Linear Polychromatic Colorings of Hypercube Faces

Evan Chen

Department of Mathematics
Massachusetts Institute of Technology, U.S.A.

evanchen@mit.edu

Submitted: Sep 14, 2016; Accepted: Dec 19, 2017; Published: Jan 12, 2018
Mathematics Subject Classifications: 05C15, 05C35

Abstract

A coloring of the ℓ-dimensional faces of Q_n is called d-polychromatic if every embedded Q_d has every color on at least one face. Denote by $p^\ell(d)$ the maximum number of colors such that any Q_n can be colored in this way. We provide a new lower bound on $p^\ell(d)$ for $\ell > 1$.

Keywords: polychromatic, coloring, hypercube

1 Introduction

Denote by Q_n the n-dimensional hypercube on 2^n vertices.

Definition 1.1. For $\ell \geqslant 0$, a Q_ℓ*-coloring* of Q_n is a coloring of each of the the ℓ-dimensional faces of Q_n with one of $r \geqslant 1$ colors. For $d \geqslant \ell$, such a coloring is called *d-polychromatic* if every embedded Q_d contains all r colors.

For $d \geqslant \ell \geqslant 1$, we denote by $p^\ell(d)$ the maximum r for which a d-polychromatic Q_ℓ-coloring is possible on every hypercube Q_n, for all $n \geqslant d$.

The case $\ell = 1$ was first introduced in 2007 by Alon, Krech, and Szabó in [1]. They prove the following result.

Theorem ([1, Theorem 4])**.** *For any $d \geqslant 1$,*

$$\binom{d+1}{2} \geqslant p^1(d) \geqslant \left\lfloor \frac{(d+1)^2}{4} \right\rfloor .$$

The lower bound is done through a construction which in this paper will be called the *basic construction*, described in Section 2. It was then shown by Offner in 2008 that in fact this construction is sharp.

Theorem ([4])**.** *For any $d \geqslant 1$, we have*

$$p^1(d) = \left\lfloor \frac{(d+1)^2}{4} \right\rfloor .$$

Alon, Krech, and Szabó also suggest in [1] the problem of examining $p^\ell(d)$. In 2015, Ozkahya and Stanton [5] gave a direct generalization of the basic construction to prove the following.

Theorem ([5])**.** *For any $d, \ell \geqslant 1$, let $0 < r \leqslant \ell+1$ be such that $r \equiv d+1 \pmod{\ell+1}$. Then*

$$\binom{d+1}{\ell+1} \geqslant p^\ell(d) \geqslant \left\lceil \frac{d+1}{\ell+1} \right\rceil^r \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{\ell+1-r}.$$

Henceforth, denote the right-hand side as

$$p^\ell_{\text{bas}}(d) := \left\lceil \frac{d+1}{\ell+1} \right\rceil^r \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{\ell+1-r}$$

for brevity. For $\ell = 1$ this coincides with the result of [1]. It is then natural to wonder whether an analog of Offner's result holds for $\ell > 1$. In a few small cases it was recently shown this is not the case; Goldwasser, Lidicky, Martin, Offner, Talbot and Young prove in [3] the following result.

Theorem ([3, Theorems 20 and 21])**.** *We have $p^2(3) = 3$ and $p^2(4) \geqslant 5$.*

In contrast $p^2_{\text{bas}}(3) = 2$ and $p^2_{\text{bas}}(4) = 4$.

In the present paper we show the following more general result.

Theorem 1.2. *For $d \geqslant 4$, we have*

$$p^2(d) \geqslant \begin{cases} (k^2+1)(k+1) & d = 3k \\ (k^2+k+1)(k+1) & d = 3k+1 \\ (k^2+k+1)(k+2) & d = 3k+2. \end{cases}$$

In particular,

$$p^2(d) > p^2_{\text{bas}}(d).$$

Our construction is by a so-called *linear coloring*, defined at the end of Section 2. For concreteness, Table 1 lists the values of the construction for $4 \leqslant d \leqslant 12$, as well as the bounds given by $p^2_{\text{bas}}(d)$ and $\binom{d+1}{3}$.

This easily implies that unlike $\ell = 1$, we have $p^\ell(d) > p^\ell_{\text{bas}}(d)$ for any $d > \ell > 1$ (with the $d = \ell+1$ case following from $p^2(3) = 3$). We state this formally as the following corollary.

d	$p^2_{\text{bas}}(d)$	Thm. 1.2	$\binom{d+1}{3}$
$d = 4$	$4 = 2 \cdot 2 \cdot 1$	$6 = 3 \cdot 2$	10
$d = 5$	$8 = 2 \cdot 2 \cdot 2$	$9 = 3 \cdot 3$	20
$d = 6$	$12 = 3 \cdot 2 \cdot 2$	$15 = 5 \cdot 3$	35
$d = 7$	$18 = 3 \cdot 3 \cdot 2$	$20 = 5 \cdot 4$	56
$d = 8$	$27 = 3 \cdot 3 \cdot 3$	$28 = 7 \cdot 4$	84
$d = 9$	$36 = 4 \cdot 3 \cdot 3$	$40 = 10 \cdot 4$	120
$d = 10$	$48 = 4 \cdot 4 \cdot 3$	$52 = 13 \cdot 4$	165
$d = 11$	$64 = 4 \cdot 4 \cdot 4$	$65 = 13 \cdot 5$	220
$d = 12$	$80 = 5 \cdot 4 \cdot 4$	$85 = 17 \cdot 5$	286

Table 1: For $4 \leqslant d \leqslant 12$, the values of $p^2_{\text{bas}}(d)$, $\binom{d+1}{3}$, and the construction provided by Theorem 1.2.

Corollary 1.3. *For any $\ell > 1$ we have*

$$\limsup_{d \to \infty} \left(p^\ell(d) - p^\ell_{\text{bas}}(d) \right) = \infty.$$

The rest of the paper is structured as follows. In Section 2 we present the background theory and information for the problem, and in Section 3 we prove the construction which gives the bound in Theorem 1.2. Finally in Section 4 we mention some upper bounds on the number of colors possible in a linear polychromatic coloring.

2 Simple and linear colorings

It is conventional to refer to the vertices of Q_n with n-dimensional binary strings, and to represent an embedded Q_k by writing $*$ in the corresponding coordinates. For example, in Q_8 the embedded Q_2 whose four vertices are 01000011, 01001011, 01100011, 01101011, is typically represented by

$$01{*}0{*}011.$$

We say that a Q_ℓ-coloring is *simple* if the color of each Q_d depends only on the number of 1's in the $d+1$ regions (possibly empty) delimited by the $*$'s. For example, in a simple 2-polychromatic coloring of Q_7, the faces $01{*}0{*}011$ and $10{*}0{*}101$ would be assigned the same color.

The following generalization of [1, Claim 10] (present also as [3, Lemma 18] and [5, Claim 6]) shows that in fact it suffices to only consider simple colorings. The proof is a nice application of the Ramsey theorem.

Theorem 2.1. *Let $d \geqslant \ell \geqslant 1$ and assume $r \leqslant p^\ell(d)$. Then for every $n \geqslant d$, there is a simple d-polychromatic Q_ℓ-coloring of Q_n with r colors.*

Thus for the purposes of coloring, we can consider an embedded Q_k in Q_n as a sequence of nonnegative integers $(a_0, a_1, \dots, a_k)$ such that a_i denotes the number of 1's between the ith and $(i+1)$st star. For example, 01∗0∗011 can be identified with $(1, 0, 2)$. In light of this a Q_ℓ-coloring with colors from a set S can be thought of as a function

$$\chi : \mathbb{Z}_{\geqslant 0}^{\ell+1} \twoheadrightarrow S.$$

We can now motivate the so-called basic colorings as follows.

Definition 2.2. For $n \geqslant d \geqslant \ell \geqslant 1$, choose positive integers $m_0, m_1, \dots, m_\ell$ with sum $d+1$ and consider the coloring

$$\chi : \mathbb{Z}_{\geqslant 0}^{\ell+1} \twoheadrightarrow \bigoplus_{i=0}^{\ell} \mathbb{Z}/m_i$$

by projection. This induces a Q_ℓ-coloring of every Q_n with $m_0 m_1 \dots m_\ell$ colors.

We call any coloring of this form a *basic* coloring.

Example 2.3. Let $d = 14$, $\ell = 2$, $m_0 = m_1 = m_2 = 5$. We claim this gives a basic 14-polychromatic Q_2-coloring

$$\chi : \mathbb{Z}_{\geqslant 0}^{3} \twoheadrightarrow \mathbb{Z}/5 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/5$$

with $5^3 = 125$ colors.

Consider an embedded Q_{14} in some Q_n, which can be thought of as a sequence of 14 stars. Select the 5th and 9th star as follows, and denote the remaining bits by $\varepsilon_1, \dots, \varepsilon_{12}$, as shown below.

$$\begin{array}{cccccccccccccc} * & * & * & * & \boxed{*} & * & * & * & * & \boxed{*} & * & * & * & * \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 & & \varepsilon_9 & \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} \end{array}$$

This gives $2^{12} = 4096$ choices of Q_2 faces in our embedded Q_{14}. We claim that all colors are present among just these faces.

Let x, y, z denote the number of 1's from the ambient Q_n present in the three regions cut out by the boxed stars. Then, we wish to show that

$$\chi\left(x + \varepsilon_1 + \cdots + \varepsilon_4,\ y + \varepsilon_5 + \cdots + \varepsilon_8,\ z + \varepsilon_9 + \cdots + \varepsilon_{12}\right)$$

achieves all colors, which is obvious since $\varepsilon_i + \varepsilon_{i+1} + \varepsilon_{i+2} + \varepsilon_{i+3}$ takes all possible values modulo 5.

More generally, as shown in [5, Theorem 1], every basic coloring is indeed seen to be d-polychromatic. The lower bound $p^\ell_{\text{bas}}(d)$ now follows by taking the m_i such that $|m_i - m_j| \leqslant 1$ for all $1 \leqslant i < j \leqslant \ell$.

Definition 2.4. More generally, a *linear coloring* is one where the colors are selected from some (finite) abelian group Z, and which is induced by an additive map

$$\chi : \mathbb{Z}_{\geqslant 0}^{\ell+1} \twoheadrightarrow Z.$$

3 A family of linear colorings

We now exhibit a family of linear d-polychromatic Q_ℓ-colorings.

Theorem 3.1. *Let $n > t$ be positive integers. Set either*

- $m = t^2 + 1$ *and* $d = 2t + n - 1$*, where* $t \geqslant 2$*, or*
- $m = t^2 + t + 1$ *and* $d = 2t + n$*, where* $t \geqslant 1$*.*

Then the coloring

$$\chi : \mathbb{Z}^3_{\geqslant 0} \twoheadrightarrow \mathbb{Z}/m \oplus \mathbb{Z}/n \quad \text{given by} \quad (p, q, r) \mapsto (p - tq, p + q + r)$$

is a linear d-polychromatic Q_2-coloring with mn colors.

Proof. We begin by addressing the first case $m = t^2+1$, $d = 2t+n-1$. Let $Z = \mathbb{Z}/m\oplus\mathbb{Z}/n$.

Fix an embedding Q_d, which as usual we think of as a sequence of d stars embedded in an ambient string of 1's and 0's. We can represent this with the diagram

$$x_0 \quad \underbrace{* \quad x_1 \quad * \quad \cdots \quad * \quad x_{d-1} \quad *}_{d \text{ stars}} \quad x_d$$

where x_i denotes the number of 1's in the region delimited by those two stars.

First, consider the family of squares cut out by the star pattern

$$*^{t-1} \quad \boxed{*} \quad *^{t-1} \quad \boxed{*} \quad *^{n-1}$$

where we consider the squares formed when all the bits other than the tth and $2t$th bit are assigned a particular value. For example, the square

$$\underbrace{0 \cdots 0}_{t-1 \text{ 0's}} \quad \boxed{*} \quad \underbrace{0 \cdots 0}_{t-1 \text{ 0's}} \quad \boxed{*} \quad \underbrace{0 \cdots 0}_{n-1 \text{ 0's}}$$

is assigned color $(X, S) \in Z$ where $X = (x_0 + \cdots + x_{t-1}) - t(x_t + \cdots + x_{2t-1}) \pmod m$ and $S = x_0 + \cdots + x_d \pmod n$.

Now suppose we vary the choice of assigned bits. First consider the last $n - 1$ stars. Since $\{0, 1, \ldots, n - 1\}$ covers all residues modulo n, we see that the second coordinate is arbitrary, even regardless of the choices of the first $2(t - 1)$ stars. Moreover, the first coordinate doesn't depend on the choice of these last $n - 1$ stars.

So we focus on the first coordinate. Let $0 \leqslant u \leqslant t-1$ and $0 \leqslant v \leqslant t-1$ be the number of 1's we select in the first and second regions, respectively. (Thus the first coordinate receives color $X + u - tv$.) The values of $u - tv$ (modulo m) are given in the table

$u - tv$	$u = 0$	$u = 1$	$\cdots$	$u = t - 1$
$v = 0$	0	1	$\cdots$	$t - 1$
$v = 1$	$t^2 - t + 1$	$t^2 - t + 2$	$\cdots$	t^2
$v = 2$	$t^2 - 2t + 1$	$t^2 - 2t + 2$	$\cdots$	$t^2 - t$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$v = t - 1$	$t + 1$	$t + 2$	$\cdots$	$2t.$

Thus, we see that we achieve exactly the colors with first coordinate in the set $X + \{0, 1, \ldots, t-1, t+1, t+2, \ldots, t^2\}$ so the colors not present are exactly those whose first coordinate is

$$X + t \pmod{m}.$$

Next, consider the family

$$*^t \quad \boxed{*} \quad *^{t-2} \quad \boxed{*} \quad *^{n-1}$$

and this time define $Y = (x_0 + \cdots + x_t) - t(x_{t+1} + \cdots + x_{2t-1}) \pmod{n}$, which is the first coordinate of the analogous all-zero color. Again, consider varying the choice of assigned bits, this time with $u \in \{0, 1, \ldots, t\}$ and $v \in \{0, \ldots, t-2\}$. The values of $u - tv$ are given in the table

$u - tv$	$u = 0$	$u = 1$	$\cdots$	$u = t-1$	$u = t$
$v = 0$	0	1	$\cdots$	$t-1$	t
$v = 1$	$t^2 - t + 1$	$t^2 - t + 2$	$\cdots$	t^2	$t^2 + 1$
$v = 2$	$t^2 - 2t + 1$	$t^2 - 2t + 2$	$\cdots$	$t^2 - t$	$t^2 - t + 1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$v = t-2$	$2t + 1$	$2t + 2$	$\cdots$	$3t$	$3t + 1.$

So by the same argument as in the previous case, the colors not present are exactly those whose first coordinate is in the set

$$Y + \{t+1, t+2, \ldots, 2t\} \pmod{m}.$$

If $Y - X \notin \{1, 2, \ldots, t\}$ then we are now done. Let $\delta = Y - X$ and henceforth assume $Y - X \in \{1, 2, \ldots, t\}$. We denote by $k = X + t = Y + t + \delta$, and call any color of the form $(k, \bullet)$ a "critical color." We wish to show all n critical colors are present on some other face.

We consider the two families

$$*^{t-1} \quad \boxed{*} \quad *^{t} \quad \boxed{*} \quad *^{n-2}$$
$$*^{t} \quad \boxed{*} \quad *^{t-1} \quad \boxed{*} \quad *^{n-2}$$

which we will call the "first" family and the "second" family. Let $C = -tx_{2t} \pmod{m}$. As before, the all-zero squares in these families receive the colors $(X + C, S) \in Z$ and $(Y + C, S) \in Z$, respectively.

Define u and v as before and now let $0 \leqslant w \leqslant n-2$ denote the number of 1's in the rightmost region. Again, we can exhibit two tables for u and v defined as before: for the first family we obtain a table

$u - tv$	$u = 0$	$u = 1$	$\cdots$	$u = t-1$
$v = 0$	0	1	$\cdots$	$t-1$
$v = 1$	$t^2 - t + 1$	$t^2 - t + 2$	$\cdots$	t^2
$v = 2$	$t^2 - 2t + 1$	$t^2 - 2t + 2$	$\cdots$	$t^2 - t$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$v = t-1$	$t + 1$	$t + 2$	$\cdots$	$2t$
$v = t$	1	2	$\cdots$	$t.$

and for the second family we obtain a table

$u - tv$	$u = 0$	$u = 1$	$\cdots$	$u = t-1$	$u = t$
$v = 0$	0	1	$\cdots$	$t-1$	t
$v = 1$	$t^2 - t + 1$	$t^2 - t + 2$	$\cdots$	t^2	$t^2 + 1$
$v = 2$	$t^2 - 2t + 1$	$t^2 - 2t + 2$	$\cdots$	$t^2 - t$	$t^2 - t + 1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$v = t-2$	$2t + 1$	$2t + 2$	$\cdots$	$3t$	$3t + 1$
$v = t-1$	$t + 1$	$t + 2$	$\cdots$	$2t$	$2t + 1.$

Note that every possible first coordinate is represented in both tables.

Set $h = k - (X + C)$. Then the entries equal to h in the first table correspond to choices (u_1, v_1) which yield squares of critical color (regardless of the choice of w). In fact, as we vary w the critical colors which are obtained are $(k, u_1 + v_1 + S + w)$, which is exactly the sequence of colors

$$(k, u_1 + v_1 + S),\ (k, u_1 + v_1 + S + 1),\ \ldots,\ (k, u_1 + v_1 + S + n - 2).$$

Thus the only critical color not present is $(k, u_1 + v_1 + S - 1)$.

Similarly, the entries equal to $h + \delta$ in the second table correspond to choices (u_2, v_2) which yield squares in the second family with color $(k, \bullet)$ (again regardless of the choice of w). For such a choice of (u_2, v_2), by the same logic, the only critical color not present is $(k, u_2 + v_2 + S - 1)$.

So the problem reduces to the following. For arbitrary h and $1 \leqslant \delta \leqslant t$, we need to show there exist $0 \leqslant u_1 \leqslant t - 1$, $0 \leqslant v_1 \leqslant t$, $0 \leqslant u_2 \leqslant t$, and $0 \leqslant v_2 \leqslant t - 1$ so that

$$u_1 - tv_1 \equiv h \pmod{m} \tag{1}$$

$$u_2 - tv_2 \equiv h + \delta \pmod{m} \tag{2}$$

$$u_1 + v_1 \not\equiv u_2 + v_2 \pmod{n}. \tag{3}$$

Intuitively, one can see this geometrically from the earlier tables. The quantities $u_i + v_i \pmod{n}$ correspond to "northeast diagonals" in the table, which are "spaced apart" (since $n > t$) in such a way that a perturbation by $\delta < t$ must move any h into a different diagonal.

We formalize this intuition in the following calculations.

- In the case $h = t$, take $(u_1, v_1) = (t - 1, t)$ and $(u_2, v_2) = (\delta - 1, t - 1)$. Then $(u_1 + v_1) - (u_2 + v_2) = t + 1 - \delta$, which is not divisible by n since $n > t$ and $1 \leqslant \delta \leqslant t$.

- In the case $h = t - \delta$, take $(u_1, v_1) = (t - \delta, 0)$ and $(u_2, v_2) = (t, 0)$. Then $(u_1 + v_1) - (u_2 + v_2) = -\delta$, again not divisible by n.

- Now assume neither h nor $h+\delta$ is equal to t. Then we can pick (u_1, v_1) and (u_2, v_2) satisfying (1) and (2), and actually $u_1, v_1, u_2, v_2 \in \{0, 1, \ldots, t-1\}$. Let $A = u_2 - u_1$ and $B = v_1 - v_2$, so $A, B \in [-(t-1), t-1]$ Now, subtracting (1) from (2) gives

$$A + tB \equiv \delta \pmod{m}.$$

We have on one hand that $A+tB \leqslant t-1+t(t-1) < m < m+\delta$. On the other hand if $B \neq -(t-1)$ we also have $A+tB \geqslant -(t-1)+t(-t+2) \geqslant -t^2+t+1 > -m+\delta$. So there are only two possibilities: either

$$(A, B) = (\delta, 0) \quad \text{or} \quad (A, B) = (-1-t+\delta, -(t-1)).$$

In both cases, $A \neq B$ and $|A-B| \leqslant \delta \leqslant t < n$, hence

$$A \not\equiv B \pmod{n}$$

which yields (3).

Having completed all cases, this completes the proof of the situation $m = t^2+1$, $d = 2t+n-1$.

The case where $m = t^2+t+1$ and $d = 2t+n$ is virtually identical, and so we will merely give a brief overview. The idea this time is to consider first the two families

$$*^{t-1} \quad \boxed{*} \quad *^{t} \quad \boxed{*} \quad *^{n-1}$$

$$*^{t} \quad \boxed{*} \quad *^{t-1} \quad \boxed{*} \quad *^{n-1}$$

in order to once again reduce to a set of n missing colors. Then one considers the family

$$*^{t-1} \quad \boxed{*} \quad *^{t+1} \quad \boxed{*} \quad *^{n-2}$$

$$*^{t} \quad \boxed{*} \quad *^{t} \quad \boxed{*} \quad *^{n-2}$$

in the same manner as before. □

Proof of Theorem 1.2. In Theorem 3.1, take the following choices of parameters:

- If $d = 3k$, take $t = k$, $m = t^2+1$, $n = k+1$.
- If $d = 3k+1$, take $t = k$, $m = t^2+t+1$, $n = k+1$.
- If $d = 3k+2$, take $t = k$, $m = t^2+t+1$, $n = k+2$. □

Proof of Corollary 1.3. The result is immediate by Theorem 1.2 for $\ell = 2$.

For any general $\ell > 2$, let $d+1 = m_0+m_1+\cdots+m_\ell$ where $m_i \in \mathbb{Z}$ and $|m_i - m_j| \leqslant 1$ for any i and j. Let

$$\chi_0 : \mathbb{Z}_{\geqslant 0}^{\ell-2} \twoheadrightarrow \bigoplus_{j=0}^{\ell-3} \mathbb{Z}/m_j$$

denote the basic coloring on $m_0 + \cdots + m_{\ell-3} - 1$ stars, and let

$$\chi_1 : \mathbb{Z}^3_{\geqslant 0} \twoheadrightarrow Z$$

denote the coloring in Theorem 1.2 on $m_{\ell-2} + m_{\ell-1} + m_\ell - 1$ stars.

Then we can consider a coloring

$$\chi : \mathbb{Z}^{\ell+1}_{\geqslant 0} \twoheadrightarrow \left(\bigoplus_{j=0}^{\ell-3} \mathbb{Z}/m_j \right) \oplus Z$$

defined by $\chi_0 \oplus \chi_1$, that applies χ_0 to the first $\ell - 2$ components and χ_1 on the last three. By construction χ also gives a d-polychromatic coloring, and the corollary follows. $\square$

Example 3.2. To illustrate Corollary 1.3, suppose $d = 12$ and $\ell = 4$. Pick $m_0 = m_1 = m_2 = 3$ and $m_3 = m_4 = 2$; then the coloring χ_0 has $p^1_{\mathrm{bas}}(5) = 9$ colors, while χ_1 has 15 colors (as in Theorem 1.2), and so the coloring $\chi_0 \oplus \chi_1 = 9 \cdot 15 = 135$ colors. On the other hand, $p^4_{\mathrm{bas}}(12) = 3^3 \cdot 2^2 = 108$ colors, according to the Theorem from [5].

4 Upper bounds

We do not have at present any upper bound for $p^2(Q_d)$ other than the simple $\binom{d+1}{3}$ bound. In this section we briefly mention an upper bound for the number of colors in a *linear* d-polychromatic coloring.

Specifically, we use the geometry of numbers to prove the following.

Theorem 4.1. *Let $\chi : \mathbb{Z}^3_{\geqslant 0} \twoheadrightarrow Z$ be a linear d-polychromatic coloring. For d sufficiently large, we have*

$$|Z| < \frac{26}{27} \binom{d+1}{3}.$$

Proof. Let $N = |Z|$. Extend χ to a map $\mathbb{Z}^3 \twoheadrightarrow Z$ of abelian groups. Then consider $\mathbb{Z}^3$ as a tetrahedral lattice Λ_0 in $\mathbb{R}^3$. In this case, the kernel of χ is a lattice Λ of index N in $\mathbb{Z}^3$.

Let $n = d - 2$. Now if we consider the coloring of Q_d itself by χ (or really any embedding of Q_d into Q_N with all ambient bits zero), we see that the colors present are precisely those $\chi(x, y, z)$ where $x + y + z \leqslant n$, $x, y, z \in \mathbb{Z}_{\geqslant 0}$. Thus we obtain a regular tetrahedron T of side length n in which all colors are present.

On the other hand suppose that Λ contains a nonzero vector v which fits inside a regular tetrahedron of side length $s > 0$. Therefore for any $p \in \mathbb{Z}^3$, χ assigns the same color to both p and $p + v$. In particular, this implies all the colors are present in a frustum of T with height s layers; this gives a bound of

$$N \leqslant \binom{d+1}{3} - \binom{d+1-s}{3}. \tag{4}$$

Now let c be the length of the shortest nonzero vector in Λ. Then since a tetrahedron has height equal to $\sqrt{2/3}$ times its side length, we may take

$$s = \left\lceil \sqrt{3/2}c \right\rceil . \tag{5}$$

Next we bring in the theory of sphere packing. Observe that if we construct spheres of diameter c centered at each point in Λ, then we have obtained a packing of spheres in $\mathbb{R}^3$. We have $\det(\Lambda) = N \det(\Lambda_0)$, but Λ_0 is known to be an optimal packing of 3-spheres (see e.g. [2]), and so from this we deduce that

$$0 < c \leqslant \sqrt[3]{N}. \tag{6}$$

Collating (4), (5), (6) together we deduce the inequality

$$c \leqslant \sqrt[3]{\binom{d+1}{3} - \binom{d - \sqrt{3/2}c}{3}}.$$

Thus, we have

$$\begin{aligned} 6c^3 &\leqslant (d^3 - d) + \left(\sqrt{3/2}c - d\right)\left(\sqrt{3/2}c - (d-1)\right)\left(\sqrt{3/2}c - (d-2)\right) \\ &= \sqrt{27/8}c^3 - 9/2(d-1)c^2 + (3d^2 - 6d + 2)\sqrt{3/2}c + (3d^2 - 3d). \end{aligned}$$

We can rewrite this as

$$\left(6 - \sqrt{\frac{27}{8}}\right)\left(\frac{c}{d}\right)^3 + \frac{9}{2}\left(\frac{c}{d}\right)^2 - 3\sqrt{\frac{3}{2}}\left(\frac{c}{d}\right) \leqslant O\left(\frac{1}{d}\right).$$

Solving the resulting quadratic, we see that for sufficiently large d we have $c/d \leqslant 0.5434$, and thus $s < 0.5434\sqrt{3/2}d + 1 < 0.666d$. Finally, using (4) we have

$$\begin{aligned} N &\leqslant \binom{d+1}{3} - \binom{0.334d + 1}{3} \\ &< \left(1 - \left(\frac{1}{3}\right)^3\right)\binom{d+1}{3} \\ &= \frac{26}{27}\binom{d+1}{3} \end{aligned}$$

again for d sufficiently large. $\square$

It would be interesting if any stronger upper bounds could be proven for polychromatic colorings, linear or otherwise.

Acknowledgments

This research was funded by NSF grant 1358659 and NSA grant H98230-16-1-0026 as part of the 2016 Duluth Research Experience for Undergraduates (REU).

The author thanks Joe Gallian for supervising the research and for suggesting the problem, as well as helpful comments on early drafts of the paper. The author would also like to acknowledge the anonymous referee for several corrections and suggestions on the paper.

References

[1] Noga Alon, Anja Krech, and Tibor Szabó. Turán's theorem in the hypercube. *SIAM J. Discrete Math.*, 21(1):66–72 (electronic), 2007.

[2] John Horton Conway and Neil J. A. Sloane. *Sphere Packings, Lattices and Groups.* Grundlehren der mathematischen Wissenschaften v. 290. Springer, 3ed. edition, 1998.

[3] John Goldwasser, Bernard Lidicky, Ryan R. Martin, David Offner, John Talbot, and Michael Young. Polychromatic colorings on the hypercube, 2016, 1603.05865.

[4] David Offner. Polychromatic colorings of subcubes of the hypercube. *SIAM J. Discrete Math.*, 22(2):450–454, 2008.

[5] Lale Özkahya and Brendon Stanton. On a covering problem in the hypercube. *Graphs Combin.*, 31(1):235–242, 2015.