Some new groups which are not CI-groups with respect to graphs

Ted Dobson

Department of Mathematics and Statistics
Mississippi State University
PO Drawer MA Mississippi State, MS 39762 U.S.A.

dobson@math.msstate.edu

and

IAM
University of Primorska
6000 Koper, Slovenia

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Abstract

A group G is a CI-group with respect to graphs if two Cayley graphs of G are isomorphic if and only if they are isomorphic by a group automorphism of G. We show that an infinite family of groups which include $D_n \times F_{3p}$ are not CI-groups with respect to graphs, where p is prime, $n \neq 10$ is relatively prime to $3p$, D_n is the dihedral group of order n, and F_{3p} is the nonabelian group of order $3p$.

Keywords: Cayley graph; CI-group; isomorphism

The Cayley isomorphism problem has been studied extensively for the last 50 years. This problem is actually two different, but highly related problems. The most general version asks for necessary and sufficient conditions to determine isomorphism between two Cayley (di)graphs of a group G. Usually what is meant by "necessary and sufficient conditions" is for an explicit and minimal list L (which may or may not depend on the specific Cayley (di)graphs under consideration) of elements of S_G, and which satisfies the statement two Cayley (di)graphs of G are isomorphic if and only if they are isomorphic by an element on the list L. The less general problem specifies the minimal list L in advance as the group automorphisms of G, and asks for which G this list is necessary and sufficient. The choice of group automorphisms as the minimal list is because the image of a Cayley (di)graph of G under a group automorphism is also a Cayley (di)graph of G, and so one must check whether group automorphisms are (di)graph isomorphisms.

In this paper, we contribute to the second problem by considering an infinite family $\mathcal{F}$ of groups which include $D_n \times F_{3p}$, where D_n is the dihedral group of order n and F_{3p} is the nonabelian group of order $3p$, with $n \neq 10$ relatively prime to $3p$ and p a prime. We show that testing only the group automorphisms of a group G in $\mathcal{F}$ is not sufficient to test for isomorphism among Cayley graphs of G. Using the standard terminology for this problem (defined below), we show these groups are not CI-groups with respect to graphs. We remark that word *graph* here is chosen deliberately, as this fact is already know for Cayley *digraphs* of these groups [10].

Definition 1. Let G be a group and $S \subset G$ such that $1 \notin S$ and $S = S^{-1} = \{s^{-1} : s \in S\}$. Define a **Cayley graph of $\boldsymbol{G}$**, denoted $\mathrm{Cay}(G, S)$, to be the graph with $V(\mathrm{Cay}(G, S)) = G$ and $E(\mathrm{Cay}(G, S)) = \{\{g, gs\} : g \in G, s \in S\}$. We call S the **connection set of $\mathbf{Cay}(\boldsymbol{G}, \boldsymbol{S})$**.

Definition 2. A group $G \leqslant S_X$, where S_X is the **symmetric group on the set $\boldsymbol{X}$**, is **transitive** if whenever $x, y \in X$ then there exists $g \in G$ with $g(x) = y$.

Define $g_L : G \mapsto G$ by $g_L(h) = gh$. It is straightforward to verify $g_L \in \mathrm{Aut}(\mathrm{Cay}(G, S))$, and so $G_L = \{g_L : g \in G\} \leqslant \mathrm{Aut}(\mathrm{Cay}(G, S))$. Here $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is the group of all automorphisms of $\mathrm{Cay}(G, S)$. The group G_L is the **left regular representation of $\boldsymbol{G}$**. It is easy to see that $G_L \leqslant S_G$ is a transitive group, and so Cayley graphs are vertex-transitive graphs, that is, graphs whose automorphism group is transitive on their vertex-set. More generally, a transitive group $H \leqslant S_X$ is **regular** if the stabilizer in H of a point is trivial. Equivalently, $|H| = |X|$. This is where the word "regular" in "left regular representation" comes from.

Definition 3. Let G be a group. We say G is a **CI-group with respect to graphs** if whenever $S, T \subset G$ with $S^{-1} = S$ and $T^{-1} = T$, then $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are isomorphic if and only there exists a group automorphism $\alpha \in \mathrm{Aut}(G)$ with $\alpha(\mathrm{Cay}(G, S)) = \mathrm{Cay}(G, T)$.

It is easy to show that if $\alpha \in \mathrm{Aut}(G)$, then $\alpha(\mathrm{Cay}(G, S)) = \mathrm{Cay}(G, \alpha(S))$. Thus if testing isomorphisms between two Cayley graphs of G, the group automorphisms of G must be checked. The notion of a graphical regular representation or GRR of a group G will be crucial in our construction.

Definition 4. A **graphical regular representation or GRR** of a group G is a Cayley graph Γ of G such that $\mathrm{Aut}(\Gamma) = G_L$.

All groups which have a GRR are known, see [7]. There are two infinite families of groups G which do not have GRR's, namely abelian groups and generalized dicyclic groups (the interested reader is referred to [7] for the definition of a generalized dicyclic group). Additionally, there are 13 groups of small order not in these two infinite families which do not have GRR's, and one of these groups, namely D_{10}, will play a role in this paper. We now define some groups which will be of interest in this paper.

Definition 5. Let M be an abelian group such that every Sylow p-subgroup of M is elementary abelian. Denote the largest order of any element of M by $\exp(M)$. Let

$n \in \{2,3,4,8\}$ be relatively prime to $|M|$. Set $E(n,M) = \mathbb{Z}_n \ltimes_\phi M$, where if n is even then $\phi(g) = g^{-1}$, while if $n = 3$ then $\phi(g) = g^\ell$, where ℓ is an integer satisfying $\ell^3 \equiv 1 \pmod{\exp(M)}$ and $\gcd(\ell(\ell-1), \exp(M)) = 1$.

If $M = \mathbb{Z}_p$, and $3|(p-1)$ then $E(3,\mathbb{Z}_p)$ is the nonabelian group of order $3p$, which we denote by F_{3p} (as this group is a Frobenious group). Similarly, $E(2,\mathbb{Z}_n)$ is the dihedral group of order $2n$. The next result is a combination of results of Li, Lu, and Pálfy [9], and Somlai [11], and lists all possible CI-groups with respect to graphs. Not every group in this result is known to be a CI-group with respect to graphs - see [6] for a recent list of the known CI-groups with respect to graphs.

Theorem 6. *Let G be a CI-group with respect to graphs.*

1. *If G does not contain elements of order 8 or 9, then $G = H_1 \times H_2 \times H_3$, where the orders of H_1, H_2, and H_3 are pairwise relatively prime, and*
   (a) *H_1 is an abelian group, and each Sylow p-subgroup of H_1 is isomorphic to $\mathbb{Z}_p^k$ for $k < 2p+3$ or $\mathbb{Z}_4$;*
   (b) *H_2 is isomorphic to one of the groups $E(2,M)$, $E(4,M)$, Q_8, or 1;*
   (c) *H_3 is isomorphic to one of the groups $E(3,M)$, A_4, or 1.*
2. *If G contains elements of order 8, then $G \cong E(8,M)$ or $\mathbb{Z}_8$.*
3. *If G contains elements of order 9, then G is one of the groups $\mathbb{Z}_2 \ltimes \mathbb{Z}_9$, $\mathbb{Z}_4 \ltimes \mathbb{Z}_9$, $\mathbb{Z}_9 \ltimes \mathbb{Z}_2^2$, or $\mathbb{Z}_2^n \times \mathbb{Z}_9$, with $n \leqslant 5$.*

Before turning to our results, we need some additional terms and notation.

Definition 7. Let $G \leqslant S_X$ be transitive, where X is a set. A subset $B \subseteq X$ is a **block** of G if whenever $g \in G$, then $g(B) \cap B = \emptyset$ or B. If B is a block of G, then $g(B)$ is also a block of G for every $g \in G$. The set $\{g(B) : B \in \mathcal{B}\}$ is an **invariant partition of G**.

Definition 8. Let $G \leqslant S_X$ be transitive with invariant partition $\mathcal{B}$. An element $g \in G$ induces a permutation $g/\mathcal{B}$ on $\mathcal{B}$ given by $g/\mathcal{B}(B) = B'$ if and only if $g(B) = B'$. We set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$.

Definition 9. Let Γ be a vertex-transitive graph and $G \leqslant \mathrm{Aut}(\Gamma)$ be transitive with invariant partition $\mathcal{B}$. Define the **block quotient graph of Γ with respect to $\mathcal{B}$**, denoted $\Gamma/\mathcal{B}$, to be the graph with vertex set $\mathcal{B}$ and edge set $\{\{B,B'\} : B \neq B' \in \mathcal{B} \textit{ and } uv \in E(\Gamma) \textit{ for some } u \in B \textit{ and } v \in B'\}$.

Intuitively, $\Gamma/\mathcal{B}$ is obtained by identifying all the vertices in each block of $\mathcal{B}$, then eliminating loops and multiple edges. Additionally, if Γ is a vertex-transitive graph with $G \leqslant \mathrm{Aut}(\Gamma)$ transitive with an invariant partition $\mathcal{B}$, then $G/\mathcal{B} \leqslant \mathrm{Aut}(\Gamma/\mathcal{B})$. We will need the following technical lemma.

Lemma 10. *Let G be a group and $\Gamma_1 = \mathrm{Cay}(G, S_1) \cong \mathrm{Cay}(G, S_2) = \Gamma_2$ with a regular group $R \leqslant \mathrm{Aut}(\mathrm{Cay}(G, S_1)) \cap \mathrm{Aut}(\mathrm{Cay}(G, S_2))$. Let $g \in G$. If R is a CI-group with respect to graphs, then Γ_1 and Γ_2 are isomorphic by and element of S_G that normalizes R and fixes g.*

Proof. By [8, Lemma 3.7.2], we have Γ_1 and Γ_2 are isomorphic to Cayley graphs of R. Then there exists δ_i such that $\delta_i(\Gamma_i)$ is a Cayley graph of R, $i = 1, 2$. Choose δ_i such that $\delta_i^{-1} R \delta_i = R_L$. This can be done as $R \leqslant \mathrm{Aut}(\Gamma_1) \cap \mathrm{Aut}(\Gamma_2)$ (as in the proof of [8, Lemma 3.7.2]). As R is a CI-group with respect to graphs, there exists $\alpha \in \mathrm{Aut}(R)$ with $\alpha(\delta_1(\Gamma_1)) = \delta_2(\Gamma_2)$. Then $\delta_2^{-1}\alpha\delta_1 : \Gamma_1 \mapsto \Gamma_2$ is an isomorphism, and as α normalizes R_L by [1, Corollary 4.2B], $\delta_2^{-1}\alpha\delta_1$ normalizes R. Let $\delta_2^{-1}\alpha\delta_1(g) = g'$ and $r \in R$ such that $r(g') = g$. Then $r\delta_2^{-1}\alpha\delta_1$ is an isomorphism from Γ_1 to Γ_2 which normalizes R and fixes g. □

Theorem 11. *Let $p \geqslant 5$ be prime such that $3|(p-1)$, G be a group of order relatively prime to $3p$ that has a GRR whose connection set contains a non self-inverse element, and F_{3p} be the nonabelian group of order $3p$. Then $G \times F_{3p}$ is not a CI-group with respect to graphs.*

Proof. We will construct graphs Γ and Γ' that are isomorphic graphs whose automorphism groups contain a common regular subgroup $R \cong G \times F_{3p}$ but are not isomorphic by an element that normalizes R and fixes a point. The result will follow by Lemma 10. We will construct these graphs as Cayley graphs of $K = G \times \mathbb{Z}_3 \times \mathbb{Z}_p$.

We define all permutations we shall have need of: Let $\omega, \beta \in \mathbb{Z}_p^*$ with ω of order 3, and $\langle\beta\rangle = \mathbb{Z}_p^*$. For $h \in G$, define $\bar{h}_L, \rho, \tau, \tau', \hat{\omega}, \hat{\beta}, \iota, \Psi : K \mapsto K$ by $\bar{h}_L(g, i, j) = (hg, i, j)$, $\rho(g, i, j) = (g, i, j+1)$, $\tau(g, i, j) = (g, i+1, j)$, $\tau'(g, i, j) = (g, i+1, \omega j)$, $\hat{\omega}(g, i, j) = (g, i, \omega j)$, $\hat{\beta}(g, i, j) = (g, i, \beta j)$, $\iota(g, i, j) = (g, -i, -j)$, $\Psi(g, 0, j) = (g, 0, j)$, $\Psi(g, 1, j) = \Psi(g, 1, j-1)$, and $\Psi(g, 2, j) = (g, 1, j-1-\omega)$. Note that $(G \times \mathbb{Z}_3 \times \mathbb{Z}_p)_L = \langle \bar{h}_L, \rho, \tau : h \in G\rangle$, and let

$$R = \langle \bar{h}_L, \rho, \tau' : h \in G\rangle \cong G_L \times \langle \rho, \tau'\rangle \cong (G \times F_{3p})_L.$$

Set $\tau'\tau^{-1} = \hat{\omega}$. As $\{\bar{h}_L : h \in G\}$ is the unique normal subgroup of R of order $|G|$ which is relatively prime to $3p$, we see $\{\bar{h}_L : h \in G\}$ characteristic in R. Similarly, $\langle \rho, \tau'\rangle$ is also characteristic in R, and $\langle \rho, \tau\rangle$ is characteristic in K_L. As the orbits of $\langle \rho, \tau'\rangle$ and $\langle \rho, \tau\rangle$ are identical, R and K_L both have a unique invariant partition $\mathcal{B}$ with $|G|$ blocks of size $3p$. Also, $\mathrm{Aut}(R) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(F_{3p})$ and $\mathrm{Aut}(K) = \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_p)$.

By hypothesis, there exists a graph $\mathrm{Cay}(G, T)$ with $\mathrm{Aut}(\mathrm{Cay}(G, T)) = G_L$ and $t \in T \subset G$ is not self-inverse. Then $T^{-1} = T$, and there exists a minimal set $V \subset T$ with $T = \{v, v^{-1} : v \in V\}$ and $t \in V$. Note $|V| < |T|$ as $t \in T$. By [13, Theorem 1.2] there exists a digraph $\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_p, U)$ such that $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_p, U)) = \langle (x, y) \mapsto (-x, -\omega y) + (a, b) : a \in \mathbb{Z}_3, b \in \mathbb{Z}_p\rangle \cong \mathbb{Z}_6 \ltimes \mathbb{Z}_{3p}$, and as the map $(x, y) \mapsto (-x, -y)$ is contained in $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_p, U))$, the digraph $\mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_p, U)$ is a graph. Let

$$S = \{(1_G, u) : u \in U\} \cup \{(v, 1, \omega^k), (v, 1, \omega^k)^{-1} : v \in V, k = 0, 1, 2\} \subseteq K.$$

Then $S^{-1} = S$ and $\Gamma = \mathrm{Cay}(K, S)$ is a Cayley graph of K.

Let $\Gamma' = \iota(\Gamma)$. As $\iota \in \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_p) = \mathrm{Aut}(K)$, we see $\Gamma' = \mathrm{Cay}(K, \iota(S))$ is isomorphic to Γ and

$$\iota(S) = \{(1_G, u) : u \in U\} \cup \{(v, 2, -\omega^k), (v^{-1}, 1, \omega^k) : v \in V, k = 0, 1, 2\}$$

as $U = -U$. As $t \in V$ and $t \neq t^{-1}$, we see $\iota(S) \neq S$ and so $\Gamma \neq \Gamma'$.

Note that to determine if an automorphism of K is contained in the automorphism group of a Cayley graph of K, it suffices to check that the automorphism of K fixes the connection set of the Cayley graph. It is straightforward to verify that $\hat{\omega}(S) = S$ and $\hat{\omega}(\iota(S)) = \iota(S)$, and so $\hat{\omega} \in \mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(\Gamma')$ as $\hat{\omega} \in \mathrm{Aut}(K)$. As $\tau \in K_L \leqslant \mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(\Gamma')$, and $\tau'\tau^{-1} = \hat{\omega}$, we have $\tau' \in \mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(\Gamma')$, and so $R \leqslant \mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(\Gamma')$. Recalling $R \cong (G \times F_{3p})_L$, we see $\mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(\Gamma')$ contains a regular subgroup isomorphic to $G \times F_{3p}$.

Towards a contradiction, suppose $\alpha \in S_{G \times \mathbb{Z}_3 \times \mathbb{Z}_p}$ fixes $(1_G, 0, 0)$, normalizes R, and $\alpha(\Gamma) = \Gamma'$. As α normalizes R and $\mathcal{B}$ is the unique invariant partition of R with blocks of size $3p$, we see $\mathcal{B}$ is also an invariant partition of $\langle \alpha, R \rangle$. This follows as $\alpha(\mathcal{B})$ is then an invariant partition of R, and so $\alpha(\mathcal{B}) = \mathcal{B}$. Then $\alpha/\mathcal{B} : \Gamma/\mathcal{B} \mapsto \Gamma'/\mathcal{B}$ is an isomorphism that normalizes G_L and fixes 1_G, and so by [1, Corollary 4.2B] is an automorphism of G. As $\Gamma/\mathcal{B} \cong \Gamma'/\mathcal{B} = \mathrm{Cay}(G, T)$, we see $\alpha/\mathcal{B} \in \mathrm{Aut}(\mathrm{Cay}(G, T)) = G_L$, and so $\alpha/\mathcal{B} = 1$.

It is clear Ψ and $\hat{\beta}$ centralize $\{\bar{h}_L : h \in G\}$ and by [4, Lemma 2.5] Ψ and $\hat{\beta}$ normalize $\langle \rho, \tau' \rangle$. Then $\Psi, \hat{\beta} \in N_{S_K}(R)$. As $\alpha/\mathcal{B} = 1$, we conclude by [4, Lemma 2.5] and [1, Corollary 4.2B] that $\alpha \in \langle \Psi, \hat{\beta} \rangle$. As $\langle \Psi \rangle \triangleleft \langle \Psi, \hat{\beta} \rangle$, we may write $\alpha = \Psi^a \hat{\beta}^b$ where $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_p^*$. As α maps the neighbors of $(1_G, 0, 0)$ to the neighbors of $(1_G, 0, 0)$,

$$\alpha(\{(t, 1, \omega^k j) : k = 0, 1, 2\}) = \{(t, 1, \beta^b \omega^k j - a) : k = 0, 1, 2\} \subseteq \iota(S).$$

However, no element of the form $(t, 1, j)$ is contained in $\iota(S)$ for any $j \in \mathbb{Z}_p$, a contradiction. The result follows by Lemma 10. □

Corollary 12. *Let M be an abelian group that contains an element of prime order $p \geqslant 5$. Let $H = Q_8$, $E(2, M')$, or $E(4, M')$, where M' is an abelian group of order relatively prime to 6 and $|M|$. If $H \neq D_{10}$ then $H \times E(3, M)$ is not a CI-group with respect to graphs.*

Proof. There exists $N \triangleleft M$ such that $M/N \cong \mathbb{Z}_p$ and, as ϕ (as in the definition of $E(3, M)$) fixes every subgroup of M, $N \triangleleft E(3, M)$. Then $E(3, M)/N \cong F_{3p}$. As quotients of CI-groups with respect to graphs are CI-groups with respect to graphs [5, Theorem 8], it suffices to show that $H \times F_{3p}$ is not a CI-group with respect to graphs. The groups Q_8 and $E(4, M')$ have a GRR [7] which is of course connected. As Q_8 contains a unique subgroup of order 2 and is a 2-group, any generating set of Q_8 must contain an element of order 4. As $E(4, M')$ contains a unique subgroup H of index 2 that contains the unique element of $E(4, M')$ of order 2, any generating set of $E(4, M')$ must also contain an element of order 4. The result follows by Theorem 11 in the cases $H = Q_8$ or $E(4, M')$.

Suppose $H = E(2, M')$. As $H = E(2, M') \neq D_{10}$, either $|M'|$ is divisible by a prime q other than 5 or by 25. Note that $q \neq 2$ or 3 as $|M'|$ is relatively prime to 6. Then there exists $L \lhd H \times F_{3p}$ such that $(H \times F_{3p})/L \cong D_{2q} \times F_{3p}$, where either $q = 25$ or $q \geqslant 7$ is prime. Again by [5, Theorem 8] it suffices to show that $D_{2q} \times F_{3p}$ is not a CI-group with respect to graphs. As D_{2q} has a GRR whose connection set contains an element of order q by the proof of [12, Theorem 2], the result follows in this case by Theorem 11. □

Noting that if $H_3 = A_4$ then $H_2 = 1$ in Theorem 6, combining Corollary 12 with Theorem 6 we have the following improvement to Theorem 6.

Corollary 13. *Let G be a CI-group with respect to graphs.*

1. *If G does not contain elements of order 8 or 9, then $G = H_1 \times H_2 \times H_3$, where the orders of H_1, H_2, and H_3 are pairwise relatively prime, and*
   - *(a) H_1 is an abelian group, and each Sylow p-subgroup of H_1 is isomorphic to $\mathbb{Z}_p^k$ for $k < 2p + 3$ or $\mathbb{Z}_4$;*
   - *(b) H_2 is isomorphic to one of the groups $E(2, M)$, $E(3, M)$, $E(4, M)$, Q_8, A_4, or 1;*
   - *(c) H_3 is isomorphic to one of the groups D_{10}, or 1.*
2. *If G contains elements of order 8, then $G \cong E(8, M)$ or $\mathbb{Z}_8$.*
3. *If G contains elements of order 9, then G is one of the groups $\mathbb{Z}_2 \ltimes \mathbb{Z}_9$, $\mathbb{Z}_4 \ltimes \mathbb{Z}_9$, $\mathbb{Z}_9 \ltimes \mathbb{Z}_2^2$, or $\mathbb{Z}_2^n \times \mathbb{Z}_9$, with $n \leqslant 5$.*

We remark that it has been shown that $E(3, p)$ is a CI-group with respect to graphs [2], and some groups $H_1 \times E(3, M)$ with $H_1 \neq 1$ as in the above result are CI-groups with respect to graphs [3, Theorem 22].

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