Two-Vertex Generators of Jacobians of Graphs

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Abstract

We give necessary and sufficient conditions under which the Jacobian of a graph is generated by a divisor that is the difference of two vertices. This answers a question posed by Becker and Glass and allows us to prove various other propositions about the order of divisors that are the difference of two vertices. We conclude with some conjectures about these divisors on random graphs and support them with empirical evidence.

1 Introduction

Given a finite, undirected, connected multigraph G without loops, a *divisor* is an assignment of integer values to the vertices. The *degree* of a divisor is the sum of these values. The *Jacobian* of G, denoted $\mathrm{Jac}(G)$, is a finite abelian group defined as the quotient of the degree zero divisors by an equivalence relation determined by chip-firing on the graph (see Section 2.1). Recent work of several authors [6, 5, 17] considers the likelihood that a random graph has cyclic Jacobian. It has been conjectured in [5] based on a Cohen-Lenstra heuristic and empirical evidence that the probability of cyclic Jacobian in an Erdős-Rényi random graph goes to $\prod_{i=1}^{\infty} \zeta(2i+1)^{-1} \approx 0.79$ as the number of vertices goes to infinity. Indeed, Wood in [17] proved this to be an upper bound, but no nontrivial lower bound on this probability is known.

The central object of study in this paper is the divisor δ_{xy} which is -1 at vertex x, 1 at vertex y, and 0 elsewhere. The collection of all δ_{xy} together generate the whole Jacobian because they generate the group of degree zero divisors, and the Jacobian is a quotient of that group. Thus δ_{xy} presents an ideal candidate for a generator of $\mathrm{Jac}(G)$.

In [12, 5.1], Lorenzini constructs a graph G' by removing a preexisting edge (x, y) from G and proves that the condition $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G')|) = 1$ implies that the Jacobians of G and G' are both cyclic. Lorenzini's proof does not establish explicit generators for these Jacobians, however. More recently, Becker and Glass [3, Open Question 2.8] ask whether δ_{xy} generates $\mathrm{Jac}(G)$ under the same assumption that $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G')|) = 1$.

In this paper we resolve the question of [3] with an affirmative answer, provide a stronger result, and also prove its converse. The approach we take allows us to treat the cases of adding and removing an edge between x and y similarly. Our work strengthens the original theorem of Lorenzini [12, 5.1] and the question of Becker-Glass in two ways: 1) we consider the gcd of $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G')|$ in general rather than only in the relatively prime case, and 2) we show that δ_{xy} is an explicit generator of the Jacobian under certain conditions.

Theorem 1.1. *Fix vertices x, y in G. Let G_1 be G with an edge added between x and y. Then*

$$[Jac(G) : \langle \delta_{xy} \rangle] \;\Big|\; \gcd(|Jac(G)|, |Jac(G_1)|)$$

and

$$\gcd(|Jac(G)|, |Jac(G_1)|) \;\Big|\; [Jac(G) : \langle \delta_{xy} \rangle]^2.$$

Moreover, the same results hold for G_1, namely

$$[Jac(G_1) : \langle \delta_{xy} \rangle] \;\Big|\; \gcd(|Jac(G)|, |Jac(G_1)|)$$

and

$$\gcd(|Jac(G)|, |Jac(G_1)|) \;\Big|\; [Jac(G_1) : \langle \delta_{xy} \rangle]^2.$$

Notice that because the same result holds for G_1, we can get an analogous result for G with an edge removed by exchanging the roles of G and G_1 and applying this theorem. Also, note that δ_{xy} is used to represent an element of $\mathrm{Jac}(G)$ and an element of $\mathrm{Jac}(G_1)$, this is a somewhat abusive notation.

For more general and precise statements of this theorem, see Theorems 3.3 and 3.4. Theorem 1.1 implies the following corollary.

Corollary 1.2. *Let G, x, y, and G_1 be as in Theorem 1.1. The following are equivalent:*

- *δ_{xy} generates $Jac(G)$.*
- *δ_{xy} generates $Jac(G_1)$.*
- *$\gcd(|Jac(G)|, |Jac(G_1)|) = 1$.*

These results relate the order of δ_{xy} to deleting and inserting edges in G. Our main theorem is proven in Section 3. In Section 4 we show that similar results hold under edge contraction. In Section 5 we move on to considering lower bounds on the order of δ_{xy}. Finally, with the hope of eventually extending statements about δ_{xy} to random graphs we provide some conjectures and empirical evidence in Section 6 relating to the probability that some δ_{xy} generates the Jacobian.

2 Background

2.1 Chip-firing and the Jacobian of graphs

Unless specified otherwise, throughout this paper we will assume that any graph G is a connected, undirected multigraph without loops. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of G respectively, and let $n = |V(G)|$ be the number of vertices. For a vertex $v \in V(G)$, let $\mathrm{val}(v)$ denote the valency of v — i.e. the number of edges incident to v.

Following [1], a *divisor* $D \in \mathbb{Z}^n$ on G is a formal $\mathbb{Z}$-linear combination of the vertices of G. Divisors are often interpreted as assignments of an integer number of "chips" to each of the vertices of G. The *degree* of a divisor $\deg(D) = \sum_{v \in V(G)} D(v)$ is the divisor's total number of chips. $\mathrm{Div}(G)$ denotes the group of all divisors on G where the group law is addition of divisors as vectors in $\mathbb{Z}^n$, and $\mathrm{Div}^0(G)$ denotes the subgroup of divisors with degree zero.

There is an equivalence relation on the elements of $\mathrm{Div}(G)$ based on the well-known *chip-firing game*, which is defined as follows. Given any divisor $D \in \mathrm{Div}(G)$, let $K_{v,w}$ denote the number of edges between two vertices $v, w \in V(G)$. We can *fire* a vertex v by sending one chip from v along each of its incident edges to adjacent vertices. This operation produces a new divisor D', given by:

$$D'(w) = \begin{cases} D(v) - \mathrm{val}(v) & \text{if } w = v \\ D(w) + K_{v,w} & \text{if w is adjacent to v} \\ D(w) & \text{otherwise.} \end{cases}$$

Note that any chip-firing move at a given vertex can be reversed by firing at all of the other vertices. We define a *firing script* to be a vector $\sigma \in \mathbb{Z}^n$ whose entries specify the number of times each vertex in G should be fired. We say that two divisors D_1 and D_2 are *equivalent* if there exists a firing script taking one to the other. We denote this relation by $\sim$. Note that the degree of a divisor is invariant under chip-firing. The set of *principal divisors* $\mathrm{Prin}(G)$ is the set of divisors equivalent to the divisor with zero chips on every vertex, which we will call the *zero divisor*.

The *Jacobian* $\mathrm{Jac}(G) := \mathrm{Div}^0(G)/\mathrm{Prin}(G)$ is the group of equivalence classes of divisors on G with degree zero. The Jacobian is sometimes also referred to as the *sandpile group* or the *critical group*. If $D \in \mathrm{Div}^0(G)$, we denote the equivalence class containing D by $[D]$. $\mathrm{Jac}(G)$ is always finite for a connected graph, and its order is equal to the number of spanning trees on G by the matrix-tree theorem (see [1], or [7]).

The Jacobian of a graph is equal to the torsion subgroup of the cokernel of the graph's *Laplacian*, which is an $n \times n$ integer matrix denoted by L and defined as follows. Let Δ be the diagonal matrix with (i,i)–entry equal to $\mathrm{val}(v_i)$, and let A be the adjacency matrix of G. Then $L = \Delta - A$. L gives a map from the space of firing scripts to the space of principal divisors: if σ is a firing script, then $L\sigma$ is the principal divisor obtained by applying σ to the zero divisor (again, see [1] or [7]).

For a given choice of $i \in \{1, \ldots, n\}$ the *reduced Laplacian* $\tilde{L}$ is L with the ith row and ith column removed. $\tilde{L}$ is invertible if G is connected, regardless of the choice of i, and furthermore we have $\det(\tilde{L}) = |\mathrm{Jac}(G)|$.

In similar fashion, if $D \in \mathrm{Div}^0(G) \subset \mathbb{Z}^n$ then we define the *reduced divisor* $\tilde{D} \in \mathbb{Z}^{n-1}$ to be D with the ith entry deleted. If $D \in \mathrm{Div}^0(G)$, then D is uniquely specified by $n-1$ of its entries, since the final entry must be the negative sum of the other $n-1$ entries.

Likewise, since the all-ones vector generates the kernel of L, for any firing script σ there is a unique firing script σ_0 such that the ith entry of σ_0 is zero and $L\sigma = L\sigma_0$. We define the *reduced firing script* $\tilde{\sigma} \in \mathbb{Z}^{n-1}$ of σ to be σ_0 with the ith entry deleted. Unless specified otherwise, we will always let $i = n$ be the index deleted when referring to a reduced Laplacian, divisor, or firing script. Note that $\widetilde{L\sigma} = \tilde{L}\tilde{\sigma}$ as expected.

Notation 2.1. *Throughout this paper, we will let m denote $|Jac(G)|$. Because we are interested in when $Jac(G)$ is cyclic we will compare $Jac(G)$ with $\mathbb{Z}/m\mathbb{Z}$. When we use the name m as opposed to $|Jac(G)|$ or $\det \tilde{L}$, we are thinking about m as the modulus of $\mathbb{Z}/m\mathbb{Z}$.*

Notation 2.2. *There is a bijection ϕ between the degree zero divisors and the reduced divisors which we denote $\phi(D) = \tilde{D}$ and a bijection ρ between the firing scripts mod the all-ones vector and the reduced firing scripts which we denote $\rho(\sigma) = \tilde{\sigma}$.*

2.2 Monodromy weights

In [15], Shokrieh considers a symmetric, bilinear map from $\mathrm{Jac}(G) \times \mathrm{Jac}(G)$ to $\mathbb{Q}/\mathbb{Z}$ known as the monodromy pairing. To prove our main results, we will utilize related maps from $\mathrm{Jac}(G)$ to $\mathbb{Z}/m\mathbb{Z}$ that we will call monodromy weights. The relationship between

monodromy weights and Shokrieh's monodromy pairing will be made explicit in section 2.3.

Since $\mathrm{Div}^0(G)$ is a subgroup of $\mathrm{Div}(G)$, which is isomorphic to $\mathbb{Z}^n$, any homomorphism $\phi : \mathrm{Div}^0(G) \to \mathbb{Z}/m\mathbb{Z}$ can be written as a dot product $\phi(D) = w \cdot D \pmod m$ of integral vectors, where $D \in \mathrm{Div}^0(G)$ and $w \in \mathbb{Z}^n$. The entries of w correspond to an assignment of integer weights to vertices of G; hence, we call w a *weight vector* on G. There are many possible weight vectors w representing a given homomorphism ϕ: we can add any multiple of m to the weight on a given vertex without changing ϕ, and we can also add a constant vector to w without changing ϕ.

While any arbitrary weight vector $w \in \mathbb{Z}^n$ represents a homomorphism

$$\phi : \mathrm{Div}^0(G) \to \mathbb{Z}/m\mathbb{Z},$$

this homomorphism descends to a well-defined map from $\mathrm{Jac}(G)$ to $\mathbb{Z}/m\mathbb{Z}$ if and only if $w \cdot D \equiv 0 \pmod m$ for all $D \in \mathrm{Prin}(G)$. We call such a weight vector a *monodromy weight* on G. The following proposition provides a method of finding monodromy weights.

Proposition 2.3. *Given a graph G with Laplacian L, a vector w satisfies*

$$Lw \equiv 0 \pmod m$$

if and only if it is a monodromy weight.

Proof. The principal divisors are exactly the divisors that are of the form $L\sigma$ for some firing script σ. Thus w is a monodromy weight if and only if $w \cdot L\sigma \equiv 0 \pmod m$ for all σ. Because L is symmetric this is equivalent to saying $Lw \cdot \sigma \equiv 0 \pmod m$ for all σ. This happens exactly if $Lw \equiv 0 \pmod m$. □

Proposition 2.3 gives us an idea of how to find monodromy weights. However, actually solving the equation is not as simple, since L is singular over the integers. Furthermore, infinitely many monodromy weights represent the same homomorphism from $\mathrm{Jac}(G)$ to $\mathbb{Z}/m\mathbb{Z}$; we would like to identify a set of representatives among the monodromy weights, one for each homomorphism.

If w is a monodromy weight representing a homomorphism ϕ, then by adding a constant vector to w we can make its nth entry zero without changing the homomorphism w represents, obtaining a new monodromy weight w_0. Let $\tilde{w}$ denote the first $n-1$ entries of w_0; we call $\tilde{w}$ a *reduced monodromy weight* representing ϕ. The following two propositions address the first of the problems mentioned above, providing a means of solving for the monodromy weights.

Proposition 2.4. *Let $w \in \mathbb{Z}^n$ be a weight vector. Then w is a monodromy weight if and only if $\tilde{L}\tilde{w} \equiv 0 \pmod m$.*

Proof. Observe that since w and w_0 represent the same homomorphism, $Lw \equiv 0 \pmod m$ if and only if $Lw_0 \equiv 0 \pmod m$. The first $n-1$ entries of Lw_0 are simply $\tilde{L}\tilde{w}_0$. The last row of L is the negative sum of the first $n-1$ rows by the definition of the Laplacian;

hence, if the dot product of w_0 with each of the first $n-1$ rows of L is zero modulo m, the dot product of w_0 with the nth row must also be zero, implying that $Lw_0 \equiv 0 \pmod{m}$. Thus we see that $Lw_0 \equiv 0 \pmod{m}$ if and only if $\tilde{L}\tilde{w}_0 \equiv 0 \pmod{m}$. □

Proposition 2.5. *Any reduced monodromy weight $\tilde{w}$ is a solution to the equation $\tilde{L}\tilde{w} = m\tilde{D}$ for $\tilde{w}$ over the integers for some $D \in \mathrm{Div}^0(G)$.*

Proof. First note that $\tilde{L}$ is invertible over the rationals, so there is a unique solution $\tilde{w} = m\tilde{L}^{-1}\tilde{D}$ for $\tilde{w}$. We have $\tilde{L}\tilde{w} = \tilde{L}m\tilde{L}^{-1}\tilde{D} = m\tilde{D} \equiv 0 \pmod{m}$, so by Proposition 2.4 $\tilde{w}$ is a reduced monodromy weight. □

Proposition 2.5 gives us a concrete way of finding reduced monodromy weights. We now address the second problem mentioned earlier by showing that if we consider reduced monodromy weights modulo m, they correspond bijectively with the elements of $\mathrm{Jac}(G)$.

Proposition 2.6. *Let $D_1, D_2 \in \mathrm{Div}^0(G)$ and let $\tilde{w}_1, \tilde{w}_2$ be derived from D_1, D_2 as described in Proposition 2.5. Then $\tilde{w}_1 \equiv \tilde{w}_2 \pmod{m}$ if and only if $D_1 \sim D_2$.*

Proof. If $D_1 \sim D_2$, then $\tilde{D}_1 = \tilde{L}\tilde{\sigma} + \tilde{D}_2$ for some reduced firing script $\tilde{\sigma}$. It is easy to check that $\tilde{w}_1 = \tilde{w}_2 + m\tilde{\sigma}$, which gives $\tilde{w}_1 \equiv \tilde{w}_2 \pmod{m}$. Conversely, if $\tilde{w}_1 \equiv \tilde{w}_2 \pmod{m}$ then $\tilde{w}_1 = \tilde{w}_2 + mv$ for some $v \in \mathbb{Z}^{n-1}$. By the derivation of $\tilde{w}_1$ and $\tilde{w}_2$, this means that $m\tilde{L}^{-1}(\tilde{D}_1 - \tilde{D}_2) = mv$, so we have $\tilde{D}_1 - \tilde{D}_2 = \tilde{L}v$, which implies that $D_1 \sim D_2$. □

Notation 2.7. *For any square integer matrix A we let C^A be the integer matrix of cofactors of A. In the case of $A = \tilde{L}$ we know that $\det(\tilde{L}) = m \neq 0$ and so $C^{\tilde{L}} = m\tilde{L}^{-1}$.*

Let $\mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$ be the group of homomorphisms $\phi : \mathrm{Jac}(G) \to \mathbb{Z}/m\mathbb{Z}$, let K be the group of reduced monodromy weights taken modulo m. Propositions 2.5 and 2.6 tell us that there is an isomorphsim $F : \mathrm{Jac}(G) \to K$ given by $[D] \mapsto C^{\tilde{L}}\tilde{D} \pmod{m}$, which suggests that $\mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$ may also be isomorphic to $\mathrm{Jac}(G)$. This is indeed the case, and is in fact a manifestation of a far more general and well-known result: any finite abelian group is (non-canonically) isomorphic to its Pontryagin dual.

We now highlight a particular isomorphism from $\mathrm{Jac}(G)$ to $\mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$. Given $\tilde{w} \in K$, let $\phi_{\tilde{w}} \in \mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$ be the associated homomorphism mapping

$$[D] \mapsto \tilde{w} \cdot \tilde{D} \pmod{m};$$

let $\Phi : K \to \mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$ be the map $\tilde{w} \mapsto \phi_{\tilde{w}}$.

Proposition 2.8. *The composition $\Phi \circ F : Jac(G) \to K \to \mathrm{Hom}(Jac(G), \mathbb{Z}/m\mathbb{Z})$ is an isomorphism between $Jac(G)$ and $\mathrm{Hom}(Jac(G), \mathbb{Z}/m\mathbb{Z})$.*

Proof. We already know that F is an isomorphism, so we only need to show that Φ is an isomorphism. The discussion from the beginning of this subsection through Proposition 2.4 shows that Φ is surjective. Towards seeing that Φ is injective, suppose that $\phi_{\tilde{w}_1} = \phi_{\tilde{w}_2}$. Then $(\tilde{w}_1 - \tilde{w}_2) \cdot \tilde{D} \equiv 0 \pmod{m}$ for all $\tilde{D} \in \mathbb{Z}^{n-1}$. Since $\tilde{D}$ can be chosen arbitrarily, this implies that $\tilde{w}_1 - \tilde{w}_2 \equiv 0 \pmod{m}$. □

Note that the uniqueness of the reduced monodromy weight in K representing a given homomorphism $\phi \in \mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$ implies that any two monodromy weights represent the same homomorphism if and only if their difference taken modulo m is a constant vector. Proposition 2.8 can also be used to obtain information about the cardinality of a minimum generating set for $\mathrm{Jac}(G)$. For any abelian group Γ we let $\mathrm{rk}(\Gamma)$ denote the minimum cardinality of a generating set for Γ. For any integer matrix A let $\mathrm{rk}_m(A) := \mathrm{rk}(\mathrm{Im}(A$ reduced modulo $m))$, which is the "rank of A over $\mathbb{Z}/m\mathbb{Z}$", and let $\mathrm{nul}_m(A) := \mathrm{rk}(\mathrm{Ker}(A$ reduced modulo $m))$, which is the "nullity of A over $\mathbb{Z}/m\mathbb{Z}$". Then we can derive the following corollary.

Corollary 2.9. *Over $\mathbb{Z}/m\mathbb{Z}$, we have $nul_m(\tilde{L}) = rk_m(C^{\tilde{L}}) = rk(Jac(G))$.*

Proof. In the context of this proof, we will consider all matrices and vectors to be taken modulo m. We know that $F : \mathrm{Jac}(G) \to K$ given by $[D] \mapsto C^{\tilde{L}}\tilde{D} \pmod m$ is an isomorphism, so $\mathrm{rk}(\mathrm{Jac}(G)) = \mathrm{rk}(K) = \mathrm{rk}(\mathrm{Im}(F)) = \mathrm{rk}_m(C^{\tilde{L}})$. If $\tilde{u} \in \ker \tilde{L}$, then by Proposition 2.4 $\tilde{u} \in K$, so by Proposition 2.8 $\tilde{u} \in \mathrm{Im}\, C^{\tilde{L}}$. Conversely, if $\tilde{u} \in \mathrm{Im}\, C^{\tilde{L}}$, then over $\mathbb{Z}^{n-1}$ we have $\tilde{u} + m\tilde{v} = m\tilde{L}^{-1}\tilde{w}$ for some $\tilde{v}, \tilde{w} \in \mathbb{Z}^{n-1}$. Thus $\tilde{L}\tilde{u} = m(\tilde{w} - \tilde{L}\tilde{v}) \equiv 0 \pmod m$, so $\tilde{u} \in \ker \tilde{L}$. Consequently, we have $\ker \tilde{L} = \mathrm{Im}\, C^{\tilde{L}}$, which implies that $\mathrm{nul}_m(\tilde{L}) = \mathrm{rk}_m(C^{\tilde{L}})$. □

Example 2.10. *Let G be the cycle graph on six vertices. Note that $Jac(G) \cong \mathbb{Z}/6\mathbb{Z}$, so $m = 6$. $\tilde{L}$ and $C^{\tilde{L}}$ are the following over $\mathbb{Z}/6\mathbb{Z}$:*

$$\tilde{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad C^{\tilde{L}} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 2 & 0 & 4 & 2 \\ 3 & 0 & 3 & 0 & 3 \\ 2 & 4 & 0 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Let col_i denote the ith column of $C^{\tilde{L}}$. Observe that modulo 6, we have $col_i = i \cdot col_1$, which implies that $rk_m(C^{\tilde{L}}) = 1$. Since $Jac(G)$ is cyclic, it has a generating set of cardinality 1, so likewise $rk(Jac(G)) = 1$. Finally, it can easily be verified that the kernel of $\tilde{L}$ over $\mathbb{Z}/6\mathbb{Z}$ is generated by the vector $v = (1, 2, 3, 4, 5)$, so $nul_m(\tilde{L}) = 1$ as well.

2.3 Relationship to the monodromy pairing

We now make the relationship between monodromy weights and the monodromy pairing explicit. Let M be any generalized inverse of the Laplacian matrix. Shokrieh's pairing is given by $\langle \cdot\,, \cdot \rangle : \mathrm{Jac}(G) \times \mathrm{Jac}(G) \to \mathbb{Q}/\mathbb{Z}$, evaluated as follows:

$$\langle D_1, D_2 \rangle = D_1^T M D_2.$$

Using the isomorphism $\Phi \circ F : \mathrm{Jac}(G) \to \mathrm{Hom}(\mathrm{Jac}(G), \mathbb{Z}/m\mathbb{Z})$ we can define a map $\varphi : \mathrm{Jac}(G) \times \mathrm{Jac}(G) \to \mathbb{Z}/m\mathbb{Z}$ by $([D_1], [D_2]) \mapsto \Phi \circ F([D_1])([D_2])$. The map can be computed

as follows. Let D_1, D_2 be representatives of elements of $\mathrm{Jac}(G)$. Then $\varphi([D_1],[D_2])$ is given by

$$\tilde{D}_1^T C^{\tilde{L}} \tilde{D}_2 \pmod{m}.$$

In this computation, the reduction modulo m of $C^{\tilde{L}}\tilde{D}_2$ is $F([D_2])$, and multiplying by $\tilde{D}_1^T$, i.e. taking the dot product with $\tilde{D}_1$, is then just the application of the corresponding homomorphism to $[D_1]$.

This map is, in essence, the same as the map defined by Shokrieh, differing only by a multiplicative factor of m and the use of any generalized inverse in the monodromy pairing instead of $\tilde{L}^{-1}$. Since the matrix obtained from $\tilde{L}^{-1}$ by making the nth row and columns all 0 is a generalized inverse of L, we see that the two pairings are really the same.

3 Edge deletion and insertion

In this section, we provide the proofs of the two main theorems from the introduction. Recall from section 2 that $m = |\mathrm{Jac}(G)| = \det \tilde{L}$. First we provide a lemma relating the order of a divisor to the monodromy weights.

Lemma 3.1. *Let $D \in \mathrm{Div}^0(G)$ be a divisor, $\tilde{w} = C^{\tilde{L}}\tilde{D} \in \mathbb{Z}^{n-1}$ with each component taking a value at least 0 and less than m. Let $\phi : \mathbb{Z}^{n-1} \to \mathbb{Z}/m\mathbb{Z}$ be the map induced by inner product with $\tilde{w}$. Then*

$$|[D]|_{Jac(G)} = |Im(\phi)| = m/\gcd(m, \tilde{w}) = \det \tilde{L}/\gcd(\det \tilde{L}, C^{\tilde{L}}\tilde{D})$$

where $\gcd(m, \tilde{w})$ denotes the gcd of m and the entries of the vector $\tilde{w}$, and $|\cdot|_{Jac(G)}$ gives the order of an element of $Jac(G)$.

Proof. The order of $[D]$ is the smallest integer k such that the divisor kD is linearly equivalent to the zero divisor $\mathbf{0}$. Thus we have $kD - L\sigma = \mathbf{0}$ for some firing script σ. Thus $\tilde{L}\tilde{\sigma} = k\tilde{D}$, and multiplying both sides by $\tilde{L}^{-1}$ gives $k\tilde{L}^{-1}\tilde{D} = \tilde{\sigma}$. Since σ is a firing script, it must be an integer vector, thus k is the smallest integer such that $k\tilde{L}^{-1}\tilde{D}$ is an integer vector.

Then since k is minimal, the gcd of the entries of $\tilde{\sigma}$ and k is 1. Since $\tilde{w} = C^{\tilde{L}}\tilde{D}$, we can write $\tilde{w} = (m/k)\tilde{\sigma}$, and thus $\gcd(m, \tilde{w}) = m/k$. So, $\mathrm{Im}(\phi)$ is the subgroup of $\mathbb{Z}/m\mathbb{Z}$ generated by (m/k) since we can write any multiple of m/k as an inner product of a reduced divisor with $\tilde{w}$ by Bezout's identity, where the reduced divisor defines a linear combination of the entries of $\tilde{w}$. Therefore, $|\mathrm{Im}(\phi)| = m/(m/k) = k = m/\gcd(m, \tilde{w})$.

The last equality follows because $\tilde{w} = C^{\tilde{L}}\tilde{D}$. □

Before proving our first main theorem, we provide one last lemma.

Lemma 3.2. *Let G be a multigraph with at least three vertices and with an edge e between the vertices x and y. Let $\tilde{L}$ be the Laplacian of G reduced by the vertex x. Then $C^{\tilde{L}}_{yy}$ is the number of spanning trees of G that include the edge e between x and y.*

Lemma 3.2 is an immediate consequence of Theorem 4.7 in [2], which is a generalization of Kirchhoff's Matrix Tree Theorem [10]. Note that if there is no edge between vertices x and y in G, then $C_{yy}^{\tilde{L}}$ is equal to the number of new spanning trees which G would gain if an edge were added between x and y.

We can use these lemmas to answer the question of when a divisor supported on only two vertices is a generator. The following theorem provides and generalizes an affirmative answer to a question by Becker and Glass [3, Open Question 2.8] based off of a theorem of Lorenzini [12, 5.1].

Theorem 3.3. *Let G be a connected multigraph, and G_1 the multigraph obtained by deleting any integer k_{xy} of the edges between the vertices x and y (where negative k_{xy} represents adding edges). Let S be the subgroup of $Jac(G)$ generated by $[\delta_{xy}]$. Then*

$$[Jac(G) : S] \mid \gcd(|Jac(G)|, |Jac(G_1)|).$$

Note that since we allow k_{xy} to be positive or negative, the result would be the same if we took S to be the subgroup of $\mathrm{Jac}(G_1)$ generated by $[\delta_{xy}]$.

Proof. First suppose that k_{xy} is non-negative so that G_1 is obtained by deleting edges from G. As in Lemma 3.2, let x correspond to the row and column in our definition of the reduced Laplacian of both G denoted $\tilde{L}$ and G_1 denoted $\tilde{L}_1$. By Lemma 3.2, we have that $C_{yy}^{\tilde{L}}$ is the number of spanning trees using a specific x, y edge. So, $k_{xy}C_{yy}^{\tilde{L}}$ gives the number of spanning trees using any of the k_{xy} edges that are removed to form G_1. The number of spanning trees of G is the sum of the number of spanning trees of G_1 and the number of spanning trees including any one of the k_{xy} edges removed from G to form G_1. Thus, by the matrix tree theorem:

$$\det \tilde{L} = \det \tilde{L}_1 + k_{xy}C_{yy}^{\tilde{L}}.$$

If k_{xy} is negative, then we can view G as obtained from G_1 by removing $-k_{xy}$ edges. So the above computation shows that $\det \tilde{L}_1 = \det \tilde{L} + (-k_{xy})C_{yy}^{\tilde{L}_1}$. Since $\tilde{L}_1$ and $\tilde{L}$ differ only in the (y, y)th entry, we have $C_{yy}^{\tilde{L}_1} = C_{yy}^{\tilde{L}}$. Putting these together we see that $\det \tilde{L} = \det \tilde{L}_1 + k_{xy}C_{yy}^{\tilde{L}}$. So we have shown that for any integer k_{xy}, $\det \tilde{L} = \det \tilde{L}_1 + k_{xy}C_{yy}^{\tilde{L}_1}$.

Moreover, $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = \gcd(\det \tilde{L}, \det \tilde{L}_1)$, so that

$$\gcd(\det \tilde{L}, C_{yy}^{\tilde{L}}) \mid \gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|).$$

Note that this statement would be equality if $k_{xy} = \pm 1$, which is guaranteed if we force G to be simple.

Now let $w = C^{\tilde{L}}\tilde{\delta}_{xy} = C_{yy}^{\tilde{L}}\tilde{\delta}_{xy}$, noting that the reduced δ_{xy} is just the indicator vector of vertex y. Then, by Lemma 3.1 we have that

$$|[\delta_{xy}]|_{\mathrm{Jac}(G)} = \det \tilde{L} / \gcd(\det \tilde{L}, C^{\tilde{L}}\tilde{\delta}_{xy}).$$

Putting these facts together, we get that

$$[\mathrm{Jac}(G) : S] = \frac{\det \tilde{L}}{|[\delta_{xy}]|_{\mathrm{Jac}(G)}} = \gcd(\det \tilde{L}, C^{\tilde{L}}\tilde{\delta}_{xy}) \;\Big|\; \gcd(\det \tilde{L}, C^{\tilde{L}}_{yy}) \;\Big|\; \gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|).$$ □

Following the statement of this theorem, it is natural to ask if the converse is also true. That is, if $[\delta_{xy}]$ is a generator of the Jacobian, are the orders of the Jacobians of G and G_1 relatively prime? The following theorem answers this question.

Theorem 3.4. *Let G be a connected multigraph, and G_1 the graph obtained by removing k_{xy} edges between x and y, where a negative value of k_{xy} corresponds to adding edges. Let S be the subgroup of $Jac(G)$ generated by $[\delta_{xy}]$. Moreover, assume that $\gcd(|Jac(G)|, k_{xy}) = 1$ (which is guaranteed if G is simple, since $k_{xy} = \pm 1$). Then*

$$\gcd(|Jac(G)|, |Jac(G_1)|) \;\Big|\; [Jac(G) : S]^2.$$

Proof. As in Theorem 3.3 we choose $\tilde{L}$ to be the reduced Laplacian, reducing at vertex x. Recall that $m = |\mathrm{Jac}(G)|$.

Note that since $\gcd(m, k_{xy}) = 1$ and $\det \tilde{L} = \det \tilde{L}_1 + k_{xy} C^{\tilde{L}}_{yy}$ as in the proof of the previous theorem, we have that

$$\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = \gcd(m, |C^{\tilde{L}}_{yy}|). \tag{1}$$

Let $\tilde{w} = C^{\tilde{L}}\tilde{\delta}_{xy}$, where $C^{\tilde{L}}$ is the cofactor matrix of the reduced Laplacian. Then let $\phi : \mathrm{Jac}(G) \to \mathbb{Z}/m\mathbb{Z}$ be the map induced by inner product with w. Now, note that ϕ is a group homomorphism, so $m = |\mathrm{Im}(\phi)| \cdot |\ker(\phi)|$. So, we also have that

$$|\ker(\phi)| = m/|\mathrm{Im}(\phi)|. \tag{2}$$

Now, let $\phi_S : S \to \mathbb{Z}/m\mathbb{Z}$ be the map ϕ restricted to the subgroup generated by $[\delta_{xy}]$. Note that since $w = C^{\tilde{L}}\tilde{\delta}_{xy}$, we have that the weight on the yth vertex is exactly $C^{\tilde{L}}_{yy}$. Thus the image of S (the subgroup consisting of multiples of δ_{xy}, ie multiples with weight only on y in the reduced Laplacian) under the map ϕ is the subgroup consisting of multiples of $\gcd(m, |C^{\tilde{L}}_{yy}|)$ in $\mathbb{Z}/m\mathbb{Z}$. The order of this subgroup is $m/\gcd(m, |C^{\tilde{L}}_{yy}|) = |\mathrm{Im}(\phi_S)|$, and thus

$$\gcd(m, |C^{\tilde{L}}_{yy}|) = m/|\mathrm{Im}(\phi_S)|. \tag{3}$$

Observe that this is similar to but not the same as Lemma 3.1, since now we have $\gcd(m, |C^{\tilde{L}}_{yy}|)$ in the denominator rather than $\gcd(m, \tilde{w})$.

Since φ is also a homomorphism, we have $|S| = |\mathrm{Im}(\phi_S)| \cdot |\ker(\phi_S)|$. This gives us that

$$|\mathrm{Im}(\phi_S)| = |S|/|\ker(\phi_S)|. \tag{4}$$

The last fact we need is that $\ker(\phi_S)$ is a subgroup of $\ker(\phi)$. Thus, we have that for some integer c that

$$|\ker(\phi_S)| = |\ker(\phi)|/c. \tag{5}$$

Now we have that by (1), (3), and (4)

$$\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = \gcd(m, |C_{yy}^{\tilde{L}}|) = \frac{m}{|\mathrm{Im}(\phi_S)|} = \frac{m}{|S|/|\ker(\phi_S)|}.$$

By lemma 3.1 we have that $|S| = |\mathrm{Im}(\phi)|$. So, with (5) and (2) we conclude that

$$\begin{aligned}\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) &= \frac{m}{|\mathrm{Im}(\phi)|/|\ker(\phi_S)|} = \frac{m}{|\mathrm{Im}(\phi)|}|\ker(\phi_S)| \\ &= \frac{m}{|\mathrm{Im}(\phi)|}\left(\frac{|\ker(\phi)|}{c}\right) = \frac{m}{|\mathrm{Im}(\phi)|}\left(\frac{m/|\mathrm{Im}(\phi)|}{c}\right) \\ &= \frac{(m/|\mathrm{Im}(\phi)|)^2}{c} = \frac{[\mathrm{Jac}(G) : S]^2}{c}.\end{aligned}$$

This proves the theorem for G and S, and in addition, the multiplicative factor is the index of $\ker(\phi|_S)$ in $\ker(\phi)$. □

From the above theorems we can now give a precise condition for when $[\delta_{xy}]$ is a generator of $\mathrm{Jac}(G)$ by considering the case when $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G_1)|) = 1$, proving Corollary 1.2 from the introduction.

4 Edge contraction

Theorems 3.3 and 3.4 shed some light on the behavior of the Jacobian when an edge is removed from the graph. It would be natural to ask what happens when a vertex is removed. This is the subject of the following corollary and proposition resulting from our main theorem.

Corollary 4.1. *Let G be a simple graph and let $e = (x, y)$ be an edge. Let G/e denote the graph obtained by identifying the vertices x and y. Let S be as defined in Theorem 3.3. Then $[Jac(G) : S] \mid gcd(|Jac(G)|, |Jac(G/e)|)$.*

Proof. Let $T(G)$ denote the number of spanning trees of G. As explained in [11], $T(G)$ obeys the following recurrence: $T(G) = T(G - e) + T(G/e)$, where $G - e$ denotes the graph obtained from G upon removal of the edge e. Thus $\gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G/e)|) = \gcd(|\mathrm{Jac}(G)|, |\mathrm{Jac}(G - e)|)$, and the result follows by Theorem 3.3. □

Remark 4.2. *This corollary essentially is saying that $T(G/(x, y)) = C_{yy}^{\tilde{L}}$ where $\tilde{L}$ is L with the x row and column deleted. This can be seen by comparing the above recurrence with the equation for $\det \tilde{L}$ in the proof of Theorem 3.3.*

It would then be natural to ask whether $\mathrm{Jac}(G/e)$ is cyclic if the orders of the Jacobians of G and G/e are relatively prime. Theorem 5.1 of [12] proves that in this case $\mathrm{Jac}(G)$ is cyclic, and in Lemma 6.2 of [13] this result is extended to show that $\mathrm{Jac}(G-e)$ is cyclic. We now apply Theorem 3.3 to show that on an undirected graph, $\mathrm{Jac}(G/e)$ must be cyclic as well.

Proposition 4.3. *Let G be a simple graph, and e, G/e defined as above. Let z be the vertex obtained by identifying x and y. If $|Jac(G)|$ and $|Jac(G/e)|$ are relatively prime and*

$$D_x(v) = \begin{cases} 0 & (v,x) \notin E(G) \\ (val_G(x)-1) & v = z \\ -1 & (v,x) \in E(G), v \neq z, \end{cases}$$

then $Jac(G/e)$ is cyclic and $[D_x]$ generates it.

Proof. Consider some arbitrary $D' \in \mathrm{Div}^0(G/e)$ and let $D \in \mathrm{Div}^0(G)$ such that $\forall v \neq x, y$, $D(v) = D'(v)$. Then since the values sum to 0, we have $D'(z) = D(x) + D(y)$. Consider the firing script σ taking D to an equivalent multiple of δ_{xy}, and let $\sigma(v)$ denote the number of times the vertex v is fired. Without loss of generality let $\sigma(x) = 0$. Now let σ' be a set of firing moves on G/e such that $\sigma'(z) = \sigma(y)$ and $\sigma'(v) = \sigma(v)$ otherwise. Let D'_1 be the divisor obtained from D' after the firing of σ'. Then we have

$$\begin{aligned} D'_1(v) &= D'(v) - \mathrm{val}_{G/e}(v)\sigma'(v) + \sum_{(u,v)\in G/e} \sigma'(u) \\ &= \begin{cases} 0 & (v,x) \notin G \\ -\sigma'(z)(\mathrm{val}_G(x)-1) & v = z \\ \sigma'(z) & (v,x) \in G, v \neq z. \end{cases} \end{aligned}$$

Thus $D'_1 = -\sigma'(z)D_x$, where D_x is as defined in the statement of the proposition.

Thus every $D' \in \mathrm{Div}^0(G/e)$ is a equivalent to a multiple of D_x. Hence $\mathrm{Jac}(G/e)$ is generated by $[D_x]$, thus it is cyclic. □

Note that the form of D_x depends on a choice of ordering of x and y. The following remark gives the relation between the generators if the ordering is reversed.

Remark 4.4. *Let D_y be the divisor obtained by reversing the ordering of x and y such that $[D_y]$ generates $Jac(G)$. Then $D_x \sim -D_y$. To see this, note that*

$$D_y(v) = \begin{cases} 0 & (v,y) \notin E(G) \\ (val_G(y)-1) & v = z \\ -1 & (v,y) \in E(G), v \neq z. \end{cases}$$

Firing once at z gives us

$$D'_y(v) = \begin{cases} 0 & (v,x) \notin E(G) \\ -(val_G(x)-1) & v = z \\ 1 & (v,x) \in E(G), v \neq z, \end{cases}$$

which is exactly $-D_x(v)$.

5 Bounding the order of $[\delta_{xy}]$ below

The following proposition bounds the maximal order of some $[\delta_{xy}]$.

Proposition 5.1. *Let G be a connected multigraph with $n = |V(G)|$ vertices and $\epsilon = |E(G)|$ edges.*

1. *There exist vertices x, y connected by an edge e such that $|[\delta_{xy}]|_{Jac(G)} \geqslant \epsilon/(n-1)$.*
2. *Moreover, if G is 2-edge-connected, then there exist vertices x, y connected by an edge e such that $|[\delta_{xy}]|_{Jac(G)} \geqslant \epsilon/(\epsilon - n + 1)$.*

Note that in the first inequality the bound is tight for a spanning tree on n vertices, and in the second the bound is tight for an n-cycle.

Proof. Recall from Lemma 3.2 that $C^{\tilde{L}}_{yy}$ gives exactly the number of spanning trees of G containing the edge e. There are $\det \tilde{L}$ spanning trees, each containing $n-1$ edges, thus there are a total of $(n-1)\det \tilde{L}$ instances of edges over all spanning trees of G. Since there are ϵ edges in total, using the pigeonhole principle, we obtain the following bound: there exists an edge $e = (x, y)$ such that $C^{\tilde{L}}_{yy} \leqslant (\det \tilde{L})(n-1)/\epsilon$. By Theorem 3.3, we have $|\mathrm{Jac}(G)|/|[\delta_{xy}]|_{\mathrm{Jac}(G)} \leqslant \gcd(\det \tilde{L}, C^{\tilde{L}}_{yy}) \leqslant C^{\tilde{L}}_{yy} \leqslant (\det \tilde{L})(n-1)/\epsilon$. The first of the above inequalities follows.

For the second, since G is 2-edge-connected, for all vertices x, y connected by an edge e we have $\det \tilde{L} > C^{\tilde{L}}_{yy}$ since there exists some spanning tree not containing the edge e. Using the pigeonhole principle again, we can bound $C^{\tilde{L}}_{yy}$ from below: there are some vertices x, y connected by an egde e such that $\det \tilde{L} > C^{\tilde{L}}_{yy} \geqslant (\det \tilde{L})(n-1)/\epsilon$. Thus $\gcd(\det \tilde{L}, C^{\tilde{L}}_{yy}) \leqslant \det \tilde{L} - C^{\tilde{L}}_{yy} \leqslant \det \tilde{L} - (\det \tilde{L})(n-1)/\epsilon$ and the second inequality follows. □

With stronger assumptions on G we can find better bounds on the order of $[\delta_{xy}]$. A graph is *biconnected* if any vertex can be removed while leaving a connected graph on $n-1$ vertices, which is equivalent to saying the graph cannot be decomposed into the wedge product of two proper subgraphs that are not just vertices. Lemma 27 of [8] gives a lower bound for $|[\delta_{xy}]|_{\mathrm{Jac}(G)}$ in a biconnected graph as $\mathrm{val}(x)$, and the maximum valency of any vertex is bounded from below by $2\epsilon/n$. This is almost always better than the bounds of Corollary 5.1, but ours hold more generally. In fact, for a biconnected graph we are able to strengthen the result of [8].

Proposition 5.2. *Given a biconnected simple graph G and an edge (x, y),*

$$|[\delta_{xy}]|_{Jac(G)} \geqslant val(x) + \frac{val(x) - 1}{val(y) - 1}.$$

This is always stronger than the result of [8], and equality is attained when G is a complete graph.

Proof. Let a be a positive integer such that $a\delta_{xy} \sim 0$, and let σ be the firing script taking $a\delta_{xy}$ to 0. Let S be the set of vertices s such that $\sigma(s)$ is minimum. Without loss of generality by Convention 2.2, we can set $\sigma(s) = 0$ for all $s \in S$. We have 3 cases:
Case 1: $y \in S$
This case is not possible since the initial value on y is a. If $\sigma(y) = 0$, σ can only increase the value of the divisor on y, thus in this case we cannot have that $a\delta_{xy} + L\sigma = \mathbf{0}$.
Case 2: $\exists\ z \in S, z \neq x, y$.
In this case z does not fire, and hence for the value of the divisor to remain at 0 we require that all neighbors of z do not fire as well. We repeat this argument for all neighbors of z that are not x or y. Since G is biconnected there exists a path from each vertex z to y not passing through x. Thus we eventually conclude that $\sigma(y) = 0$. This is a contradiction since we initially assumed that $\sigma(y) > 0$.
Case 3: $S = \{x\}$.
We can assume that for all $v \neq x$, $\sigma(v) > 0$, since otherwise this would reduce to one of the earlier cases. Thus each neighbor of x fires at least once. For the value on y to be zero after firing, y must fire at least $a/\mathrm{val}(y)$ times. In fact, each neighbor of y that is not x must fire at least once (else we have Case 2), thus y needs to fire at least $(a + \mathrm{val}(y) - 1)/\mathrm{val}(y)$ times. Since $a = \sum \sigma(v)$ over all neighbors v of x, we have $a \geqslant \mathrm{val}(x) - 1 + (a + \mathrm{val}(y) - 1)/\mathrm{val}(y) = \mathrm{val}(x) + (a - 1)/\mathrm{val}(y)$. Further manipulation gives the result. □

Remark 5.3. *Much of this proof follows immediately from the fact that σ is an integer-valued harmonic function on the graph G with source y and sink x. This necessarily implies that y and x are the unique maximum and minimum of σ respectively. The bound follows soon after.*

We have a similar result for multigraphs.

Proposition 5.4. *Given a biconnected multigraph G and vertices x, y connected by an edge,*

$$|[\delta_{xy}]|_{Jac(G)} \geqslant (val(x) - 1)\frac{val(y)}{val(y) - 1}.$$

Proof. The proof proceeds similarly, except this time we can only bound the number of times y fires by $a/\mathrm{val}(y)$. The earlier bound does not hold, since firing on other vertices may not increase the number of chips on y. This occurs when all of y's edges are connected to x, such as on a multigraph with two vertices x and y and multiple edges between them. Thus we have $a \geqslant \mathrm{val}(x) - 1 + a/\mathrm{val}(y)$. Solving this gives the result. □

Note that as long as x is chosen to be the vertex with higher valency, this result is always stronger than that of [8].

6 Conjectures and experimental results on random graphs

6.1 Random graph conjectures

As in the introduction, we consider Erdős-Rényi random graphs $G_{n,p}$ on n vertices where each edge occurs with probability p. One of the original motivations of our research was to use facts about generators to find a lower bound on the probability that the Jacobian of a graph is cyclic. As mentioned previously, it has been proven in [17] that $\prod_{i=1}^{\infty} \zeta(2i+1)^{-1}$ is an upper bound on the probability. However, no lower bound is known. It is conceivable that better understanding of when $[\delta_{xy}]$ generates the Jacobian will shed light on a lower bound.

By Corollary 1.2 we have that $[\delta_{xy}]$ is a generator of $\mathrm{Jac}(G)$ exactly when $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G_1)|$ are relatively prime. The question then becomes how the orders of these two groups depend on each other. Two positive integers chosen uniformly at random from the integers less than n are coprime with probability $\zeta(2)^{-1}$ as n goes to infinity [9]. We know that $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G_1)|$ are not independent. For example, if $\mathrm{Jac}(G)$ is not cyclic, Corollary 1.2 tells us that the orders cannot be relatively prime. However, when $\mathrm{Jac}(G)$ is cyclic, there is no obvious relationship between $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G_1)|$. So, the question becomes whether $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G_1)|$ behave like random integers when $\mathrm{Jac}(G)$ is cyclic and we choose an edge (x, y) at random to remove (or add if it is not an edge of the graph). Performing various simulations led us to the following conjecture.

Conjecture 6.1. *Let p be fixed. For each graph, fix vertices x and y. Then*

$$\lim_{n\to\infty} \mathbb{P}\left([\delta_{xy}] \text{ generates } Jac(G_{n,p}) \mid Jac(G_{n,p}) \text{ cyclic}\right) = \zeta(2)^{-1} \approx 0.607927$$

If this is true and there is a reasonable amount of independence across choices of x and y, we would expect to be able to find some edge x, y that allows us to make $|\mathrm{Jac}(G)|$ and $|\mathrm{Jac}(G_1)|$ coprime. This leads us to the following conjecture.

Conjecture 6.2. *For any fixed p,*

$$\lim_{n\to\infty} \mathbb{P}\left(\exists\ [\delta_{xy}] \text{ generator of } Jac(G_{n,p}) \mid Jac(G_{n,p}) \text{ cyclic}\right) = 1$$

Or equivalently,

$$\lim_{n\to\infty} \mathbb{P}\left(\exists\ [\delta_{xy}] \text{ generator of } Jac(G_{n,p})\right) = \lim_{n\to\infty} \mathbb{P}\left(Jac(G_{n,p}) \text{ cyclic}\right)$$

This conjecture says that the existence of a $[\delta_{xy}]$ generator is almost surely equivalent to $\mathrm{Jac}(G_{n,p})$ having a cyclic Jacobian. Interestingly, Conjecture 6.1 (if true) would seem to imply that any fixed divisor of the form δ_{xy} is as likely to generate $\mathrm{Jac}(G)$ as a group element of $\mathrm{Jac}(G)$ chosen uniformly at random (as experiments and heuristics suggest that this latter probability should also converge to $\zeta(2)^{-1}$). Although this would mean such divisors are not especially likely to be generators, it at least seems to assure us that if we wanted to analyze the behavior of random elements of $\mathrm{Jac}(G)$, the elements of the form δ_{xy} seem to already provide a reasonable approximation. Lastly, note that our conjectures do not depend on p, as long as p is fixed. This follows [17], [5], and [6] which give results about the likelihood of cyclic Jacobians in random graphs that do not depend on p.

6.2 Experimental data

To provide some support beyond intuition for Conjectures 6.1 and 6.2, we conducted simulations using SageMath [16] for various values of n (available on GitHub at [4]). For each n we conducted 10,000 trials where each trial consisted of generating $G_{n,p}$ with a cyclic Jacobian and checking the desired property.

To sample G from the conditional distribution of $G_{n,p}$ with cyclic Jacobian, each trial we repeatedly generated copies of $G_{n,p}$ until we found a G with cyclic Jacobian. The desired properties were then checked by using the Smith normal form to calculate the orders of $\mathrm{Jac}(G)$ and $\mathrm{Jac}(G_1)$ and applying Corollary 1.2. In the below tables, we report the probability that these properties held in our experiments of 10,000 trials for various values of p and n.

$\mathbb{P}(\text{fixed } [\delta_{xy}] \text{ generates } \mathrm{Jac}(G))$

n	$p = 0.1$	$p = 0.5$	$p = 0.9$
20	0.4976	0.6121	0.2927
40	0.5857	0.6092	0.5331
60	0.6089	0.6073	0.6036

$\mathbb{P}(\exists\ [\delta_{xy}] \text{ that generates } \mathrm{Jac}(G))$

n	$p = 0.1$	$p = 0.5$	$p = 0.9$
20	0.9979	1.0	0.9600
40	1.0	1.0	0.9991
60	1.0	1.0	1.0

Here G is a graph sampled from the conditional distribution of $G_{n,p} \mid \mathrm{Jac}(G_{n,p})$ cyclic. Note that the values of 1.0 indicate that the property held for every trial in our sample. So, while there are simple examples that can be constructed for every n with no $[\delta_{xy}]$ generator, they are unlikely to be generated by $G_{n,p}|\mathrm{Jac}(G_{n,p})$ cyclic and therefore did not occur in our sample.

6.3 Conjecture on the order of $[\delta_{xy}]$

In Section 5, we proved some lower bounds on the order of some $[\delta_{xy}]$ in the graph. However, by experimentation these do not seem to be the best possible lower bounds. We conjecture that a better bound can be found if we assert that a graph is biconnected. This condition prevents the construction of pathological counterexamples using wedge products and gives, based on computations of Dhruv Ranganathan and Jeffrey Yu [14], the following conjecture.

Conjecture 6.3. *Let G be a biconnected, simple graph with n vertices. Fix a vertex x. Then there exists a vertex y such that $|[\delta_{xy}]|_{Jac(G)} \geqslant n$.*

We further tested this conjecture on random graphs of varying size and on all biconnected graphs with fewer than nine vertices. We could not find any counterexamples. One way to see why the biconnected assumption is necessary is to let G be the wedge of two triangles, a "bowtie". Then $|V(G)| = 5$ but $\mathrm{Jac}(G) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, so $|[\delta_{xy}]|_{\mathrm{Jac}(G)} \leqslant 3$, so the desired result would not hold for G.

An affirmative resolution of this conjecture could provide some leverage over finding classes of graphs with cyclic Jacobian. Moreover, following the work of [8], this conjecture would prove that for any positive integer n there exists an integer k_n such that for all $k > k_n$ there is no biconnected graph G with $\mathrm{Jac}(G) \cong (\mathbb{Z}/n\mathbb{Z})^k$.

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