The cycle polynomial of a permutation group

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Abstract

The cycle polynomial of a finite permutation group G is the generating function for the number of elements of G with a given number of cycles:

$$F_G(x) = \sum_{g \in G} x^{c(g)},$$

where $c(g)$ is the number of cycles of g on Ω. In the first part of the paper, we develop basic properties of this polynomial, and give a number of examples.

In the 1970s, Richard Stanley introduced the notion of reciprocity for pairs of combinatorial polynomials. We show that, in a considerable number of cases, there is a polynomial in the reciprocal relation to the cycle polynomial of G; this is the *orbital chromatic polynomial* of Γ and G, where Γ is a G-invariant graph, introduced by the first author, Jackson and Rudd. We pose the general problem of finding all such reciprocal pairs, and give a number of examples and characterisations: the latter include the cases where Γ is a complete or null graph or a tree.

The paper concludes with some comments on other polynomials associated with a permutation group.

1 The cycle polynomial and its properties

The *cycle index* of a permutation group G acting on a set Ω of size n is a polynomial in n variables which keeps track of all the cycle lengths of elements. If the variables are $s_1, \ldots, s_n$, then the cycle index is given by

$$Z_G(s_1, \ldots, s_n) = \sum_{g \in G} \prod_{i=1}^{n} s_i^{c_i(g)},$$

where $c_i(g)$ is the number of cycles of length i in the cycle decomposition of G. (It is customary to divide this polynomial by $|G|$ but we prefer not to do so here.)

Define the *cycle polynomial* of a permutation group G to be $F_G(x) := Z_G(x, x, \ldots, x)$; that is

$$F_G(x) = \sum_{g \in G} x^{c(g)},$$

where $c(g)$ is the number of cycles of g on Ω (including fixed points). Clearly the cycle polynomial is a monic polynomial of degree n.

Proposition 1. *If a is an integer, then $F_G(a)$ is a multiple of $|G|$.*

Proof. Consider the set of colourings of Ω with a colours (that is, functions from Ω to $\{1, \ldots, a\}$. There is a natural action of G on this set. A colouring is fixed by an element $g \in G$ if and only if it is constant on the cycles of g; so there are $a^{c(g)}$ colourings fixed by g. Now the orbit-counting Lemma shows that the number of orbits of G on colourings is

$$\frac{1}{|G|} \sum_{g \in G} a^{c(g)};$$

and this number is clearly a positive integer. The fact that $f(a)$ is an integer for all a follows from [7, Proposition 1.4.2]. □

Note that the combinatorial interpretation of $F_G(a)/|G|$ given in the proof of Proposition 1 is the most common application of Pólya's theorem.

Proposition 2. *$F_G(0) = 0$; $F_G(1) = |G|$; and $F_G(2) \geqslant (n + 1)|G|$, with equality if and only if G is transitive on sets of size i for $0 \leqslant i \leqslant n$.*

Proof. The first assertion is clear.

There is only one colouring with a single colour.

If there are two colours, say red and blue, then the number of orbits on colourings is equal to the number of orbits on (red) subsets of Ω. There are $n+1$ possible cardinalities of subsets, and so at least $n + 1$ orbits, with equality if and only if G is is transitive on sets of size i for $0 \leqslant i \leqslant n$. (Groups with this property are called set-transitive and were determined by Beaumont and Petersen [1]; there are only the symmetric and alternating groups and four others with $n = 5, 6, 9, 9$.) □

Now we consider values of F_G on negative integers. Note that the *sign* $\mathrm{sgn}(g)$ of the permutation g is $(-1)^{n-c(g)}$; a permutation is even or odd according as its sign is $+1$ or -1. If G contains odd permutations, then the even permutations in G form a subgroup of index 2.

Proposition 3. *If G contains no odd permutations, then F_G is an even or odd function according as n is even or odd; in other words,*

$$F_G(-x) = (-1)^n F_G(x).$$

Proof. The degrees of all terms in F_G are congruent to n mod 2. □

In particular, we see that if G contains no odd permutations, then $F_G(x)$ vanishes only at $x = 0$. However, for permutation groups containing odd permutations, there may be negative roots of F_G.

Theorem 4. *Suppose that G contains odd permutations, and let N be the subgroup of even permutations in G. Then, for any positive integer a, we have $0 \leqslant (-1)^n F_G(-a) < F_G(a)$, with equality if and only if G and N have the same number of orbits on colourings of Ω with a colours.*

Proof. Let Δ denote the set of a-colourings of Ω. Say that an orbit $\mathcal{O}$ of G on Δ is split if the G- and N-orbits of ω do not coincide for some (and hence any) $\omega \in \mathcal{O}$. The number of split orbits is the number of N-orbits on Δ less the number of G-orbits on Δ, namely

$$\frac{1}{|G|}\sum_{g\in N} 2\cdot|\operatorname{Fix}_\Delta(g)| - \frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}_\Delta(g)|,$$

where $\operatorname{Fix}_\Delta(g)$ denotes the number of fixed points of g on Δ. This, in turn, is equivalent to

$$\frac{1}{|G|}\sum_{g\in G}\operatorname{sgn}(g)a^{c(g)} = \frac{1}{|G|}(-1)^n F_G(-a),$$

and the result follows from this. □

Proposition 5. *If G is a permutation group containing odd permutations, then the set of negative integer roots of F_G consists of all integers $\{-1, -2, \ldots, -a\}$ for some $a \geqslant 1$.*

Proof. $F_G(-1) = 0$, since G and N have equally many orbits (namely 1) on colourings with a single colour.

Now suppose that $F_G(-a) = 0$, so that G and N have equally many orbits on colourings with a colours; thus every G-orbit is an N-orbit. Now every colouring with $a-1$ colours is a colouring with a colours, in which the last colour is not used; so every G-orbit on colourings with $a-1$ colours is an N-orbit, and so $F_G(-a+1) = 0$. The result follows. □

The property of having a root $-a$ is preserved by overgroups:

Proposition 6. *Suppose that G_1 and G_2 are permutation groups on the same set, with $G_1 \leqslant G_2$. Suppose that $F_{G_1}(-a) = 0$, for some positive integer a. Then also $F_{G_2}(-a) = 0$.*

Proof. It follows from the assumption that G_1 (and hence also G_2) contains odd permutations. Let N_1 and N_2 be the subgroups of even permutations in G_1 and G_2 respectively. Then $N_2 \cap G_1 = N_1$, and so $N_2 G_1 = G_2$. By assumption, G_1 and N_1 have the same orbits on a-colourings. Let K be an a-colouring, and $g \in G_2$; write $g = hg'$, with $h \in N_2$ and $g' \in G_1$. Now Kh and Khg' are in the same G_1-orbit, and hence in the same N_1-orbit; so there exists $h \in N_1$ with $Kg = Khg' = Khh'$. Since $hh' \in N_2$, we see that the G_2-orbits and N_2-orbits on a-colourings are the same. Hence $F_{G_2}(-a) = 0$. □

The cycle polynomial has nice behaviour under direct product, which shows that the property of having negative integer roots is preserved by direct product.

Proposition 7. *Let G_1 and G_2 be permutation groups on disjoint sets Ω_1 and Ω_2. Let $G = G_1 \times G_2$ acting on $\Omega_1 \cup \Omega_2$. Then*

$$F_G(x) = F_{G_1}(x) \cdot F_{G_2}(x).$$

In particular, the set of roots of F_G is the union of the sets of roots of F_{G_1} and F_{G_2}.

Proof. This can be done by a calculation, but here is a more conceptual proof. It suffices to prove the result when a positive integer a is substituted for x. Now a G-orbit on a-colourings is obtained by combining a G_1-orbit on colourings of Ω_1 with a G_2-orbit of colourings of Ω_2; so the number of orbits is the product of the numbers for G_1 and G_2. □

The result for the wreath product, in its imprimitive action, is obtained in a similar way.

Proposition 8.

$$F_{G \wr H}(x) = |G|^m F_H(F_G(x)/|G|).$$

Proof. Again it suffices to prove that, for any positive integer a, the equation is valid with a substituted for x.

Let Δ be the domain of H, with $|\Delta| = m$. An orbit of the base group G^m on a-colourings is an m-tuple of G-orbits on a-colourings, which we can regard as a colouring of Δ, from a set of colours whose cardinality is the number $F_G(a)/|G|$ of G-orbits on a-colourings. Then an orbit of the wreath product on a-colourings is given by an orbit of H on these $F_G(a)/|G|$-colourings, and so the number of orbits is $(1/|H|)F_H(F_G(a)/|G|)$. Multiplying by $|G \wr H| = |G|^m|H|$ gives the result. □

Corollary 9. *If n is odd, $m > 1$, and $G = S_n \wr S_m$, then $F_G(x)$ has roots $-1, \ldots, -n$.*

Proof. In Proposition 12 to come, we show that $F_{S_n}(x) = x(x+1)\cdots(x+n-1)$. Therefore $F_{S_n}(x)$ divides $F_G(x)$, so we have roots $-1, \ldots, -n+1$. Also, there is a factor

$$|S_n|(F_{S_n}(x)/|S_n| + 1) = x(x+1)\cdots(x-n+1) + n!.$$

Substituting $x = -n$ and recalling that n is odd, this is $-n! + n! = 0$. □

The next corollary shows that there are imprimitive groups with arbitrarily large negative roots.

Corollary 10. *Let n be odd and let G be a permutation group of degree n which contains no odd permutations. Suppose that $F_G(a)/|G| = k$. Then, for $m > k$, the polynomial $F_{G \wr S_m}(x)$ has a root $-a$.*

Proof. By Proposition 3, $F_G(-a)/|G| = -k$. Now for $m > k$, the polynomial $F_{S_m}(x)$ has a factor $x + k$; the expression for $F_{G \wr S_m}$ shows that this polynomial vanishes when $x = -a$. □

The main question which has not been investigated here is:

What about non-integer roots?

It is clear that $F_G(x)$ has no positive real roots; so if G contains no odd permutations, then $F_G(x)$ has no real roots at all, by Proposition 3. When $G = A_n$ we have the following result which was communicated to us by Valentin Féray [4].

Theorem 11. *If $F_{A_n}(a) = 0$ for some complex number a then $\Re(a) = 0$.*

Proof. We show that this result is a particular case of [8, Theorem 3.2]. In the notation of that theorem, set $d := n-1$ and $g(t) := t^d + 1$ so that the hypotheses are clearly satisfied and $m = 0$. Now

$$P(q) = (E^d + 1)\prod_{i=0}^{n-1}(q+i) = \prod_{i=0}^{n-1}(q-i) + \prod_{i=0}^{n-1}(q+i) = 2F_{A_n}(q),$$

where E is the backward shift operator on polynomials in q given by $Ef(q) = f(q-1)$. [8, Theorem 3.2(a)] now delivers the result. □

2 Some examples

Proposition 12. *For each n we have,*

$$F_{S_n}(x) = \prod_{i=0}^{n-1}(x+i).$$

Proof. By induction and Proposition 6, $F_{S_n}(-a) = 0$ for each $0 \leqslant a \leqslant n-2$. Since $F_{S_n}(x)$ is a polynomial of degree n and the coefficient of x in $F_{S_n}(x)$ is $(n-1)!$ we must have that $n-1$ is the remaining root.

Several other proofs of this result are possible. We can observe that the number of orbits of S_n on a-colourings of $\{1,\ldots,n\}$ (equivalently, n-tuples chosen from the set of colours with order unimportant and repetitions allowed) is $\binom{n+a-1}{n}$. Or we can use the fact that the number of permutations of $\{1,\ldots,n\}$ with k cycles is the unsigned *Stirling number of the first kind* $u(n,k)$, whose generating function is well known to be

$$\sum_{k=1}^{n} u(n,k)x^k = x(x+1)\cdots(x+n-1).$$ □

Proposition 13. *For each n we have*

$$F_{C_n}(x) = \sum_{d|n}\phi(d)x^{n/d},$$

where ϕ is Euler's totient function.

Proof. This is a consequence of the well-known formula for the cycle index of a cyclic group. ☐

Proposition 14. *Let p be an odd prime and G be the group* $\mathrm{PGL}_2(p)$ *acting on a set of size $p+1$. Then $F_G(x)$ is given by*

$$\frac{p(p+1)}{2}x^2 F_{C_{p-1}}(x) + \frac{p(p-1)}{2}F_{C_{p+1}}(x) + (p^2-1)(x^2 - x^{p+1})$$

Proof. Each semisimple element of $\mathrm{GL}_2(p)$ either has eigenvalues in $\mathbb{F}_p$ and lies in a torus isomorphic to $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ or else it has eigenvalues in $\mathbb{F}_{p^2} \backslash \mathbb{F}_p$ and lies in a torus isomorphic to $\mathbb{F}_{p^2}^\times$. If T is either kind of torus then $N_{\mathrm{GL}_2(p)}(T)$ is generated by T and an automorphism which inverts each element, so there are $|\,\mathrm{GL}_2(p)|/2|T|$ conjugates of T. Distinct tori intersect in the subgroup of scalar matrices $Z(\mathrm{GL}_2(p))$. Hence, ignoring the identity, the images of semisimple elements of $\mathrm{GL}_2(p)$ in G can be partitioned into $|\,\mathrm{GL}_2(p)|/2(p-1)^2 = p(p+1)/2$ tori of the first type, each generated by an element of cycle type $(p-1, 1^2)$ and $|\,\mathrm{GL}_2(p)|/2(p^2-1) = p(p-1)/2$ tori of the second type, each generated by an element of cycle type $(p+1)$. This leaves the elements of order p, all self-centralizing and of cycle type $(p, 1)$. Therefore

$$F_G(x) = x^2\frac{p(p+1)}{2}F_{C_{p-1}}(x) + \frac{p(p-1)}{2}F_{C_{p+1}}(x) + (p-1)(p+1)x^2 - ax^{p+1},$$

where $a = p^2 - 1$ corrects for over-counting the identity. ☐

3 Reciprocal pairs

Richard Stanley, in a 1974 paper [6], explained (polynomial) combinatorial reciprocity thus:

> A *polynomial reciprocity theorem* takes the following form. Two combinatorially defined sequences $S_1, S_2, \ldots$ and $\bar{S}_1, \bar{S}_2, \ldots$ of finite sets are given, so that the functions $f(n) = |S_n|$ and $\bar{f}(n) = |\bar{S}_n|$ are polynomials in n for all integers $n \geqslant 1$. One then concludes that $\bar{f}(n) = (-1)^d f(-n)$, where $d = \deg f$.

We will see that, in a number of cases, the cycle polynomial satisfies a reciprocity theorem.

3.1 The orbital chromatic polynomial

First, we define the polynomial which will serve as the reciprocal polynomial in these cases. A *(proper) colouring* of a graph Γ with q colours is a map from the vertices of Γ to the set of colours having the property that adjacent vertices receive different colours. Note that, if Γ contains a loop (an edge joining a vertex to itself), then it has no proper colourings. Birkhoff observed that, if there are no loops, then the number of colourings with q colours is the evaluation at q of a monic polynomial $P_\Gamma(x)$ of degree equal to the number of vertices, the *chromatic polynomial* of the graph.

Now suppose that G is a group of automorphisms of Γ. For $g \in G$, let Γ/g denote the graph obtained by "contracting" each cycle of g to a single vertex; two vertices are joined by an edge if there is an edge of Γ joining vertices in the corresponding cycles. The chromatic polynomial $P_{\Gamma/g}(q)$ counts proper q-colourings of Γ fixed by g. If any cycle of g contains an edge, then Γ/g has a loop, and $P_{\Gamma/g} = 0$. Now (with a small modification of the definition in [3]) we define the *orbital chromatic polynomial* of the pair (Γ, G) to be

$$P_{\Gamma,G}(x) = \sum_{g\in G} P_{\Gamma/g}(x). \tag{1}$$

The orbit-counting Lemma immediately shows that $P_{\Gamma,G}(q)/|G|$ is equal to the number of G-orbits on proper q-colourings of Γ.

Now, motivated by Stanley's definition, we say that the pair (Γ, G), where Γ is a graph and G a group of automorphisms of Γ, is a *reciprocal pair* if

$$P_{\Gamma,G}(x) = (-1)^n F_G(-x),$$

where n is the number of vertices of Γ.

Remark 15. An alternative definition of reciprocality is obtained using the polynomial

$$\bar{P}_{\Gamma,G}(x) = \sum_{g\in G} \operatorname{sgn}(g)\bar{P}_{\Gamma/g}(x)$$

where $\bar{P}_\Delta(x) = (-1)^n P(-x)$ is the dual chromatic polynomial, first defined by Stanley [6] enumerating certain coloured acyclic orientations of Δ. It is a straightforward exercise to see that reciprocality is equivalent to

$$\bar{P}_{\Gamma,G}(x) = F_G(x).$$

Problem Find all reciprocal pairs.

This problem is interesting because, as we will see, there are a substantial number of such pairs, for reasons not fully understood. In the remainder of the paper, we present the evidence for this, and some preliminary results on the above problem.

A basic result about reciprocal pairs is the following.

Lemma 16. *Suppose that (G, Γ) is a reciprocal pair. Then the number of edges of Γ is the sum of the number of transpositions in G and the number of transpositions (i, j) in G for which $\{i, j\}$ is a non-edge.*

Proof. Whitney [9] showed that the leading terms in the chromatic polynomial of a graph Γ with n vertices and m edges are $x^n - mx^{n-1} + \cdots$. This follows from the inclusion-exclusion formula for the chromatic polynomial: x^n is the total number of colourings of the vertices of Γ, and for each edge $\{i, j\}$, the number of colourings in which i and j have the same colour is x^{n-1}. So in the formula (1) for $P_{\Gamma,G}(x)$, the identity element of G contributes $x^n - mx^{n-1} + \cdots$. The only additional contributions to the coefficient of x^{n-1} in $P_{\Gamma,G}(x)$ come from elements g of G such that only a single edge of Γ is contracted to obtain Γ/g and these are transpositions. A transposition (i, j) makes a non-zero contribution if and only if $\{i, j\}$ is a non-edge. So the coefficient of x^{n-1} is $-m + t^0(G)$, where $t^0(G)$ is the number of transpositions with this property.

On the other hand, the coefficient of x^{n-1} in $F_G(x)$ is the number of permutations in G with $n-1$ cycles, that is, the total number $t(G)$ of transpositions. So the coefficient in $(-1)^n F_G(-x)$ is $-t(G)$.

Equating the two expressions gives $m = t(G) + t^0(G)$, as required. □

We remark that the converse to Lemma 16 does not hold: consider the group $G = S_3 \wr S_3$ acting on 3 copies of K_3 (see Proposition 8: but note that $(3K_3, S_3 \wr C_3)$ is a reciprocal pair, by Proposition 21). We also observe the following corollary to Lemma 16.

Corollary 17. *If Γ is not a complete graph and (Γ, G) is a reciprocal pair then Γ has at most $\frac{(n-1)^2}{2}$ edges.*

Proof. If Γ has $\binom{n}{2} - \delta$ edges then by Lemma 16,

$$\binom{n}{2} - \delta = t(G) + t^0(G) \leqslant t(G) + \delta.$$

If $0 < \delta < \frac{n-1}{2}$ then

$$t(G) \geqslant \binom{n}{2} - 2\delta > \binom{n}{2} - (n-1) = \binom{n-1}{2}.$$

It is well-known that a permutation group of degree n containing at least $\binom{n-1}{2} + 1$ transpositions must be the full symmetric group. But this implies that Γ is a complete graph, a contradiction. □

According to Lemma 16, if Γ is not a null graph and (Γ, G) is a reciprocal pair, then G contains transpositions. Now as is well-known, if a subgroup G of S_n contains a transposition, then the transpositions generate a normal subgroup N which is the direct product of symmetric groups whose degrees sum to n. (Some degrees may be 1, in which case the corresponding factor is absent.)

A G-invariant graph must induce a complete or null graph on each of these sets. Moreover, between any two such sets, we have either all possible edges or no edges.

Suppose that $n_1, \ldots, n_r$ are the sizes of the N-orbits carrying complete graphs and $m_1, \ldots, m_s$ the orbits containing null graphs. Then Lemma 16 shows that the total number of edges of the graph is

$$\sum_{i=1}^{r} \binom{n_i}{2} + 2 \sum_{j=1}^{s} \binom{m_j}{2}.$$

The first term counts edges within N-orbits, so the second term counts edges between different N-orbits.

3.2 Examples

Proposition 18. *The following hold:*

(a) Let Γ be a null graph, and G a subgroup of the symmetric group S_n. Then $P_{\Gamma,G}(x) = F_G(x)$.

(b) Let Γ be a complete graph, and G a subgroup of the symmetric group S_n. Then $P_{\Gamma,G}(x) = x(x-1)\cdots(x-n+1)$, independent of G.

Proof. (a) The chromatic polynomial of a null graph on n vertices is x^n. So, if $g \in G$ has $c(g)$ cycles, then Γ/g is a null graph on $c(g)$ vertices. Thus

$$P_{\Gamma,G}(x) = \sum_{g \in G} x^{c(g)} = F_G(x).$$

(b) In the formula (1) for $P_{\Gamma,G}(x)$, $P_{\Gamma/g}(x)$ is 0 unless g is the identity element of G, when it is $x(x-1)\cdots(x-n+1)$. □

Corollary 19. *The following hold:*

(a) If Γ is a null graph, then (Γ, G) is a reciprocal pair if and only if G contains no odd permutations.

(b) If Γ is a complete graph, then (Γ, G) is a reciprocal pair if and only if G is the symmetric group.

Proof. (a) This follows from Proposition 3.

(b) We saw in the preceding section that, if $G = S_n$, then $F_G(x) = x(x+1)\cdots(x+n-1)$. Thus, we see that (G, Γ) is a reciprocal pair if and only if G is the symmetric group. □

Proposition 20. *Let Γ be the disjoint union of graphs $\Gamma_1, \ldots, \Gamma_r$, and G the direct product of groups $G_1, \ldots, G_r$, where $G_i \leqslant \mathrm{Aut}(\Gamma_i)$. Then*

$$P_{\Gamma,G}(x) = \prod_{i=1}^{r} P_{\Gamma_i,G_i}(x).$$

In particular, if (Γ_i, G_i) is a reciprocal pair for $i = 1, \ldots, r$, then (Γ, G) is a reciprocal pair.

The proof is straightforward; the last statement follows from Proposition 7. The result for wreath products is similar:

Proposition 21. *Let Γ be the disjoint union of m copies of the n-vertex graph Δ. Let $G \leqslant \mathrm{Aut}(\Delta)$, and H a group of permutations of degree m. Then*

$$P_{\Gamma,G\wr H}(x) = |G|^m F_H(P_{\Delta,G}(x)/|G|).$$

In particular, if (Δ, G) is a reciprocal pair and H contains no odd permutations, then $(\Gamma, G \wr H)$ is a reciprocal pair.

Proof. Given q colours, there are $P_{\Delta,G}(q)/|G|$ orbits on colourings of each copy of Δ; so the overall number of orbits is the same as the number of orbits of H on an m-vertex null graph with $P_{\Delta,G}(q)/|G|$ colours available.

For the last part, the hypotheses imply that $P_{\Delta,G}(q) = (-1)^n F_G(-q)$, and that the degree of each term in F_H is congruent to m mod 2. So the expression evaluates to $|G|^m F_H(F_G(-x)/|G|)$ if either m or n is even, and the negative of this if both are odd; that is, $(-1)^{mn}|G|^m F_H(F_G(-x)/|G|)$. □

We now give some more examples of reciprocal pairs.

Example 22. Let Γ be a 4-cycle, and G its automorphism group, the dihedral group of order 8. There are 4 edges in Γ, and 2 transpositions in G, each of which interchanges two non-adjacent points (an opposite pair of vertices of the 4-cycle); so the equality of the lemma holds. Direct calculation shows that

$$P_{\Gamma,G}(x) = x(x-1)(x^2-x+2), \qquad F_G(x) = x(x+1)(x^2+x+2),$$

so $P_{\Gamma,G}(x) = (-1)^n F_G(-x)$ holds in this case.

The 4-cycle is also the complete bipartite graph $K_{2,2}$. We note that, for $n > 2$, $(K_{n,n}, S_n \wr S_2)$ is not a reciprocal pair. This can be seen from the fact that $F_{S_n \wr S_2}(x)$ has factors $x, x+1, \ldots, x+n-1$, whereas $K_{n,n}$ has chromatic number 2 and so $x-2$ is not a factor of $P_{K_{n,n},G}(x)$ for any $G \leqslant S_n \wr S_2$.

We do not know whether other complete multipartite graphs support reciprocal pairs.

Example 23. Let Γ be a path with 3 vertices and G its automorphism group which is cyclic of order 2. Direct calculation shows that

$$P_{\Gamma,G}(x) = x^2(x-1), \qquad F_G(x) = x^2(x+1).$$

This graph is an example of a star graph, for which we give a complete analysis in the next section.

Example 24. Write N_n for the null graph on n vertices and let Γ be the disjoint union of K_m and N_n together with all edges in between and set $G = S_m \times S_n \leqslant \mathrm{Aut}(\Gamma)$. Then Γ has $\binom{m}{2} + mn$ edges and G has $\binom{m}{2} + \binom{n}{2}$ transpositions of which $\binom{n}{2}$ correspond to non-edges in Γ. Thus, according to Lemma 16 we need $n = m+1$. Now the only elements $g \in G$ which give non-zero contribution to $P_{\Gamma,G}(x)$ lie in the S_n component. We get:

$$P_{\Gamma,G}(x) = x(x-1)\cdots(x-(m-1)) \cdot \sum_{g \in S_n} (x-m)^{c(g)}.$$

Using Proposition 18(b) and $m = n - 1$ this becomes

$$(-1)^m F_{S_m}(-x) F_{S_n}(-x) \cdot (-1)^n,$$

which is equal to $-F_G(-x)$ by Proposition 7.

4 Reciprocal pairs containing a tree

In this section we show that the only trees that can occur in a reciprocal pair are stars, and we determine the groups that can be paired with them.

Theorem 25. *Suppose Γ is a tree and (Γ, G) is a reciprocal pair. Let n be the number of vertices in Γ and assume $n \geqslant 3$. The following hold:*

(a) n is odd;

(b) Γ is a star;

(c) $(C_2)^k \leqslant G \leqslant C_2 \wr S_k$ where $n = 2k + 1$.

Conversely any pair (Γ, G) which satisfies conditions (a)-(c) is reciprocal.

In what follows we assume the following:

- Γ is a tree;
- (Γ, G) is a reciprocal pair;
- n is the number of vertices in Γ and $n \geqslant 3$.

Note that any two vertices interchanged by a transposition are non-adjacent. For suppose that a transposition flips an edge $\{v, w\}$. If the tree is central, then there are paths of the same length from the centre to v and w, creating a cycle. If it is bicentral, then the same argument applies unless $\{v, w\}$ is the central edge, in which case the tree has only two vertices, a contradiction.

Lemma 26. *If n is odd then (Γ, G) is a reciprocal pair if and only if*

$$x(F_G(x - 1) + F_G(-x)) = F_G(-x). \tag{2}$$

Proof. The chromatic polynomial of a tree with r vertices is easily seen to be $x(x-1)^{r-1}$. Hence

$$P(\Gamma/g) = x(x - 1)^{c(g)-1}$$

for each $g \in G$ and then

$$P_{\Gamma,G}(x) = \sum_{g \in G} P(\Gamma/g) = \sum_{g \in G} x(x - 1)^{c(g)-1} = \frac{x}{x - 1} F_G(x - 1).$$

Rearranging (and using that n is odd) yields the Lemma. □

Lemma 27. *G has $\frac{n-1}{2}$ transpositions; in particular, n is odd.*

Proof. As in Lemma 16, let $t(G)$ be the number of transpositions in G and $t^0(G)$ be the number of transpositions (i, j) in G for which $i \not\sim j$ in Γ. If G fixes an edge (u, v) of Γ then $(u, v) \in G$ implies $n = 2$, a contradiction. Thus $(u, v) \notin G$, and every transposition in G is a non-edge. Hence $t^0(G) = t(G)$ so $2t(G) = n - 1$ by Lemma 16. □

Lemma 28. *$(C_2)^k \leqslant G \leqslant C_2 \wr S_k$ where $n = 2k + 1$.*

Proof. The transpositions in a permutation group G generate a normal subgroup H which is a direct product of symmetric groups. If there are two non-disjoint transpositions in G, one of the direct factors is a symmetric group with degree at least 3, and hence $F_H(x)$ has a root -2 by Propositions 12 and 7. Then by Proposition 6, $F_G(x)$ has a root -2. By Lemma 26 with $x = 2$,

$$0 = F_G(-2) = 2(F_G(1) + F_G(-2)) = 2F_G(1) = 2|G|,$$

a contradiction. So the transpositions are pairwise disjoint, and generate a subgroup $(C_2)^k$ with $n = 2k + 1$ by Lemma 27. Thus the conclusion of the lemma holds. □

Lemma 29. *Γ is a star.*

Proof. Let v be the unique fixed point of G. By Lemma 28, for each $u \neq v$ there exists a unique vertex u' with $(u, u') \in G$. This is possible only if each u has distance 1 from v. Hence Γ is a star. □

Proof of Theorem 25. (a),(b) and (c) follow from Lemmas 27, 29 and 28 respectively. Conversely, suppose that (a),(b) and (c) hold. Then $G = C_2 \wr K$ for some permutation group K of degree k. By Proposition 8, $F_G(x) = x \cdot 2^k F_K(x(x + 1)/2)$. Now it is clear that

$$-\frac{F_G(-x)}{x} = 2^k \cdot F_K\left(\frac{x(x-1)}{2}\right) = \frac{F_G(x-1)}{x-1},$$

so that (2) holds and we deduce from Lemma 26 that (Γ, G) is a reciprocal pair. Our proof is complete. □

Given a set of reciprocal pairs $(\Gamma_1, G_1), \ldots, (\Gamma_m, G_m)$ with each Γ_i a star we can take direct products and wreath products (using Propositions 20 and 21) to obtain reciprocal pairs (Γ, G) with Γ a forest. We do not know whether all such pairs arise in this way.

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