

Zero sum partition of Abelian groups into sets of the same order and its applications

Sylwia Cichacz

Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30, 30-059 Kraków, Poland

cichacz@agh.edu.pl

Submitted: Apr 19, 2017; Accepted: Jan 15, 2018; Published: Jan 25, 2018

Mathematics Subject Classifications: 05C25, 05C78

Abstract

We will say that an Abelian group Γ of order n has the *m-zero-sum-partition property* (*m-ZSP-property*) if m divides n , $m \geq 2$ and there is a partition of Γ into pairwise disjoint subsets A_1, A_2, \dots, A_t , such that $|A_i| = m$ and $\sum_{a \in A_i} a = g_0$ for $1 \leq i \leq t$, where g_0 is the identity element of Γ .

In this paper we study the *m-ZSP* property of Γ . We show that Γ has the *m-ZSP* property if and only if $m \geq 3$ and $|\Gamma|$ is odd or Γ has more than one involution. We will apply the results to the study of group distance magic graphs as well as to generalized Kotzig arrays.

Keywords: Abelian group, zero sum partition, group distance magic labeling, Kotzig arrays

1 Introduction

Assume Γ is an Abelian group of order n with the operation denoted by $+$. For convenience we will write ka to denote $a + a + \dots + a$ (where the element a appears k times), $-a$ to denote the inverse of a and we will use $a - b$ instead of $a + (-b)$. Moreover, the notation $\sum_{a \in S} a$ will be used as a short form for $a_1 + a_2 + a_3 + \dots$, where a_1, a_2, a_3, \dots are all elements of the set S . The identity element of Γ will be denoted by g_0 . Recall that any group element $\iota \in \Gamma$ of order 2 (i.e., $\iota \neq g_0$ and $2\iota = g_0$) is called an *involution*.

In [13] Kaplan, Lev and Roditty introduced the notion of zero-sum partitions of subsets in Abelian groups. Let Γ be an Abelian group and let A be a finite subset of $\Gamma - \{g_0\}$, with $|A| = n - 1$. We shall say that A has the *zero-sum-partition property* (*ZSP-property*) if for every partition $n - 1 = r_1 + r_2 + \dots + r_t$ of $n - 1$, with $r_i \geq 2$ for $1 \leq i \leq t$ and for any possible positive integer t , there is a partition of A into pairwise disjoint subsets

A_1, A_2, \dots, A_t , such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = g_0$ for $1 \leq i \leq t$. When Γ is finite, we shall say that Γ has the ZSP-property if $A = \Gamma - \{g_0\}$ has the ZSP-property.

The following theorem for cyclic groups was proved in [13].

Theorem 1 ([13]). *The group \mathbb{Z}_n has the ZSP-property if and only if n is odd.*

Moreover, Kaplan, Lev and Roditty showed that if Γ is a finite Abelian group of an even order n such that the number of involutions in Γ is different from 3, then Γ does not have the ZSP-property [13]. Their results along with results proved by Zeng [18] give necessary and sufficient conditions for the ZSP-property for a finite Abelian group.

Theorem 2 ([13, 18]). *Let Γ be a finite Abelian group. Then Γ has the ZSP-property if and only if either Γ is of an odd order or Γ contains exactly three involutions.*

They used those results to study anti-magic trees [13, 18]. In [4] the ZSP-property of cyclic groups of an odd order was used for determining the group irregularity strength of disconnected graphs.

We will say that an Abelian group Γ of order n has the *m-zero-sum-partition property* (*m-ZSP-property*) if m divides n , $m \geq 2$ and there is a partition of Γ into pairwise disjoint subsets A_1, A_2, \dots, A_t , such that $|A_i| = m$ and $\sum_{a \in A_i} a = g_0$ for $1 \leq i \leq t$.

A similar problem, where the orders of subsets are not important, is called the *modular sumset partition problem*. This problem was studied in [14]. A non-increasing sequence $\langle m_1, \dots, m_k \rangle$ of positive integers is said to be *n-realizable* if the set $\{1, 2, \dots, n\}$ can be partitioned into k mutually disjoint subsets X_1, X_2, \dots, X_k such that $\sum_{x \in X_i} x = m_i$ for each $1 \leq i \leq k$. Lladó and Moragas considered the modular version of the problem and, by using the polynomial method introduced by Alon [2], they proved that all sequences in \mathbb{Z}_p of length $k \leq (p-1)/2$ are realizable for any prime $p \geq 3$ [14]. The study of *n-realizable* sequences was motivated by the ascending subgraph decomposition problem posed by Alavi, Boals, Chartrand, Erdős and Oellerman [1].

Consider a simple graph G whose order we denote by $n = |G|$. We denote by $V(G)$ the vertex set and $E(G)$ the edge set of G . The *open neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x , and the degree $d(x)$ of x is $|N(x)|$, the order of the neighborhood of x . In this paper we also investigate distance magic labelings, which belong to a large family of magic-type labelings. Generally speaking, a magic-type labeling of a graph $G = (V, E)$ is a mapping from V, E , or $V \cup E$ to a set of labels which most often is a set of integers or group elements. Then the weight of a graph element is typically the sum of labels of the neighboring elements of one or both types. When the weight of each element is required to be equal, then we speak about magic-type labeling; when the weights are all different (or even form an arithmetic progression), then we speak about an antimagic-type labeling. Probably the best known problem in this area is the *antimagic conjecture* by Hartsfield and Ringel [11], which claims that the edges of every graph except K_2 can be labeled by integers $1, 2, \dots, |E|$ so that the weight of each vertex is different.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $\ell: V \rightarrow \{1, 2, \dots, n\}$ with the property that there is a positive integer k such that

$$w(x) = \sum_{y \in N(x)} \ell(y) = k \text{ for every } x \in V(G).$$

If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*. O'Neal and Slater obtained a formula for the vertex magic constant in terms of a fractional domination parameter of the graph, which implies the uniqueness of the magic constant [17]. The following result was proved in [16].

Observation 3 ([16]). *There is no distance magic r -regular graph with r odd.*

Froncek in [10] defined the notion of group distance magic graphs, i.e., the graphs allowing a bijective labeling of vertices with elements of an Abelian group resulting in constant sums of neighbor labels.

A Γ -*distance magic labeling* of a graph $G = (V, E)$ with $|V| = n$ is a bijection ℓ from V to an Abelian group Γ of order n such that the weight $w(x) = \sum_{y \in N(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*. A graph G is called a *group distance magic graph* if there exists a Γ -distance magic labeling for every Abelian group Γ of order $|V(G)|$.

The connection between distance magic graphs and Γ -distance magic graphs is the following. Let G be a distance magic graph of order n with the magic constant μ' . If we replace the label n in a distance magic labeling for the graph G by the label 0, then we obtain a \mathbb{Z}_n -distance magic labeling for the graph G with the magic constant $\mu \equiv \mu' \pmod{n}$. Hence every distance magic graph with n vertices admits a \mathbb{Z}_n -distance magic labeling. However, a \mathbb{Z}_n -distance magic graph on n vertices is not necessarily a distance magic graph. Moreover, there are some graphs that are not distance magic while at the same time they are group distance magic (see [5]).

A general theorem for Γ -distance magic labeling similar to Observation 3 was proved recently.

Theorem 4 ([7]). *Let G be an r -regular graph on n vertices, where r is odd. There does not exist an Abelian group Γ of order n with exactly one involution ι such that G is Γ -distance magic.*

The following theorem was proved in [6].

Theorem 5 ([6]). *Let G be a graph of order $n \equiv 2 \pmod{4}$ with all vertices of odd degree. Then there is no Abelian group Γ of order n such that G is a Γ -distance magic graph.*

Notice that the constant sum partitions of a group Γ lead to complete multipartite Γ -distance magic labeled graphs. For instance, the partition $\{0\}, \{1, 2, 4\}, \{3, 5, 6\}$ of the group \mathbb{Z}_7 with constant sum 0 leads to a \mathbb{Z}_7 -distance magic labeling of the complete tripartite graph $K_{1,3,3}$ (see [5, 6]).

The paper is organized as follows. In Section 2 we give some preliminaries on Abelian groups. In Section 3 we show that Γ has the m -ZSP property if and only if $|\Gamma|$ is odd or Γ has more than one involution. So far there was only one known example of a Γ -distance magic labeling of an odd regular graph for a group Γ which has more than one involution, namely $K_{3,3,3,3}$ with $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ (see [7]). In Section 4 we use the m -ZSP property of Abelian groups to show an infinite family of odd regular graphs possessing Γ -distance magic labeling for groups Γ with more than one involution. Finally, in Section 5 we introduce a generalization of Kotzig arrays and give necessary and sufficient conditions for their existence.

2 Preliminaries

A non-trivial finite group has elements of order 2 (involutions) if and only if the order of the group is even. The fundamental theorem of finite Abelian groups states that a finite Abelian group Γ of order n can be expressed as the direct product of cyclic subgroups of prime-power order. This implies that

$$\Gamma \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} \quad \text{where} \quad n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

and p_i for $i \in \{1, 2, \dots, k\}$ are not necessarily distinct primes. This product is unique up to the order of the direct product. When p is the number of these cyclic components whose order is a multiple of 2, then Γ has $2^p - 1$ involutions. In particular, if $n \equiv 2 \pmod{4}$, then $\Gamma \cong \mathbb{Z}_2 \times \Lambda$ for some Abelian group Λ of odd order $n/2$. Moreover every cyclic group of an even order has exactly one involution. Since the properties and results in this paper are invariant under the isomorphism between groups, we only need to consider one group in each isomorphism class.

The sum of all elements of a group Γ is equal to the sum of its involutions and the identity element. The following lemma was proved in [9] (see [9], Lemma 8).

Lemma 6 ([9]). *Let Γ be an Abelian group.*

- *If Γ has exactly one involution ι , then $\sum_{g \in \Gamma} g = \iota$.*
- *If Γ has no involutions, or more than one involution, then $\sum_{g \in \Gamma} g = g_0$.*

Zeng proves a lemma which plays an important role in the proof of the main result (see [18], Lemma 2.1).

Lemma 7 ([18]). *Let Γ be a finite Abelian group of an odd order or Γ contains exactly three involutions. Let $\text{Bij}(\Gamma)$ denote the set of all bijections from Γ to itself. Then there exist $\phi, \varphi \in \text{Bij}(\Gamma)$ (not necessarily distinct) such that $g + \phi(g) + \varphi(g) = g_0$ for every $g \in \Gamma$.*

3 Zero sum partition

Let the identity element of a group Λ be denoted by a_0 . By $\langle g \rangle$ we denote the subgroup generated by g in the group Γ . A quotient group Γ modulo H for a subgroup H of Γ will be denoted by Γ/H .

For convenience, let \mathcal{G} denote the set consisting of all Abelian groups which are of odd order or contain more than one involution.

We will start with some lemmas.

Lemma 8. *Let Γ be an Abelian group with involution set $I^* = \{\iota_1, \iota_2, \dots, \iota_{2^p-1}\}$, $p > 1$ and let $I = I^* \cup \{g_0\}$. Then there exists a partition $A = \{A_1, A_2, \dots, A_{2^{p-2}}\}$ of I such that $|A_i| = 4$, $\sum_{a \in A_i} a = g_0$ for $i \in \{1, 2, \dots, 2^{p-2}\}$.*

Proof. Let $\iota_0 = g_0$. Recall that $I = \{\iota_0, \iota_1, \dots, \iota_{2^p-1}\}$ is a subgroup of Γ . Moreover, notice that $\Lambda = \{\iota_0, \iota_1, \iota_2, \iota_1 + \iota_2\}$ is a subgroup of I . Therefore there exists a coset decomposition of I into $\iota_{j_1} + \Lambda, \iota_{j_2} + \Lambda, \dots, \iota_{j_{2^{p-2}}} + \Lambda$ for $\iota_{j_k} \in I$, $k = 1, 2, \dots, 2^{p-2}$. Set $A_k = \iota_{j_k} + \Lambda$ for $k = 1, 2, \dots, 2^{p-2}$. Obviously $\sum_{\iota \in A_k} \iota = \iota_0$. \square

Using the same method as in the proof of Lemma 7, we can obtain the following lemma.

Lemma 9. *Let Γ be a finite Abelian group of odd order or Γ contains more than one involution. Let $\text{Bij}(\Gamma)$ denote the set of all bijections from Γ to itself. Then there exist $\phi, \varphi \in \text{Bij}(\Gamma)$ (not necessarily distinct) such that $g + \phi(g) + \varphi(g) = g_0$ for every $g \in \Gamma$.*

Proof. First we prove the following claim. Suppose that $\Gamma_1, \Gamma_2 \in \mathcal{G}$ and the assertion of the lemma is true for Γ_1, Γ_2 . Then the assertion holds for $\Gamma = \Gamma_1 \times \Gamma_2$. Assume the bijections for Γ_1 are ϕ_1 and φ_1 , whereas those for Γ_2 are ϕ_2 and φ_2 . Then

$$\phi = (\phi_1, \phi_2) : \Gamma_1 \times \Gamma_2 \rightarrow \Gamma_1 \times \Gamma_2, \quad (a_1, a_2) \mapsto (\phi_1(a_1), \phi_2(a_2))$$

and

$$\varphi = (\varphi_1, \varphi_2) : \Gamma_1 \times \Gamma_2 \rightarrow \Gamma_1 \times \Gamma_2, \quad (a_1, a_2) \mapsto (\varphi_1(a_1), \varphi_2(a_2)).$$

By the assertion above and Lemma 7, and noting that this lemma is invariant under isomorphism, it suffices to prove it for $\Gamma = \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^\kappa}$ with $\alpha, \beta, \kappa \geq 1$.

The proof is by induction on $|\Gamma|$. We deal with three base cases. For $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the table below gives the desired bijections.

g	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
$\phi(g)$	(0,0,1)	(1,1,1)	(0,1,0)	(1,0,0)	(0,0,0)	(1,1,0)	(0,1,1)	(1,0,1)
$\varphi(g)$	(0,0,1)	(1,1,0)	(0,0,0)	(1,1,1)	(1,0,0)	(0,1,1)	(1,0,1)	(0,1,0)

Whereas for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ the table:

g	(0,0,0)	(0,1,0)	(1,0,0)	(1,1,0)	(0,0,1)	(0,1,1)	(1,0,1)	(1,1,1)
$\phi(g)$	(0,1,0)	(1,1,2)	(1,0,1)	(0,0,3)	(1,0,3)	(0,0,1)	(0,1,3)	(1,1,1)
$\varphi(g)$	(0,1,0)	(1,0,2)	(0,0,3)	(1,1,1)	(1,0,0)	(0,1,2)	(1,1,0)	(0,0,2)
g	(0,0,2)	(0,1,2)	(1,0,2)	(1,1,2)	(0,0,3)	(0,1,3)	(1,0,3)	(1,1,3)
$\phi(g)$	(0,0,2)	(1,0,0)	(1,1,3)	(0,1,1)	(0,0,0)	(1,0,2)	(1,1,0)	(0,1,2)
$\varphi(g)$	(0,0,0)	(1,1,2)	(0,1,3)	(1,0,1)	(0,0,1)	(1,1,3)	(0,1,1)	(1,0,3)

Suppose that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$. Then for $g = (i, j, 0)$ and $g = (i, j, 4)$ we use the table for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where in the last coordinate we put 4 instead of 1.

If $g = (i, j, a)$ for $a \notin \{0, 4\}$, then set a triple

$$(a, b, c) \in \{(2, 3, 3), (7, 2, 7), (5, 6, 5), (6, 1, 1), (1, 5, 2), (3, 7, 6)\},$$

and for $g = (i, j, a)$ we have the following table.

g	(0,0,a)	(0,1,a)	(1,0,a)	(1,1,a)
$\phi(g)$	(0,0,b)	(1,0,b)	(1,1,b)	(0,1,b)
$\varphi(g)$	(0,0,c)	(1,1,c)	(0,1,c)	(1,0,c)

Now we proceed to the induction part. Suppose first that $\Gamma = \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^\kappa}$ with $\beta, \kappa \geq 2$. Then there exists a subgroup $\Gamma_0 = \langle 1 \rangle \times \langle 2 \rangle \times \langle 2 \rangle \in \mathcal{G}$ of Γ such that $\Gamma/\Gamma_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By the induction hypothesis, there are $\phi_0, \varphi_0 \in \text{Bij}(\Gamma_0)$ such that $a + \phi_0(a) + \varphi_0(a) = g_0$ for every $a \in \Gamma_0$. Choose a set of coset representatives for Γ/Γ_0 to be $\{0, c, d, -c - d\}$. Note that

$$(c + \Gamma_0) \cup (d + \Gamma_0) \cup (-c - d + \Gamma_0) = \bigcup_{b \in \Gamma_0} \{c + b, d + \phi_0(b), -c - d + \varphi_0(b)\},$$

and every subset $\{c + b, d + \phi_0(b), -c - d + \varphi_0(b)\}$ is zero-sum. Therefore $\Gamma = \Gamma_0 \cup (\bigcup_{i=1}^{|\Gamma_0|} \{a_{i,1}, a_{i,2}, a_{i,3}\})$ where $a_{i,1} + a_{i,2} + a_{i,3} = g_0$ for $1 \leq i \leq |\Gamma_0|$. Thus define ϕ and φ as follows.

$$\phi(a) = \begin{cases} \phi_0(a) & \text{if } a \in \Gamma_0, \\ a_{i,2} & \text{if } a = a_{i,1} \text{ for some } 1 \leq i \leq |\Gamma_0|, \\ a_{i,3} & \text{if } a = a_{i,2} \text{ for some } 1 \leq i \leq |\Gamma_0|, \\ a_{i,1} & \text{if } a = a_{i,3} \text{ for some } 1 \leq i \leq |\Gamma_0|, \end{cases}$$

and

$$\varphi(a) = \begin{cases} \varphi_0(a) & \text{if } a \in \Gamma_0, \\ a_{i,3} & \text{if } a = a_{i,1} \text{ for some } 1 \leq i \leq |\Gamma_0|, \\ a_{i,1} & \text{if } a = a_{i,2} \text{ for some } 1 \leq i \leq |\Gamma_0|, \\ a_{i,2} & \text{if } a = a_{i,3} \text{ for some } 1 \leq i \leq |\Gamma_0|. \end{cases}$$

Suppose now that $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^\kappa}$. The cases when $\kappa \leq 3$ have been shown in the base case. Thus we may assume that $\kappa \geq 4$ and there exists a subgroup $\Gamma_0 =$

$\langle 1 \rangle \times \langle 1 \rangle \times \langle 8 \rangle \in \mathcal{G}$ such that $\Gamma/\Gamma_0 \cong \mathbb{Z}_8$. By the induction hypothesis, there are $\phi_0, \varphi_0 \in \text{Bij}(\Gamma_0)$ such that $a + \phi_0(a) + \varphi_0(a) = g_0$ for every $a \in \Gamma_0$. Since

$$\mathbb{Z}_8 = \{0, 4\} \cup \{1, 2, -3\} \cup \{-1, -2, 3\},$$

we can choose a set of coset representatives for Γ/Γ_0 to be $\{0, e, c, d, -c-d, -c, -d, c+d\}$ with $2e \in \Gamma_0$ (for example $e = (0, 0, 4)$). Therefore $\Gamma_1 = \Gamma_0 \cup (e + \Gamma_0)$ (for example if $e = (0, 0, 4)$ then $\Gamma_1 = \langle 1 \rangle \times \langle 1 \rangle \times \langle 4 \rangle$) is a group and moreover $\Gamma_1 \in \mathcal{G}$. Thus there exist $\phi_1, \varphi_1 \in \text{Bij}(\Gamma_1)$ such that $a + \phi_1(a) + \varphi_1(a) = g_0$ for every $a \in \Gamma_1$. Similarly as above, we have

$$(c + \Gamma_0) \cup (d + \Gamma_0) \cup (-c - d + \Gamma_0) = \bigcup_{b \in \Gamma_0} \{c + b, d + \phi_0(b), -c - d + \varphi_0(b)\},$$

and

$$(-c + \Gamma_0) \cup (-d + \Gamma_0) \cup (c + d + \Gamma_0) = \bigcup_{b \in \Gamma_0} \{-c + b, -d + \phi_0(b), c + d + \varphi_0(b)\}.$$

Hence $\Gamma = \Gamma_1 \cup (\cup_{i=1}^{2|\Gamma_0|} \{a_{i,1}, a_{i,2}, a_{i,3}\})$ where $a_{i,1} + a_{i,2} + a_{i,3} = g_0$ for $1 \leq i \leq 2|\Gamma_0|$. Define ϕ and φ as follows.

$$\phi(a) = \begin{cases} \phi_1(a) & \text{if } a \in \Gamma_1, \\ a_{i,2} & \text{if } a = a_{i,1} \text{ for some } 1 \leq i \leq 2|\Gamma_0|, \\ a_{i,3} & \text{if } a = a_{i,2} \text{ for some } 1 \leq i \leq 2|\Gamma_0|, \\ a_{i,1} & \text{if } a = a_{i,3} \text{ for some } 1 \leq i \leq 2|\Gamma_0|, \end{cases}$$

and

$$\varphi(a) = \begin{cases} \varphi_1(a) & \text{if } a \in \Gamma_1, \\ a_{i,3} & \text{if } a = a_{i,1} \text{ for some } 1 \leq i \leq 2|\Gamma_0|, \\ a_{i,1} & \text{if } a = a_{i,2} \text{ for some } 1 \leq i \leq 2|\Gamma_0|, \\ a_{i,2} & \text{if } a = a_{i,3} \text{ for some } 1 \leq i \leq 2|\Gamma_0|. \end{cases}$$

This completes the proof. □

Lemma 10. *An Abelian group Γ of order n has the $(2m)$ -ZSP-property if and only if $(2m)|n$, $m \geq 2$ and $\Gamma \in \mathcal{G}$.*

Proof. We will show the necessity first. Suppose that Γ has the $(2m)$ -ZSP-property and A_1, A_2, \dots, A_t is the desired partition. Therefore the order of Γ is even and divisible by $2m$. Hence Γ has at least one involution. Obviously $m > 1$. Assume now that there is exactly one involution $\iota \in \Gamma$. On one hand $\sum_{g \in \Gamma} g = \sum_{i=1}^t \sum_{a \in A_i} a = g_0$, on the other $\sum_{g \in \Gamma} g = \iota$ by Lemma 6, a contradiction.

Let Γ be an Abelian group with involution set $I^* = \{\iota_1, \iota_2, \dots, \iota_{2^p-1}\}$, $p > 1$ and let $I = I^* \cup \{g_0\}$. Then there exists a partition $\{I_1, I_2, \dots, I_{2^{p-2}}\}$ of I such that $|I_i| = 4$, $\sum_{a \in I_i} a = g_0$ for $i \in \{1, 2, \dots, 2^{p-2}\}$ by Lemma 8. Note that all elements in the set $\Gamma - I$ can be split into disjoint pairs $B_i = \{g_i, -g_i\}$ for $i = 1, 2, \dots, (|\Gamma| - 2^p)/2$.

It is easy to see now that for $2m \equiv 0 \pmod{4}$ there exists a partition $A = \{A_1, A_2, \dots, A_{|\Gamma|/2m}\}$ of Γ such that $|A_i| = 2m$ and $\sum_{a \in A_i} a = g_0$ for $i \in \{1, 2, \dots, |\Gamma|/2m\}$.

Recall that Γ can be expressed as a direct product of cyclic subgroups of prime-power order. When p is the number of these cyclic components whose order is a power of 2, then Γ has $2^p - 1$ involutions. Note that for $2m \equiv 2 \pmod{4}$, because $m \geq 3$ is odd we obtain $\Gamma \cong \mathbb{Z}_{2^{\alpha_1}} \times \mathbb{Z}_{2^{\alpha_2}} \times \dots \times \mathbb{Z}_{2^{\alpha_p}} \times \Lambda$ for $\alpha_1, \alpha_2, \dots, \alpha_p \geq 1$ and some Abelian group Λ of an odd order divisible by m . Therefore $(|\Gamma| - 2^p) = |\Gamma - I| \geq (m - 1)|I| = (m - 1)2^p$. Hence $(|\Gamma| - 2^p)/2 \geq 2^p > 2^{p-2} = |I|/4$ and one can check that there exists a partition $A = \{A_1, A_2, \dots, A_{|\Gamma|/2m}\}$ of Γ such that $|A_i| = 2m$, $\sum_{a \in A_i} a = g_0$ for $i \in \{1, 2, \dots, |\Gamma|/2m\}$. \square

Lemma 11. *An Abelian group Γ of order n has the $(2m + 1)$ -ZSP-property if and only if $m \geq 1$, $(2m + 1)|n$ and $\Gamma \in \mathcal{G}$.*

Proof. We will show the necessity first. The assumptions $m \geq 1$ and $(2m + 1)|n$ are obvious. Suppose that Γ has exactly one involution ι and the $(2m + 1)$ -ZSP-property. Let A_1, A_2, \dots, A_t be the desired partition. On one hand $\sum_{g \in \Gamma} g = \sum_{i=1}^t \sum_{a \in A_i} a = g_0$, on the other $\sum_{g \in \Gamma} g = \iota$ by Lemma 6, a contradiction.

Assume that $|\Gamma|$ is odd. Let $t = n/(2m + 1)$. There exists a partition A'_1, A'_2, \dots, A'_t of the set $\Gamma - \{g_0\}$ such that $|A'_1| = 2m$ and for every $2 \leq i \leq t$, $|A'_i| = 2m + 1$ with $\sum_{a \in A'_i} a = g_0$ by Theorem 2. We obtain the desired partition A_1, A_2, \dots, A_t by setting $A_1 = A'_1 \cup \{g_0\}$ and $A_i = A'_i$ for every $2 \leq i \leq t$.

Suppose now that Γ has $2^p - 1$ involutions for some positive integer $p > 1$. Therefore $\Gamma \cong \Gamma_0 \times \Lambda$ for $\Gamma_0 \cong \mathbb{Z}_{2^{\alpha_1}} \times \mathbb{Z}_{2^{\alpha_2}} \times \dots \times \mathbb{Z}_{2^{\alpha_p}}$ for $\alpha_1, \alpha_2, \dots, \alpha_p \geq 1$ and some Abelian group Λ of odd order divisible by $(2m + 1)$. Let $t = |\Lambda|/(2m + 1)$. Obviously $\Gamma_0, \Lambda \in \mathcal{G}$. Hence Λ has the $(2m + 1)$ -ZSP-property and there exists a partition A_1, A_2, \dots, A_t of the group Λ such that for every $1 \leq i \leq t$, $|A_i| = 2m + 1$ with $\sum_{a \in A_i} a = a_0$. Denote the h -th element in the set A_i by $a_{i,h}$.

Define a function $\gamma: \Gamma_0 \rightarrow \Gamma_0$ such that $\gamma(w) = (2m - 1)w$ for all $w \in \Gamma_0$. Note that $\gamma \in \text{Bij}(\Gamma_0)$ (and moreover it is an automorphism), because $\gcd(2^\alpha, 2m - 1) = 1$ for any positive integer α . Recall that there exist $\phi, \varphi \in \text{Bij}(\Gamma_0)$ such that $g + \phi(g) + \varphi(g) = g_0^0$ for every $g \in \Gamma_0$ (g_0^0 is the identity element of Γ_0) by Lemma 7. Let $f: \mathbb{Z}_{|\Gamma_0|} \rightarrow \Gamma_0$ be a bijection. Define A_i^j by replacing an element $a_{i,h}$ in Λ by $a_{i,1}^j = (\phi(f(j)), a_{i,1})$, $a_{i,2}^j = (\varphi(f(j)), a_{i,2})$, $a_{i,h}^j = (\gamma^{-1}(f(j)), a_{i,h})$ for $j = 0, 1, \dots, |\Gamma_0| - 1$, $i = 1, 2, \dots, t$, $h = 3, 4, \dots, 2m + 1$. Observe that $\sum_{h=3}^{2m+1} a_{i,h}^j = (f(j), a_0)$ for $j = 0, 1, \dots, |\Gamma_0| - 1$, $i = 1, 2, \dots, t$. Therefore $A_1^0, \dots, A_t^0, A_1^1, \dots, A_t^1, \dots, A_1^{|\Gamma_0|-1}, \dots, A_t^{|\Gamma_0|-1}$ is a partition of the group Γ such that for every $1 \leq i \leq t$, $0 \leq j \leq |\Gamma_0| - 1$ we have $|A_i^j| = 2m + 1$ with $\sum_{a \in A_i^j} a = (g_0^0, a_0) = g_0$. \square

By above Lemmas 10 and 11 the following theorem is true.

Theorem 12. *An Abelian group Γ of order n has the m -ZSP-property if and only if $m \geq 3$, $m|n$ and $\Gamma \in \mathcal{G}$.*

Since any group $\Gamma \in \mathcal{G}$ has the m -ZSP property for any $m \geq 3$, we state the following conjecture.

Conjecture 13. Let $\Gamma \in \mathcal{G}$ be of an even order n . For every partition $n - 1 = r_1 + r_2 + \dots + r_t$ of $n - 1$, with $r_i \geq 3$ for $1 \leq i \leq t$ and for any possible positive integer t , there is a partition of $\Gamma - \{g_0\}$ into pairwise disjoint subsets A_1, A_2, \dots, A_t , such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = g_0$ for $1 \leq i \leq t$.

Recently, it was proved that the conjecture is true for $t = 3$, see [6].

4 Group distance labeling of odd regular graphs

The *lexicographic product* or *graph composition* $G \circ H$ of graphs G and H is a graph such that the vertex set of $G \circ H$ is the Cartesian product $V(G) \times V(H)$; and any two vertices (u, v) and (x, y) are adjacent in $G \circ H$ if and only if either u is adjacent with x in G or $u = x$ and v is adjacent with y in H . Note that $G \circ H$ and $H \circ G$ are not isomorphic in general. One can imagine obtaining $G \circ H$ by blowing up each vertex of G into a copy of H .

By results proved in [3] (see Theorem 2.2, [3]) we have the following observation.

Observation 14 ([3]). *If G is a regular graph, then the lexicographic product $G \circ \overline{K}_{2n}$ is group distance magic for any $n \geq 1$.*

When G is an arbitrary graph (not necessarily regular), we can use the m -ZSP-property of $\Gamma \in \mathcal{G}$ to obtain the following.

Theorem 15. *If G is a graph of order t , then the lexicographic product $G \circ \overline{K}_n$ has a Γ -distance magic labeling for $n \geq 3$ and any $\Gamma \in \mathcal{G}$ of order nt .*

Proof. Let G be a graph with the vertex set $V(G) = \{v_0, v_1, \dots, v_{t-1}\}$, and let $V(\overline{K}_{2n+1}) = \{x_0, x_1, \dots, x_{n-1}\}$. Since $\Gamma \in \mathcal{G}$, there exists a partition A_1, A_2, \dots, A_t of Γ such that $|A_i| = n$ with $\sum_{a \in A_i} a = g_0$ for every $1 \leq i \leq t$ by Theorem 12. Label the vertices of the i -th copy of \overline{K}_n using elements from the set A_i for $i = 1, 2, \dots, t$. Therefore $w(v_i, x_j) = g_0$. \square

Theorem 15 implies immediately the following observation.

Observation 16. *If G is a graph of odd order t , then the lexicographic product $G \circ \overline{K}_{2n+1}$ is group distance magic for $n \geq 1$.*

Observe that if G is an odd regular graph, then the lexicographic product $G \circ \overline{K}_{2n+1}$ is also an odd regular graph. Thus the order of the graph $G \circ \overline{K}_{2n+1}$ is even by handshaking lemma. Recall that $K_{3,3,3,3} \cong K_4 \circ \overline{K}_3$ which is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ -distance magic [7]. From Theorems 4 and 15 we obtain the following theorem showing that there exist infinitely many odd regular Γ -distance magic graphs where Γ is a group with more than one involution.

Theorem 17. *If G is an odd regular graph of order t , then the lexicographic product $G \circ \overline{K}_{2n+1}$ has a Γ -distance magic labeling for $n \geq 1$ and $|\Gamma| = (2n + 1)t$ if and only if Γ has more than one involution.*

For bipartite Eulerian graphs (i.e., a graph with all vertices of even degrees) such that the partition sets have the same cardinality, we have the following theorem.

Theorem 18. *If G is an Eulerian bipartite graph of order $t \equiv 2 \pmod{4}$ with partition sets of the same cardinality, then $G \circ \overline{K}_{2n+1}$ is group distance magic for $n \geq 1$.*

Proof. Notice that the partition sets V_0, V_1 of G have the same cardinality $k = t/2$. Let $V_0 = \{v_0, v_1, \dots, v_{k-1}\}$, $V_1 = \{w_0, w_1, \dots, w_{k-1}\}$ whereas $V(\overline{K}_{2n+1}) = \{x_0, x_1, \dots, x_{2n}\}$. Let Γ be any Abelian group of order $(2n + 1)t$. Observe that $\Gamma \cong \mathbb{Z}_2 \times \Lambda$ for some Abelian group Λ of odd order $(2n + 1)t/2$ because $t \equiv 2 \pmod{4}$.

Since $2n + 1 \geq 3$, there exists a partition A_1, A_2, \dots, A_k of the group Λ such that for every $1 \leq i \leq k$, $|A_i| = 2n + 1$ with $\sum_{a \in A_i} a = a_0$ by Theorem 12. Define A_i^0 by replacing each element a in Λ by $(0, a)$ and analogously A_i^1 by replacing each element a in Λ by $(1, a)$. Label the vertices of the i -th copy of \overline{K}_{2n+1} in V_0, V_1 using elements from the set A_i^0, A_i^1 , respectively for $i = 1, 2, \dots, k$.

Notice that $w(x) = g_0$ for any $x \in V(G \circ \overline{K}_{2n+1})$ because x has an even degree. \square

Recall that any regular bipartite graph G has the partition sets with the same cardinality, therefore by above Observation 14 and Theorem 18 we easily obtain the following observation.

Observation 19. *If G is an r -regular bipartite graph of order $t \equiv 2 \pmod{4}$ and $n \geq 2$, then $G \circ \overline{K}_n$ is group distance magic if and only if rn is even.*

Proof. Assume first that rn is odd, then $nt \equiv 2 \pmod{4}$ and by Theorem 5 there does not exist a group Γ such that $G \circ \overline{K}_n$ admits a Γ -distance labeling. If n is even then we are done by Observation 14. In the last case n odd and r even we apply Theorem 18. \square

5 Γ -Kotzig arrays

In [15] Marr and Wallis give a definition of a *Kotzig array* as a $j \times k$ grid, where each row is a permutation of $\{0, 1, \dots, k - 1\}$ and each column has the same sum. Note that a Latin square is a special case of a Kotzig array.

Lemma 20 ([15]). *A Kotzig array of size $j \times k$ exists if and only if $j > 1$ and $j(k - 1)$ is even.*

Kotzig arrays play important role in graph labelings. There are many constructions based on Kotzig arrays of various magic-type labelings of copies and products of regular graphs [8, 12, 15].

For an Abelian group Γ of order k we define a Γ -Kotzig array of size $j \times k$ as a $j \times k$ grid, where each row is a permutation of elements of Γ and each column has the same sum. Denote by $x_{i,j}$ the entry in the i -th row and j -th column of the array.

Lemma 21. A Γ -Kotzig array of size $2 \times k$ exists for any Abelian group Γ .

Proof. Let $\Gamma = \{g_0, g_1, \dots, g_{k-1}\}$. Let $x_{1,i} = g_i$, $x_{2,i} = -g_i$ for $i = 0, 1, \dots, k-1$. \square

Lemma 22. A Γ -Kotzig array of size $3 \times k$ exists for any Abelian group $\Gamma \in \mathcal{G}$.

Proof. Let $\Gamma = \{g_0, g_1, \dots, g_{k-1}\}$. There exist $\phi, \varphi \in \text{Bij}(\Gamma)$ such that $g + \phi(g) + \varphi(g) = g_0$ for every $g \in \Gamma$ by Lemma 9. Let $x_{1,i} = g_i$, $x_{2,i} = \phi(g_i)$, $x_{3,i} = \varphi(g_i)$. \square

Theorem 23. A Γ -Kotzig array of size $j \times k$ exists if and only if $j > 1$ and j is even or $\Gamma \in \mathcal{G}$.

Proof. Obviously for $j = 1$ there does not exist a Γ -Kotzig array. Assume first j is even. To construct a Γ -Kotzig array of size $j \times k$, we simply take $j/2$ of Γ -Kotzig arrays of size $2 \times k$ and “glue” them into an array $j \times k$.

Suppose j is odd and there exists a Γ -Kotzig array for a group Γ containing only one involution ι . Recall that $kg = g_0$ for any $g \in \Gamma$ and for some $\mu \in \Gamma$ we have $x_{1,i} + x_{2,i} + \dots + x_{j,i} = \mu$ for any $i = 1, 2, \dots, k$. On one hand $\sum_{i=1}^k \sum_{l=1}^j x_{l,i} = k\mu = g_0$. On the other $\sum_{i=1}^k \sum_{l=1}^j x_{l,i} = \sum_{l=1}^j \sum_{i=1}^k x_{l,i} = j \sum_{g \in \Gamma} g = j\iota = \iota$, a contradiction.

Assume now that $\Gamma \in \mathcal{G}$. To construct a Γ -Kotzig array of size $j \times k$, we simply take $(j-3)/2$ of Γ -Kotzig arrays of size $2 \times k$, one Γ -Kotzig array of size $3 \times k$ and “glue” them into an array $j \times k$. \square

References

- [1] Y. Alavi, A.J. Boals, G. Chartrand, P. Erdős and O. Oellerman, The ascending subgraph decomposition problem, *Congressus Numeratium*, 58:7–14, 1987.
- [2] N. Alon, Combinatorial Nullstellensatz, *Combin. Probab. Comput.* 8(12):7–29, 1999.
- [3] M. Anholcer, S. Cichacz, I. Peterin, A. Tepeh, Group distance magic labeling of direct product of graphs, *Ars Mathematica Contemporanea* 9:93–108, 2015.
- [4] M. Anholcer, S. Cichacz, Group irregular labelings of disconnected graphs, *Contributions to Discrete Mathematics* 12(2):158–166, 2017.
- [5] S. Cichacz, Note on group distance magic complete bipartite graphs, *Central European Journal of Mathematics* 12(3):529–533, 2014 [doi:10.2478/s11533-013-0356-z](https://doi.org/10.2478/s11533-013-0356-z).
- [6] S. Cichacz, On zero sum-partition of Abelian groups into three sets and group distance magic labeling, *Ars Mathematica Contemporanea* 13(2):417–425, 2017.
- [7] S. Cichacz, D. Froncek, Distance magic circulant graphs, *Discrete Mathematics* 339(1):84–94, 2016 [doi:10.1016/j.disc.2015.07.002](https://doi.org/10.1016/j.disc.2015.07.002).
- [8] S. Cichacz, D. Froncek, I. Singgih, Vertex magic total labelings of 2-regular graphs, *Discrete Mathematics* 340(1):3117–3124, 2017 [doi:10.1016/j.disc.2016.06.022](https://doi.org/10.1016/j.disc.2016.06.022).
- [9] D. Combe, A.M. Nelson, W.D. Palmer, Magic labellings of graphs over finite abelian groups, *Australasian Journal of Combinatorics* 29:259–271, 2004.

- [10] D. Froncek, Group distance magic labeling of Cartesian products of cycles, *Australasian Journal of Combinatorics* 55:167–174, 2013.
- [11] N. Hartsfield, G. Ringel, Pearls in Graph Theory, pp. 108–109, *Academic Press*, Boston (1990) (Revised version 1994).
- [12] N. Inayah, A. Lladó, J. Moragas, Magic and antimagic H -decompositions, *Discrete Mathematics* 312:1367–1371, 2012, [doi:10.1016/j.disc.2011.11.041](https://doi.org/10.1016/j.disc.2011.11.041).
- [13] G. Kaplan, A. Lev, Y. Roditty, On zero-sum partitions and anti-magic trees, *Discrete Mathematics* 309(8):2010–2014, 2009, [doi:10.1016/j.disc.2008.04.012](https://doi.org/10.1016/j.disc.2008.04.012).
- [14] A. Lladó, J. Moragas, On the modular sumset partition problem, *European Journal of Combinatorics* 33(4): 427–434, 2012, [doi:10.1016/j.ejc.2011.09.001](https://doi.org/10.1016/j.ejc.2011.09.001).
- [15] A.M. Marr, W.D. Wallis, *Magic Graphs*, 2nd ed., Springer, 2013.
- [16] M. Miller, C. Rodger and R. Simanjuntak, Distance magic labelings of graphs, *Australasian Journal of Combinatorics*, 28:305–315, 2003.
- [17] A. O’Neal, P.J. Slater, Uniqueness of vertex magic constants, *SIAM J. Discrete Math.* 27:708–716, 2013, [doi:10.1137/110834421](https://doi.org/10.1137/110834421).
- [18] X. Zeng, On zero-sum partitions of abelian groups. *Integers* 15 Paper No. A44: 16 pp, 2015.