Graphical Mahonian statistics on words

Amy Grady Svetlana Poznanović*

Department of Mathematical Sciences
Clemson University
Clemson, SC 29634, U.S.A.

{agrady,spoznan}@clemson.edu

Submitted: Jul 1, 2016; Accepted: Dec 19, 2017; Published: Jan 12, 2018
Mathematics Subject Classifications: 05A05

Abstract

Foata and Zeilberger defined the graphical major index, maj_U, and the graphical inversion index, inv_U, for words over the alphabet $\{1, 2, \ldots, n\}$. These statistics are a generalization of the classical permutation statistics maj and inv indexed by directed graphs U. They showed that maj_U and inv_U are equidistributed over all rearrangement classes if and only if U is *bipartitional*. In this paper we strengthen their result by showing that if maj_U and inv_U are equidistributed on a single rearrangement class then U is *essentially bipartitional*. Moreover, we define a graphical sorting index, sor_U, which generalizes the sorting index of a permutation. We then characterize the graphs U for which sor_U is equidistributed with inv_U and maj_U on a single rearrangement class.

1 Introduction

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a sequence of nonnegative integers. We will denote by $\mathcal{R}(\boldsymbol{\alpha})$ the set of permutations of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n}\}$, i.e., $\mathcal{R}(\boldsymbol{\alpha})$ is the set of all words over the alphabet $\{1, 2, \ldots, n\}$ containing α_i occurrences of the letter i for all $i = 1, 2, \ldots, n$. For $w = x_1 x_2 \ldots x_m \in \mathcal{R}(\boldsymbol{\alpha})$, the *inversion number* is defined as

$$\mathrm{inv}\, w = \sum_{1 \leqslant i < j \leqslant m} \mathcal{X}(x_i > x_j),$$

and the *major index* is defined as

$$\mathrm{maj}\, w = \sum_{i=1}^{m-1} i \mathcal{X}(x_i > x_{i+1}),$$

*Supported by NSF grant DMS-1312817.

where $\mathcal{X}$ is the characteristic function defined as $\mathcal{X}(A) = 1$ when A is true and $\mathcal{X}(A) = 0$ when A is false. The set of all positions i such that $x_i > x_{i+1}$ is known as the *descent set* of w, $\operatorname{Des} w$, and its cardinality is denoted by $\operatorname{des} w$. So, $\operatorname{maj} w = \sum_{i \in \operatorname{Des} w} i$.

The generating function for permutations by number of inversions goes back to Rodrigues [16] and the generalization to multisets is due to MacMahon [11]. MacMahon also showed [10, 12] that maj and inv are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. Namely,

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\operatorname{inv} w} = \sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\operatorname{maj} w} = \begin{bmatrix} \alpha_1 + \alpha_2 + \cdots + \alpha_n \\ \alpha_1, \alpha_2, \ldots, \alpha_n \end{bmatrix}$$

where

$$\begin{bmatrix} \alpha_1 + \alpha_2 + \ldots + \alpha_k \\ \alpha_1, \alpha_2, \ldots, \alpha_k \end{bmatrix} = \frac{[\alpha_1 + \alpha_2 + \cdots + \alpha_k]!}{[\alpha_1]![\alpha_2]! \ldots [\alpha_k]!}$$

is the q-multinomial coefficient and $[n]! = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$ is the q-factorial.

In honor of MacMahon, all permutation statistics that share the same distribution are called Mahonian. These two classical Mahonian statistics have been generalized in various ways. Some examples are Kadell's weighted inversion number [8], the r-major index introduced by Rawlings [15], the statistics introduced by Clarke [2], and the maj-inv statistics of Kasraoui [9]. The generalization that we will be considering in this paper is due to Foata and Zeilberger [4]. They defined graphical statistics (graphical inversions and graphical major index) parametrized by a general directed graph U and they described the graphs U for which these statistics are equidistributed on *all* rearrangement classes.

A *directed graph* or a *binary relation* on $X = \{1, 2, \ldots, n\}$ is any subset U of the Cartesian product $X \times X$. For each such directed graph U, we have the following statistics defined on each word $w = x_1 x_2 \ldots x_m$ over the alphabet X:

$$\begin{aligned} \operatorname{inv}_U w &= \sum_{1 \leqslant i < j \leqslant m} \mathcal{X}((x_i, x_j) \in U), \\ \operatorname{Des}_U w &= \{i \colon 1 \leqslant i \leqslant m, (x_i, x_{i+1}) \in U\}, \\ \operatorname{des}_U w &= |\operatorname{Des}_U|, \\ \operatorname{maj}_U w &= \sum_{i \in \operatorname{Des}_U w} i. \end{aligned}$$

An *ordered bipartition* of X is a sequence $(B_1, B_2, \ldots, B_k)$ of nonempty disjoint subsets of X such that $B_1 \cup B_2 \cup \cdots \cup B_k = X$, together with a sequence $(\beta_1, \beta_2, \ldots, \beta_k)$ of elements equal to 0 or 1. If $\beta_i = 0$ we say the subset B_i is *non-underlined*, and if $\beta_i = 1$ we say the subset B_i is *underlined*.

A relation U on $X \times X$ is said to be *bipartitional*, if there exists an ordered bipartition $((B_1, B_2, \ldots, B_k), (\beta_1, \beta_2, \ldots, \beta_k))$ such that $(x, y) \in U$ if and only if either $x \in B_i$, $y \in B_j$ and $i < j$, or x and y belong to the same underlined block B_i. Bipartitional relations were introduced in [4] as an answer to the question "When are inv_U and maj_U equidistributed over all rearrangement classes?".

Theorem 1 ([4]). *The statistics* inv_U *and* maj_U *are equidistributed on each rearrangement class* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if the relation* U *is bipartitional.*

In particular, if U is bipartitional with blocks $((B_1, \dots, B_k), (\beta_1, \dots, \beta_k))$ then

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{inv}_U w} = \sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{maj}_U w} = \begin{bmatrix} |\boldsymbol{\alpha}| \\ m_1, \dots, m_k \end{bmatrix} \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)} q^{\beta_j \binom{m_j}{2}}. \tag{1}$$

Here and later we use the notation

$$\begin{aligned} |\boldsymbol{\alpha}| &= \alpha_1 + \cdots + \alpha_n, \\ m_i &= |B_i|, \\ \boldsymbol{\alpha}(B) &= (\alpha_{i_1}, \dots, \alpha_{i_l}) \text{ if } B_i = \{i_1, \cdots, i_l\} \text{ with } i_1 < \cdots < i_l. \end{aligned}$$

A similar result was proved in [3], where the definition of graphical inversions and major index is modified to allow different behavior of the letters at the end of the word. Hetyei and Krattenthaler [7] showed that the poset of bipartitional relations ordered by inclusions has nice combinatorial properties. Han [6] showed that bipartitional relations U can also be characterized as relations U for which both U and its complement are transitive. In particular, we will use Han's formulation of this characterization as stated in [6].

Theorem 2 ([6]). *U is bipartional if and only if the following two properties hold:*

(i) $(x, y) \in U, (y, z) \in U \implies (x, z) \in U$

(ii) $(x, y) \in U, (z, y) \notin U \implies (x, z) \in U.$

Here we do two different things. First, we strengthen Foata and Zeilberger's result by showing that the equidistribution of inv_U and maj_U on a *single* rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$ implies that U is essentially bipartitional (Theorem 3). Second, we define a graphical sorting index on words, a statistic which generalizes the sorting index for permutations [13]. We then describe the directed graphs U for which sor_U is equidistributed with inv_U and maj_U on a fixed class $\mathcal{R}(\boldsymbol{\alpha})$ (Theorem 4).

In the next section we define the terminology we need and state the main results. Then we prove Theorem 3 and Theorem 4 in Section 3 and Section 4, respectively.

2 Preliminaries and Main Results

It will be convenient to refer to $U \subseteq X \times X$ as a directed graph and a binary relation interchangeably and use language related to both terms. For example, in some places we will use the notation $x \geqslant_U y$ or $x \to y$ to represent the directed edge $(x, y) \in U$. Also, we will say x is related to y if $(x, y) \in U$ or $(y, x) \in U$.

In this paper we will be considering the distribution of inv_U and maj_U over a fixed rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$. Notice that if the multiplicity α_x of $x \in X$ is 1, then the pair

(x,x) contributes neither to inv_U nor to maj_U. Therefore, omitting or adding such pairs to U doesn't change these two statistics over $\mathcal{R}(\boldsymbol{\alpha})$. For that purpose, we define U to be *essentially bipartitional relative to* $\boldsymbol{\alpha}$ if there are disjoint sets $I \subseteq X$ and $J \subseteq X$ such that

(1) $\alpha_x = 1$ for all $x \in I \cup J$ and

(2) $(U \setminus \{(x,x) \colon x \in I\}) \cup \{(x,x) \colon x \in J\}$ is bipartitional.

Theorem 3. *The statistics* inv_U *and* maj_U *are* equi**di**stributed *over* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if the relation* U *is essentially bipartitional relative to* $\boldsymbol{\alpha}$*.*

In view of the comment preceding the theorem, the "if" part of Theorem 3 follows from Theorem 1. We prove the "only if" in Section 3.

The third Mahonian statistic we will consider is the *sorting index* introduced by Peterson [13] and also studied independently by Wilson [17]. Every permutation $\sigma \in S_n$ can uniquely be decomposed as a product of transpositions, $\sigma = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k)$, such that $j_1 < j_2 < \cdots < j_k$ and $i_1 < j_1, i_2 < j_2, \ldots, i_k < j_k$, for some $k < n$. The sorting index is defined by

$$\operatorname{sor} \sigma = \sum_{r=1}^{k} (j_r - i_r).$$

The desired transposition decomposition can be found using the Straight Selection Sort algorithm. The algorithm first places n in the n-th position by applying a transposition, then places $n-1$ in the $(n-1)$-st position by applying a transposition, etc. For example, for $\sigma = 2413576$, we have

$$2413576 \overset{(67)}{\to} 2413567 \overset{(24)}{\to} 2314567 \overset{(23)}{\to} 2134567 \overset{(12)}{\to} 1234567$$

and, therefore, $\operatorname{sor} \sigma = (2-1) + (3-2) + (4-2) + (7-6) = 5$.

The sorting index has been extended to labeled forests by the authors [5]. It can also be naturally extended to words $w \in \mathcal{R}(\boldsymbol{\alpha})$ by using a stable variant of Straight Selection Sort which reorders the letters into a weakly increasing sequence. At each step transpositions are applied to place all the n's at the end, then all the $n-1$'s to the left of them, etc, so that for each $x \in X$, the α_x copies of x stay in the same relative order they were right before they were "processed". Then we define $\operatorname{sor} w$ to be the sum of the number of positions each element moved during the sorting. For example, applying this sorting algorithm to $w = 143123123$ yields

$$143123123 \to 133123124 \to 133122134 \to 131122334 \to 121123334 \to 111223334 \qquad (2)$$

and thus $\operatorname{sor} w = 7 + 2 + 4 + 4 + 2 = 19$.

We define a graphical sorting index that depends on U using the same sorting algorithm but at each step, when sorting x, we only count how many elements y such that $(x,y) \in U$ it "jumps over". More formally, to compute $\operatorname{sor}_U w$ for $w = x_1 x_2 \ldots x_m$:

- Begin with $i = m$, and $\operatorname{sor}_U w = 0$.

- Consider the largest element in the first i letters of w with respect to the integer order. If there are multiple copies of the largest element, let x_j, $j \leqslant i$ be the rightmost one.
- For each $h = j+1, j+2, \ldots, i$, if $(x_j, x_h) \in U$ increase $\operatorname{sor}_U w$ by 1.
- Interchange x_j with x_i and keep using the notation $w = x_1 x_2 \ldots x_m$.
- Repeat this process for $i = m-1, \ldots, 1$.

For example, consider the sorting index of the same word $w = 143123123$ under the relation $U = \{(4,3),(3,3),(3,1),(2,3),(1,1)\}$. The sorting steps are the same as given in (2) and thus $\operatorname{sor}_U w = 3+1+2+2+0 = 8$. In particular, if U is the natural integer order $U = \{(x,y)\colon x > y\}$ then $\operatorname{sor} w = \operatorname{sor}_U w$. Our second main result is the following.

Theorem 4. *The statistics* sor_U*,* inv_U *and* maj_U *are equidistributed on a fixed rearrangement class* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if the relation* U *has the following properties.*

1. *U is bipartitional with no underlined blocks.*
2. *If $(x,y) \in U$ then $x > y$.*
3. *All but the last block of U are of size at most 2.*
4. *If U has k blocks $B_1, \ldots, B_k$ and $|B_i| = 2$ for some $1 \leqslant i \leqslant k-1$ then $\alpha_{\max B_i} = 1$.*

We give the proof of Theorem 4 in Section 4.

3 The Proof of Theorem 3

The proof of Theorem 3 is based on a series of lemmas that we prove next. The first two of them describe how the distribution of maj_U and inv_U over $\mathcal{R}(\boldsymbol{\alpha})$ are related for general U. This will lead us to define special words in $\mathcal{R}(\boldsymbol{\alpha})$ which we call *maximal chain words*. Then we will show that when inv_U and maj_U are equidistributed over $\mathcal{R}(\boldsymbol{\alpha})$, the chains that are the building blocks of the same maximal chain word are nicely related to each other. We use this to show that U and U^c have to satisfy the properties of Theorem 2 modulo some relations (x,x) with $\alpha_x = 1$.

We begin with a simple but very useful observation.

Lemma 5. *The statistics* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$ *if and only if* maj_{U^c} *and* inv_{U^c} *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*.*

Proof. This follows from the fact that for every $w \in \mathcal{R}(\boldsymbol{\alpha})$,

$$\operatorname{maj}_U w + \operatorname{maj}_{U^c} w = \binom{|\boldsymbol{\alpha}|}{2} = \operatorname{inv}_U w + \operatorname{inv}_{U^c} w. \qquad \square$$

Lemma 6. *For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and any relation U on $X = \{1, 2, \ldots, n\}$,*

$$\max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{maj}_U\, w \geqslant \max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{inv}_U\, w.$$

Proof. We will use induction on $|\boldsymbol{\alpha}|$. It's clear that the statement holds when $|\boldsymbol{\alpha}| = 1$. Assume that it holds for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leqslant m$.

Consider a rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$ such that $|\boldsymbol{\alpha}| = m + 1$ and a relation U on $[n]$. Let $(\boldsymbol{\alpha}, U)$ be a directed graph with vertex set $\{1^{\alpha_1}, \ldots, n^{\alpha_n}\}$ and a directed edge $x \to y$ whenever $(x, y) \in U$. Let $x_1 \to x_2 \to \cdots \to x_n$ be a directed path in $(\boldsymbol{\alpha}, U)$ of maximal possible length. This means we have a descending chain $x_1 \geqslant_U x_2 \geqslant_U \cdots \geqslant_U x_l$ of maximal possible length that uses at most α_i copies of each $i \in [n]$. Set $\boldsymbol{\alpha}' = (\alpha'_1, \ldots, \alpha'_n)$ where

$$\alpha'_i = \alpha_i - \sum_{j=1}^{l} \mathcal{X}(x_j = i).$$

Let u' be a word that maximizes maj_U on the rearrangement class $\mathcal{R}(\boldsymbol{\alpha}')$. One can easily verify that for the word $u = u' x_1 x_2 \cdots x_l$ in $\mathcal{R}(\boldsymbol{\alpha})$ we have

$$\mathrm{maj}_U\, u = \mathrm{maj}_U\, u' + \frac{(l-1)(2m+2-l)}{2}. \tag{3}$$

To bound $\max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{inv}_U\, w$, first suppose there is an element $y \in (\boldsymbol{\alpha}', U)$ such that for all $i = 1, 2, \ldots, l$ we have $(y, x_i) \in U$ or $(x_i, y) \in U$. If $(y, x_1) \in U$ then $y \to x_1 \to x_2 \to \cdots \to x_l$ is a longer directed path in $(\boldsymbol{\alpha}, U)$, therefore $(y, x_1) \notin U$ and $(x_1, y) \in U$. Similarly, if $(x_l, y) \in U$ we can form the longer directed path $x_1 \to x_2 \to \cdots \to x_l \to y$ in $(\boldsymbol{\alpha}, U)$; thus we must have $(x_l, y) \notin U$ and $(y, x_l) \in U$. However, this implies that there are elements x_i and x_{i+1} such that $(x_i, y), (y, x_{i+1}) \in U$, which yields a longer directed path $x_1 \to x_2 \to \cdots \to x_i \to y \to x_{i+1} \to \cdots \to x_l$. Therefore, every $y \in (\boldsymbol{\alpha}', U)$ is related to at most $l - 1$ elements in the chain $x_1 \to \cdots \to x_l$.

Now consider a word $v \in \mathcal{R}(\boldsymbol{\alpha})$ and the corresponding word $v' \in \mathcal{R}(\boldsymbol{\alpha}')$ obtained by deleting $x_1, \ldots, x_l$. By the argument in the previous paragraph, the $m + 1 - l$ letters in v' create at most $(m+1-l)(l-1)$ graphical inversions with $x_1, \ldots, x_l$. Therefore, by (3) and the induction hypothesis,

$$\begin{aligned}
\max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{inv}_U\, w &\leqslant \max_{w' \in \mathcal{R}(\boldsymbol{\alpha}')} \mathrm{inv}_U\, w + (m+1-l)(l-1) + \binom{l}{2} && (4)\\
&= \max_{w' \in \mathcal{R}(\boldsymbol{\alpha}')} \mathrm{inv}_U\, w + \frac{(l-1)(2m+2-l)}{2} && (5)\\
&\leqslant \max_{w' \in \mathcal{R}(\boldsymbol{\alpha}')} \mathrm{maj}_U\, w + \frac{(l-1)(2m+2-l)}{2} && (6)\\
&\leqslant \max_{w \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{maj}_U\, w. && (7)
\end{aligned}$$

□

The proof of Lemma 6 also shows that a word $w = w_k w_{k-1} \cdots w_1$ with the property $\mathrm{maj}_U(w) \geqslant \max_{v \in \mathcal{R}(\boldsymbol{\alpha})} \mathrm{inv}_U\, v$ can be constructed by greedily "peeling off" directed paths (i.e. descending chains) of maximal length from $(\boldsymbol{\alpha}, U)$ and ordering them from right to left, forming the subwords $w_1, w_2, \ldots, w_k$ in that order. These kind of words will be used in the proofs that follow and when the relation U is understood, we will call such words *maximal chain words* in $\mathcal{R}(\boldsymbol{\alpha})$.

Moreover, if inv_U and maj_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$, equalities hold in (4), (6), and (7). Exploiting this, one can derive conclusions of how the elements from different chains in the maximal chain words are related to each other if maj_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. We list the properties that will be important later in a series of three lemmas.

Lemma 7. *Suppose* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*. Let* $w = w_k w_{k-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ *be a maximal chain formed from the maximal chains* $w_1, w_2, \ldots, w_k$*. Then*

(i) For each of the maximal descending chains $w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j}$

$$(x_r, x_s) \in U \quad \textit{or} \quad (x_s, x_r) \in U \quad \textit{for all} \quad i_{j-1} + 1 \leqslant r < s \leqslant i_j, \tag{8}$$

(ii) Each letter y *in a maximal descending chain* w_i*,* $i > j$*, is related to exactly* $|w_j| - 1$ *elements from* $w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j}$*, i.e., there is a unique* $r \in \{i_{j-1}+1, \ldots, i_j\}$ *such that* $(y, x_r) \notin U$ *and* $(x_r, y) \notin U$*. Moreover,* $(x_s, y) \in U$ *for* $i_{j-1} + 1 \leqslant s < r$ *and* $(y, x_s) \in U$ *for* $r < s \leqslant i_j$*.*

Proof. Condition (i) is necessary for equality to hold in (4). The property (ii) also follows from the fact that equality holds in (4) and the definition of a maximal chain word which implies that the chain w_j is be the longest one that can be formed among the letters in $w_k w_{k-1} \cdots w_j$. □

The following lemma shows that if maj_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$ the elements in the maximal chains can be reordered, if necessary, so that within each of them the following property holds: if x precedes y in the same chain of a maximal chain word then $(x, y) \in U$.

Lemma 8. *If* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*, then there exists a maximal chain word* $w = w_k w_{k-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ *with subwords* w_i *formed from descending chains such that for any* $w_j = x_{i_{j-1}+1} x_{i_{j-1}+2} \cdots x_{i_j}$ *we have*

$$(x_r, x_s) \in U \quad \textit{for all} \quad i_{j-1} + 1 \leqslant r < s \leqslant i_j. \tag{9}$$

Proof. Since the equality in (4) holds, the elements $x_1, x_2, \ldots, x_l$ in the maximal chain can be arranged so that they form $\binom{l}{2}$ graphical inversions, which implies the statement in the lemma. □

Lemma 9. *Suppose* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$*. Let* $w = w_k w_{k-1} \cdots w_1$ *be a maximal chain word in* $\mathcal{R}(\boldsymbol{\alpha})$ *for* U *with maximal chains* $w_1, \ldots, w_k$*. If* $(x, y) \in U$ *and* $(y, x) \in U$ *for some* $x \neq y$*, then the* α_x *copies of* x *and the* α_y *copies of* y *are all in the same chain* w_i*.*

Proof. Without loss of generality, suppose there is an x that appears in a chain w_{j_1} and a y that appears in the chain w_{j_2}, $j_1 > j_2$. Consider the chain $w_{j_2} : b_1 \geqslant_U b_2 \geqslant_U \ldots \geqslant_U b_{l-1} \geqslant_U y \geqslant_U b_{l+1} \geqslant_U \ldots \geqslant_U b_m$. By Lemma 7, there is exactly one $i \in \{1, 2, \ldots, m\}$ such that $(x, b_i), (b_i, x) \notin U$, $(b_1, x), \ldots, (b_{i-1}, x) \in U$, $(x, b_{i+1}), \ldots, (x, b_m) \in U$. If $l < i$ then the chain $b_1 \geqslant_U b_2 \geqslant_U \cdots > b_{l-1} \geqslant_U x \geqslant_U y \geqslant_U b_{l+1} \geqslant_U \ldots \geqslant_U b_m$ is a longer chain than w_{j_2} and if $l > i$ then $b_1 \geqslant_U b_2 \geqslant_U \ldots \geqslant_U b_{l-1} \geqslant_U y \geqslant_U x \geqslant_U b_{l+1} \geqslant_U \ldots \geqslant_U b_m$ is a longer chain than w_{j_2}. This contradicts the definition of a maximal chain word. $\square$

The remaining part of this section is devoted to proving that the relations U and U^c are both transitive modulo some relations (x, x) with $\alpha_x = 1$.

Lemma 10. *Suppose* maj_U *and* inv_U *are equidistributed on* $\mathcal{R}(\boldsymbol{\alpha})$. *If* $(x, y), (y, x) \in U$ *and* $\alpha_x > 1$ *then* $(x, x) \in U$.

Proof. Since $(x, y), (y, x) \in U$, by Lemma 9, all the x's and y's must be in the same maximal chain of a maximal chain word. In particular, since two x's are in the same chain, part (i) of Lemma 7 implies that $(x, x) \in U$. $\square$

Lemma 11. *Suppose* maj_U *and* inv_U *are equidistributed over* $\mathcal{R}(\boldsymbol{\alpha})$ *and let* x *and* y *be two distinct elements of* X *such that* $(x, y), (y, x) \in U$. *For every* $z \in \{1^{\alpha_1}, \ldots, n^{\alpha_n}\} \setminus \{x, y\}$, *we have*

$$(z, x) \in U \text{ if and only if } (z, y) \in U$$

and

$$(x, z) \in U \text{ if and only if } (y, z) \in U.$$

Proof. If $z = x$ then $\alpha_x > 1$ and the claim follows from Lemma 10. The same is true if $z = y$. So, suppose $z \neq x$, $z \neq y$. Because of symmetry, it suffices to prove

$$(z, x) \in U \implies (z, y) \in U \tag{10}$$

$$(x, z) \in U \implies (y, z) \in U \tag{11}$$

To see (10), suppose that $(z, x) \in U, (z, y) \notin U$. We consider two cases.

Case 1: $(y, z) \notin U$. Let $w = w_t w_{t-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ be a maximal chain word that satisfies (9). By Lemma 9, x and y are in the same chain w_i of w. By Lemma 7, z is in a different chain w_j and by Lemma 9, $(x, z) \notin U$. If $j > i$, notice that, by Lemma 7, x cannot precede y in w_i, so w_i must be of the form $w_i = b_1 \cdots b_k y b_{k+1} \cdots b_l x b_{l+1} \cdots b_m$. Then $b_1 \cdots b_k z b_{k+1} \cdots b_l x y b_{l+1} \cdots b_m$ is a descending chain longer than w_i. If $j < i$, then $w_j = b_1 \cdots b_k z b_{k+1} \cdots b_l$. By part (ii) of Lemma 7, $(b_k, x), (y, b_{k+1}) \in U$, which implies that $b_1 \cdots b_k x y b_{k+1} \cdots b_l$ is a descending chain longer than w_j.

Case 2: $(y, z) \in U$. By Lemma 5, maj_{U^c} and inv_{U^c} are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. Let $w = w_t w_{t-1} \cdots w_1 \in \mathcal{R}(\boldsymbol{\alpha})$ be a maximal chain word for U^c that satisfies (9). Suppose x, y, z are in the chains w_i, w_j, w_k, respectively. By Lemma 7, $i \neq j$, $i \neq k$. If $i < j, k$ and $w_i = b_1 \cdots b_l x b_{l+1} \cdots b_m$ then a different maximal chain word w' could be constructed by taking the same chains $w_1, \ldots, w_{i-1}$ as in w and replacing w_i by $b_1 \cdots b_l y b_{l+1} \cdots b_m$.

Since $(z, y) \in U^c$, it follows from Lemma 7 that z is not in relation U^c with some b_r, $r \leqslant l$ and therefore $(z, x) \in U^c$, which contradicts $(z, x) \in U$. The similar argument holds if $j < i, k$. If $k < i, j$ and $w_k = b_1 \cdots b_l z b_{l+1} \cdots b_m$ then y is not in relation U^c with some b_r, $r > l$, and a different maximal chain word for U^c could be formed by replacing w_k with $b_1 \cdots b_l z b_{l+1} \cdots b_{r-1} y b_{r+1} \cdots b_m$. Part (ii) of Lemma 7 now implies that $(z, x) \in U^c$, which contradicts $(z, x) \in U$. Finally, if $j = k < i$, then since $(z, x) \notin U^c$ and $(x, y), (y, x) \notin U^c$, Lemma 7 implies that $(x, z) \in U^c$ and y precedes z in w_j. Therefore, $(y, z) \in U^c$, which contradicts $(y, z) \notin U$.

The implication (11) can be proved by considering completely analogous cases, so we omit it here. □

Proof of Theorem 3. Assume inv_U and maj_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. Let the symmetric part of U be

$$S(U) = \{(x, y) \in X \times X : (x, y), (y, x) \in U \text{ for some } y \neq x\}$$

and let

$$X_U = \{x \in X : (x, y) \in S(U) \text{ for some } y \in X\}.$$

Let

$$U' = (U \cup \{(x, x) \colon x \in X_U, \alpha_x = 1\}) \setminus \{(x, x) \colon x \notin X_U, \alpha_x = 1\}.$$

We will show that U' is bipartitional using the characterization given by Theorem 2, which will imply that U is essentially bipartitional relative to $\boldsymbol{\alpha}$.

To show that U' is transitive, suppose $(x, y), (y, z) \in U'$.

First consider the case when x, y, z are all different. If $(y, x) \in U$ or $(z, y) \in U$, then $(x, z) \in U$ by Lemma 11. Hence $(x, z) \in U'$. If $(y, x), (z, y), (x, z) \notin U$ then let $w \in \mathcal{R}(\boldsymbol{\alpha})$ be a maximal chain word for U^c. If x, y, z all appear in the same chain w_i, by Lemma 8, the elements in w_i can be reordered to give a sequence $z_1, \ldots, z_l$ such that $(z_r, z_s) \notin U$ for all $1 \leqslant r < s \leqslant l$. This is possible only if z precedes x and $(z, x) \notin U$. Applying Lemma 11 to U^c, we get that $(x, y), (y, z) \notin U$ which contradicts the starting assumption. If not all x, y, z appear in the same maximal chain w_i, assume, without loss of generality, that x is the one that appears in the rightmost chain of the maximal chain word $w \in \mathcal{R}(\boldsymbol{\alpha})$ for U^c. Suppose y does not appear in w_i. Let t_y be the unique letter in w_i (guaranteed by Lemma 7) not related to y in U^c. Then another maximal chain word can be constructed in which t_y in the maximal chain w_i is replaced by y. Repeating this argument, we see that one can construct a maximal chain word for U^c in which x, y, z are all in the same maximal chain, which we saw is impossible.

If not all x, y, z are different, one only needs to consider the case $x = z \neq y$. Then $x \in X_U$. If $\alpha_x = 1$, $(x, x) \in U'$ by definition. Otherwise $\alpha_x > 1$ and $(x, x) \in U$ by Lemma 10.

To show that U' has the second property from Theorem 2, assume that $(x, y) \in U'$ and $(z, y), (x, z) \notin U'$. If all x, y, z are different, then by the previous argument applied to U^c, we get $(x, y) \notin U$, which contradicts the assumption $(x, y) \in U$. The only case left to be considered is $x = y \neq z$. Then $(x, x) \in U'$ and $(x, z), (z, x) \notin U$. If $\alpha_x > 1$, then Lemma 10 applied to U^c yields $(x,x) \notin U$, which contradicts $(x,x) \in U'$. If $\alpha_x = 1$, then by the definition of U', $x \in X_U$. This means that $(x,w),(w,x) \in U$ for some $w \neq x$. Then x,z,w, are all different and by the preceding argument we get that $(w,x) \in U$, $(z,x) \notin U$ implies $(w,z) \in U$. But then Lemma 11 applied to U yields $(x,z) \in U$, a contradiction. □

4 Graphical Sorting Index

In this section we will prove Theorem 4. The "if" part follows from the following proposition and (1), while the "only if" part follows from Lemma 14 and Lemma 16.

Assume U satisfies the properties of Theorem 4 and has blocks $B_1,\ldots,B_k$. To each word $w \in \mathcal{R}(\boldsymbol{\alpha})$, we associate a pair of two sequences: a sequence of partitions and a sequence of nonnegative integers. This map is a generalization of the B-code defined for permutations [1, 14]. Precisely, we define a map $\phi : \mathcal{R}(\boldsymbol{\alpha}) \longrightarrow A$, where A is a set of pairs

$$((b_{1,1} \geqslant b_{1,2} \geqslant \cdots \geqslant b_{1,m_1}; b_{2,1} \geqslant \cdots \geqslant b_{2,m_2}; \ldots ; b_{k,1} \geqslant \cdots \geqslant b_{k,m_k}), (p_1,p_2,\ldots,p_k))$$

satisfying

(1°) for $i < k$ in each partition $b_{i,1} \geqslant \cdots \geqslant b_{i,m_i} \geqslant 0$ each part has size $b_{i,j} \leqslant m_{i+1} + m_{i+2} + \cdots + m_k$, $1 \leqslant j \leqslant m_i$, while $b_{k,j} = 0$ for $1 \leqslant j \leqslant m_k$,

(2°) $p_i = 0$ if $|B_i| = 1$ and $1 \leqslant p_i \leqslant m_i$ if $|B_i| = 2$.

For $w = x_1 \cdots x_l \in \mathcal{R}(\boldsymbol{\alpha})$, $\phi(w)$ is computed as follows.

(1) Set $j = 1$.

(2) If $B_j = \{y_1, y_2\}$ has two integers $y_2 > y_1$ then let $p_j = i$ be the position of y_2 in the subword of w formed by the elements of B_j. Otherwise set $p_j = 0$.

(3) Sort the elements of the block B_j and form the partition $b_{j,1} \geqslant \cdots \geqslant b_{j,m_j} \geqslant 0$ from the contributions to $\operatorname{sor} w$ (listed in nonincreasing order) by the elements of B_j. Keep calling the partially sorted word w.

(4) If $j < k$ increase j by 1 and go to step (2). Otherwise stop.

Consider, for example, the bipartitional binary relation

$$U = \{(5,3),(5,2),(5,1),(4,3),(4,2),(4,1),(3,2),(3,1)\}$$

with blocks $B_1 = \{5,4\}, B_2 = \{3\}, B_3 = \{2,1\}$ and $\beta_1 = \beta_2 = \beta_3 = 0$. Take the word $w = 42345411 \in \mathcal{R}(2,1,1,3,1)$. Since the subword formed by the 4's and the 5 is 4454, we have $p_1 = 3$. The steps for sorting the 4's and the 5 are

$$4234541 \xrightarrow{+1} 42341415 \xrightarrow{+1} 42341145 \xrightarrow{+2} 42311445 \xrightarrow{+4} 12314445$$

and, therefore, the first partition in $\phi(w)$ is $4 \geqslant 2 \geqslant 1 \geqslant 1$. Then $p_2 = 0$ and sorting the 3 yields 12134445, therefore the second partition is 1. Finally, $p_3 = 2$ and

$$\phi(w) = ((4 \geqslant 2 \geqslant 1 \geqslant 1; 1; 0 \geqslant 0 \geqslant 0), (3, 0, 2)).$$

Since the parts of the partitions in the $\phi(w)$ represent contributions to the sorting index, the bound for their size $b_{i,j} \leqslant m_{i+1} + m_{i+2} + \cdots + m_k$ easily follows. Therefore, the $\phi(w)$ is clearly a map from $\mathcal{R}(\boldsymbol{\alpha})$ to the set of pairs of sequences of partitions and integers which satisfy (1°) and (2°), which we claim is a bijection. For describing the inverse, the crucial observation is that for blocks of size 2, $B_j = \{y_1 < y_2\}$, the contribution to the sorting index is given by b_{j,p_j}. Then given

$$((b_{1,1} \geqslant b_{1,2} \geqslant \cdots \geqslant b_{1,m_1}; b_{2,1} \geqslant \cdots \geqslant b_{2,m_2}; \ldots; b_{k,1} \geqslant \cdots \geqslant b_{k,m_k}), (p_1, p_2, \ldots, p_k))$$

which satisfies (1°) and (2°), the corresponding word $w \in \mathcal{R}(\boldsymbol{\alpha})$ is constructed as follows.

(1) Let $j = k$ and w be the empty word.

(2) Add to the end of w the elements of B_j with their multiplicities, listed in nondecreasing order $x_{j,1} x_{j,2} \cdots x_{j,m_j}$.

(3) If $|B_j| = 1$, then for $i = 1, \ldots, m_j$, swap $x_{j,i}$ with the element of w which is $b_{j,i}$ places to the left of $x_{j,i}$.

(4) If $B_j = \{y_1 < y_2\}$, then let $b'_{j,1} \geqslant \cdots \geqslant b'_{j,m_j-1}$ be the partition obtained from $b_{j,1} \geqslant \cdots \geqslant b_{j,m_j}$ by deleting the part b_{j,p_j}. Then for $i = 1, \ldots, m_j - 1$, swap $x_{j,i}$ with the element of w which is $b'_{j,i}$ places to the left of $x_{j,i}$. Finally, swap $x_{j,m_j} = y_2$ with the element in w which is $b_{j,p_j} + m_j - p_j$ positions to its left. (After this step there are b_{j,p_j} elements from $B_{j+1}, \ldots, B_k$ and $m_j - p_j$ elements from B_j to the right of y_2.)

(5) If $j > 1$ decrease j by 1 and go to step (2). Otherwise stop.

Proposition 12. *If U satisfies the properties of Theorem 4 and has blocks $B_1, \ldots, B_k$ then*

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\operatorname{sor}_U w} = \begin{bmatrix} |\alpha| \\ m_1, \ldots, m_k \end{bmatrix} \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)}.$$

Proof. The $\phi(w)$ is designed so that $\operatorname{sor}_U w = \sum_{i=1}^{k} \sum_{j=1}^{m_i} b_{i,j}$. The bijection described above then yields the generating function for sor_U. Let $p(j,k,n)$ denote the number of partitions of n into at most k parts, with largest part at most j. It is known that $\sum_{n \geqslant 0} p(j,k,n) q^n = \begin{bmatrix} j+k \\ j \end{bmatrix}$. The block B_j contributes

$$\binom{m_j}{\boldsymbol{\alpha}(B_j)} \sum_{n \geqslant 0} p(m_{j+1} + m_{j+2} \cdots + m_n, m_j, n) q^n = \binom{m_j}{\boldsymbol{\alpha}(B_j)} \begin{bmatrix} m_j + m_{j+1} \cdots + m_n \\ m_j \end{bmatrix}$$

to $\sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\operatorname{sor}_U w}$, where the leading binomial coefficient counts the number of possible values of p_j. Thus we have

$$\sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\operatorname{sor}_U w} = \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)} \begin{bmatrix} m_j + m_{j+1} \cdots + m_n \\ m_j \end{bmatrix} = \begin{bmatrix} |\alpha| \\ m_1, \ldots, m_k \end{bmatrix} \prod_{j=1}^{k} \binom{m_j}{\boldsymbol{\alpha}(B_j)}. \qquad \square$$

In particular, we get the generating function for the standard sorting index for words.

Corollary 13.

$$\sum_{w\in\mathcal{R}(\boldsymbol{\alpha})} q^{\operatorname{sor} w} = \begin{bmatrix} |\alpha| \\ m_1, \ldots, m_k \end{bmatrix}.$$

Finally, we prove the "only if" part of Theorem 4 via the following few lemmas.

Lemma 14. *If* sor_U, maj_U, *and* inv_U *are equidistributed over a fixed rearrangement class* $\mathcal{R}(\boldsymbol{\alpha})$ *then the relation* U *must be a subset of the integer order modulo relations* (x, x).

Proof. Suppose the statistics sor_U, maj_U, and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$. By Theorem 3, U must be essentially bipartitional relative to α. That means that there are subsets $I, J \subset \{x\colon \alpha_{x=1}\}$ such that $U' = (U \setminus \{(x,x)\colon x \in I\}) \cup \{(x,x)\colon x \in J\}$ is bipartitional. Without loss of generality we may assume that I, J are chosen so that U' does not have underlined blocks $\{x\}$ of size 1 such that $\alpha_x = 1$. We claim that U' is a subset of the natural order.

First we will show that there are no underlined blocks in U'. Suppose the contrary. Then there exist elements x and y such that $(x, y), (y, x) \in U'$ ($x \neq y$ or y is a second copy of the same element with $\alpha_x > 1$). Because we have both (x, y) and (y, x) in U' every word $w \in \mathcal{R}(\boldsymbol{\alpha})$ has at least one U'-inversion. Therefore the minimum inv_U over the rearrangement class $\mathcal{R}(\boldsymbol{\alpha})$ is 1. On the other hand, $\operatorname{sor}_U 11\cdots 122\cdots 2\cdots nn\cdots n = 0$. This is a contradiction, and thus there are no underlined blocks in U'.

Now assume that U' is not a subset of the natural integer order. Then there exist at least two elements such that $(x, y) \in U'$, but $y > x$ with respect to the natural order. Let $B_1, B_2, \ldots, B_k$ be the blocks of U'. Now consider the words created by placing the elements of B_1 in some order followed by the elements of B_2 placed to the right of B_1 and continue the process until the elements of B_k in some order are the last elements of the word. The words of this type will have inv_U equal to the number of edges in the graph $(\boldsymbol{\alpha}, U')$ as defined in the proof of Lemma 6. Therefore, the maximum inv_U is bounded below by the number of edges in $(\boldsymbol{\alpha}, U')$ (it is in fact equal to the number of edges in $(\boldsymbol{\alpha}, U')$). In the sorting algorithm, however, elements are only sorted over elements that are smaller than them with respect to the natural order. Therefore x will never jump over y, and thus the relation (x, y) will never contribute to the sorting index. Since each edge of the graph $(\boldsymbol{\alpha}, U')$ contributes at most 1 to sor_U, we conclude that the maximum of sor_U on $\mathcal{R}(\boldsymbol{\alpha})$ is less than the maximum of inv_U. This is a contradiction, and U' must be a subset of the natural order. $\square$

The next inequality will be used to prove the remaining part of Theorem 4.

Lemma 15. *For $a, b \in \mathbb{Z}_{\geqslant 1}$*

$$\sum_{i=0}^{\min\{a,b\}} \binom{a}{i} \leqslant \binom{a+b}{b}$$

and equality holds if and only if $b = 1$.

Proof. If $a \leqslant b$ then using the Vandermonde's Identity we have

$$\sum_{i=0}^{\min\{a,b\}} \binom{a}{i} = \sum_{i=0}^{a} \binom{a}{i} \leqslant \sum_{i=0}^{a} \binom{a}{i}\binom{b}{a-i} = \binom{a+b}{b}$$

and equality holds if and only if $a = b = 1$. Similarly, if $a > b$ then

$$\sum_{i=0}^{\min\{a,b\}} \binom{a}{i} = \sum_{i=0}^{b} \binom{a}{i} \leqslant \sum_{i=0}^{b} \binom{a}{i}\binom{b}{b-i} = \binom{a+b}{b} \qquad \square$$

Lemma 16. *Suppose U is a bipartitional relation with blocks $B_1, \ldots, B_k$, none of which are underlined, such that sor_U, maj_U, and inv_U are equidistributed over $\mathcal{R}(\boldsymbol{\alpha})$. Then for every $1 \leqslant i < k$, $|B_i| \leqslant 2$ and if the equality $|B_i| = 2$ holds then $\alpha_{\max B_i} = 1$.*

Proof. By Lemma 14, the blocks $B_1, \ldots, B_k$ are consecutive intervals with $n \in B_1$ and $1 \in B_k$. If $k = 1$ there is nothing to prove, so suppose $k > 1$.

Let $\mathrm{i}(B_1, \ldots B_k)$ and $\mathrm{s}(B_1, \ldots, B_k)$ denote the number of words in $\mathcal{R}(\boldsymbol{\alpha})$ that maximize inv_U and sor_U, respectively. Let $B_1 = \{s, s+1, \ldots, n\}$, $s \leqslant n-1$. The words in $\mathcal{R}(\boldsymbol{\alpha})$ that maximize inv_U are exactly those formed by a permutation of the elements of B_1 (with their multiplicities) followed by a permutation of the elements from B_2, etc. So, $\mathrm{i}(B_1, \ldots, B_k) = \prod_{i=1}^{k} \binom{m_i}{\boldsymbol{\alpha}(B_i)}$.

On the other hand, if $w \in \mathcal{R}(\boldsymbol{\alpha})$ maximizes sor_U then after sorting the n's, one obtains a word $w' \in \mathcal{R}(\boldsymbol{\alpha}')$ that maximizes sor_U for $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1})$. The map $w \to w'$ is not one-to-one. One can write $w' = uv$ where u is the longest prefix of w' formed by elements of B_1. Then the number of words w that yield w' is at most $\sum_{i=0}^{\min\{|u|,\alpha_n\}} \binom{|u|}{i}$. Namely, such a w can be obtained by appending the α_n copies of n to w' and then swapping the leftmost i copies of n with i letters from u and the remaining $\alpha_n - i$ copies of n with the first $\alpha_n - i$ letters of v.

Since, by Lemma 15,

$$\sum_{i=0}^{\min\{|u|,\alpha_n\}} \binom{|u|}{i} \leqslant \binom{|u| + \alpha_n}{\alpha_n} \leqslant \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n}$$

with equality when $\alpha_n = 1$, we have

$$\mathrm{s}(B_1, \ldots, B_k) \leqslant \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n} \mathrm{s}(B_1 \setminus \{n\}, \ldots, B_k),$$

where $\mathrm{s}(B_1 \setminus \{n\}, \ldots, B_k)$ is the number of words in $\mathcal{R}(\boldsymbol{\alpha}')$ that maximize sor_U. So, inductively, we get

$$\mathrm{s}(B_1, \ldots, B_k) \leqslant \binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_s, \ldots, \alpha_{n-1}, \alpha_n} \mathrm{s}(B_2, \ldots, B_k) \leqslant \prod_{i=1}^{k} \binom{m_i}{\boldsymbol{\alpha}(B_i)} = \mathrm{i}(B_1, \ldots, B_k).$$

Since we have equalities everywhere, $\alpha_n = 1$. We also get that $\mathrm{s}(B_1 \setminus \{n\}, \ldots, B_k) = \mathrm{i}(B_1 \setminus \{n\}, \ldots, B_k)$ and by the same argument, $\alpha_n = \alpha_{n-1} = \cdots = \alpha_{s+1} = 1$.

Now consider a permutation p of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, (s-1)^{\alpha_{s-1}}\}$ which maximizes sor_U. By appending α_s copies of s to p and then swapping them with the first α_s letters of p we get the word

$$\underbrace{ss\cdots s}_{\alpha_s} p'.$$

One can readily see that the word

$$w' = (n-1)\underbrace{ss\cdots s}_{\alpha_s - 1} p' s(s+1)(s+2)\cdots(n-2) \in \mathcal{R}(\boldsymbol{\alpha}')$$

maximizes sor_U over $\mathcal{R}(\boldsymbol{\alpha}')$. Also, there are exactly $\alpha_s + 1$ words w in $\mathcal{R}(\boldsymbol{\alpha})$ that maximize sor_U which can be obtained from w', namely,

$$\begin{aligned}
&n\underbrace{ss\cdots s}_{\alpha_s-1} p' s(s+1)(s+2)\cdots(n-2)(n-1),\\
&(n-1)n\underbrace{ss\cdots s}_{\alpha_s-2} p' s(s+1)(s+2)\cdots(n-2)s,\\
&(n-1)sn\underbrace{ss\cdots s}_{\alpha_s-3} p' s(s+1)(s+2)\cdots(n-2)s,\\
&\cdots\\
&(n-1)\underbrace{ss\cdots s}_{\alpha_s-2} np' s(s+1)(s+2)\cdots(n-2)s,\\
&(n-1)\underbrace{ss\cdots s}_{\alpha_s-1} np'' s(s+1)(s+2)\cdots(n-2)a,
\end{aligned}$$

where a is the first letter of p'. However, as we saw above, if sor_U and inv_U are equidistributed on $\mathcal{R}(\boldsymbol{\alpha})$, each word w' corresponds to exactly $\binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n}$ words w. So,

$$\binom{\alpha_n + \alpha_{n-1} + \cdots + \alpha_s}{\alpha_n} = \alpha_s + 1$$

and therefore $s = n - 1$.

This proves that either $B_1 = \{n-1, n\}$ with $\alpha_n = 1$ or $B_1 = \{n\}$. Since the block is of this form, reasoning as in the proof of Proposition 12 one can see that

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{sor}_U w} = \binom{m_1}{\boldsymbol{\alpha}(B_1)} \begin{bmatrix} m_1 + m_2 \cdots + m_n \\ m_j \end{bmatrix} \sum_{w \in \mathcal{R}(\boldsymbol{\alpha}'')} q^{\mathrm{sor}_U w},$$

where $\mathcal{R}(\boldsymbol{\alpha}'')$ is the set of all permutations of the elements of $B_2, \ldots, B_k$ with the multiplicities given by $\boldsymbol{\alpha}$. Since

$$\sum_{w \in \mathcal{R}(\boldsymbol{\alpha})} q^{\mathrm{inv}_U w} = \binom{m_1}{\boldsymbol{\alpha}(B_1)} \begin{bmatrix} m_1 + m_2 \cdots + m_n \\ m_j \end{bmatrix} \sum_{w \in \mathcal{R}(\boldsymbol{\alpha}'')} q^{\mathrm{inv}_U w},$$

we conclude that sor_U and inv_U are equdistributed on $\mathcal{R}(\boldsymbol{\alpha}'')$ and inductively, we get that each of the remaining blocks $B_2, \ldots, B_{k-1}$ has either size 1 or size 2 with the multiplicity of the largest element being 1. $\square$

This completes the proof of Theorem 4.

References

[1] W.Y.C. Chen, G.Z. Gong, and J.J.F. Guo. The sorting index and permutation codes. *Adv. Appl. Math.*, 50(3):367–389, 2013.

[2] B. Clarke. A note on some Mahonian statistics. *Sém. Lothar. Combin.*, 53:B53a, 2005.

[3] D. Foata and C. Krattenthaler. Graphical major indices, II. *Sém. Lothar. Combin.*, 34:B34k, 1995.

[4] D. Foata and D. Zeilberger. Graphical major indices. *J. Comput. Appl. Math.*, 68(1):79–101, 1996.

[5] A. Grady and S. Poznanović. Sorting index and Mahonian-Stirling pairs for labeled forests. *Adv. Appl. Math.*, 80:93–113, 2016.

[6] G.-N. Han. Ordres bipartitionnaires et statistiques sur les mots. *Electronic J. Combin.*, 3(2):#R3, 1996.

[7] G. Hetyei and C. Krattenthaler. The poset of bipartitions. *European J. Combin.*, 32(8):1253–1281, 2011.

[8] K.W.J. Kadell. Weighted inversion numbers, restricted growth functions, and standard Young tableaux. *J. Combin. Theory Ser. A*, 40(1):22–44, 1985.

[9] A. Kasraoui. A classification of Mahonian maj-inv statistics. *Adv. Appl. Math.*, 42(3):342–357, 2009.

[10] P.A. MacMahon. The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects. *Amer. J. Math.*, 281–322, 1913.

[11] P.A. MacMahon. Two applications of general theorems in combinatory analysis:(1) to the theory of inversions of permutations;(2) to the ascertainment of the numbers of terms in the development of a determinant which has amongst its elements an arbitrary number of zeros. *Proc. Lond. Math. Soc.*, 2(1):314–321, 1917.

[12] P.A. MacMahon. *Combinatory Analysis.* Chelsea Publishing Co., New York, 1960.

[13] T.K. Petersen. The sorting index. *Adv. Appl. Math.*, 47(3):615–630, 2011.

[14] S. Poznanović. The sorting index and equidistribution of set-valued statistics over restricted permutations. *J. Combin. Theory Ser. A*, 125:254–272, 2014.

[15] D. Rawlings. The r-major index. *J. Combin. Theory Ser. A*, 31(2):175–183, 1981.

[16] O. Rodrigues. Note sur les inversions, ou dérangements produits dans les permutations. *J. de Math.*, 4(1839):236–240, 1839.

[17] M.C. Wilson. An interesting new Mahonian permutation statistic. *Electron. J. Combin.*, 17(1):#R147, 2010.