Smaller subgraphs of minimum degree k

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Abstract

In 1990, Erdős, Faudree, Rousseau and Schelp proved that for $k \geqslant 2$ every graph with $n \geqslant k+1$ vertices and $(k-1)(n-k+2)+\binom{k-2}{2}+1$ edges contains a subgraph of minimum degree k on at most $n-\sqrt{n/6k^3}$ vertices. They conjectured that it is possible to remove at least $\epsilon_k n$ many vertices and remain with a subgraph of minimum degree k, for some $\epsilon_k > 0$. We make progress towards their conjecture by showing that one can remove at least $\Omega(n/\log n)$ many vertices.

1 Introduction

It is easy to show that every graph on $n \geqslant 4$ vertices with at least $2n-2$ edges contains a subgraph of minimum degree 3. More generally, any graph on $n \geqslant k+1$ vertices with at least $t_k(n) := (k-1)(n-k+2)+\binom{k-2}{2}$ edges[1] contains a subgraph of minimum degree k, for all $k \geqslant 2$. This statement is best possible in two ways: (1) there exist graphs on $n \geqslant k+1$ vertices with $t_k(n)-1$ edges which do not contain a subgraph of minimum degree k, and (2) there exist graphs on n vertices with $t_k(n)$ edges without a subgraph of minimum degree k on fewer than n vertices. For example the wheel $W(1,n) = K_1 + C_{n-1}$ (where $+$ denotes the graph join operation) has exactly $2n-2$ edges and minimum degree 3, but

[1] Technically, we may relax the first condition to $n \geqslant k-1$, since for $n \in \{k-1, k\}$ the condition on the number of edges cannot be satisfied. However, there seems no point in doing so. For $n = k-2$, the statement is wrong.

contains no proper induced subgraph with minimum degree 3. A similar construction is available for all k (consider the generalized wheel $W(k-2,n) = K_{k-2} + C_{n-k+2}$).

Erdős conjectured that the presence of even a single additional edge allows one to find a much smaller subgraph:

Conjecture 1.1 (Erdős [1, 2]). *For every $k \geqslant 2$ there exists an $\epsilon_k > 0$ such that every graph on $n \geqslant k+1$ vertices and $t_k(n)+1$ edges contains a subgraph of minimum degree k with at most $(1-\epsilon_k)n$ vertices.*

As far as we know, the only progress on this conjecture is the following theorem due to Erdős, Faudree, Rousseau and Schelp from 1990:

Theorem 1.2 (Erdős, Faudree, Rousseau, Schelp [2]). *For $k \geqslant 2$, let G be a graph on $n \geqslant k+1$ vertices and $t_k(n)+1$ edges. Then G contains a subgraph of order at most $n - \lfloor\sqrt{n/(6k^3)}\rfloor$ and minimum degree at least k.*

Here, we will show that it is possible to replace the $\sqrt{n/6k^3}$ by $\Omega(n/\log n)$:

Theorem 1.3. *For $k \geqslant 2$, let G be a graph on $n \geqslant k+1$ vertices and $t_k(n)+1$ edges. Then G contains a subgraph of order at most $n - n/(8(k+1)^5 \log_2 n)$ and minimum degree at least k.*

We remark that very recently the complete proof of Conjecture 1.1 was given by Sauermann [3]. Her proof is partially based on the ideas introduced in this paper.

We use standard graph theoretic notation. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$. We write v_G for the number of vertices and e_G for the number of edges in G. $V_i(G)$ denotes the set of vertices of G with degree exactly i. Similarly $V_{\leqslant i}(G)$ denotes the set of vertices of degree at most i. For a vertex v, we denote its neighborhood by $\Gamma_G(v)$ and its degree by $\deg_G(v)$ (we omit the subscript G if it is clear from the context). We write $E_G(A,B)$ (or $E(A,B)$) for the set of edges in G with one endpoint in A and another in B. The minimum degree of a graph G is denoted by $\delta(G)$.

2 Proof of Theorem 1.3

The proof will use induction on the number of vertices of G. In the base case $n = k+1$ it is easy to check that $t_k(n)+1 = \binom{k+1}{2}+1$, so the theorem holds vacuously. Assume now that the theorem holds for all graphs G on $k+1 \leqslant n' < n$ vertices.

If G contains a non-empty set A of at most $n-k-1$ vertices with the property that at most $(k-1)|A|$ edges of G intersect A, then $G-A$ has (1) $n-|A| \geqslant k+1$ vertices and (2) at least $t_k(n)+1-(k-1)|A| = t_k(n-|A|)+1$ edges. In this case, by induction, $G-A$ (and hence G) contains a subgraph with minimum degree k and at most

$$n - |A| - \frac{n-|A|}{8(k+1)^5 \log_2(n-|A|)} \leqslant n - \frac{n}{8(k+1)^5 \log_2 n}$$

vertices, and we are done. From now on, we will assume that every non-empty set of at most $n-k-1$ vertices intersects at least $(k-1)|A|+1$ edges. In particular, G has minimum degree at least k.

The following notion is important for the rest of the proof:

Definition 2.1 (Good set)**.** *By a* good set*, we mean any set of vertices of G constructed according to the following rules:*

1. *If v has degree k, then $\{v\}$ is good.*
2. *If A is good and $v \notin A$ is such that all but at most $k-1$ neighbors of v belong to A, then $A \cup \{v\}$ is good.*
3. *If A and B are both good and if G contains an edge that meets both A and B, then $A \cup B$ is good (this is the case if $A \cap B \neq \varnothing$ or if $E(A,B) \neq \varnothing$).*

We say that a good set is maximal *if it is not properly contained in another good set.*

The relevance of this notion for our problem is partly due to the following claim.

Claim 2.2. *The following statements hold:*

(i) Every good set C with $|C| \leqslant n-k-1$ intersects at most $(k-1)|C|+1$ edges of G.

(ii) If C is a good set with $|C| \leqslant n-k-1$, then $G-C$ contains a subgraph of minimum degree at least k.

(iii) If C and C' are maximal good sets and $C \neq C'$, then $C \cap C' = \varnothing$ and $E_G(C,C') = \varnothing$.

Proof. The statement (i) is proved by induction on the rules 1, 2, and 3 in the definition of a good set. Let $g(X)$ denote the number of edges intersecting a set X. Let C be a good set of size at most $n-k-1$. If C is produced by rule 1, we automatically have $g(C) \leqslant k = (k-1)|C|+1$. If C is produced by rule 2 or by the $A \cap B = \varnothing$ case of rule 3, then the bound on $g(C)$ follows easily by induction. The only difficult case is when $C = A \cup B$ for two smaller good sets A and B where $A \cap B \neq \varnothing$ (the other case of rule 3). In this case, we use the assumption that $g(A \cap B) \geqslant (k-1)|A \cap B|+1$ and the induction hypothesis for A and B to get

$$\begin{aligned} g(C) &\leqslant g(A) + g(B) - g(A \cap B) \\ &\leqslant (k-1)|A| + 1 + (k-1)|B| + 1 - (k-1)|A \cap B| - 1 \\ &= (k-1)(|A| + |B| - |A \cap B|) + 1 \\ &= (k-1)|C| + 1. \end{aligned}$$

For (ii), suppose that $|C| \leqslant n-k-1$, so $v_{G-C} = n - |C| \geqslant k+1$. Because C intersects at most $(k-1)|C|+1$ edges, we have

$$e_{G-C} \geqslant e_G - (k-1)|C| - 1 \geqslant t_k(n) - (k-1)|C| = t_k(v_{G-C}),$$

where the last equality follows from the definition $t_k(n) = (k-1)(n-k+2) + \binom{k-2}{2}$. Then (ii) follows because every graph G' with $v_{G'} \geqslant k+1$ and $e_{G'} \geqslant t_k(v_{G'})$ contains a subgraph of minimum degree k. Statement (iii) follows immediately from the definition of a good set (rule 3). □

We now handle the case where some good set is very large. Since every good set is obtained by application of one of the rules given in Definition 2.1, it is clear that if C is a good set of size at least two, then there is a good subset $C' \subseteq C$ of size $|C|/2 \leqslant |C'| \leqslant |C| - 1$. In particular, if some good set C has size at least $n/(k+2)$, then there also exists a good set $C' \subseteq C$ satisfying $n/(2k+4) \leqslant |C'| \leqslant n/(k+2) \leqslant n-k-1$, where the last inequality holds since $n > k+1$. Then Claim 2.2 (ii) implies that $G - C'$ has a subgraph of minimum degree k, and so G has a subgraph of order at most $n - n/(2k+4) \leqslant n - n/(8(k+1)^5 \log_2 n)$ with minimum degree k, and we are done. From now on, we may thus assume that every good set has size at most $n/(k+2) \leqslant n-k-1$.

The rest of the proof is split into two cases depending on the number of vertices of degree exactly k. Already in the proof of Theorem 1.2, Erdős, Faudree, Rousseau, and Schelp observed that Conjecture 1.1 holds if the number of vertices of degree k is not too large:

Lemma 2.3 (Lemma 4 in [2])**.** *For $k \geqslant 2$, let G be a graph on n vertices and $t_k(n)+1$ edges with $\delta(G) \geqslant k$. If for some $0 < \alpha < 1/(2k)$, G has at most αn vertices of degree k, then G has a subgraph H of order at most $n - (1-2\alpha k)n/(8k^2)$ with $\delta(H) \geqslant k$.*

Set $\alpha := 1/(2k+2)$. The exact value of α is not too important, but we need $\alpha < 1/(2k)$ and the given value seems convenient. If G contains fewer than αn vertices of degree k, then, by Theorem 2.3, G contains a subgraph of order at most

$$n - (1-2\alpha k)n/(8k^2) = n - \frac{n}{8k^2(k+1)} \leqslant n - \frac{n}{8(k+1)^5 \log_2 n}$$

with minimum degree k, and we are done. So from now on assume that G contains at least αn vertices of degree k.

To motivate the rest of the proof, we now briefly discuss the proof strategy that was used by Erdős, Faudree, Rousseau, and Schelp in [2]. Let x be some parameter with $0 < x < n$, which we will optimize later. If there is a good set of size at least x, then Claim 2.2 (ii) implies that we can find a subgraph of minimum degree k on at most $n - x$ vertices (the case where the good set is larger than $n-k-1$ is already excluded). Otherwise, every good set is smaller than x. Now we run an algorithm which constructs a chain of subgraphs $G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots$ greedily as follows: if $V(H_i)$ contains a maximal good set C such that $H_i - C$ has minimum degree at least k, then we let $H_{i+1} = H_i - C$; otherwise $H_{i+1} = H_i$. One can show, using the statements in Claim 2.2 and the fact that every maximal good set has size at most x, that this algorithm removes at least $\Omega(n/x)$ good sets, and thus produces a graph with $n - \Omega(n/x)$ vertices, which has minimum degree at least k by construction. By setting x to the optimal value $\sqrt{n}$, we thus obtain a subgraph with minimum degree k of size $n - \Theta(\sqrt{n})$. In our proof, we avoid the case distinction based on the maximum size of a good set. However, we also construct a small subgraph of G by removing maximal good sets. More precisely, we start by choosing a collection $\mathcal{F}$ of maximal good sets that covers $\Omega(n/\log n)$ vertices and such that all good sets in $\mathcal{F}$ are of comparable sizes; the existence of this collection is guaranteed by Claim 2.4 further below. Next, we remove a positive fraction of the sets in $\mathcal{F}$ from G in such a way that the remaining graph still contains a subgraph of minimum degree k. The main technical statement that makes this possible is Claim 2.5 below. Since the sets in $\mathcal{F}$ all have similar sizes, this means that we remove $\Omega(n/\log n)$ vertices, completing the proof. We now turn to the details.

Claim 2.4. *There exists a collection $\mathcal{F}$ of maximal good sets such that*

$$n > \sum_{C\in\mathcal{F}} |C| \geqslant \frac{\alpha n}{\log_2 n}$$

and such that for any two $C, C' \in \mathcal{F}$, we have $|C|/2 \leqslant |C'| \leqslant 2|C|$.

Proof. Let $\mathcal{F}$ denote the collection of all maximal good sets. Since G contains at least αn vertices of degree k, and since every such vertex is contained in one of the good sets in $\mathcal{F}$, we have $\sum_{C\in\mathcal{F}}|C| \geqslant \alpha n$. For every $1 \leqslant i \leqslant \log_2 n$, let $\mathcal{F}_i \subseteq \mathcal{F}$ denote the subfamily of all $C \in \mathcal{F}$ with $2^{i-1} \leqslant |C| \leqslant 2^i$. By averaging, there exists i such that $\sum_{C\in\mathcal{F}_i}|C| \geqslant \alpha n/(\log_2 n)$. If $\sum_{C\in\mathcal{F}_i}|C| < n$, then $\mathcal{F}_i$ is a collection with the desired properties. Otherwise we remove a single good set, say C^*, from $\mathcal{F}_i$. Since each good set has size at most $n/(k+2)$, we have

$$\sum_{C\in\mathcal{F}_i\setminus\{C^*\}} |C| \geqslant n - n/(k+2) \geqslant \alpha n/(\log_2 n),$$

where we used $\alpha = 1/(2k+2) \leqslant 1 - 1/(k+2)$. Then $\mathcal{F}_i \setminus \{C^*\}$ is a collection with the desired properties. □

Let $\mathcal{F}$ be a collection as in the statement of Claim 2.4. For every $v \in V(G)$, we define $\mathcal{F}_*(v) \subseteq \mathcal{F}$ to be the set of all good sets $C \in \mathcal{F}$ such that C contains a neighbor of v in G. Moreover, we let $\mathcal{F}(v) \subseteq \mathcal{F}_*(v)$ be any subcollection of size $k+1$ if $|\mathcal{F}_*(v)| > k+1$ and we let $\mathcal{F}(v) = \mathcal{F}_*(v)$ otherwise.

Claim 2.5. *There is a set $S \subseteq V(G)$ of size at most $2|\mathcal{F}|+k^2$ with the following property: for every subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that for all $s \in S$ we have $|\mathcal{F}' \cap \mathcal{F}(s)| \leqslant 1$, the graph $G - \bigcup_{C\in\mathcal{F}'} C$ contains a subgraph of minimum degree k.*

We postpone the proof of Claim 2.5 to the end of the proof. For now, assume that we have a set S as in the claim. Our goal is to find a subcollection $\mathcal{F}' \subseteq \mathcal{F}$ of size $\Omega(|\mathcal{F}|)$ containing at most one good set from every $\mathcal{F}(s)$. To find such a collection $\mathcal{F}'$, we construct an auxiliary 'conflict graph' $G_{\mathcal{F}}$ on the vertex set $\mathcal{F}$ by adding a clique on

$\mathcal{F}(s)$ for every $s \in S$. Note that we are looking for an independent set in $G_{\mathcal{F}}$. Because $|\mathcal{F}(s)| \leqslant k+1$ holds by construction and since $|S| \leqslant 2|\mathcal{F}| + k^2$, we have

$$e_{G_{\mathcal{F}}} \leqslant |S|\binom{k+1}{2} \leqslant \frac{2(|\mathcal{F}|+k^2)(k+1)^2}{2} \leqslant |\mathcal{F}|(1+k^2)(k+1)^2 \leqslant |\mathcal{F}|((k+1)^4 - 1/2).$$

By Turán's theorem, every graph on n vertices with at most cn edges contains an independent set of size at least $n/(2c+1)$. Thus, $G_{\mathcal{F}}$ contains an independent set $\mathcal{F}' \subseteq \mathcal{F}$ of size at least $|\mathcal{F}|/(2(k+1)^4)$. Because any two $C, C' \in \mathcal{F}$ satisfy $|C|/2 \leqslant |C'| \leqslant 2|C|$, and since $\sum_{C \in \mathcal{F}} |C| \geqslant \alpha n/(\log_2 n)$, we have

$$\sum_{C \in \mathcal{F}'} |C| \geqslant \frac{\alpha n}{4(k+1)^4 \log_2 n}.$$

Then $G - \bigcup_{C \in \mathcal{F}'} C$ has at most $n - \alpha n/(4(k+1)^4 \log_2 n)$ vertices and contains a subgraph of minimum degree k, by the defining property of S. Recalling that $\alpha = 1/(2k+2)$, this completes the proof of the theorem. It remains to prove Claim 2.5.

2.1 Proof of Claim 2.5

For the proof of the claim, we need the following definition and lemma.

Definition 2.6 (**(H, S, k)**-cover)**.** *Suppose that H is a graph and that $S \subseteq V(H)$ is a subset of its vertices. Given $k \geqslant 2$, a graph $\tilde{H}$ is called an (H, S, k)*-cover *if it contains H as a subgraph and if $V_{\leqslant k-1}(\tilde{H}) \subseteq V(H) \setminus S$.*

Lemma 2.7. *Suppose that H is a graph that does not contain a subgraph of minimum degree k, for $k \geqslant 2$. Then there exists a subset $S \subseteq V_{\leqslant k-1}(H)$ of cardinality at most $2(k-1)v_H - 2e_H$ such that every (H, S, k)-cover contains a subgraph of minimum degree k.*

Proof. For a graph H, we define the function

$$\phi(H) := 2(k-1)v_H - 2e_H - \sum_{w \in V_{\leqslant k-1}(H)} (k-1-\deg_H(w)). \tag{1}$$

By induction on the number of vertices of H, we will prove the following statement, which is slightly stronger than the claim of the lemma: if H has no subgraph of minimum degree k, then there is a subset $S \subseteq V_{\leqslant k-1}(H)$ of size at most $\phi(H)$ such that every (H, S, k)-cover contains a subgraph of minimum degree k.

In the base case $v_H = 1$, we can let S be the set containing the single vertex of H. In this case, we have $|S| \leqslant k-1 = \phi(H)$, and every (H, S, k)-cover has minimum degree at least k by definition.

If $v_H \geqslant 2$, then the fact that H does not contain a subgraph of minimum degree k implies that there is a vertex v with $\deg(v) \leqslant k-1$. Let us write $H' := H - v$. Then H' is a graph without a subgraph of minimum degree k, and it has fewer vertices than H.

Hence, by induction, it contains a set $S' \subseteq V_{\leqslant k-1}(H')$ of size $|S'| \leqslant \phi(H')$ such that every (H', S', k)-cover has a subgraph of minimum degree k. We now define the set S by

$$S := (S' \cup I_v) \setminus V_k(H),$$

where

$$I_v = \begin{cases} \{v\} & \text{if } \deg(v) \leqslant k-2 \text{ or } \Gamma(v) \cap S' \neq \varnothing, \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that from the definition of S it follows that if $v \notin S$, then we have $S = S'$. Furthermore, since $S' \subseteq V_{\leqslant k-1}(H')$ and by definition S does not contain vertices of degree k, we also have $S \subseteq V_{\leqslant k-1}(H)$. To check that S is not too large, note that

$$\begin{aligned} \phi(H) - \phi(H') &= 2(k-1) - 2\deg_H(v) - \big(k - 1 - \deg_H(v) - |\Gamma(v) \cap V_{\leqslant k-1}(H)|\big) \\ &= (k-1) - \deg_H(v) + |V_{\leqslant k-1}(H) \cap \Gamma(v)| \geqslant 0. \end{aligned}$$

Therefore, if $v \notin S$, then $|S| \leqslant |S'| \leqslant \phi(H') \leqslant \phi(H)$. If $v \in S$, then we distinguish two cases:

1. If $\deg(v) = k-1$ and $|V_{\leqslant k-1}(H) \cap \Gamma(v)| = 0$, then there must exist a vertex $u \in S' \cap \Gamma(v)$ with degree at least k. By definition of S', u has degree at most $k-1$ in $H' = H - v$ and thus u has degree exactly k in H. In particular, we have $u \in S' \setminus S$ and thus $|S| \leqslant |S'| \leqslant \phi(H') \leqslant \phi(H)$.

2. Otherwise, we either have $\deg(v) \leqslant k-2$ or we have $\deg(v) = k-1$ and $|V_{\leqslant k-1}(H) \cap \Gamma(v)| \geqslant 1$. In both cases, we have $\phi(H) - \phi(H') \geqslant 1$ and so $|S| \leqslant |S'| + 1 \leqslant \phi(H') + 1 \leqslant \phi(H)$.

Now suppose that $\tilde{H} \supseteq H$ is an (H, S, k)-cover. We claim that then either $\tilde{H}$ is an (H', S', k)-cover or $\tilde{H} - v$ is an (H', S', k)-cover. In both cases, the induction hypothesis implies that $\tilde{H}$ has a subgraph of minimum degree k. To show this, we first recall that by the definition of an (H, S, k)-cover, we have $V_{\leqslant k-1}(\tilde{H}) \subseteq V(H) \setminus S$. We distinguish two cases. If $\deg_{\tilde{H}}(v) \geqslant k$, then actually $V_{\leqslant k-1}(\tilde{H}) \subseteq V(H) \setminus (S \cup \{v\}) = V(H') \setminus S$. Moreover, by construction of S, all vertices of $V(H')$ that belong to $S' \setminus S$ must have degree k in H (and so degree at least k in $\tilde{H}$). Therefore we have $V_{\leqslant k-1}(\tilde{H}) \subseteq V(H') \setminus S'$. In other words, $\tilde{H}$ is (H', S', k)-cover. Otherwise, we have $\deg_{\tilde{H}}(v) \leqslant k-1$. Then $v \notin S$ and thus $\deg_H(v) = k-1 = \deg_{\tilde{H}}(v)$, $\Gamma_H(v) \cap S' = \varnothing$ and $S' = S$. It moreover follows that $\Gamma_{\tilde{H}}(v) = \Gamma_H(v) \subseteq V(H') \setminus S'$. These observations show that

$$\begin{aligned} V_{\leqslant k-1}(\tilde{H} - v) &\subseteq (V_{\leqslant k-1}(\tilde{H}) \setminus \{v\}) \cup \Gamma_{\tilde{H}}(v) \\ &\subseteq (V(H) \setminus (S \cup \{v\})) \cup \Gamma_{\tilde{H}}(v) \\ &= V(H') \setminus S'. \end{aligned}$$

Thus $\tilde{H} - v$ is an (H', S', k)-cover, completing the proof. □

Using this lemma, we now prove Claim 2.5. Consider the graph $H \coloneqq G - \bigcup_{C \in \mathcal{F}} C$ obtained by removing all sets in $\mathcal{F}$ from G. We have $v_H = n - \sum_{C \in \mathcal{F}} |C| > 0$. By Claim 2.2 (i), every good set C intersects at most $(k-1)|C|+1$ edges, and so

$$e_H \geqslant t_k(n) + 1 - \sum_{C \in \mathcal{F}} \big((k-1)|C|+1\big) = t_k(v_H) - |\mathcal{F}| + 1. \qquad (2)$$

If H contains a subgraph of minimum degree k, then we are done because we can simply choose $S = \varnothing$. Otherwise, we apply Theorem 2.7 to H to obtain a set $S \subseteq V_{\leqslant k-1}(H)$ of size

$$|S| \leqslant 2(k-1)v_H - 2e_H \leqslant 2(k-1)(k-2) - 2\binom{k-2}{2} + 2|\mathcal{F}| - 2 \leqslant 2|\mathcal{F}| + k^2$$

(using (2) and the definition of $t_k(v_H)$ to bound e_H) such that every (H, S, k)-cover contains a subgraph of minimum degree k. To complete the proof of the claim, suppose that $\mathcal{F}' \subseteq \mathcal{F}$ contains at most one set from each set $\mathcal{F}(s)$ where $s \in S$. It is enough to show that the graph $G - \bigcup_{C \in \mathcal{F}'} C$ is an (H, S, k)-cover. Note first that since each element of $\mathcal{F}$ is a maximal good set, the elements of $\mathcal{F}$ are pairwise disjoint and for any two distinct $C, C' \in \mathcal{F}$, there are no edges between C and C' in G. Since G has minimum degree at least k by assumption, this means in particular that $V_{\leqslant k-1}(G - \bigcup_{C \in \mathcal{F}'} C) \subseteq V(H)$. Furthermore, every $s \in S$ has degree at least k in $G - \bigcup_{C \in \mathcal{F}'} C$, for one of the following two reasons:

1. Either $\mathcal{F}' \cap \mathcal{F}(s) = \varnothing$ or $|\mathcal{F}(s) \setminus \mathcal{F}'| \geqslant k$ in which case the claim directly follows.
2. Or $\mathcal{F}(s) = \mathcal{F}_*(s)$ and we have removed exactly one good set from $\mathcal{F}(s)$, say $\mathcal{F}' \cap \mathcal{F}(s) = \{\tilde{C}\}$. Then the degree of s in $G - \tilde{C}$ (which equals the degree in $G - \bigcup_{C \in \mathcal{F}'} C$) must be at least k by the maximality of $\tilde{C}$.

Thus $V_{\leqslant k-1}(G - \bigcup_{C \in \mathcal{F}'} C) \subseteq V(H) \setminus S$, so $G - \bigcup_{C \in \mathcal{F}'} C$ is an (H, S, k)-cover. This completes the proof of the claim.

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