A Rational Catalan Formula for $(m, 3)$-Hikita Polynomials

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Abstract

Hikita polynomials are the combinatorial side of the rational shuffle theorem. Building upon a recent formula for $(m, 3)$-Catalan polynomials, we prove a formula for $(m, 3)$-Hikita polynomials in terms of Catalan polynomials. This formula shows a surprising relation among coefficients of Hikita polynomials and implies deeper recursive relations and proves the q, t-symmetry of $(m, 3)$-Hikita polynomials.

Keywords: Catalan, Dyck path, parking function, shuffle theorem, rational shuffle theorem

1 Introduction

In the early 1990's Garsia and Haiman introduced an important sum of functions in $\mathbb{Q}(q, t)$, the q, t-Catalan polynomial $C_n(q, t)$, which has since been shown to have interpretations in terms of algebraic geometry and representation theory. These classical q, t-Catalan polynomials are given by

$$C_n(q, t) = \sum_{\pi} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}, \tag{1}$$

where the sums are over all Dyck paths π from $(0, 0)$ to (n, n) and the statistics dinv and area arise from the shape and size of the Dyck path. For an overview of the classical q, t-Catalan polynomials and Dyck paths, see [GH96, GH02, HHL$^+$05, Hag08].

Recently, a valuable generalization of the classic q, t-Catalan polynomial has come to light [Hik12]. For positive integers m, n that are coprime, these (m, n)-*rational* q, t-Catalan polynomials have a similar description to the classic case

$$C_{m,n}(q, t) = \sum_{\pi} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}, \tag{2}$$

where the sum is over all *rational* Dyck paths π from $(0,0)$ to (m,n).

A rational Dyck path is a path in the $m \times n$ integer lattice which proceeds by north and east steps from $(0,0)$ to (m,n) and which always remains weakly above the main diagonal $y = \frac{m}{n}x$. The collection of cells lying above a Dyck path π always forms an English Ferrers diagram $\lambda(\pi)$. The dinv and area statistics on rational Dyck paths arise in a similar way to the statistics on square Dyck paths and will be defined explicitly in Section 1.2.

The classical Catalan polynomials can be realized as a special case of the rational Catalan polynomials, specifically

$$C_n(q,t) = C_{n+1,n}(q,t).$$

So the classical dinv and area statistics can be defined in terms of the rational statistics.

In 2015, Kaliszewski and Li gave an explicit formula of $(3,n)$-rational and $(n,3)$-rational q,t-Catalan polynomials [KL15, KL16] for n not divisible by 3,

$$C_{3,n}(q,t) = C_{n,3}(q,t) = \sum_{0 \leqslant i < n/3} s_{(n-1-2i,i)}(q,t). \tag{3}$$

The function $s_{(b,a)}(q,t)$ where $b > a$ is the Schur function $s_{(b,a)}$ with the evaluations $x_1 = q, x_2 = t$, and $x_i = 0$ for $i > 2$. For example,

$$s_{(5,2)}(q,t) = q^5t^2 + q^4t^3 + q^3t^4 + q^2t^5.$$

In 2005, Haglund, Haiman, Loehr, Remmel, and Ulyanov conjectured that certain generalizations of Catalan polynomials involving *parking functions* are the image of the Macdonald eigen-operator, specifically

$$(-1)^n \nabla e_n(X) = \sum_{P \in \mathsf{PF}(n)} t^{\mathsf{area}(P)} q^{\mathsf{dinv}(P)} F_{\mathsf{iDes}(P)}(X) \tag{4}$$

where ∇ is the operator satisfying

$$\nabla \tilde{H}_\mu(x;q,t) = t^{n(\lambda)} q^{n(\lambda')} \tilde{H}_\mu(x;q,t)$$

for modified Macdonald polynomials $\{\tilde{H}_\mu\}_\mu$ [HHL+05] and the F are Gessel's fundamental quasi-symmetric basis,

$$F_S(X) = \sum_{\substack{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \\ i_\ell = i_{\ell+1} \Rightarrow \ell \in S}} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The area statistic on parking functions is the area of the underlying Dyck path and the dinv statistic is obtained by inversions along attacking cells. The *shuffle theorem*, Equation (4), was later proven by Carlsson and Mellit [CM15].

In [Hik12], Hikita extended parking functions and their statistics to the (m,n)-rational case and defined polynomials

$$\mathcal{H}_{m,n}(X;q,t) = \sum_{P \in \mathsf{PF}(m,n)} t^{\mathsf{area}(P)} q^{\mathsf{dinv}(P)} F_{\mathsf{iDes}(P)}(X), \tag{5}$$

where the sum is over all (m, n)-rational parking functions. Similar to how an n-Dyck path can be realized as a rational $(n + 1, n)$-Dyck path, an n-parking function can be realized as a rational $(n + 1, n)$-parking function.

For each (m, n)-rational Dyck path there is an (m, n)-rational parking function with the same statistics, so the (m, n)-rational q, t-Catalan polynomial appears within $\mathcal{H}_{m,n}(X; q, t)$. Specifically, $C_{m,n}(q, t)$ is the coefficient of $F_\emptyset$ in the expansion given in Equation (5).

The shuffle theorem was generalized by Gorsky and Neguţ [GN15] using operators $P_{k,n}$ on symmetric functions. This so-called *rational shuffle conjecture*, later changed to the rational shuffle theorem after a proof by Mellit [Mel16], is

$$P_{k,n} \cdot 1 = \mathcal{H}_{m,n}(X; q, t).$$

While the shuffle theorem and rational shuffle theorem allow us to describe the images of the ∇ and $P_{k,n}$ operators in terms of combinatorial objects, much is still unknown about these objects and the structure of the Hikita polynomials. For example, it is known from the algebra involved that the parking functions exhibit a symmetry in their q and t statistics when the iDes statistic is fixed, i.e.

$$\mathcal{H}(X; q, t) = \mathcal{H}(X; t, q).$$

However, there is no bijective proof of this fact. Even in the best studied case, the classic Catalan polynomials, there is no simple bijection that exchanges area and dinv. Also, while it is known that Hikita's polynomials are symmetric functions in the x-indeterminates, there isn't a conjecture as to a cancellation-free or positive Schur expansion.

The main result of this paper is to study the properties of the $(m, 3)$-Hikita polynomials, showing the q, t-symmetry of the parking functions and the positive Schur expansion in these cases. The first step in exploring Hikita polynomials was the cases when $m = 2$ or $n = 2$, which was completed by Leven [Lev14]. Our work continues in this process, exploring the next case and we hope that our work will inspire further results regarding these properties in more general cases.

We begin the paper by discussing rational Dyck paths and their combinatorics, including the Fast dinv algorithm. We then introduce a useful function, the modified Catalan polynomials, which we will use in later parts of the paper.

In Section 3 we discuss rational parking function combinatorics and their connection to affine permutations. We then prove that if m, n are coprime then the coefficient of $F_{[n-1]}$ in $\mathcal{H}_{m,n}(X; q, t)$ is

$$\begin{cases} 0 & \text{if } m < n \\ C_{m-n,n}(q, t) & \text{if } m > n \end{cases}. \tag{6}$$

While this can be seen as a result of repeated applications of the Macdonald eigen-operator ∇, this paper contains the first combinatorial proof of this fact.

In Section 4 we explore a relation between the coefficients of the two- and three-row Hikita polynomials $\mathcal{H}_{m,2}(X; q, t)$ and $\mathcal{H}_{m,3}(X; q, t)$ and the two- and three-row Catalan polynomials $C_{m,2}(q, t)$ and $C_{m,3}(q, t)$. We will prove formulas:

$$\mathcal{H}_{m,2}(X; q, t) = C_{m,2}(q, t)s_{(1,1)}(X) + C_{m-2,2}(q, t)s_{(2)}(X)$$

and

$$\mathcal{H}_{m,3}(X;q,t) = \\ C_{m,3}(q,t)s_{(1,1,1)}(X) + \left(K_{m-1,3}(q,t) + K_{m-2,3}(q,t)\right)s_{(2,1)}(X) + C_{m-3,3}(q,t)s_3(X), \tag{7}$$

where

$$K_{m,3}(q,t) = \sum_{0\leqslant i<m/3} s_{(n-1-2i,i)}(q,t).$$

Note that for m not divisible by 3,

$$K_{m,3}(q,t) = C_{m,3}(q,t),$$

by Equation (3). Equation (7) and the more general result of Equation (6) suggest that there may be a more general way to describe all Hikita polynomials as sums of rational q,t-Catalan polynomials.

This new formula and the q,t-symmetry of $C_{k,3}(q,t)$, [GM14, KL15], imply the q,t-symmetry of $\mathcal{H}_{m,3}(X;q,t)$. Then, in Section 5 we will explicitly show this symmetry by constructing a bijection on the set of $(m,3)$-parking functions that exchanges area and dinv while holding the inverse descent set fixed.

1.1 Rational Dyck Paths

Suppose that m,n are positive coprime integers. Construct the (m,n)-*lattice* by drawing a rectangular integer lattice in $\mathbb{R}^2$ whose southwest corner lies on the origin and whose northeast corner lies on the point (m,n). The cell whose northeast corner lies on the point (u,v) will be referred to as cell (u,v). Thus we can define the i^{th} row as the set of cells

$$\{(u,i)|1\leqslant u\leqslant n\}$$

and the j^{th} column as the set of cells

$$\{(j,v)|1\leqslant v\leqslant m\}.$$

The (m,n)-*diagram* is the (m,n)-lattice where each cell (u,v) contains an integer a that satisfies

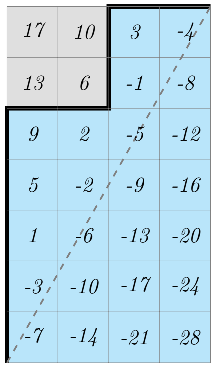
$$a = (v-1)m - un. \tag{8}$$

We call a the *rank* of cell (u,v), denoted by $\gamma_{m,n}(u,v)$ or simply $\gamma(u,v)$ when the parameters are clear from context. Since m and n are coprime, there are no duplicate ranks.

An (m,n)-*rational Dyck path* or (m,n)-Dyck path is path on the (m,n)-diagram that begins at $(0,0)$ and ends at (m,n). The path can only consist of northward and eastward steps and must always lie weakly above the diagonal $y=\frac{n}{m}x$.

An (m,n)-Dyck path partitions the cells within the (m,n)-diagram into two sets. Since the path must lie weakly above the diagonal, one of the sets will always contain the southeast corner of the (m,n)-diagram. We say that any of the cells in this set are *below the path*. The other set of cells, which may be empty, is *above the path*. We say that a cell is *on the path* if its western edge is part of the path. We will say that a rank is above, below, or on the path if the cell containing it is above, below, or on the path; respectively.

Example 1. *A* $(4,7)$*-Dyck path:*

$\pi =$ 

.

The cells that are above the path are colored gray and those that are below the path are colored cyan. For example, rank 6 is above the path, while rank 5 is below the path. The ranks that are on the path are $-7,-3,-1,1,3,5,$ *and* 9.

Here, for example, $\gamma(1,5)=9$ *and* $\gamma(4,3)=-20$.

The set of cells above the path always has the shape of a Ferrers diagram (in English notation). This is because the only allowed moves are northward steps and eastward steps, so the number of cells above the path in each row must be weakly increasing from bottom to top.

1.2 Statistics on Rational Dyck Paths

Let π be an (m,n)-Dyck path for coprime m and n. We partition the cells containing positive ranks into three sets, $\mathsf{Area}(\pi), \mathsf{Dinv}(\pi)$ and $\mathsf{Skip}(\pi)$. Define

$$\mathsf{Area}(\pi) = \{(x,y) : (x,y) \text{ is below path } \pi \text{ and contains a positive rank}\}. \tag{9}$$

For the cells above the path define

$$\mathsf{Dinv}(\pi) = \left\{ (x,y) \text{ above the path} : \frac{\mathsf{arm}(x,y)}{\mathsf{leg}(x,y)+1} < \frac{m}{n} < \frac{\mathsf{arm}(x,y)+1}{\mathsf{leg}(x,y)} \right\}, \tag{10}$$

where $\mathsf{arm}(x,y)$ is the number of cells above π and strictly east of (x,y) and $\mathsf{leg}(x,y)$ is the number of cells above π and strictly south of (x,y). We interpret division by zero to be infinity. Define the Skip set to be the remaining cells above the path:

$$\mathsf{Skip}(\pi) = \{(x,y) \text{ above the path} : (x,y) \notin \mathsf{Dinv}(\pi)\}\,. \tag{11}$$

Set

$$\mathsf{area}(\pi) = |\mathsf{Area}(\pi)|, \qquad \mathsf{dinv}(\pi) = |\mathsf{Dinv}(\pi)|, \qquad \mathsf{skip}(\pi) = |\mathsf{Skip}(\pi)|.$$

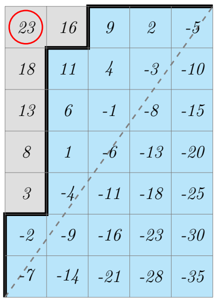
The idea is that when counting the cells that contribute to dinv we should *skip* any that do not satisfy the inequality in (10).

Presented in [KL15, KL16], the *fast* dinv algorithm gives a faster and simpler way to determine whether or not a cell is in Dinv. For a given cell (x,y) above an (m,n)-Dyck path π, let $(x,y)^{\downarrow}$ be the southernmost cell in the same column as (x,y) that is above the path and $(x,y)^{\Downarrow}$ be the northernmost cell in the same column as (x,y) that is below the path. Similarly, let $(x,y)^{\rightarrow}$ be the easternmost cell in the same row as (x,y) that is above the path and $(x,y)^{\Rightarrow}$ be the westernmost cell in the same row as (x,y) that is below the path. So $(x,y)^{\Rightarrow}$ is exactly one cell to the east of $(x,y)^{\rightarrow}$ and $(x,y)^{\Downarrow}$ is exactly one cell south of $(x,y)^{\downarrow}$.

Theorem 2 (Fast dinv). *Suppose π is an (m,n)-Dyck path and let (x,y) be a cell above the path in π. The cell (x,y) is in $\mathsf{Dinv}(\pi)$ if and only if*

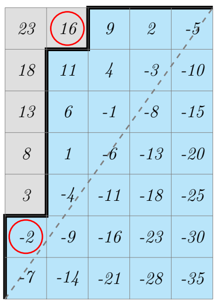
$$\gamma[(x,y)^{\rightarrow}] > \gamma[(x,y)^{\Downarrow}] \qquad \textit{and} \qquad \gamma[(x,y)^{\downarrow}] > \gamma[(x,y)^{\Rightarrow}]. \tag{12}$$

Example 3. *Consider cell $(1,7)$ of the $(5,7)$-Dyck path:*

$\pi =$ 

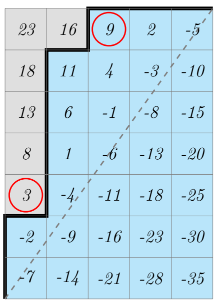
.

We compute and get $\gamma[(1,7)^{\rightarrow}] = 16 > \gamma[(1,7)^{\Downarrow}] = -2$:

$\pi =$ 

.

But $\gamma[(1,7)^{\downarrow}] = 3 < \gamma[(1,7)^{\Rightarrow}] = 9$:

$\pi =$ 

.

So $(1,7) \notin \mathsf{Dinv}(\pi)$.

Define the rational Catalan polynomial

$$C_{m,n}(q,t) = \sum_{\pi} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}$$

where the sum is over all (m,n)-Dyck paths.

2 Modified Catalan Polynomials, $K_{m,3}(q,t)$

For m a positive integer, define

$$K_{m,3}(q,t) = \sum_{0 \leqslant i < m/3} s_{m-1-2i,i}(q,t).$$

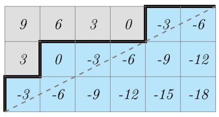
where s is the Schur basis for symmetric functions. When m is not divisible by 3

$$K_{m,3}(q,t) = C_{m,3}(q,t) = C_{3,m}(q,t),$$

so $K_{m,3}$ are the generating functions for rational Dyck paths.

The functions $K_{3k,3}(q,t)$ can be realized as generating functions for modified $(3k,3)$-rational Dyck paths. When a rank appears more than once in the $(3k,3)$-diagram then the one appearing further to the east is considered slightly larger and only *positive* ranks can lie above the path.

Example 4. *The Dyck path*

$\pi =$ 

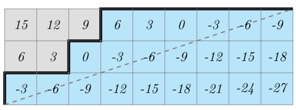
is a valid modified $(6,3)$*-Dyck path because the rank in cell* $(4,3)$ *is slightly larger than the* 0*-rank in cell* $(2,2)$*, and is therefore positive.*

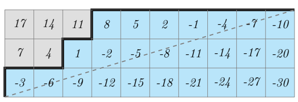
Rather than showing a direct proof in the style of [KL16], we will make the observation that there is a bijection between the set of modified $(3k,3)$-Dyck paths and the set of $(3k+1,3)$-Dyck paths where the rank 1 is below the path. One can see the bijection by superimposing the $(3k+1,3)$-diagram over the $(3k,3)$-diagram and noting that

$$\gamma_{3k,3}(x,y) > \gamma_{3k,3}(u,v) \qquad \Longleftrightarrow \qquad \gamma_{3k+1,3}(x,y) > \gamma_{3k+1,3}(u,v).$$

Thus the dinv and area statistics match precisely.

Example 5. *Consider*

$\pi =$ 

versus $\pi =$ 

.

Proposition 6. *Suppose that m is a positive integer that is not divisible by 3 and π is an $(m,3)$-Dyck path. Then for any cell in the middle row $(x,2)$,*

$$(x,2) \in \mathsf{Dinv}(\pi).$$

Proof. Observe that

$$\gamma[(x,2)^{\downarrow}] = \gamma(x,2) > \gamma[(x,2)^{\Rightarrow}]$$

and

$$\gamma[(x,2)^{\rightarrow}] \geqslant -2 > -3x = \gamma[(x,2)^{\Downarrow}].$$

The result follows from Theorem 2. □

Let π_i be the $(3k+1,k)$-Dyck path where rank 1 is above the path and with exactly $k+i$ cells in the top row that are above the path. Suppose that $1 \leqslant j_1 \leqslant i < j_2 \leqslant k$. Then

$$\begin{aligned}\gamma[(j_1,3)^{\Rightarrow}] &= \gamma(k+i+1,3)\\ &= \gamma(i,3) - 3k - 3\\ &< \gamma(j_1,3) - 3k - 3\\ &< \gamma(j_1,3) - 3k - 1\\ &= \gamma(j_1,3) - m\\ &= \gamma[(j_1,3)^{\Downarrow}]\end{aligned}$$

and $\gamma[(j_1,3)^{\Downarrow}] < 0 < \gamma[(j_1,3)^{\rightarrow}]$. Therefore, by Theorem 2, $(j_1,3) \in \mathsf{Dinv}(\pi_i)$. Similarly,

$$\gamma[(k+j_1,3)^{\Rightarrow}] < \gamma(k+j_1,3) = \gamma[(k+j_1,3)^{\downarrow}]$$

and $\gamma[(k+j_1,3)^{\Downarrow}] < 0 < \gamma[(k+j_1,3)^{\rightarrow}]$ so $(k+j_1,3) \in \mathsf{Dinv}(\pi_i)$. However,

$$\begin{aligned}\gamma[(j_2,3)^{\Rightarrow}] &= \gamma(k+i+1,3)\\ &= \gamma(i,3) - 3k - 3\\ &\geqslant \gamma(j_2,3) - 3k\\ &> \gamma(j_2,3) - m\\ &= \gamma[(j_2,3)^{\downarrow}]\end{aligned}$$

so $(j_2,3) \notin \mathsf{Dinv}(\pi_i)$.

Therefore, each path π_i contributes term $q^{k+2i}t^{k-i}$ to $C_{3k+1,3}(q,t)$. In general, the $(3k+1,3)$-Dyck paths where 1 lies above the path contribute

$$\sum_{i=0}^{k} q^{k+2i}t^{k-i} = \sum_{i=0}^{k} q^{3k-2i}t^{i}$$

to $C_{3k+1,3}(q,t)$. Note that in the bijection, a $(3k+1,3)$ path with rank 1 below the path is mapped to a modified $(3k,3)$ path. Since the rank 1 is transformed into rank 0, that cell no longer contributes to the area of the image. So

$$\begin{aligned}\frac{C_{3k+1,3}(q,t)-\sum_{i=0}^{k} q^{3k-2i}t^{i}}{t} &= \frac{\sum_{i=0}^{k} s_{3k-2i,i}(q,t)-\sum_{i=0}^{k} q^{3k-2i}t^{i}}{t} \\ &= \sum_{0\leqslant i<k} s_{3k-1-2i,i}(q,t) \\ &= K_{3k,3}(q,t).\end{aligned}$$

3 Rational Parking Functions

Let m and n be coprime. An (m,n)-parking function is an (m,n)-Dyck path with the integers $1,\ldots,n$ written on the path so that within any particular column the entries are increasing from bottom to top.

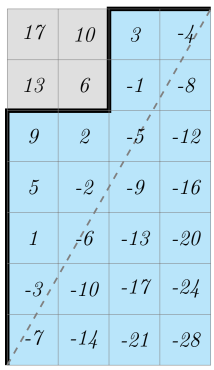
For an (m,n)-Dyck path π, let $R(\pi)=\{r_1,\ldots,r_n\}$ be the set of ranks on the path of π written in increasing order. In this language an (m,n)-parking function can be realized as a pair (π,σ) where π is an (m,n)-Dyck path and σ is a permutation such that if $r_{\sigma^{-1}(i)}=k$ and $r_{\sigma^{-1}(j)}=k+m$ in $R(\pi)$ then $i<j$.

Each pair (π,σ) can be uniquely represented by using inline or window notation:

$$\left[r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}, \ldots, r_{\sigma^{-1}(n)}\right].$$

For a more classical handling of (m,n)-parking functions, see [BGLX16].

Example 7. *Consider the* $(4,7)$*-Dyck path:*

$\pi =$ 

.

The set of $(4,7)$*-parking functions includes:*

$$[-7,-3,-1,1,3,5,9], \qquad [-7,-1,3,-3,1,5,9], \qquad \textit{and} \qquad [-1,-7,-3,1,5,9,3],$$

all supported by the displayed Dyck path, π*. Notice that for each parking function (window) above, the ranks appearing in column 1 of the diagram, namely* $\{-7,-3,1,5,9\}$ *appear in increasing order from left to right.*

3.1 Statistics on Parking Functions

The construction of the Hikita polynomial from parking functions is analogous to the construction of the Catalan polynomial from Dyck paths. The first step is to extend the Dyck path statistics to the set of parking functions.

Given an (m, n)-parking function $P = (\pi, \sigma)$ we define the area to be the area of the underlying Dyck path and the inverse descent set is the inverse descent set of the permutation σ, i.e.

$$\mathsf{area}(P) = \mathsf{area}(\pi) \qquad \text{and} \qquad \mathsf{iDes}(P) = \mathsf{iDes}(\sigma).$$

If we look at the window W for P we can compute the inverse descents of the parking function by computing the descents of the window,

$$\mathsf{iDes}(P) = \mathsf{Des}(W).$$

Let m-bounded inversions be pairs:

$$\mathsf{Inv}(P) = \{(i, j) | i < j \text{ and } \sigma(j) < \sigma(i) < \sigma(j) + m\} \tag{13}$$

and $\mathsf{inv^m}(P) = |\mathsf{Inv}(P)|$. Note the reversed role of i and j from the traditional tdinv statistic (see below). Define the dinv of a parking function to be

$$\mathsf{dinv}(P) = \mathsf{dinv}(\pi) - \mathsf{inv^m}(P). \tag{14}$$

Remark 8. The definition of dinv for parking functions is given in [GM13, BGLX16] as

$$\mathsf{dinv}(P) = \mathsf{dinv}(\pi) + \mathsf{tdinv}(P) - \mathsf{maxtdinv}(\pi)$$

where tdinv is the number of pairs

$$\{(i, j) | i < j \text{ and } \sigma(i) < \sigma(j) < \sigma(i) + m\}$$

and $\mathsf{maxtdinv}$ is the largest tdinv of all parking functions associated to the path. It is not difficult to see that the definition given on line (14) is equivalent.

This allows us to define the Hikita polynomials

$$\mathcal{H}_{m,n}(X; q, t) = \sum_{\mathsf{PF}} t^{\mathsf{area(PF)}} q^{\mathsf{dinv(PF)}} F_{\mathsf{iDes(PF)}}(X)$$

where the sum is over all (m, n)-parking functions and $\{F_\alpha\}$ is the set of Gessel's fundamental quasisymmetric functions.

It is immediate by the descriptions of the statistics that if P is a parking function on Dyck path π with window

$$\sigma = [w_1, w_2, \ldots, w_n]$$

where $w_i < w_{i+1}$ for all i, then

$$\mathsf{dinv}(P) = \mathsf{dinv}(\pi).$$

Example 9. *Recall the* $(4,7)$*-parking function* $\sigma = [-7,-1,3,-3,1,5,9]$ *from Example 7. By looking at the area of the underlying path,* $\mathsf{area}(\pi) = 4$ *we immediately have that* $\mathsf{area}(\sigma) = 4$*. Similarly,* $\mathsf{dinv}(\pi)$ *can be quickly computed to be 3. Since the* m*-bounded inversions are*

$$(2,4) \quad \textit{and} \quad (3,5),$$

$\mathsf{dinv}(\sigma) = 3 - 2 = 1$.

Recall that the inverse descent set of the parking function is equal to the descents in the window. Therefore, $\mathsf{iDes}(\sigma) = \{3\}$.

3.2 Affine Permutations and Parking Functions

An affine n-permutation is a permutation σ of Z such that

$$\sigma(i+n) = \sigma(i) + n$$

for all $i \in \mathbb{Z}$ and

$$\sum_{i=1}^{n} \sigma(i) = \frac{n(n+1)}{2}.$$

An affine n-permutation is m-bounded if $\sigma^{-1}(i) < \sigma^{-1}(i+m)$ for all $i \in \mathbb{Z}$.

There is a correspondence between (m,n)-parking functions and m-bounded affine n-permutations, see [GMV14]. Suppose that P is a parking function. To obtain the affine permutation from the window notation of P, add $\kappa = n + 1 - \mathsf{area}(P)$ to each rank:

$$\sigma = [w_1, w_2, \ldots, w_n] \longrightarrow [w_1 + \kappa, w_2 + \kappa, \ldots, w_n + \kappa] = \hat{\sigma}.$$

These affine permutations are m-bounded because for all i,

$$\sigma^{-1}(i) < \sigma^{-1}(i+m).$$

The statistics of the parking function can be recovered from the affine permutation. Let a be the smallest letter appearing in the window of $\hat{\sigma}$; then

$$\mathsf{area}(P) = 1 - a.$$

An m-bounded inversion is a pair (i,j) such that

$$i < j, \quad 1 \leqslant j \leqslant n, \quad \text{and} \quad \hat{\sigma}(j) < \hat{\sigma}(i) < \hat{\sigma}(j) + m.$$

Let the number of m-bounded inversions of $\hat{\sigma}$ be denoted $\mathsf{inv}^{\mathsf{m}}(\hat{\sigma})$, which gives

$$\mathsf{dinv}(P) = \frac{(m-1)(n-1)}{2} - \mathsf{inv}^{\mathsf{m}}(\hat{\sigma}).$$

Example 10. *Recall that* $\sigma = [-7, -1, 3, -3, 1, 5, 9]$ *is a* $(4, 7)$*-parking function from Example 7. The corresponding affine* 7*-permutation is*

$$\hat{\sigma} = [-3, 3, 7, 1, 5, 9, 13],$$

obtained by adding $\kappa = 4$ *to each entry in the window.*

The area of σ *can be recovered from* $\hat{\sigma}$ *as* $1 - (-3) = 4$*. And the* m*-bounded inversions of* $\hat{\sigma}$ *are*

$$(-7, 1), \quad (-4, 1), \quad (-2, 1), \quad (-1, 4), \quad (0, 2), \quad (0, 5),$$

which can be seen by extending $\hat{\sigma}$*'s window to the left:*

$$\hat{\sigma} = \dots, -1, -10, -4, 0, -6, -2, 2, 6, [-3, 3, 7, 1, 5, 9, 13],$$

and

$$(2, 4), \quad \text{and} \quad (3, 5).$$

So

$$\mathsf{dinv}(\sigma) = \frac{(7-1)(4-1)}{2} - 8 = 1.$$

Similarly, $\mathsf{iDes}(\sigma)$ *can be read from the window of* $\hat{\sigma}$ *since a constant shift doesn't change descents. So* $\mathsf{iDes}(\sigma) = \{3\}$*.*

3.3 The $F_{[n-1]}$-term

Let m and n be coprime. The highest order term $F_{[n-1]}$ of $\mathcal{H}_{m,n}$ occurs when the parking function consists of a Dyck path π and the longest permutation $w_0 \in \mathfrak{S}_n$. This can only happen when for each rank $k \in R(\pi)$, $k + m$ does not appear in $R(\pi)$. Another way of saying this is that each rank on the path π must lie in a distinct column of the (m, n)-diagram.

Since this can only happen when $m \geqslant n$, we have the following proposition:

Proposition 11. *The coefficient of* $F_{[n-1]}$ *is 0 in all* $\mathcal{H}_{m,n}$ *where* $m < n$*.*

Theorem 12. *For any* m, n *coprime with* $m > n$*, the coefficient of* $F_{[n-1]}$ *in* $\mathcal{H}_{m,n}$ *is* $C_{m-n,n}$*.*

Proof. Suppose that π is an (m, n)-Dyck path with $m > n$ coprime and each rank on the path in a distinct column. The term in the $F_{[n-1]}$-coefficient in $\mathcal{H}_{m,n}$ arising from π corresponds to the parking function

$$\sigma = \{w_1, w_2, \dots, w_n\}$$

where $w_i > w_{i+1}$ for all i.

This parking function corresponds to an affine permutation, which can be factored into an affine part and a standard permutation $\hat{\sigma} = (\eta, \omega_0)$. We claim that (η, id) is $(m - n)$-bounded and

$$\frac{(m-n-1)(n-1)}{2} - \mathsf{inv}^{\mathsf{m-n}}(\eta, id) = \frac{(m-1)(n-1)}{2} - \mathsf{inv}^{\mathsf{m}}(\eta, \omega_0). \tag{15}$$

To show that (η, id) is $(m-n)$-bounded first note that the window of (η, ω_0) is decreasing. So if i appears in the window of (η, ω_0) then $i+m$ must appear right of the window. Therefore $i+m-n$ must either appear in the window or right of the window. But the window of (η, id) is increasing, so $i+m-n$ must appear to the right of i. Since this holds for all i in the window, it holds for all i.

To prove Equation (15) we will show that

$$\mathsf{inv}^{\mathsf{m}}(\eta, \omega_0) - \mathsf{inv}^{\mathsf{m-n}}(\eta, id) = \binom{n}{2}.$$

We do this by noting that for every pair (i, j) in the window of (η, id) with $i < j$, if $j > i + m$, there is a k_j to the left of the window with $i + m - n < k_j < i + m$. □

Since for degree 2, $F_{\emptyset} = s_{(1,1)}$ and $F_{\{1\}} = s_{(2)}$ this gives:

Corollary 13. *For any odd* $m > 0$,

$$\mathcal{H}_{m,2}(X; q, t) = C_{m,2}(q, t)s_{(1,1)}(X) + C_{m-2,2}(q, t)s_{(2)}(X).$$

4 The Hikita Polynomials $\mathcal{H}_{m,3}$

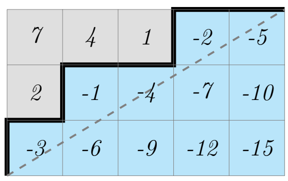
Using Theorem 12 we can expand

$$\mathcal{H}_{m,3}(q, t) = C_{m,3}(q, t)F_{\emptyset}(X) + p_1^m(q, t)F_{\{1\}}(X) + p_2^m(q, t)F_{\{2\}}(X) + C_{m-3,3}(q, t)F_{\{1,2\}}(X)$$

for some polynomials p_1^m and p_2^m. We will continue by investigating the polynomials $p_1^m(q, t)$ and $p_2^m(q, t)$.

Consider the set of $(m, 3)$-Dyck paths for m not divisible by 3. We can uniquely describe an $(m, 3)$-Dyck path by stating the number of cells above the path in rows 1 and 2, respectively. We will denote this by $D_m(k, \ell)$, where $0 \leqslant k < 2m/3$, $0 \leqslant \ell < m/3$ and $\ell \leqslant k$.

Example 14.

$D_5(3, 1) =$ 

.

For a fixed m we will partition the set of $(m, 3)$-Dyck paths into various types, based on k and ℓ:

- Type 0: $\{D_m(0, 0)\}$,
- Type 1: $\{D_m(k, k) : k < m/3\}$,
- Type 2a: $\{D_m(k, 0) : k < m/3\}$,

- Type 2b: $\{D_m(k,0) : k > m/3\}$,
- Type 3a: $\{D_m(k,\ell) : 0 < \ell < k, k < m/3\}$,
- Type 3b: $\{D_m(k,\ell) : 0 < \ell, k > m/3\}$.

Define the polynomials

$$P_y^m(q,t) = \sum_{\pi \text{ type } y} t^{\mathsf{area}(\pi)} q^{\mathsf{dinv}(\pi)},$$

so that

$$C_{m,3}(q,t) = \sum_{y\in\{0,1,2a,2b,3a,3b\}} P_y^m(q,t).$$

Since there is only one Dyck path of type 0,

$$P_0^m(q,t) = t^{m-1}.$$

Recall from Proposition 6 that any cell in the middle row of a $(m,3)$-Dyck path π is in $\mathsf{Dinv}(\pi)$. We will now explore which cells in the top row are in $\mathsf{Dinv}(\pi)$ and how the parking functions on π affect the statistics.

4.1 The polynomial P_1^m

Let m not be divisible by 3. Let π be an $(m,3)$-Dyck path of type 1 and let $(x,3)$ be any cell above the path in the top row. Consider that

$$\begin{aligned}\gamma[(x,3)^{\Rightarrow}] &= 2m-3-3k\\ &> m-3x\\ &= \gamma(x,2) = \gamma[(x,3)^{\downarrow}],\end{aligned}$$

since $x < k < m/3$. So by Theorem 2 no cells in the top row are in $\mathsf{Dinv}(\pi)$, and therefore

$$P_1^m(q,t) = \sum_{1\leqslant k<m/3} t^{m-1-2k} q^k.$$

There are three parking functions associated to any $(m,3)$-Dyck path of type 1:

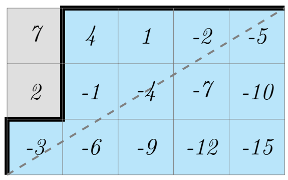
$$p_1 = [-3,a,a+m], \qquad p_2 = [a,-3,a+m], \qquad p_3 = [a,a+m,-3].$$

If p_i are the parking functions associated to the Dyck path $D_m(k,k)$ then $a = m-3-3k < m-3$. Note that $\mathsf{Des}(p_1) = \emptyset, \mathsf{Des}(p_2) = \{1\}$ and $\mathsf{Des}(p_3) = \{2\}$.

So $(1,2)$ is an m-bounded inversion in p_2 and $(1,3)$ is an m-bounded inversion in p_3. Therefore

$$\mathsf{dinv}(p_2) = \mathsf{dinv}(p_3) = \mathsf{dinv}(D_m(k,k)) - 1. \tag{16}$$

Example 15. *The type 1* $(5,3)$*-Dyck path is:*

$D_5(1,1) =$  ,

so

$$P_1^5(q,t) = q^1 t^2.$$

There are three associated parking functions:

$$[-3,-1,4], \qquad [-1,-3,4], \qquad [-1,4,-3]$$

which contribute terms

$$q^1 t^2 F_{\emptyset}, \qquad t^2 F_{\{1\}}, \qquad t^2 F_{\{2\}}$$

to $\mathcal{H}_{5,3}(X;q,t)$*, respectively.*

4.2 The polynomial P_{2a}^m

Let m not be divisible by 3. Let π be an $(m,3)$-Dyck path of type $2a$ and let $(x,3)$ be any cell above the path in the top row. Consider that

$$\begin{aligned}\gamma[(x,3)^{\rightarrow}] &= 2m - 3k \\ &> m - 3x \\ &= \gamma(x,2) = \gamma[(x,3)^{\Downarrow}],\end{aligned}$$

since $x < k < m/3$. In addition,

$$\gamma[(x,3)^{\downarrow}] = \gamma(x,3) > \gamma[(x,3)^{\Rightarrow}].$$

So by Theorem 2 every cell above the path is in $\mathsf{Dinv}(\pi)$, and therefore

$$P_{2a}^m(q,t) = \sum_{1 \leqslant k < m/3} t^{m-1-k} q^k.$$

There are three parking functions associated to any $(m,3)$-Dyck path of type $2a$:

$$p_1 = [-3, m-3, b], \qquad p_2 = [-3, b, m-3], \qquad p_3 = [b, -3, m-3].$$

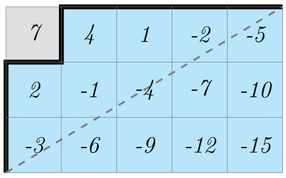
If p_i are the parking functions associated to the Dyck path $D_m(k,0)$ then $m - 3 < b = 2m - 3 - 3k < 2m - 3$, since $k < m/3$. Note that $\mathsf{Des}(p_1) = \emptyset, \mathsf{Des}(p_2) = \{2\}$ and $\mathsf{Des}(p_3) = \{1\}$.

So $(2,3)$ is an m-bounded inversion in p_2 and $(1,3)$ is an m-bounded inversion in p_3. Therefore

$$\mathsf{dinv}(p_2) = \mathsf{dinv}(p_3) = \mathsf{dinv}(D_m(k,0)) - 1. \tag{17}$$

Example 16. *The type 2a* $(5,3)$*-Dyck path is:*

$$D_5(1,0) =$$



,

so

$$P^5_{2a}(q,t) = q^1 t^3.$$

There are three associated parking functions:

$$[-3,2,4], \qquad [-3,4,2], \qquad [4,-3,2]$$

which contribute terms

$$q^1 t^3 F_{\emptyset}, \qquad t^3 F_{\{2\}}, \qquad t^3 F_{\{1\}}$$

to $\mathcal{H}_{5,3}(X;q,t)$*, respectively.*

4.3 The polynomial P^m_{2b}

Let m not be divisible by 3 and let π be an $(m,3)$-Dyck path of type $2b$ and let $(x,3)$ for $x < k - m/3$ be a cell above the path in the top row. Consider that since $k > m/3$,

$$\begin{aligned}\gamma[(x,3)^{\rightarrow}] &= 2m - 3k \\ &< m - 3x \\ &= \gamma(x,2) = \gamma[(x,3)^{\Downarrow}].\end{aligned}$$

So by Theorem 2 every cell above the path with $x < k - m/3$ is not in $\mathsf{Dinv}(\pi)$. However, every cell $(x,3)$ above the path with $x > k - m/3$ is in $\mathsf{Dinv}(\pi)$. This can be seen because $m - 3x < 2m - 3k$, reversing the inequality above, and $\gamma[(x,3)^{\downarrow}] = \gamma(x,3) > \gamma[(x,3)^{\Rightarrow}]$. Therefore,

$$P^m_{2b}(q,t) = \sum_{m/3<k<2m/3} t^{m-1-k} q^{\lceil m/3 \rceil}.$$

There are three parking functions associated to any $(m,3)$-Dyck path of type $2b$:

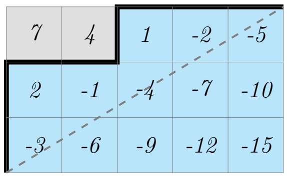
$$p_1 = [-3, m-3, b], \qquad p_2 = [-3, b, m-3], \qquad p_3 = [b, -3, m-3].$$

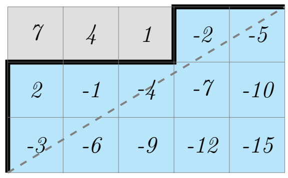
If p_i are the parking functions associated to the Dyck path $D_m(k,0)$ then $-3 < b = 2m - 3 - 3k < m - 3$, since $k > m/3$. Note that $\mathsf{Des}(p_1) = \{2\}, \mathsf{Des}(p_2) = \emptyset$ and $\mathsf{Des}(p_3) = \{1\}$.

So $(2,3)$ is an m-bounded inversion in p_1 and $(1,2)$ is an m-bounded inversion in p_3. Therefore

$$\mathsf{dinv}(p_1) = \mathsf{dinv}(p_3) = \mathsf{dinv}(D_m(k,0)) - 1. \tag{18}$$

Example 17. *The type 2b* $(5,3)$*-Dyck paths are:*

$D_5(2,0) =$ 

and $D_5(3,0) =$ 

,

so

$$P^5_{2b}(q,t) = q^2t^2 + q^2t^1.$$

$D_5(2,0)$ *has three associated parking functions:*

$$[-3,2,1], \qquad [-3,1,2], \qquad [1,-3,2]$$

which contribute terms

$$q^1t^1F_{\{2\}}, \qquad q^2t^1F_{\emptyset}, \qquad q^1t^1F_{\{1\}}$$

to $\mathcal{H}_{5,3}(X;q,t)$*, respectively.*

4.4 The polynomial P^m_{3a}

Let m not be divisible by 3. There are six parking functions associated to any $(m,3)$-Dyck path of type $3a$:

$$p_1 = [-3,a,b], \qquad p_2 = [-3,b,a], \qquad p_3 = [a,-3,b],$$

$$p_4 = [a,b,-3], \qquad p_5 = [b,-3,a], \qquad p_6 = [b,a,-3],$$

with $a < b$.

Let π be a $(m,3)$-Dyck path of type $3a$ and let $(x,3)$ be a cell above the path in the top row such that $(x,2)$ is also above the path. Since $k < m/3$,

$$\begin{aligned}\gamma[(x,3)^{\Rightarrow}] &= 2m-3-3k \\ &> m-3x \\ &= \gamma(x,2) = \gamma[(x,3)^{\downarrow}].\end{aligned}$$

By Theorem 2 every cell in the middle row that is above the path is immediately below a cell that is not in $\mathsf{Dinv}(\pi)$. But for any cell $(x,3)$ that does not have a cell immediately below it that is above the path,

$$\begin{aligned}\gamma[(x,3)^{\rightarrow}] &= 2m-3k \\ &> m-3x \\ &= \gamma(x,2) = \gamma[(x,3)^{\Downarrow}],\end{aligned}$$

and $\gamma[(x,3)^{\downarrow}] = \gamma(x,3) > \gamma[(x,3)^{\Rightarrow}]$, so it is in $\mathsf{Dinv}(\pi)$. Therefore,

$$P^m_{3a}(q,t) = \sum_{2\leqslant k<m/3}\sum_{\ell=1}^{k-1} t^{m-1-k-\ell}q^k.$$

If p_i are the parking functions associated to the Dyck path $D_m(k,\ell)$ then $a = m - 3\ell$ and $b = 2m - 3k$ with $m - 3 < b < a + m$ since $k < m/3$. Note that

$$\mathsf{Des}(p_1) = \emptyset, \qquad \mathsf{Des}(p_2) = \mathsf{Des}(p_4) = \{2\},$$

$$\mathsf{Des}(p_3) = \mathsf{Des}(p_5) = \{1\}, \qquad \mathsf{Des}(p_6) = \{1,2\}.$$

So $(2,3)$ is an m-bounded inversion in p_2, $(1,2)$ is an m-bounded inversion in p_3, and $(1,3)$ is an m-bounded inversion in p_4, and p_5. Therefore

$$\mathsf{dinv}(p_2) = \mathsf{dinv}(p_5) = \mathsf{dinv}(p_3) = \mathsf{dinv}(p_4) = \mathsf{dinv}(D_m(k,\ell)) - 1. \tag{19}$$

Example 18. *The $(3,5)$-diagram does not support any Dyck paths of type $3a$. So*

$$P^5_{3a}(q,t) = 0.$$

4.5 The polynomial P^m_{3b}

Let m not be divisible by 3 and let π be an $(m,3)$-Dyck path of type $3b$. We will break these Dyck paths into two further subcases, those with $\ell < k - m/3$ and those with $\ell > k - m/3$. In the first case, when $\ell < k - m/3$, if $\ell < x < k - m/3$ then the cell $(x,3)$ is not in $\mathsf{Dinv}(\pi)$. This is because

$$\begin{aligned}\gamma[(x,3)^{\Downarrow}] &= m - 3x \\ &> 2m - 3k \\ &= \gamma[(x,3)^{\rightarrow}]\end{aligned}$$

and Theorem 2. The cells $(x,3)$ where $x \leqslant \ell$ are all in $\mathsf{Dinv}(\pi)$ because

$$\begin{aligned}\gamma[(x,3)^{\downarrow}] &= m - 3x \\ &> 2m - 3 - 3k \\ &= \gamma[(x,3)^{\Rightarrow}]\end{aligned}$$

and

$$\begin{aligned}\gamma[(x,3)^{\rightarrow}] &\geqslant -2 \\ &> -3x \\ &= \gamma[(x,3)^{\Downarrow}]\end{aligned}$$

Therefore, these $(m,3)$-Dyck paths contribute

$$\sum_{m/3+1<k<2m/3} \left(\sum_{1\leqslant \ell < k - m/3} q^{\lceil m/3\rceil + 2\ell} t^{m-1-k-\ell} \right) \tag{20}$$

to $C_{m,3}(q,t)$.

In the second case, where $\ell > k - m/3$, then every cell $(x, 3)$ that is immediately above a cell in the middle row that is above the path and $x > k + 1 - m/3$ is not in $\mathsf{Dinv}(\pi)$. This is because

$$\begin{aligned}\gamma[(x,3)^{\downarrow}] &= m - 3x \\ &< 2m - 3 - 3k \\ &= \gamma[(x,3)^{\Rightarrow}].\end{aligned}$$

However, for every cell $(x, 3)$ with $x < k + 1 - m/3$,

$$\begin{aligned}\gamma[(x,3)^{\downarrow}] &= m - 3x \\ &> 2m - 3 - 3k \\ &= \gamma[(x,3)^{\Rightarrow}],\end{aligned}$$

and

$$\begin{aligned}\gamma[(x,3)^{\rightarrow}] &\geqslant -2 \\ &> -3x \\ &= \gamma[(x,3)^{\Downarrow}],\end{aligned}$$

so the cell is in $\mathsf{Dinv}(\pi)$, by Theorem 2. Therefore, these $(m, 3)$-Dyck paths contribute

$$\sum_{m/3<k<2m/3} \left(\sum_{k-m/3<\ell<m/3} q^{2k-\lfloor m/3 \rfloor} t^{m-1-k-\ell} \right) \tag{21}$$

to $C_{m,3}(q, t)$.

By summing Lines (20) and (21),

$$P_{3b}^{m}(q,t) = \sum_{m/3+1<k<2m/3} \left(\sum_{1\leqslant\ell<k-m/3} q^{\lceil m/3 \rceil+2\ell} t^{m-1-k-\ell} \right) + \sum_{m/3<k<2m/3} \left(\sum_{k-m/3<\ell<m/3} q^{2k-\lfloor m/3 \rfloor} t^{m-1-k-\ell} \right) .$$

The type 3b $(m, 3)$-Dyck paths also have six parking functions associated to each of them:

$$\begin{aligned} p_1 &= [-3, a, b], & p_2 &= [-3, b, a], & p_3 &= [a, -3, b], \\ p_4 &= [a, b, -3], & p_5 &= [b, -3, a], & p_6 &= [b, a, -3], \end{aligned}$$

where a is the rank appearing in row 2 and b is the rank appearing in row 1.

If p_i are the parking functions associated to the Dyck path $D_m(k,\ell)$ then $a = m - 3\ell$ and $b = 2m - 3k$ with $b < 2m - 3$ and $a, b > m - 3$. If $a < b$ then the case proceeds similarly to type $3a$:

$$\mathsf{Des}(p_1) = \emptyset, \qquad \mathsf{Des}(p_2) = \mathsf{Des}(p_4) = \{2\},$$

$$\mathsf{Des}(p_3) = \mathsf{Des}(p_5) = \{1\}, \qquad \mathsf{Des}(p_6) = \{1, 2\}.$$

Also, $(2,3)$ is an m-bounded inversion in p_2 and $(1,2)$ is an m-bounded inversion in p_3. However, $(1,3)$ and $(2,3)$ are m-bounded inversions in p_4 and $(1,2)$ and $(1,3)$ are m-bounded inversions in p_5. Therefore

$$\mathsf{dinv}(p_2) = \mathsf{dinv}(p_3) = \mathsf{dinv}(D_m(k,\ell)) - 1, \tag{22}$$
$$\mathsf{dinv}(p_4) = \mathsf{dinv}(p_5) = \mathsf{dinv}(D_m(k,\ell)) - 2. \tag{23}$$

If $b < a$ then

$$\mathsf{Des}(p_1) = \mathsf{Des}(p_6) = \{2\}, \qquad \mathsf{Des}(p_2) = \emptyset,$$

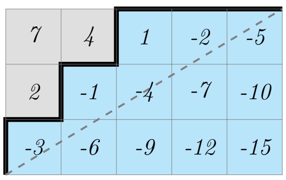
$$\mathsf{Des}(p_4) = \{1, 2\}, \qquad \mathsf{Des}(p_3) = \mathsf{Des}(p_5) = \{1\}.$$

In this case, $(2,3)$ is an m-bounded inversion of p_1 and $(1,2)$ is an m-bounded inversion of p_5. Both $(1,2)$ and $(1,3)$ are m-bounded inversions of p_3 and $(1,3)$ and $(2,3)$ are m-bounded inversions of p_6. Therefore

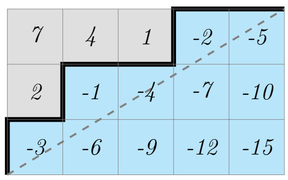
$$\mathsf{dinv}(p_1) = \mathsf{dinv}(p_5) = \mathsf{dinv}(D_m(k,\ell)) - 1, \tag{24}$$
$$\mathsf{dinv}(p_3) = \mathsf{dinv}(p_6) = \mathsf{dinv}(D_m(k,\ell)) - 2. \tag{25}$$

Example 19. *The type 3b* $(5,3)$*-Dyck paths are:*

$D_5(2,1) =$



and $D_5(3,1) =$



,

so

$$P^5_{3b}(q,t) = q^3 t^1 + q^4.$$

$D_5(2,1)$ *has six associated parking functions:*

$$[-3,-1,1], \quad [-3,1,-1], \quad [-1,-3,1], \quad [-1,1,-3], \quad [1,-3,-1], \quad [1,-1,-3]$$

which contribute terms

$$q^3 t^1 F_{\emptyset}, \quad q^2 t^1 F_{\{2\}}, \quad q^2 t^1 F_{\{1\}}, \quad q^1 t^1 F_{\{2\}}, \quad q^1 t^1 F_{\{1\}}, \quad t^1 F_{\{1,2\}},$$

to $\mathcal{H}_{5,3}(X;q,t)$*, respectively.*

4.6 The polynomials $K_{m-1,3}$ and $K_{m-2,3}$

Let m not be divisible by 3.

Lines (16)–(25) give that the coefficient of $F_{\{1\}}$ in $\mathcal{H}_{m,3}$ is

$$p_1^m = q^{-1} \cdot (P_1^m + P_{2a}^m + P_{2b}^m + 2P_{3a}^m) + (q^{-1} + q^{-2}) \cdot P_{3b}^m.$$

We claim that

$$K_{m-1,3} = q^{-1} \cdot (P_{2a}^m + P_{2b}^m + P_{3a}^m + P_{3b}^m)$$

and

$$K_{m-2,3} = q^{-1} \cdot (P_1^m + P_{3a}^m) + q^{-2} \cdot P_{3b}^m.$$

Consider $q^{-1} \cdot s_{(a,b)}(q,t)$ for $a \geqslant b > 0$,

$$\begin{aligned} q^{-1} \cdot s_{(a,b)}(q,t) &= q^{-1} \cdot \sum_{i=b}^{a} q^{a+b-i} t^i \\ &= \sum_{i=b}^{a} q^{(a-1)+b-i} t^i \\ &= q^{b-1} t^a + \sum_{i=b}^{a-1} q^{(a-1)+b-i} t^i \\ &= q^{b-1} t^a + s_{(a-1,b)}(q,t) \end{aligned}$$

where $s_{(b-1,b)}(q,t) = 0$. If we sum over each type, S,

$$\sum_S P_S^m = C_{m,3},$$

so

$$p_1^m = q^{-1} \cdot (C_{m,3} - P_0^m) + q^{-1} P_{3a}^m + q^{-2} P_{3b}^m.$$

Consider that

$$\begin{aligned} q^{-1} \cdot (C_{m,3} - t^{m-1}) &= \frac{\sum_{0 \leqslant i < m/3} s_{(m-1-2i,i)}(q,t) - t^{m-1}}{q} \\ &= \frac{s_{(m-1)}(q,t) - t^{m-1} + \sum_{1 \leqslant i < m/3} s_{(m-1-2i,i)}(q,t)}{q} \\ &= s_{(m-2)}(q,t) + \sum_{1 \leqslant i < m/3} \left(s_{(m-2-2i,i)}(q,t) + q^{i-1} t^{m-1-2i} \right) \\ &= \sum_{0 \leqslant i < m/3} s_{(m-2-2i,i)}(q,t) + \sum_{1 \leqslant i < m/3} q^{i-1} t^{m-1-2i} \\ &= K_{m-1,3}(q,t) + q^{-1} \cdot P_1^m. \end{aligned}$$

This proves that

$$p_1^m = K_{m-1,3} + q^{-1}(P_1^m + P_{3a}^m) + q^{-2} P_{3b}^m,$$

or

$$K_{m-1,3} = q^{-1} \cdot (P^m_{2a} + P^m_{2b} + P^m_{3a} + P^m_{3b}). \tag{26}$$

To address the second part, we will map a $(m, 3)$-Dyck path $D_m(k, \ell)$ with $\ell > 0$ to the $(m-2, 3)$-Dyck path $D_{m-2}(k-1, \ell-1)$. Each cell (x, y) in $D_{m-2}(k-1, \ell-1)$ has the same arm and leg as $(x+1, y)$ in $D_m(k, \ell)$ so

$$(x, y) \in \mathsf{Dinv}(D_{m-2}(k-1, \ell-1)) \quad \Longleftrightarrow \quad (x+1, y) \in \mathsf{Dinv}(D_m(k, \ell)).$$

Similarly, it is immediate that $\mathsf{area}(D_{m-2}(k-1, \ell-1)) = \mathsf{area}(D_m(k, \ell))$.

Therefore, we need to consider cells $(1, 2)$ and $(1, 3)$ in $D_m(k, \ell)$ to compute $\mathsf{dinv}(D_{m-2}(k-1, \ell-1))$. Since $(1, 2)$ is in the middle row, $(1, 2) \in \mathsf{Dinv}(D_m(k, \ell))$. The cell $(1, 3)$ is in $\mathsf{Dinv}(D_m(k, \ell))$ if and only if

$$\begin{aligned} \gamma_m[(1,3)^{\Rightarrow}] < m-3 \quad &\Longleftrightarrow \quad k > m/3 \\ &\Longleftrightarrow \quad D_m(k,\ell) \text{ is type 3b.} \end{aligned}$$

So if $D_m(k, \ell)$ is type 3b,

$$\mathsf{dinv}(D_{m-2}(k-1, \ell-1)) = \mathsf{dinv}(D_m(k, \ell)) - 2,$$

otherwise

$$\mathsf{dinv}(D_{m-2}(k-1, \ell-1)) = \mathsf{dinv}(D_m(k, \ell)) - 1.$$

Since every $(m-2, 3)$-Dyck path is the image of a $(m, 3)$ dyck path of type 1, 3a, or 3b by removing one cell from row 2 and one cell from row 3,

$$K_{m-2,3} = q^{-1}(P^m_1 + P^m_{3a}) + q^{-2} P^m_{3b}. \tag{27}$$

5 Isomorphisms Between Parking Functions and Dyck Paths

Let $\Pi(k, n)$ be the set of, possibly modified in the sense of Section 2, (k, n)-Dyck paths and $\mathsf{PF}(m, n)$ be the set of (m, n)-parking functions for m, n relatively prime. Let

$$\mathsf{PF}^S(m, n) = \{P \in \mathsf{PF}(m, n) | \mathsf{iDes}(P) = S\}.$$

We will refer to a bijection φ between sets of parking functions or Dyck paths as an *isomorphism* if

$$\mathsf{area}(P) = \mathsf{area}(\varphi(P)) \quad \text{and} \quad \mathsf{dinv}(P) = \mathsf{dinv}(\varphi(P)).$$

It is well-known that there is an isomorphism between $\mathsf{PF}^{\emptyset}(m, n)$ and $\Pi(m, n)$ by mapping a parking function to its underlying Dyck path. For $m > n$, Theorem 12 gives an isomorphism between $\mathsf{PF}^{[n-1]}(m, n)$ and $\Pi(m-n, n)$.

Section 4 suggests an isomorphism between $\mathsf{PF}^{\{1\}}(m, 3)$ and $\Pi(m-1, 3) \cup \Pi(m-2, 3)$. Note that exactly one of these sets of Dyck paths must be modified Dyck paths as per

Section 2. Let $P \in \mathsf{PF}^{\{1\}}(m,3)$ and let $\hat{\sigma} = [x,y,z]$ be the associated affine permutation. Since $\mathsf{Des}(P) = \{1\}$ it follows that $x > y, z > y$.

By [KL15] there is a unique association of $(k,3)$-Dyck paths with pairs of non-negative integers (a,d) where $a, d \geqslant k - a - d - 1$ by setting $a = \mathsf{area}(\pi)$ and $d = \mathsf{dinv}(\pi)$ for a given Dyck path π. To construct an $(m,3)$-Dyck path corresponding to a pair of such integers (a,d):

1. draw the $(m,3)$-diagram,
2. highlight the largest d ranks,
3. find the next largest rank, r,
4. highlight the next $m-1-a-d$ largest ranks that do not lie in the same row as r,
5. finally, draw the Dyck path so that all of the highlighted ranks are above the path.

This is called the *Rank Word Construction Algorithm.*

Remark 20. Note that this association applies to modified Dyck paths in the sense of Section 2. This is because we know that the $(3k,3)$-modified Catalan polynomial can be realized in terms of $(3k+1,3)$-Dyck paths. Since each term of $K_{(3k,3)}$ is a term of $C_{(3k+1,3)}$ it corresponds to a unique $(3k+1,3)$-Dyck path such that the rank 1 is not above the path. Thus to construct a $(3k,3)$-modified Dyck path we simply apply the association to the $(3k+1,3)$-diagram, obtain a $(3k+1,3)$-Dyck path, and then superimpose that Dyck path over the $(3k,3)$-diagram.

Define $\varphi : \mathsf{PF}^{\{1\}}(m,3) \to \Pi(m-1,3) \cup \Pi(m-2,3)$ by $\varphi(P) = \pi$ where

$$\mathsf{area}(\pi) = \mathsf{area}(P) \qquad \text{and} \qquad \mathsf{dinv}(\pi) = \mathsf{dinv}(P)$$

and $\pi \in \Pi(m-1,3)$ if the underlying Dyck path of P is Type 2 or Type 3 with $x < z$, otherwise $\pi \in \Pi(m-2,3)$.

5.1 Symmetry Results

In the previous section we were studying the Hikita polynomials

$$\begin{aligned}&\mathcal{H}_{m,3}(X;q,t)\\ &\quad = C_{m,3}(q,t)F_{\emptyset}(X) + p_1^m(q,t)F_{\{1\}}(X) + p_2^m(q,t)F_{\{2\}}(X) + C_{m-3,3}(q,t)F_{\{1,2\}}(X).\end{aligned}$$

By observing equations (16) – (25) we note that $p_1^m = p_2^m$, for all m. This means that $H_{m,3}$ is symmetric in X and can be written

$$\mathcal{H}_{m,3}(X;q,t) = C_{m,3}(q,t)s_{(1,1,1)}(X) + p^m(q,t)s_{(2,1)}(X) + C_{m-3,3}(q,t)s_{(3)}(X),$$

where $p^m(q,t) = p_1^m(q,t) = p_2^m(q,t)$. This necessarily pairs a parking function with inverse descent set $\{1\}$ to a unique parking function with inverse descent set $\{2\}$ with identical statistics.

In light of Equations (26) and (27), we can express p^m as

$$p^m(q,t) = K_{m-1,3}(q,t) + K_{m-2,3}(q,t).$$

So

$$\begin{aligned}&\mathcal{H}_{m,3}(X;q,t)\\ &\quad = C_{m,3}(q,t)s_{(1,1,1)}(X) + (K_{m-1,3}(q,t) + K_{m-2,3}(q,t))s_{(2,1)}(X) + C_{m-3,3}(q,t)s_{(3)}(X).\end{aligned}$$

Each of these (modified) rational Catalan polynomials is symmetric in q and t, [GM14, KL15]. That is,

$$C_{a,3}(q,t) = C_{a,3}(t,q)$$

and

$$K_{b,3}(q,t) = K_{b,3}(t,q)$$

for any b and any a not divisible by 3. Therefore the Hikita polynomials $\mathcal{H}_{m,3}$ are symmetric in q and t:

$$\mathcal{H}_{m,3}(X;q,t) = \mathcal{H}_{m,3}(X;t,q).$$

To more explicitly see this association, begin with a parking function $P = [x,y,z] \in \mathsf{PF}(m,3)$, where $\hat{P}$ is the underlying Dyck path, $\mathsf{area}(P) = a$, and $\mathsf{dinv}(P) = d$. Map P to the Dyck path π with $\mathsf{area}(\pi) = d$ and $\mathsf{dinv}(\pi) = a$ and

$$\pi \in \begin{cases} \Pi(m,3) & \text{if } \mathsf{iDes}(P) = \emptyset \\ \Pi(m-1,3) & \text{if } |\mathsf{iDes}(P)| = 1 \text{ and } \hat{P} \text{ is Type 2 or Type 3 with } x < z \\ \Pi(m-2,3) & \text{if } |\mathsf{iDes}(P)| = 1 \text{ and } \hat{P} \text{ is Type 1 or Type 3 with } x > z \\ \Pi(m-3,3) & \text{if } \mathsf{iDes}(P) = \{1,2\} \end{cases}.$$

If π is Type 0, Type 1, or Type 2, it is sufficient finish the association by choosing the unique parking function P' defined on π with $\mathsf{iDes}(P') = \mathsf{iDes}(P)$. For Type 3, then associate P to the parking function $P' = [x', y', z']$ where $x' > z'$ if and only if $x > z$.

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