|V_i| \geqslant \epsilon\tilde{n}$ for every $i \in [r]$. Thus we obtain a square of a path P which uses at least $(1-2\epsilon)\tilde{n}$ vertices from each partition of G.

This construction can be generalized to obtain a square of the cycle: before fixing the first edge e set aside sets $\tilde{V}_1 \subseteq V_1, \ldots, \tilde{V}_r \subseteq V_r$ each of size $\epsilon\tilde{n}$. Pick $e \in E(\tilde{V}_1, \tilde{V}_2)$ such that it expands (backwards) to at least a 0.51-fraction of the edges in $E(\tilde{V}_{r+1-k_0}, \tilde{V}_{r+2-k_0})$ and is good (w.r.t. $V_1 \setminus \tilde{V}_1, \ldots, V_{k_0+1} \setminus \tilde{V}_{k_0+1}$). Then embed a long path starting with e in $V(G) \setminus \bigcup \tilde{V}_i$. At any point we may decide to close it by picking the next edge such that it expands to a 0.51-fraction of the edges in $E(\tilde{V}_{r+1-k_0}, \tilde{V}_{r+2-k_0})$. With this construction we find a square of a cycle of length at least

$$r \cdot (1-3\epsilon)\tilde{n} \geqslant (1-3\epsilon)(1-\epsilon')\frac{n}{r} \geqslant (1-\mu)n$$

provided that ϵ', ϵ are chosen small enough depending on μ. □

3 Proof of Lemma 2.9

For the proof of Lemma 2.9 we need the following sparse regularity inheritance theorem:

Theorem 3.1 (Gerke, Kohayakawa, Rödl, and Steger [7]). *For $0 < \beta, \epsilon' < 1$, there exists $\epsilon_0 = \epsilon_0(\beta, \epsilon') > 0$ and $C = C(\epsilon')$ such that, for any $0 < \epsilon \leqslant \epsilon_0$ and $0 < p < 1$ every (ϵ, p)-lower-regular graph $G = (V_1 \cup V_2, E)$ satisfies that, for every $q_1, q_2 \geqslant Cp^{-1}$, the number of pairs of sets (Q_1, Q_2) with $Q_i \subseteq V_i$ and $|Q_i| = q_i$ ($i = 1, 2$) that induce an (ϵ', p)-lower-regular graph is at least*

$$\left(1 - \beta^{\min\{q_1,q_2\}}\right) \binom{|V_1|}{q_1} \binom{|V_2|}{q_2}.$$

From Theorem 3.1 we easily deduce a bound on the number of graphs in $\mathcal{G}\left(K_3, n, p, \epsilon_0\right)$ for which there exist 'many' vertices in V_1 whose neighborhood does not induce a lower regular graph of 'roughly' the expected size.

Lemma 3.2. *For every $\beta, \epsilon > 0$ there exists $\epsilon_0 > 0$ and $C > 0$ such that for all integers $n, m \in \mathbb{N}$ that satisfy $3^n n^{2n} \leqslant 2^{n^{3/2}}$ and $m \geqslant Cn^{3/2}$ the following holds. The number of graphs $G = (V_1 \cup V_2 \cup V_3, E)$ in $\mathcal{G}\left(K_3, n, m/n^2, \epsilon_0\right)$ with m edges between each two partitions for which more than ϵn vertices in $v \in V_1$ have neighborhoods in V_2, V_3 which are not of size $(1 \pm \epsilon)m/n$ or which do not induce an $(\epsilon, m/n^2)$-lower-regular subgraph in $G[V_2, V_3]$ is at most*

$$\beta^m \binom{n^2}{m}^3.$$

Proof. Write $p = m/n^2$ and define β_0 by the equation $\beta_0^{(1-\epsilon)\epsilon/2} = \beta/2$. Let $\epsilon_0 = \min\{\epsilon/4, \epsilon_0(\beta_0, \epsilon)\}$ and $C = \max\{1, C(\epsilon)/(1-\epsilon)\}$, where $\epsilon_0(\cdot, \cdot)$ and $C(\cdot)$ are the functions given by Theorem 3.1.

Consider a graph $G = (V_1 \cup V_2 \cup V_3, E)$ for which the statement fails. For any such graph we may partition V_1 into three sets $V_D, V_B, V_G \subseteq V_1$ as follows: the set V_D contains all vertices whose degree into at least one of V_2 or V_3 is not within $(1 \pm \epsilon)np$. V_B contains all vertices whose degree into both V_2 and V_3 is within $(1 \pm \epsilon)np$ but whose neighborhoods in V_2 and V_3 do not induce an (ϵ, p)-lower-regular graph in $G[V_2, V_3]$. Finally set $V_G = V_1 \setminus (V_D \cup V_B)$. Lemma 2.2 implies that $|V_D| \leqslant 2\epsilon_0 n \leqslant \epsilon n/2$. Therefore it suffices to enumerate graphs G with $|V_B| \geqslant \epsilon n/2$.

We now construct all graphs G which produce a partition with $|V_B| \geqslant \epsilon n/2$. First we pick the (ϵ_0)-regular graph spanned by V_2, V_3. There are at most $\binom{n^2}{m}$ choices.

Second we pick a partition $V = V_D \cup V_B \cup V_G$ and fix the degrees of all vertices $v \in V_1$ in $G[V_1, V_2]$ and $G[V_1, V_3]$. The number of choices is at most $3^n \cdot n^{2n}$. Finally we fix the actual neighborhoods of the vertices of V_1. For a vertex $v \in V_D \cup V_G$ the number of choices is at most

$$\binom{n}{\deg_{G[V_1,V_2]}(v)} \binom{n}{\deg_{G[V_1,V_3]}(v)}.$$

For $v \in V_B$ we have to select a neighborhood which does not induce an (ϵ, p)-lower-regular subgraph in $G[V_2, V_3]$. Since $\deg_{G[V_1,V_i]}(v) \geqslant (1-\epsilon)np \geqslant C(\epsilon)p^{-1}$ Theorem 3.1 tells us that the number of such neighborhoods is at most

$$\beta_0^{(1-\epsilon)np} \binom{n}{\deg_{G[V_1,V_2]}(v)} \binom{n}{\deg_{G[V_1,V_3]}(v)}.$$

Since $|V_B| \geqslant \epsilon n/2$ the total number of choices for the neighborhoods of the vertices in V_1 is bounded by

$$\begin{aligned} \beta_0^{(1-\epsilon)np|V_B|} \prod_{v \in V_1} \binom{n}{\deg_{G[V_1,V_2]}(v)} \binom{n}{\deg_{G[V_1,V_3]}(v)} &\leqslant \beta_0^{(1-\epsilon)\epsilon m/2} \binom{n^2}{m}^2 \\ &= \left(\frac{\beta}{2}\right)^m \binom{n^2}{m}^2, \end{aligned}$$

where the inequality follows from Vandermonde's identity. Since $3^n \cdot n^{2n} \leqslant 2^{n^{3/2}} \leqslant 2^m$ by assumption on n and m, this completes the proof. $\square$

Lemma 3.2 immediately implies the following corollary about the number of triangles spanned by almost all edges:

Corollary 3.3. *For every $\beta, \delta > 0$ there exist $\epsilon_0 > 0$ and $C > 0$ such that for $m \geqslant Cn^{3/2}$ and n sufficiently large the following holds. The number of graphs $G = (V_1 \cup V_2 \cup V_3, E)$ in $\mathcal{G}(K_3, n, m/n^2, \epsilon_0)$, with m edges between each two partitions, for which more than δm edges of $G[V_1, V_2]$ are contained in fewer than $(1-\delta)n(m/n^2)^2$ triangles is at most*

$$\beta^m \binom{n^2}{m}^3.$$

Proof. Let $p = m/n^2$ and choose ϵ small enough for $(1-\epsilon)^3 \geqslant (1-\delta)$ to hold. Denote with $V_1' \subseteq V_1$ the set of vertices $v \in V_1$ whose neighborhoods in V_2, V_3 are of size $(1 \pm \epsilon)np$ and induce an (ϵ, p)-lower-regular subgraph in $G[V_1, V_2]$. From Lemma 2.2 we deduce that every vertex in V_1' is incident to at least $(1-\epsilon)^2 np$ edges (with endpoint in V_2) which are each contained in at least $(1-\epsilon)^2 np^2 \geqslant (1-\delta)np^2$ triangles. Therefore, the total number of edges which are contained in fewer than $(1-\delta)np^2$ triangles is at most $m - |V_1'|(1-\epsilon)^2 np$. By choice of δ, the only possibility that the desired condition is not fulfilled is thus that $|V_1'| \leqslant (1-\epsilon)n$. Lemma 3.2 handles exactly this case – and thus concludes the proof if we choose C and ϵ_0 as in this lemma. $\square$

With Corollary 3.3 at hand we are now ready to prove Lemma 2.9.

Proof of Lemma 2.9. Observe first that it suffices to consider a fixed integer $k \leqslant k_{\max}$, as the lemma then follows by choosing the minimum ϵ' for all $k \leqslant k_{\max}$ and a trivial union bound argument.

We first consider property (i) of Definition 2.8. The key fact here is that we want the property to hold for *every* edge, while Corollary 3.3 guarantees this only for *'almost all'* edges. It is thus obvious what we need to do: show that we can remove edges appropriately. To this end Lemma 2.3 will come in very handy, as it shows that if we choose ϵ' small enough then we can take away edges repeatedly, while still keeping some regularity properties.

Define constants as follows: $\epsilon_0(\alpha, \epsilon)$ will be fixed at the end of the proof, but will be small enough to satisfy $(1-\epsilon) \leqslant (1-\epsilon_0)^3$. Set $\beta = (\alpha/(4e))^3$ and for $i \in [k-1]$ define $\delta_i = (\epsilon_{i-1}/4)^4/2$ and $\epsilon_i = \min\{\delta_i/4, \epsilon_{cor}(\beta, \delta_i)\}$, where $\epsilon_{cor}(\cdot,\cdot)$ denotes the function $\epsilon(\beta,\delta)$ defined in Corollary 3.3. Finally define $m_i = \lceil (1-\epsilon_i) n_0^2 p_0 \rceil$ and $\epsilon' = \epsilon_{k-1}/4$. Observe that $\epsilon_0 > \delta_1 > \epsilon_1 > \cdots > \delta_{k-1} > \epsilon_{k-1} > \epsilon'$.

We now proceed as follows: for $i = k-1$ down to $i = 1$ we remove edges from $E(V_i, V_{i+1})$ if they are not contained in enough triangles with V_{i+2} *with respect* to the edge set that survived the removal process in the previous round. That is, we remove all edges in $E(V_i, V_{i+1})$ which are contained in fewer than $(1-\epsilon) n_0 p_0^2$ triangles with V_{i+2}, only taking in account edges that are still present.

Assume first that for all $1 \leqslant i < k$ we remove at most $2\delta_i m_i$ edges. Then the resulting subgraph $\tilde{G}$ is, by construction, such that the graph satisfies property (i) of Definition 2.8. We claim that we also have that all pairs are (ϵ_0)-regular with density at least $(1-\epsilon_0)p_0$. Note that this implies that $\tilde{G} \in G(P_k^2, n_0, p_0, \epsilon_0)$. By definition of $\tilde{\mathcal{G}}(P_k^2, n_0, p_0, \epsilon')$ we know that (before removing any edges) the pair (V_i, V_{i+1}) is (ϵ')-regular (and thus $(\epsilon_0/2)$-regular) with density at least $(1-\epsilon')p_0$. If we remove at most $2\delta_i m_i \leqslant (\epsilon_0/2)^4 |E(V_i, V_{i+1)}|$ edges from $E(V_i, V_{i+1})$, then Lemma 2.3 implies that the remaining graph is (ϵ_0)-regular with density at least $(1-\epsilon'-2\delta_i)p_0 \geqslant (1-3\delta_i)p_0 \geqslant (1-\epsilon_0)p_0$.

So assume the above condition does not hold. Let i denote the largest $1 \leqslant i < k$ such that when processing $E(V_i, V_{i+1})$ we have to remove more than $2\delta_i m_i$ edges. We claim that then $G[V_i \cup V_{i+1} \cup V_{i+2}]$ contains one of the subgraphs enumerated by Corollary 3.3.

Indeed, let $G'_{i+1,i+2} \subseteq G[V_{i+1}, V_{i+2}]$ denote the subgraph obtained after removing the edges which do not satisfy property (i). If $i = k-1$ then, since we do not touch the last partition, we have $G'_{i+1,i+2} = G[V_k, V_{k+1}]$ which is trivially $(\epsilon_i/2)$-regular with density at least $(1-\epsilon_i)p_0$. If $i < k-1$ then by maximality of i the graph $G'_{i+1,i+2}$ is obtained by removing at most $2\delta_{i+1} m_{i+1} \leqslant (\epsilon_i/4)^4 |E(G[V_{i+1}, V_{i+2}])|$ edges from the $(\epsilon_i/4)$-regular graph $G[V_{i+1}, V_{i+2}]$. Therefore by Lemma 2.3 $G'_{i+1,i+2}$ is $(\epsilon_i/2)$-regular with density at least $(1-\epsilon'-2\delta_{i+1})p_0 \geqslant (1-\epsilon_i)p_0$. Let $G_{i,i+1} \subseteq G[V_i, V_{i+1}]$, $G_{i,i+2} \subseteq G[V_i, V_{i+2}]$, $G_{i+1,i+2} \subseteq G'_{i+1,i+2}$ denote spanning (ϵ_i)-regular subgraphs with exactly m_i edges each (for $m_i \gg n_0$ and $\epsilon' \leqslant \epsilon_i/2$ such subgraphs always exists, see Lemma 4.3 in [8]). Observe that to obtain $G_{i,i+1}$ we removed at most $2\epsilon_i n_0^2 p_0 \leqslant \delta_i m_i$ edges from $G[V_i, V_{i+1}]$.

Thus by choice of i there have to be at least $\delta_i m_i$ more edges in $G_{i,i+1}$ which are each contained in fewer than $(1-\epsilon) n_0 p_0^2$ triangles with V_{i+2} in $G_i \coloneqq G_{i,i+1} \cup G_{i,i+2} \cup G_{i+1,i+2}$. Since $(1-\epsilon) n_0 p_0^2 \leqslant (1-\epsilon_0)^3 n_0 p_0^2 \leqslant (1-\delta_i) n_0 (m_i/n_0^2)^2$ and $\epsilon_i \leqslant \epsilon_{cor}((\alpha/(4e))^3, \delta_i)$ we may apply Corollary 3.3 (with $\beta \leftarrow \beta = (\alpha/(4e))^3$, $\delta \leftarrow \delta_i$, $m \leftarrow m_i$) to conclude that G_i must be one of at most

$$\beta^{m_i}\binom{n_0^2}{m_i}^3,$$

graphs enumerated by Corollary 3.3. The probability that $G_{n,p}$ contains any of these graphs as a subgraph is at most

$$\binom{n}{n_0}^3\cdot\beta^{m_i}\binom{n_0^2}{m_i}^3\cdot p^{3m_i}\leqslant 2^{3n}\beta^{m_i}\left(\frac{en_0^2p}{m_i}\right)^{3m_i}\leqslant 2^{3n}\beta^{m_i}\left(\frac{2e}{a}\right)^{3m_i}=2^{3n-3m_i}.$$

Since $n_0\geqslant \eta n$ we have $m_i\gg n$ and may additionally union bound over all choices for n_0 and conclude that a.a.s. no such graph appears in $G_{n,p}$. In particular our procedure never removes more than $2\delta_i m_i$ edges and produces a graph $\tilde{G}\in\mathcal{G}\left(P_k^2,n_0,p_0,\epsilon_0\right)$ which satisfies property (i).

It remains to show that additionally the graph $\tilde{G}$ also satisfies property (ii). Fix $\epsilon_0=\min\left\{\epsilon/3,\epsilon_{lem}\left(\beta,\epsilon/2\right)\right\}$, where $\epsilon_{lem}(\cdot,\cdot)$ denotes the function $\epsilon(\beta,\epsilon)$ defined in Lemma 3.2. This choice is valid since $(1-\epsilon)\leqslant(1-(\epsilon/3))^3$. Suppose that $\tilde{G}$ fails property (ii) for some $i\in[k-1]$. We claim that then $\tilde{G}[V_i\cup V_{i+1}\cup V_{i+2}]$ must contain a subgraph enumerated by Lemma 3.2. To this end let $\tilde{G}'\subseteq\tilde{G}[V_i\cup V_{i+1}\cup V_{i+2}]$ denote a spanning subgraph in which every partition contains exactly $m_0=\lceil(1-\epsilon_0)n_0^2p_0\rceil$ edges and is $(2\epsilon_0)$-regular (as before see Lemma 4.3 in [8]).

For $\epsilon_0\leqslant\epsilon/2$ we have $(1-\epsilon/2)m_0/n_0^2\geqslant(1-\epsilon)p$ and thus every $(\epsilon/2,m_0/n_0^2)$-lower-regular graph is also (ϵ,p_0)-lower-regular. It follows that $\tilde{G}'$ must be among the at most

$$\beta^{m_0}\binom{n_0^2}{m_0}^3$$

graphs enumerated by Lemma 3.2 (with $\epsilon\leftarrow\epsilon/2$, $\beta\leftarrow\beta$ and $m\leftarrow m_0$). As before a union bound shows that a.a.s. $G_{n,p}$ does not contain such a graph and thus $\tilde{G}$ also satisfies property (ii). □

4 Proof of Lemma 2.10

Definition 2.8 already implies that every edge spanned by V_1,V_2 is contained in many copies of P_{k+4}^2. The goal of this section is to show that these paths also reach a majority of the edges spanned by V_{k+4},V_{k+5}. As a first step we show that two edges cannot span too many copies of P_k^2.

Lemma 4.1. *For every $\gamma>0$, $k>\frac{3}{2}\cdot\frac{1+\gamma}{\gamma}$ and $p=n^{-(1-\gamma)/2}$ a.a.s. no pair of edges in $G_{n,p}$ is connected by more than $2n^{k-3}p^{2k-3}$ squares of paths of length k.*

Proof. This follows directly from a theorem of Spencer on the number of graph extensions in the random graph [17]. P_k^2 rooted at both of its end-edges is strictly rooted balanced and $n^{k-3}p^{2(k-3)+3}\gg\log n$. Thus the number of squares of paths of length k connecting any two edges is concentrated around its expectation. □

The previous lemma easily implies a weaker version of Lemma 2.10 where the constant 0.52 has to be replaced with a small value that depends on α and η. And this will indeed be the first step in the proof. Next we prove a small lemma which contains two ad hoc arguments that will allow us to go from a small set of the edges in V_i, V_{i+1} to a slightly larger set of edges in V_{i+1}, V_{i+2}.

Lemma 4.2. *Let $\eta, \alpha, \gamma, \epsilon > 0$ be fixed. For $p = n^{-1/2+\gamma/2}$ in $G_{n,p}$ a.a.s. every copy of $G = (V_1 \cup V_2 \cup V_3, E) \in \tilde{\mathcal{G}}(K_3, n_0, p_0, \epsilon)$, where $n_0 \geqslant \eta n$ and $p_0 = \alpha p$, satisfies the following: for every set of edges $\tilde{E} \subseteq E(V_1, V_2)$ denote with $\triangle(\tilde{E})$ the number of edges in $E(V_2, V_3)$ which form a triangle with some edge of $\tilde{E}$ in G. Let $\tilde{V}_2 \subseteq V_2$ denote the vertices which are incident to some edge of $\tilde{E}$ and set $s = \min_{v \in \tilde{V}_2} \deg_{\tilde{E}}(v)$. Then*

$$\triangle(\tilde{E}) \geqslant \begin{cases} |\tilde{V}_2| n_0 p_0^2/(2p) & \text{if } s \geqslant \log^2(n) n_0 p / n^\gamma, \\ (1-\epsilon)^2 \left(|\tilde{V}_2| - 5\epsilon n_0 \right) n_0 p_0 & \text{if } s \geqslant 2\epsilon n_0 p_0. \end{cases}$$

Proof. Fix $v \in \tilde{V}_2$. Write $V_1^v = \Gamma_{\tilde{E}}(v)$ and let $V_3^v = V_3 \cap \Gamma(v) \cap \Gamma(V_1^v)$ denote the subset of vertices of V_3 which form a triangle with v and some edge from $\tilde{E}$. By definition of G every edge in $\tilde{E}$ is contained in at least $(1-\epsilon) n_0 p_0^2$ triangles with V_3 and therefore

$$E(V_1^v, V_3^v) \geqslant (1-\epsilon) |V_1^v| n_0 p_0^2. \tag{1}$$

A.a.s. in $G_{n,p}$ all disjoints sets A, B of sizes $|A| \geqslant \log^2(n) n_0 p_0 / n^\gamma$ and $|B| \leqslant b = n_0 p_0^2/(2p)$ have at most $(1+\epsilon)|A|bp$ edges between them. To see this, observe that $|A|bp \gg \log n \max\{|A|, b\}$ and the claim thus follows from Chernoff's inequality together with a straightforward union bound argument over all sets of size $|A|$ and at most b.

We now consider the two cases. If $s \geqslant \log^2(n) n_0 p_0 / n^\gamma$ then Equation (1) implies that $E(V_1^v, V_3^v) \geqslant (1-\epsilon)|V_1^v| n_0 p_0^2 = 2(1-\epsilon)|V_1^v| bp$. For $|V_3^v| \leqslant b$ this would contradict the bounds from the Chernoff inequality in the previous paragraph; thus $|V_3^v| \geqslant b$, implying the desired bound.

Now suppose that $s \geqslant 2\epsilon n_0 p_0$ and that v is such that its neighborhoods in V_1 and V_3 are of size $(1 \pm \epsilon) n_0 p_0$ and induce a (ϵ, p)-lower-regular subgraph. As $|V_1^v| \geqslant s$ the assumption on p and v imply $|V_1^v| \geqslant \epsilon |\Gamma(v) \cap V_1|$. Thus we can apply Lemma 2.2 to deduce that at most $\epsilon |\Gamma(v) \cap V_3|$ vertices in $\Gamma(v) \cap V_3$ have no neighbor in V_1^v. Thus $|V_3^v| \geqslant (1-\epsilon)|\Gamma(v) \cap V_3| \geqslant (1-\epsilon)^2 n_0 p_0$. By Lemma 2.2 at most $4\epsilon n_0$ vertices do have neighborhoods of the wrong size in either V_1 or V_3. By definition of $\tilde{\mathcal{G}}$ at most ϵn_0 vertices have neighborhoods which do not induce an (ϵ, p)-lower-regular subgraph. Thus $\triangle(\tilde{E}) \geqslant \left(|\tilde{V}_2| - 5\epsilon n_0 \right) (1-\epsilon)^2 n_0 p_0$, as claimed. □

With these two lemmas at hand the proof of Lemma 2.10 can be summarized as follows: Use the weak version implied by Lemma 4.1 to expand to a small fraction of the edges spanned by V_k, V_{k+1}. Then invoke Lemma 4.2 (four times!) to expand to a 0.52-fraction of the edges in V_{k+4}, V_{k+5}. The details are given below.

Lemma 2.10. *Let $\eta, \alpha, \gamma > 0$ and $k \geqslant \frac{3}{\gamma}$ be fixed and let $p = n^{-1/2+\gamma/2}$. For ϵ small enough depending on α a.a.s. every subgraph $G \subseteq G_{n,p}$ which is from $G \in \tilde{\mathcal{G}}\left(P_{k+4}^2, n_0, \alpha p, \epsilon\right)$ for some $n_0 \geqslant \eta n$ satisfies the following: all edges in $G[V_1 \cup V_2]$ expand to at least a 0.52-fraction of the edges in $G[V_{k+4} \cup V_{k+5}]$.*

Proof. Write $p_0 = \alpha p$. Since $G \in \tilde{\mathcal{G}}\left(P_{k+4}^2, n_0, p_0, \epsilon\right)$ every edge in $G[V_i \cup V_{i+1}]$ forms at least $(1-\epsilon)n_0p_0^2$ triangles with V_{i+2}. In particular every edge spanned by $V_1 \cup V_2$ is connected to $E(V_k, V_{k+1})$ by $((1-\epsilon)n_0p_0^2)^{k-1}$ copies of P_k^2. Furthermore since $k \geqslant \frac{3}{\gamma} > \frac{3}{2} \cdot \frac{1+\gamma}{\gamma}$ by Lemma 4.1 every edge in $E(V_1, V_2)$ is connected to at least

$$\frac{((1-\epsilon)n_0p_0^2)^{k-1}}{2n^{k-3}p^{2k-3}} = \Theta(n^2p) \gg n^2p/n^{\gamma/2}.$$

distinct edges in $E(V_k, V_{k+1})$ via the square of a path.

Therefore it suffices to show that every set $E_0 \subseteq E(V_k, V_{k+1})$ of size at least $n_0^2p_0/n^{\gamma/2}$ expands to at least a 0.52-fraction of $E(V_{k+4}, V_{k+5})$. To this end we will apply Lemma 4.2 four times to $G \leftarrow G[V_{k+i} \cup V_{k+i+1} \cup V_{k+i+2}]$ for $i \in \{0, 1, 2, 3\}$.

Set $s = \log^2(n)n_0p/n^\gamma$. In $G_{n,p}$ a.a.s. all degrees are bounded by $(1+o(1))np$. Therefore we can find a set of vertices $\tilde{V}_{k+1} \subseteq V_{k+1}$ of size

$$\frac{|E_0| - n_0s}{\Theta(np)} = \Theta(n^{1-\gamma/2})$$

and an edge set $\tilde{E}_0 \subseteq E_0$ such that each $v \in \tilde{V}_{k+1}$ is incident to exactly s edges of $\tilde{E}_0$.

Apply Lemma 4.2 with $G \leftarrow G[V_k \cup V_{k+1} \cup V_{k+2}]$, $\tilde{V}_2 \leftarrow \tilde{V}_{k+1}$, $\tilde{E} \leftarrow \tilde{E}_0$ and denote the set of edges which form a triangle with some edge of $\tilde{E}_0$ with $E_1 \subseteq E(\tilde{V}_{k+1}, V_{k+2})$. We have $|E_1| \geqslant \frac{p_0}{2p}|\tilde{V}_{k+1}|n_0p_0$. Denote with $\tilde{V}_{k+2} \subseteq V_{k+2}$ the set of vertices which are incident to more than s edges in E_1. A.a.s. in $G_{n,p}$ there exists no set S of size $|\tilde{V}_{k+1}| = \Theta(n^{1-\gamma/2})$ such that more than $\sqrt{n}$ vertices have degree at least $2|\tilde{V}_{k+1}|p$ into S. Therefore for ϵ small enough depending on α

$$|\tilde{V}_{k+2}| \geqslant \frac{|E_1| - \sqrt{n} \cdot \Theta(np) - n_0s}{2|\tilde{V}_{k+1}|p} \geqslant \frac{\frac{p_0}{3p}|\tilde{V}_{k+1}|n_0p_0}{2|\tilde{V}_{k+1}|p} \geqslant \frac{p_0^2}{6p^2}n_0 \geqslant 100\epsilon n_0.$$

As before pick a subset $\tilde{E}_1 \subseteq E_1$ such that every $v \in \tilde{V}_{k+2}$ is incident to exactly s edges of $\tilde{E}_1$. Apply Lemma 4.2 a second time with $G \leftarrow G[V_{k+1} \cup V_{k+2} \cup V_{k+3}]$, $\tilde{V}_2 \leftarrow \tilde{V}_{k+2}$, $\tilde{E} \leftarrow \tilde{E}_1$ and denote the set of edges which form a triangle with some edge of $\tilde{E}_1$ with $E_2 \subseteq E(V_{k+2}, V_{k+3})$. We have $|E_2| \geqslant \frac{p_0}{2p}|\tilde{V}_{k+2}|n_0p_0$. Let $\tilde{V}_{k+3} \subseteq V_{k+3}$ denote the subset of vertices which are incident to more than $s' = 2\epsilon n_0p_0$ edges in E_2. As $E(V_{k+2}, V_{k+3})$ is (ϵ)-regular, by Lemma 2.2 there are at most ϵn_0 vertices in V_{k+3} with more than $(1+\epsilon)|\tilde{V}_{k+2}|p_0$ neighbours in $\tilde{V}_{k+2}$. And by definition of (ϵ)-regularity these vertices are in total incident to at most $\epsilon n_0(1+\epsilon)|\tilde{V}_{k+2}|p_0$ edges. Therefore for ϵ sufficiently small

$$|\tilde{V}_{k+3}| \geqslant \frac{|E_2| - \epsilon n_0(1+\epsilon)|\tilde{V}_{k+2}|p_0 - n_0s'}{(1+\epsilon)|\tilde{V}_{k+2}|p_0} \geqslant \frac{\frac{p_0}{3p}|\tilde{V}_{k+2}|n_0p_0}{(1+\epsilon)|\tilde{V}_{k+2}|p_0} \geqslant \frac{p_0}{4p}n_0 \geqslant 100\epsilon n_0.$$

As before pick a subset $\tilde{E}_2 \subseteq E_2$ such that every $v \in \tilde{V}_{k+3}$ is incident to exactly s' edges of $\tilde{E}_2$. Apply Lemma 4.2 a third time with $G \leftarrow G[V_{k+2} \cup V_{k+3} \cup V_{k+4}]$, $\tilde{V}_2 \leftarrow \tilde{V}_{k+3}$, $\tilde{E} \leftarrow \tilde{E}_2$ and denote the set of edges which form a triangle with some edge of $\tilde{E}_2$ with $E_3 \subseteq E(V_{k+3}, V_{k+4})$. We have $|E_3| \geqslant (1-\epsilon)^2 \left(|\tilde{V}_{k+2}| - 5\epsilon n_0\right) n_0 p_0 \geqslant \frac{6}{7}|\tilde{V}_{k+2}| n_0 p_0$. Let $\tilde{V}_{k+4} \subseteq V_{k+4}$ denote the subset of vertices which are incident to more than $s' = 2\epsilon n_0 p_0$ edges in E_3. As before we obtain

$$|\tilde{V}_{k+4}| \geqslant \frac{|E_3| - \epsilon n_0(1+\epsilon)|\tilde{V}_{k+3}|p_0 - n_0 s'}{(1+\epsilon)|\tilde{V}_{k+3}|p_0} \geqslant \frac{\frac{5}{7}|\tilde{V}_{k+3}|n_0 p_0}{(1+\epsilon)|\tilde{V}_{k+3}|p_0} \geqslant \frac{4}{7}n_0.$$

Applying Lemma 4.2 a fourth time we see E_3 expands to $(1-\epsilon)^2(|\tilde{V}_{k+4}| - 5\epsilon n_0)n_0 p_0 > 0.53 n_0^2 p_0$ edges in $E(V_{k+4}, V_{k+5})$. This concludes the proof. □

Concluding remarks

A referee of our paper pointed out that an alternative approach to prove our result is to use the result on the counting version of the KŁR-conjecture by Conlon, Gowers, Samotij, Schacht [5]. That approach, in fact, can also be used to obtain similar results for all k-powers of an Hamilton cycle. Similarly to our result, this approach also requires that the density p is bounded away from the lower bound by a factor of n^ϵ for some positive constant $\epsilon > 0$. In [20] it was recently shown that this factor can actually be replaced by a much smaller factor of $C(\log n)^{1/k}$. It is an interesting open problem whether this factor is actually necessary or whether one could go down all the way to the trivial lower bound. Also, even though it is obvious that local resilience does not allow a spanning subgraph of a square of a Hamilton cycle, two natural questions remain: how many vertices do we actually have to miss or what additional constraints do we need to add to local resilience in order to guarantee a spanning k-th power of a Hamilton cycle.

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