Fully packed loop configurations: polynomiality and nested arches

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Submitted: Apr 12, 2017; Accepted: Jan 17, 2018; Published: Feb 16, 2018
Mathematics Subject Classifications: 05A15, 82B20, 82B23

Abstract

This article proves a conjecture by Zuber about the enumeration of fully packed loops (FPLs). The conjecture states that the number of FPLs whose link pattern consists of two noncrossing matchings which are separated by m nested arches is a polynomial function in m of certain degree and with certain leading coefficient. Contrary to the approach of Caselli, Krattenthaler, Lass and Nadeau (who proved a partial result) we make use of the theory of wheel polynomials developed by Di Francesco, Fonseca and Zinn-Justin. We present a new basis for the vector space of wheel polynomials and a polynomiality theorem in a more general setting. This allows us to finish the proof of Zubers conjecture.

Keywords: fully packed loop configurations, alternating sign matrices, wheel polynomials, nested arches, quantum Knizhnik-Zamolodchikov equations

1 Introduction

Alternating sign matrices (ASMs) are combinatorial objects with many different faces. They were introduced by Robbins and Rumsey in the 1980s and arose from generalizing the determinant. Together with Mills, they [9] conjectured a closed formula for the enumeration of ASMs of given size, first proven by Zeilberger [12]. Using a second guise of ASMs, the six vertex model, Kuperberg [8] could find a different proof for their enumeration. A more detailed account on the history of the ASM Theorem can be found in [2].

A third way of looking at ASMs are fully packed loops (FPLs). We obtain by using the FPL description a natural refined counting A_π of ASMs by means of noncrossing

*Supported by the Austrian Science Foundation FWF, START grant Y463.

matchings. Razumov and Stroganov [10] conjecturally connected FPLs to the $O(1)$ loop model, a model in statistical physics. Proven by Cantini and Sportiello [3], this connection allows a description of $(A_\pi)_{\pi\in\mathrm{NC}_n}$ as an eigenvector of the Hamiltonian of the $O(1)$ loop model, where NC_n is the set of noncrossing matchings of size n. Assuming the (at that point unproven) Razumov-Stroganov conjecture to be true, Zuber [15] formulated nine conjectures about the numbers A_π. In this paper we finish the proof of the following conjecture.

Theorem 1 ([15, Conjecture **7**]). *For noncrossing matchings $\pi_1 \in \mathrm{NC}_{n_1}$, $\pi_2 \in \mathrm{NC}_{n_2}$ and an integer m, the number of FPLs with link pattern $(\pi_1)_m\pi_2$ is a polynomial in m of degree $|\lambda(\pi_1)| + |\lambda(\pi_2)|$ with leading coefficient $\frac{f_{\lambda(\pi_1)}f_{\lambda(\pi_2)}}{|\lambda(\pi_1)|!|\lambda(\pi_2)|!}$, where f_λ denotes the number of standard Young tableaux of shape λ.*

Caselli, Krattenthaler, Lass and Nadeau [4] proved this for empty π_2 and showed that $A_{(\pi_1)_m\pi_2}$ is a polynomial for large values of m with correct degree and leading coefficient. In this paper we prove that the number $A_{(\pi_1)_m\pi_2}$ is a polynomial function in m, which is achieved without relying on the work of [4], and hence finish together with the results of [4] the proof of Theorem 1.

We conclude the introduction by sketching the theory on which the proof of Theorem 1 relies and giving an overview of this paper. In the next section we introduce the combinatorial objects and their notions.

As mentioned before the Razumov-Stroganov-Cantini-Sportiello Theorem 5 states that $(A_\pi)_{\pi\in\mathrm{NC}_n}$ is up to multiplication by a constant the unique eigenvector to the eigenvalue 1 of the Hamiltonian of the homogeneous $O(1)$ loop model. In Section 3 we present that in a special case solutions of the quantum Knizhnik-Zamolodchikov (qKZ) equations lie in the eigenspace to the eigenvalue 1 of the Hamiltonian of the inhomogeneous $O(1)$ loop model. Di Francesco and Zinn-Justin [5] could characterise the components of these solutions in a different way, namely as wheel polynomials. The specialisation of the inhomogeneous to the homogeneous $O(1)$ loop model means for wheel polynomials performing the evaluation $z_1 = \ldots = z_{2n} = 1$. Summarising, for every $\pi \in \mathrm{NC}_n$ there exists an element Ψ_π of the vector space $W_n[z]$ of wheel polynomials such that $A_\pi = \Psi_\pi(1,\ldots,1)$.

$$\begin{array}{lcccc} \text{FPLs} \overset{\mathrm{RSCS-Thm}}{\longleftrightarrow} \text{hom } O(1) & \overset{\text{specialisation}}{\longleftarrow} & \text{inhom } O(1) \overset{\mathrm{Di\ F.-Z.\ J.}}{\longleftrightarrow} & & W_n[z] \\ A_\pi = \Psi_\pi(1,\ldots,1) & \overset{\text{evaluation}}{\longleftarrow} & & & \Psi_\pi \end{array}$$

We introduce a new family of wheel polynomials D_{π_1,π_2} such that every $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ is a linear combination of D_{σ_1,σ_2} where ρ is the rotation acting on noncrossing matchings and for $i = 1, 2$ the Young diagram $\lambda(\sigma_i)$ is included in the Young diagram $\lambda(\pi_i)$.

The advantage of the wheel polynomials D_{π_1,π_2} over $\Psi_{\pi_1\pi_2}$ becomes clear in Section 4. We prove in Lemma 19 in a more general setting that $D_{\pi_1,\pi_2}(1,\ldots,1)$ is a polynomial function with degree at most $|\lambda(\pi_1)| + |\lambda(\pi_2)|$. This lemma applied in our situation and using the rotational invariance $A_\pi = A_{\rho(\pi)}$ imply the polynomiality in Theorem 1.

An extended abstract of this work was published in the Proceedings of FPSAC 2016 [1].

2 Definitions

This section should be understood as a handbook of the combinatorial objects involved in this paper.

2.1 Noncrossing matchings and Young diagrams

A *noncrossing matching* of size n consists of $2n$ points on a line labelled from left to right with the numbers $1, \dots, 2n$ together with n pairwise noncrossing arches above the line such that every point is endpoint of exactly one arch. An example can be found in Figure 1.

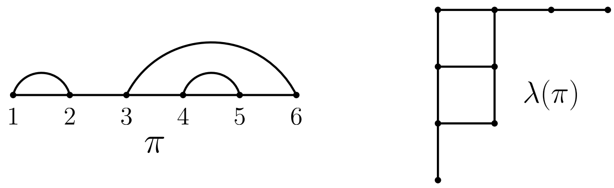

Figure 1: A noncrossing matching π of size 3 and its corresponding Young diagram $\lambda(\pi)$.

For two noncrossing matchings σ and π, denote their concatenation by $\sigma\pi$. For an integer n we define $(\pi)_n$ as the noncrossing matching π surrounded by n nested arches, see Figure 2. Define NC_n as the set of noncrossing matchings of size $2n$. It is easy to see that $|\mathrm{NC}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, where C_n is the n-th Catalan number.

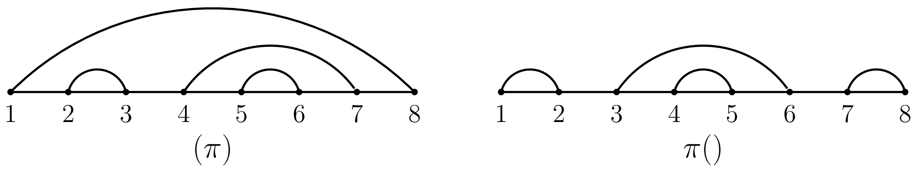

Figure 2: The noncrossing matchings (π) and $\pi()$ where π is the noncrossing matching of Figure 1.

A *Young diagram* is a finite collection of boxes, arranged in left-justified rows and weakly decreasing row-length from top to bottom. We can think of a Young diagram λ as a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, where λ_i is the number of boxes in the i-th row from top. Noncrossing matchings of size n are in bijection to Young diagrams for which the i-th row from top has at most $n - i$ boxes for $1 \leqslant i \leqslant n$. For a noncrossing matching π its corresponding Young diagram $\lambda(\pi)$ is given by the area enclosed between two paths with same start- and endpoint. The first path consists of n consecutive north-steps followed by n consecutive east-steps. We construct the second path by reading the numbers from left to right and drawing a north-step if the number labels a left-endpoint of an arch and an east-step otherwise. An example of a noncrossing matching and its corresponding Young diagram is given in Figure 1. For a given noncrossing matching π and a positive integer

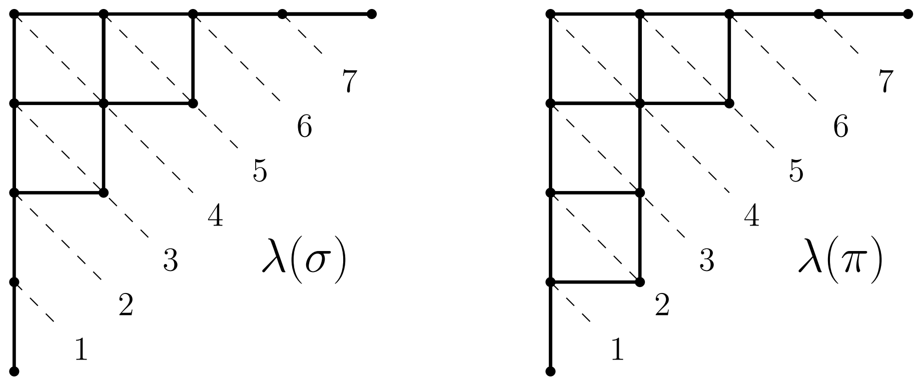

Figure 3: The matchings σ, π satisfy $\sigma \nearrow_2 \pi$.

k the Young diagrams $\lambda(\pi)$ and $\lambda((\pi)_k)$ are the same. To be able to distinguish between them we will always draw the first path of the above algorithm in the pictures of $\lambda(\pi)$.

We define a partial order on the set NC_n of noncrossing matchings via $\sigma < \pi$ iff the Young diagram $\lambda(\sigma)$ is contained in the Young diagram $\lambda(\pi)$. For $2 \leqslant j \leqslant 2n-2$ we write $\sigma \nearrow_j \pi$ if $\lambda(\pi)$ is obtained by adding a box to $\lambda(\sigma)$ on the j-th diagonal, where the diagonals are labelled as in Figure 3. This labelling of the diagonals is the second reason for drawing the consecutive north and east steps in the pictures of the Young diagrams.

2.2 The Temperley-Lieb Operators

We define first the *rotation* $\rho : \mathrm{NC}_n \to \mathrm{NC}_n$. Two numbers i and j are connected in $\rho(\pi)$ for $\pi \in \mathrm{NC}_n$ iff $i-1$ and $j-1$ are connected in π, where we identify $2n+1$ with 1. The *Temperley-Lieb operator* e_j for $1 \leqslant j \leqslant 2n$ is a map from noncrossing matchings of size n to themselves. For a given $\pi \in \mathrm{NC}_n$ the noncrossing matching $e_j(\pi)$ is obtained by deleting the arches which are incident to the points $j, j+1$ and adding an arch between $j, j+1$ and an arch between the points former connected to j and $j+1$. Thereby we identify $2n+1$ with 1. There exists also a graphical representation of the Temperley-Lieb operators. Applying e_j on a noncrossing matching π is done by attaching the diagram of e_j, depicted in Figure 4, at the bottom of the diagram of π and simplifying the paths to arches. An example for this is given in Figure 5.

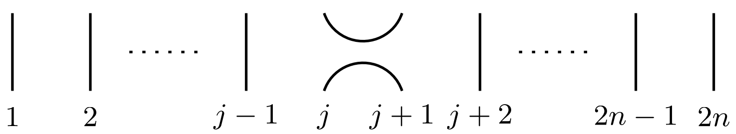

Figure 4: The graphical representation of e_j

Since noncrossing matchings of size n are in bijection with Young diagrams whose i-th row from the top has at most $n-i$ boxes, we can define e_j also for such Young diagrams

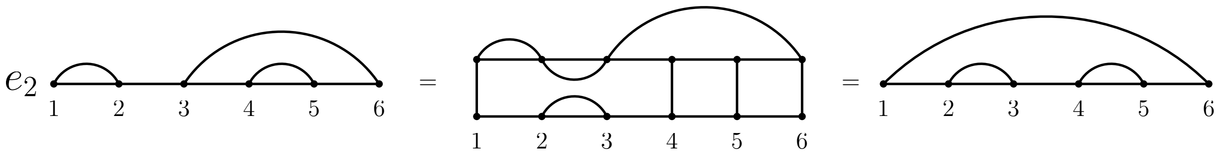

Figure 5: Calculating $e_j(\pi)$ graphically with π from the previous example.

via $e_j(\lambda(\pi)) := \lambda(e_j(\pi))$. For $1 \leqslant j \leqslant 2n-1$ the action of e_j on Young diagrams is depicted in Figure 6. The operator e_{2n} maps a Young diagram to itself iff the i-th row has less than $n-i$ boxes for all $1 \leqslant i \leqslant n-1$. Otherwise the Young diagram corresponds to a noncrossing matching of the form $(\alpha)\beta(\gamma)$, where α, β, γ are noncrossing matchings of smaller size. In this case e_{2n} maps this Young diagram to the one corresponding to the noncrossing matching $(\alpha(\beta)\gamma)$, as depicted in Figure 7.

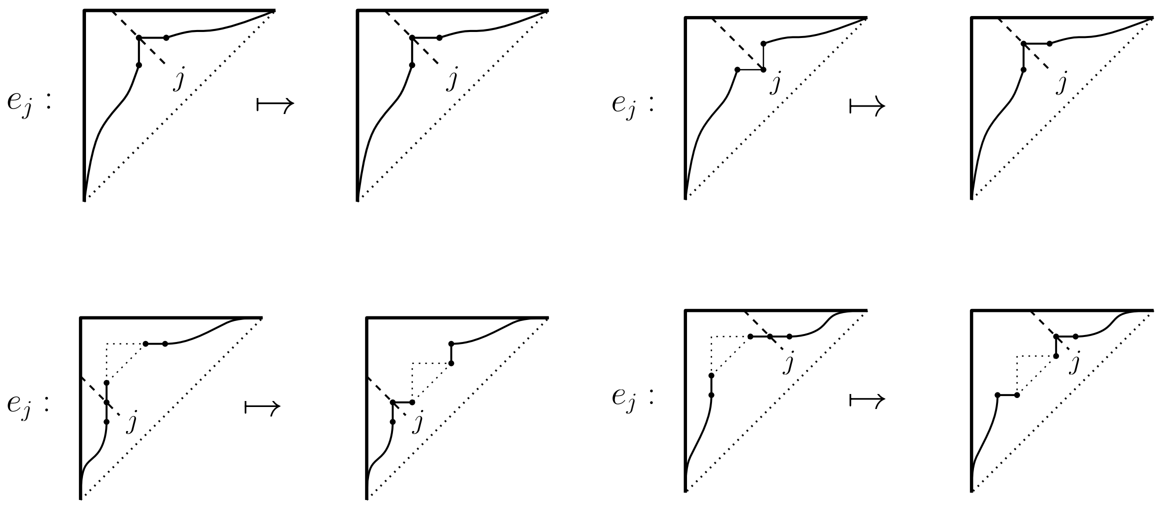

Figure 6: The action of e_j for $1 \leqslant j \leqslant 2n-1$ on Young diagrams corresponding to noncrossing matchings of size n.

The next lemma is an easy consequence of the above observations.

Lemma 2. *1. For a noncrossing matching π of size n and $2 \leqslant j \leqslant 2n-2$, the preimage $e_j^{-1}(\pi)$ is a subset of $\{\sigma | \pi \nearrow_j \sigma\} \cup \{\sigma | \sigma \leqslant \pi\}$.*

2. Let $\alpha \in \mathrm{NC}_n$, $\beta, \gamma \in \mathrm{NC}_{n'}$ be noncrossing matchings such that there exists $2 \leqslant i \leqslant 2n'-2$ with $\beta \nearrow_i \gamma$. Then the preimage $e_{2n+i}^{-1}(\alpha\beta)$ is given by

$$e_{2n+i}^{-1}(\alpha\beta) = \{\alpha\sigma | \sigma \in e_i^{-1}(\beta)\}.$$

Proof. 1. If π has no arch between j and $j+1$, then $e_j^{-1}(\pi) = \emptyset$. Figure 7 displays the action of e_j on Young diagrams and implies the statement if π has an arch between j and $j+1$.

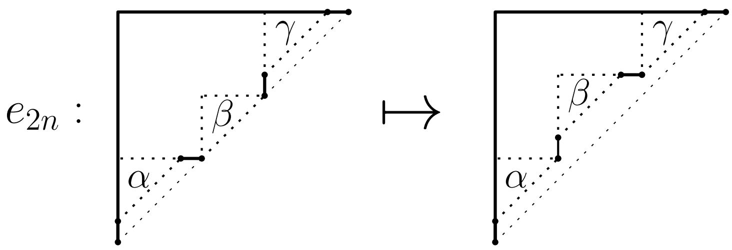

Figure 7: The action of e_{2n} on Young diagrams corresponding to noncrossing matchings of size n of the form $(\alpha)\beta(\gamma)$, where α, β, γ are noncrossing matchings.

2. Let $\sigma \in e_{2n+i}^{-1}(\alpha\beta)$ and denote by x, y the labels which are connected in σ to $2n+i$ or $2n+i+1$ respectively. By definition of e_{2n+i} the noncrossing matchings $\alpha\beta$ and σ differ only in the arches between $2n+i, 2n+i+1, x, y$. The existence of an γ with $\beta \nearrow_i \gamma$ means there exists an arch in β with left-endpoint before i and right-endpoint after i, hence surrounding $2n+i$ and $2n+i+1$. Therefore x and y must be surrounded by this arch or they are the labels of the points connected by this arch. In both cases $x, y \geqslant 2n$ which implies σ can be written as $\alpha\sigma'$ with $e_i(\sigma') = \beta$. □

The *Temperley-Lieb algebra* with parameter $\tau = -(q + q^{-1})$ of size $2n$ is generated by the Temperley-Lieb operators e_i with $1 \leqslant i \leqslant 2n$ over $\mathbb{C}$. The elements e_i, e_j satisfy for all $1 \leqslant i, j \leqslant 2n$ the following relations

$$\begin{aligned} e_i^2 &= \tau e_i, \\ e_i e_j &= e_j e_i \qquad \textit{if } 2 \leqslant |(i-j)| \leqslant 2n-2, \\ e_i e_{i\pm 1} e_i &= e_i. \end{aligned}$$

Throughout this paper we interpret $e_i v$ for some vector $v \in \{f | f : \mathrm{NC}_n \to V\}$ and a vector space V always as the action of an element of the Temperley-Lieb algebra on the vector v, where the Temperley-Lieb operators act as permutations, i. e., $e_i((v_\pi)_{\pi \in \mathrm{NC}_n}) = (v_{e_i(\pi)})_{\pi \in \mathrm{NC}_n}$.

2.3 Fully packed loop configurations

A *fully packed loop configuration* (or short *FPL*) F of size n is a subgraph of the $n \times n$ grid with n external edges on every side with the following two properties.

1. All vertices of the $n \times n$ grid have degree 2 in F.

2. F contains every other external edge, beginning with the topmost at the left side.

An FPL consists of pairwise disjoint paths and loops. Every path connects two external edges. We number the external edges in an FPL counter-clockwise with 1 up to $2n$, see Figure 8. This allows us to assign to every FPL F a noncrossing matching $\pi(F)$, where i and j are connected by an arch in $\pi(F)$ if they are connected in F. We call $\pi(F)$

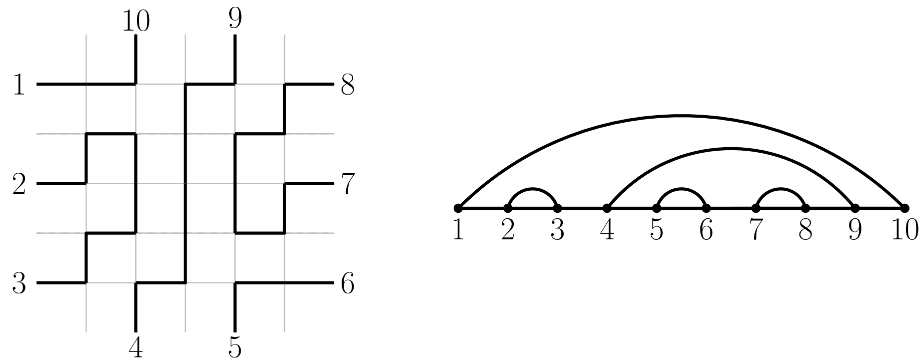

Figure 8: An example of an FPL of size 5 and its link pattern.

the *link pattern* of F and write A_π for the number of FPLs F with link pattern $\pi(F) = \pi$.

It is well known that FPLs and alternating sign matrices (ASMs) are in bijection. The number of FPLs of size n, denoted by $\mathrm{ASM}(n)$, is hence given by the ASM-Theorem [9, 12]

$$\mathrm{ASM}(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

2.4 The (in-)homogeneous $O(\tau)$ loop model

A configuration of the *inhomogeneous $O(\tau)$ loop model* of size n is a tiling of $[0, 2n] \times [0, \infty)$ with plaquettes of side length 1 depicted in Figure 9. To obtain a cylinder we identify the half-lines $\{(0, t), t \geqslant 0\}$ and $\{(2n, t), t \geqslant 0\}$. In the following we assume that the cylindrical loop percolations are filled randomly with the two plaquettes, where the probability to place the first plaquette of Figure 9 in column i is p_i with $0 < p_i < 1$ for all $1 \leqslant i \leqslant 2n$. If the probability does not depend on the column, i. e., $p_1 = \ldots = p_{2n}$, we call it the *homogeneous $O(\tau)$ loop model*. We parametrise the probabilities $p_i = \frac{qz_i - q^{-1}t}{qt - q^{-1}z_i}$ and $\tau = q + q^{-1}$. The two plaquettes in Figure 9 are interpreted to consist of two paths. By concatenating the paths of a plaquette with the paths of the neighbouring plaquettes, we see that a cylindrical loop percolation consists of noncrossing paths.

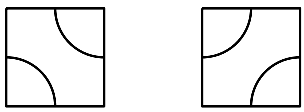

Figure 9: The two different plaquettes.

Lemma 3. *With probability 1 all paths in a random cylindrical loop percolation are finite.*

A proof for the homogeneous case can be found in [11, Lemma 1.6], the inhomogeneous case can be proven analogously. For a configuration C of the $O(\tau)$ loop model, we label

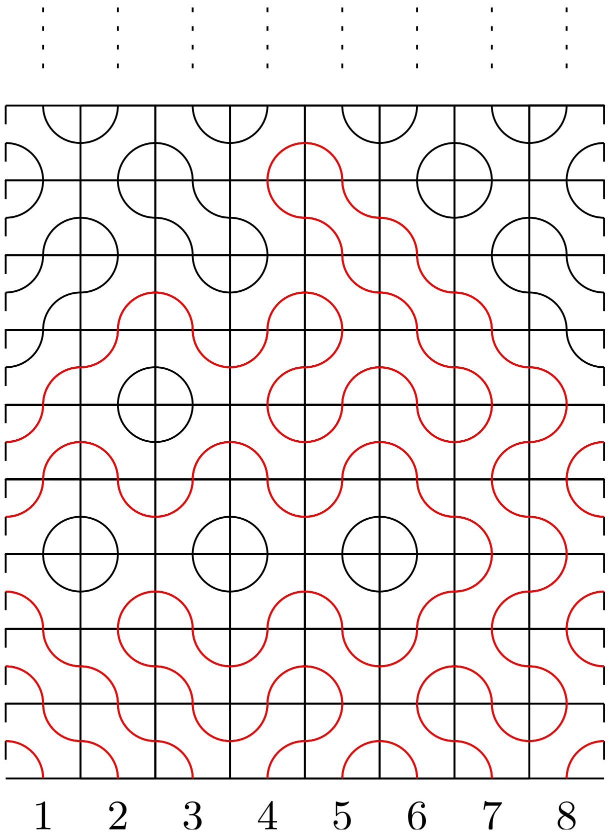

Figure 10: The beginning of a cylindrical loop percolation, where the paths starting and ending at the bottom are drawn in red.

the points $(i - \frac{1}{2}, 0)$ with i for $1 \leqslant i \leqslant 2n$. We define the *connectivity pattern* $\pi(C)$ as the noncrossing matching connecting i and j by an arch iff i and j are connected by paths in C. By the above lemma $\pi(C)$ is well defined for almost all cylindrical loop percolations C. For $\pi \in \mathrm{NC}_n$ denote by $\hat{\Psi}_\pi(t; z_1, \ldots, z_{2n})$ the probability that a configuration C has the connectivity pattern π and write $\hat{\Psi}_n(t; z_1, \ldots, z_{2n}) = (\hat{\Psi}_\pi(t; z_1, \ldots, z_{2n}))_{\pi \in \mathrm{NC}_n}$.

We define a Markov chain on the set NC_n of noncrossing matchings of size n. The transitions are given by putting $2n$ plaquettes below a noncrossing matching and simplify the paths to obtain a new noncrossing matching. An example is given in Figure 11. The probability of one transition is given by the product of the probabilities of placing the plaquettes, where placing the first plaquette of Figure 9 at the i-th position is p_i as before. We denote by $T_n(t; z_1, \ldots, z_{2n})$ the transition matrix of this Markov chain. By the Perron-Frobenius Theorem the matrix $T_n(t; z_1, \ldots, z_{2n})$ has 1 as an eigenvalue and the stationary distribution of the Markov chain is up to scaling the unique eigenvector with associated eigenvalue 1. Every configuration C of the inhomogeneous $O(\tau)$ loop model can be obtained uniquely by pushing all the plaquettes of a configuration C' one row up and filling the empty bottom row with plaquettes. Therefore the vector $\hat{\Psi}_n(t; z_1, \ldots, z_{2n})$

is the stationary distribution of this Markov chain and hence satisfies

$$T_n(t; z_1, \ldots, z_{2n}) \hat{\Psi}_n(t; z_1, \ldots, z_{2n}) = \hat{\Psi}_n(t; z_1, \ldots, z_{2n}). \tag{1}$$

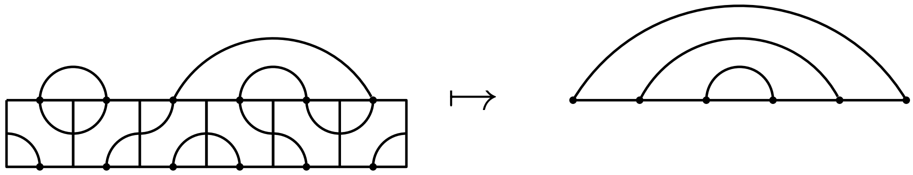

Figure 11: An example for a state transition starting with the noncrossing matching π of Figure 1. The transition probability is $p_1(1-p_2)(1-p_3)p_4p_5(1-p_6)$.

We define the Hamiltonian as the linear map $\mathcal{H}_n := \sum_{j=1}^{2n} e_j$, where e_j is interpreted as an element of the Temperley-Lieb algebra.

Theorem 4. *The stationary distribution* $\hat{\Psi}_n(t) = \hat{\Psi}_n(t; 1, \ldots, 1)$ *satisfies for* $\tau = 1$

$$\mathcal{H}_n(\hat{\Psi}_n(t)) = 2n\hat{\Psi}_n(t). \tag{2}$$

Further $\hat{\Psi}_n(t)$ *is independent of* t *and uniquely determined by* (2).

A proof of this theorem can be found for example in [11, Appendix B], however note that the matrix H_n defined there is given by $2n \cdot \mathrm{Id} - \mathcal{H}_n$.

The following theorem was conjectured by Razumov and Stroganov in [10] and later proven by Cantini and Sportiello in [3]. It creates a connection between fully packed loop configurations and the stationary distribution of the homogeneous $O(1)$ loop model.

Theorem 5 (Razumov-Stroganov-Cantini-Sportiello Theorem)**.** *Let* $n \in \mathbb{N}$*, set* $q = e^{\frac{2\pi i}{3}}$ *and* $\hat{\Psi}_\pi = \hat{\Psi}_\pi(-q; 1, \ldots, 1)$*. For all* $\pi \in \mathrm{NC}_n$ *holds*

$$\hat{\Psi}_\pi = \frac{A_\pi}{\mathrm{ASM}(n)}.$$

3 The vector space $W_n[z]$

3.1 The quantum Knizhnik-Zamolodchikov equations

In order to introduce the quantum Knizhnik-Zamolodchikov equations (qKZ-equations), we need to define first the R-matrix and the operator S_i

$$\check{R}_i(u) = \frac{(qu - q^{-1})\mathrm{Id} + (u-1)e_i}{q - q^{-1}u},$$

$$S_i(z_1, \ldots, z_{2n}) = \prod_{k=1}^{i-1} \check{R}_{i-k}\left(\frac{z_{i-k}}{q^6 z_i}\right) \rho \prod_{k=1}^{2n-i} \check{R}_{2n-k}\left(\frac{z_{2n-k+1}}{z_i}\right),$$

for $1 \leqslant i \leqslant 2n$, where e_i is understood as an element of the Temperley-Lieb algebra and ρ is the rotation as defined in section 2.2. Denote by $\Psi_n = (\Psi_\pi)_{\pi \in \mathrm{NC}_n}$ a function in $z_1, \dots, z_{2n}, q$. The level 1 *qKZ-equations* are a system of $2n$ equations

$$S_i(z_1, \dots, z_{2n})\Psi_n(t; z_1, \dots, z_{2n}) = \Psi_n(t; z_1, \dots, q^6 z_i, \dots, z_{2n}), \tag{3}$$

with $1 \leqslant i \leqslant 2n$. In the following we need the $2n+1$ equations

$$\check{R}_i\left(\frac{z_{i+1}}{z_i}\right)\Psi_n(t; z_1, \dots, z_{2n}) = \Psi_n(t; z_1, \dots, z_{i+1}, z_i, \dots, z_{2n}), \tag{4a}$$

$$\rho^{-1}\Psi_n(t; z_1, \dots, z_{2n}) = \Psi_n(t; z_2, \dots, z_{2n}, q^6 z_1), \tag{4b}$$

where $1 \leqslant i \leqslant 2n$ in (4a).

Proposition 6 ([13, section 4.1 and 4.3])**.** *1. The system of equations* (4a) *and* (4b) *imply the system of equations* (3)*.*

2. For $q = e^{\frac{2\pi i}{3}}$ and hence $\tau = 1$, it holds $S_i(z_1, \dots, z_{2n}) = T_n(z_i; z_1, \dots, z_{2n})$. By using Lagrange interpolation one can show that (3) *imply* (1)*. Since the solutions of* (1) *form a one dimensional vector space, the same is true for solutions of the system of equations* (3) *for $q = e^{\frac{2\pi i}{3}}$.*

3.2 Wheel polynomials

It turns out [5, Theorem 4] that for $q = e^{\frac{2\pi i}{3}}$ the components $\hat{\Psi}_\pi(t; z_1, \dots, z_{2n})$ of the stationary distribution of the inhomogeneous $O(1)$ loop model are up to a common factor homogeneous polynomials in $z_1, \dots, z_{2n}$ of degree $n(n-1)$ which are independent of t. In this section we characterise these homogeneous polynomials. In fact we characterise homogeneous solutions of degree $n(n-1)$ of (4a) and (4b) which are by Proposition 6 for $q = e^{\frac{2\pi i}{3}}$ also solutions of (1). The results presented here can be found in [5, 6, 7, 13, 14] and [11].

Definition 7. Let n be a positive integer and q a variable. A homogeneous polynomial $p \in \mathbb{Q}(q)[z_1, \dots, z_{2n}]$ of degree $n(n-1)$ is called *wheel polynomial* of order n if it satisfies the *wheel condition*:

$$p(z_1, \dots, z_{2n})|_{q^4 z_i = q^2 z_j = z_k} = 0, \tag{5}$$

for all triples $1 \leqslant i < j < k \leqslant 2n$. Denote by $W_n[z]$ the $\mathbb{Q}(q)$-vector space of wheel polynomials of order n.

Theorem 8 ([6, Section 4.2])**.** *The dual space $W_n[z]^*$ of $W_n[z]$ is given by*

$$W_n[z]^* = \bigoplus_{\pi \in \mathrm{NC}_n} \mathbb{Q}(q)\mathrm{ev}_\pi,$$

where ev_π is defined as $\mathrm{ev}_\pi : p(z_1, \dots, z_{2n}) \mapsto p(q^{\epsilon_1(\pi)}, \dots, q^{\epsilon_{2n}(\pi)})$ with $\epsilon_i(\pi) = -1$ iff an arch of π has a left-endpoint labelled with i and $\epsilon_i(\pi) = 1$ otherwise.

Define the linear maps $\mathbf{S}_k, \mathbf{D}_k : \mathbb{Q}(q)[z_1, \ldots, z_{2n}] \to \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ for $1 \leqslant k \leqslant 2n$ as

$$\mathbf{S}_k : \quad f(z_1, \ldots, z_{2n}) \mapsto f(z_1, \ldots, z_{k+1}, z_k, \ldots, z_{2n}), \tag{6}$$

$$\mathbf{D}_k : \quad f \mapsto \frac{qz_k - q^{-1}z_{k+1}}{z_{k+1} - z_k}(\mathbf{S}_k(f) - f). \tag{7}$$

By setting $\mathbf{D}_{k+2n} := \mathbf{D}_k$ we extend the definition of $\mathbf{D}_k$ to all integers k.
The operators $\mathbf{D}_k$ are introduced as an abbreviation for $(qz_k - q^{-1}z_{k+1})\delta_k$, where $\delta_k = \frac{1}{z_{k+1}-z_k}(\mathbf{S}_k - \mathrm{Id})$ has been used before, e. g., in [13]. One can verify easily the following Lemma.

Lemma 9. *1. The space $W_n[z]$ of all wheel polynomials of order n is closed under the action of $\mathbf{D}_k$ for $1 \leqslant k \leqslant 2n-1$. If $q = e^{\frac{2\pi i}{3}}$ the vector space $W_n[z]$ is also closed under $\mathbf{D}_{2n}$.*

2. For all $1 \leqslant k \leqslant 2n$ and all polynomials $f, g \in \mathbb{Q}(q)[z_1, \ldots, z_{2n}]$ one has

$$\mathbf{D}_k(fg) = \mathbf{D}_k(f)\mathbf{S}_k(g) + f\mathbf{D}_k(g). \tag{8}$$

The following theorem describes a very important $\mathbb{Q}(q)$-basis of $W_n[z]$.

Theorem 10 ([13, Section 4.2])**.** *Set*

$$\Psi_{()_n}(z_1, \ldots, z_{2n}) := (q - q^{-1})^{-n(n-1)} \prod_{1 \leqslant i < j \leqslant n} (qz_i - q^{-1}z_j)(qz_{n+i} - q^{-1}z_{n+j}). \tag{9}$$

Define for two noncrossing matchings σ, π with $\sigma \nearrow_j \pi$

$$\Psi_\pi := \mathbf{D}_j(\Psi_\sigma) - \sum_{\tau \in e_j^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau. \tag{10}$$

Then Ψ_π is well-defined for all $\pi \in \mathrm{NC}_n$ and satisfies

$$\Psi_{\rho^{-1}(\pi)}(z_1, \ldots, z_{2n}) = \Psi_\pi(z_2, \ldots, z_{2n}, q^6 z_1). \tag{11}$$

The set $\{\Psi_\pi, \pi \in \mathrm{NC}_n\}$ is further a $\mathbb{Q}(q)$-basis of $W_n[z]$.

The noncrossing matchings τ which appear in the sum of (10) satisfy by Lemma 2 the relation $\tau < \pi$. Hence we can use (10) to calculate the basis Ψ_π of $W_n[z]$ recursively. The vector $\Psi_n = (\Psi_\pi)_{\pi \in \mathrm{NC}_n}$ satisfies (4a). This is true since we can reformulate (4a) as

$$e_i \Psi_n = \mathbf{D}_i(\Psi_n) - (q + q^{-1})\Psi_n, \tag{12}$$

for $1 \leqslant i \leqslant 2n-1$. Let $\sigma, \pi \in \mathrm{NC}_n$ with $\sigma \nearrow_i \pi$, then the σ component of both sides in (12) is

$$\Psi_\pi - (q + q^{-1})\Psi_\sigma + \sum_{\tau \in e_i^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau = \mathbf{D}_i(\Psi_\sigma) - (q + q^{-1})\Psi_\sigma,$$

which is exactly (10). Since Ψ_n satisfies (4b) by Theorem 10, Proposition 6 states that Ψ_n is a solution of the qKZ equations and therefore for $\tau = 1$ a multiple of the stationary distribution of the inhomogeneous $O(1)$ loop model. By setting $z_1 = \ldots = z_{2n} = 1$ Theorem 5 implies $\Psi_\pi(1,\ldots,1)|_{\tau=1} = cA_\pi$ for an appropriate constant c. Because of $\Psi_{()_n}(1,\ldots,1)|_{\tau=1} = 1 = A_{()_n}$ by definition, and Theorem 5 we obtain the following theorem.

Theorem 11. *Set $q = e^{\frac{2\pi i}{3}}$ and let $\pi \in \mathrm{NC}_n$, then one has*

$$\Psi_\pi(z_1,\ldots,z_{2n}) = \mathrm{ASM}(n) \times \hat{\Psi}_n(t; z_1,\ldots,z_{2n}),$$
$$\Psi_\pi(1,\ldots,1) = A_\pi.$$

3.3 A new basis for $W_n[z]$

The following lemma is a direct consequence of the definitions of the $\mathbf{D}_i$'s and $\Psi_{()_n}$.

Lemma 12. *Let n be a positive integer, then one has*

1. $\mathbf{D}_i \circ \mathbf{D}_i = (q+q^{-1})\mathbf{D}_i$ *for* $1 \leqslant i \leqslant 2n$,
2. $\mathbf{D}_i \circ \mathbf{D}_j = \mathbf{D}_j \circ \mathbf{D}_i$ *for* $1 \leqslant i,j \leqslant 2n$ *with* $|i-j| > 1$,
3. $\mathbf{D}_{i+1} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} + \mathbf{D}_i = \mathbf{D}_i \circ \mathbf{D}_{i+1} \circ \mathbf{D}_i + \mathbf{D}_{i+1}$ *for* $1 \leqslant i \leqslant 2n$,
4. $\mathbf{D}_i(\Psi_{()_n}) = (q+q^{-1})\Psi_{()_n}$ *for* $i \notin \{n, 2n\}$.

In the following we write $\prod_{i=1}^n \mathbf{D}_i$ for the product $\mathbf{D}_1 \circ \ldots \circ \mathbf{D}_n$. Let π be a noncrossing matching with corresponding Young diagram $\lambda(\pi) = (\lambda_1,\ldots,\lambda_l)$, i. e., λ_i is the number of boxes of $\lambda(\pi)$ in the i-th row from top. We define the wheel polynomial D_π by the following algorithm. First write in every box of $\lambda(\pi)$ the number of the diagonal the box lies on. The wheel polynomial D_π is then constructed recursively by "reading" in the Young diagram $\lambda(\pi)$ the rows from top to bottom and in the rows all boxes from left to right and apply $\mathbf{D}_{\textit{number in the box}}$ to the previous wheel polynomial, starting with $\Psi_{()_n}$, which is defined in (9). For π as in Figure 12 we obtain

$$D_\pi = (\mathbf{D}_{n-3} \circ \mathbf{D}_{n-2} \circ \mathbf{D}_n \circ \mathbf{D}_{n-1} \circ \mathbf{D}_{n+3} \circ \mathbf{D}_{n+2} \circ \mathbf{D}_{n+1} \circ \mathbf{D}_n)\,(\Psi_{()_n}).$$

Alternatively we can write D_π directly as

$$D_\pi = \left(\prod_{i=1}^{l} \prod_{j=1}^{\lambda_{l+1-i}} \mathbf{D}_{n+(i-l)+(\lambda_{l+1-i}-j)}\right)(\Psi_{()_n}). \tag{13}$$

Theorem 13. *The set of wheel polynomials $\{D_\pi | \pi \in \mathrm{NC}_n\}$ is a $\mathbb{Q}(q)$-basis of $W_n[z]$. Further Ψ_π is for $\pi \in \mathrm{NC}_n$ a linear combination of D_τ's with $\tau \leqslant \pi$ and the coefficient of D_π is 1.*

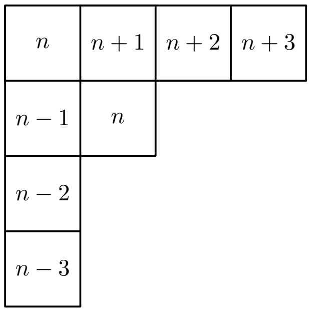

Figure 12: The numbers indicate the labels of the diagonals the boxes lie on.

Proof. We prove the second statement by induction on the number of boxes of $\lambda(\pi)$. It is by definition true for $()_n$, hence let the number $|\lambda(\pi)|$ be non-zero. Let σ be the noncrossing matching such that $\lambda(\sigma)$ is the Young diagram one obtains by deleting the rightmost box in the bottom row of $\lambda(\pi)$, and let i be the integer such that $\sigma \nearrow_i \pi$. Then Theorem 10 states

$$\Psi_\pi = \mathbf{D}_i \Psi_\sigma - \sum_{\tau \in e_i^{-1}(\sigma) \setminus \{\sigma, \pi\}} \Psi_\tau.$$

We use the induction hypothesis to express Ψ_τ and Ψ_σ as sums of $D_{\tau'}$ with $\tau' \leqslant \tau < \pi$ or $D_{\sigma'}$ with $\sigma' \leqslant \sigma < \pi$ respectively. The coefficient of D_σ in Ψ_σ is by the induction hypothesis equals to 1. Since all $\sigma' \leqslant \sigma$ satisfy the requirements of Lemma 14, this lemma implies the statement. By above arguments the set $\{D_\pi | \pi \in \mathrm{NC}_n\}$ is a $\mathbb{Q}(q)$-generating set for $W_n[z]$ of cardinality $\dim_{\mathbb{Q}(q)}(W_n[z])$, hence it is also a $\mathbb{Q}(q)$-basis. $\square$

The next lemma contains the technicalities which are needed to prove the above theorem.

Lemma 14. *Let $1 < i < 2n$ and $\sigma \in \mathrm{NC}_n$ such that the number of boxes on the i-th diagonal of $\lambda(\sigma)$ is less than the maximal possible number of boxes that can be placed there. Then $\mathbf{D}_i(D_\sigma) = D_\pi$ iff there exists a $\pi \in \mathrm{NC}_n$ with $\sigma \nearrow_i \pi$ or otherwise $\mathbf{D}_i(D_\sigma)$ is a $\mathbb{Q}(q)$-linear combination of D_τ's with $\tau \leqslant \sigma$.*

Proof. We use induction on the number of boxes of $\lambda(\sigma)$. We say that i appears in σ if there is a box in $\lambda(\sigma)$ which lies on the i-th diagonal.

1. Assume that i does not appear in σ. This implies that $i-1$ can not appear in σ. Then there are two cases:

   (a) First $i+1$ does not appear in σ. By Lemma 12 $\mathbf{D}_i$ commutes with all the $\mathbf{D}$-operators appearing in D_σ. If $i \neq n$ Lemma 12 states $\mathbf{D}_i(\Psi_{()_n}) = (q+q^{-1})\Psi_{()_n}$ and hence $\mathbf{D}_i(D_\sigma) = (q+q^{-1})D_\sigma$. The case $i = n$ implies $\sigma = ()_n$ and hence $\mathbf{D}_i(D_\sigma) = D_{(()())_{n-2}}$.

   (b) In the second case $i+1$ appears in σ. Then there is only one box on the $(i+1)$-th diagonal. This box is the leftmost box of the bottom row of $\lambda(\sigma)$. Let π

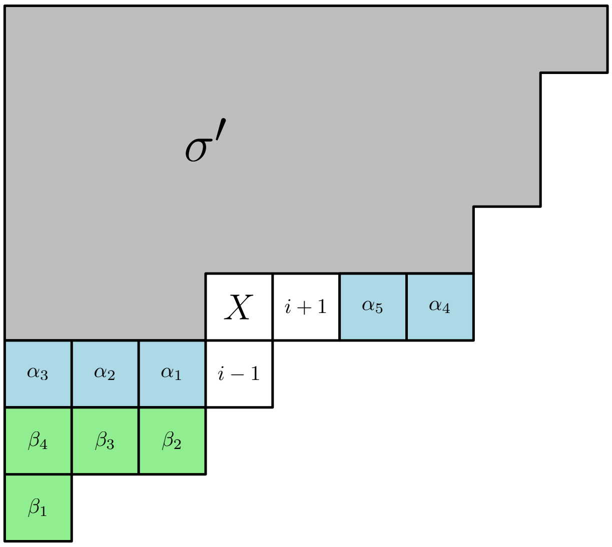

Figure 13: Schematic representation of $\lambda(\sigma)$ for σ as in the second case of the proof of Lemma 14 with $a = b = 1$.

be the noncrossing matching whose corresponding Young diagram is obtained by adding a box in a new row in $\lambda(\sigma)$, i. e., $\sigma \nearrow_i \pi$. By definition holds $D_\pi = \mathbf{D}_i(D_\sigma)$.

2. Let i appear in σ. We consider the lowest box in the i-th diagonal and call it X. Let σ' be the noncrossing matching of size n whose corresponding Young diagram $\lambda(\sigma')$ consists of all boxes above and to the left of the box X, denote by α_i with $1 \leqslant i \leqslant A$ the boxes to the right of X and in the row below but excluding the boxes in the $(i+1)$-th and $(i-1)$-th diagonal and by β_i with $1 \leqslant i \leqslant B$ the remaining boxes at the bottom. A schematic picture is given in Figure 13. Using the previous definitions we can write D_σ as

$$D_\sigma = \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \mathbf{D}_{i-1}^{b} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_{i+1}^{a} \circ \mathbf{D}_i \right) (D_{\sigma'}), \tag{14}$$

where a, b are 0 or 1.

(a) If $a = b = 0$ Lemma 12 (1,2) implies

$$\begin{aligned} \mathbf{D}_i D_\sigma &= \mathbf{D}_i \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_i \right) (D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_i^2 \right) (D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ ((q + q^{-1})\mathbf{D}_i) \right) (D_{\sigma'}) = (q + q^{-1}) D_\sigma. \end{aligned}$$

(b) For $a = b = 1$, the operator $\mathbf{D}_i$ commutes with all $\mathbf{D}_{\beta_l}$. As Figure 13 shows and by the assumptions on σ there exists a noncrossing matching π with $\sigma \nearrow_i \pi$. Hence one has

$$\mathbf{D}_i(D_\sigma) = \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \mathbf{D}_i \circ \mathbf{D}_{i-1} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_{i+1} \circ \mathbf{D}_i\right)(D_{\sigma'}) = D_\pi.$$

(c) For $a = 1, b = 0$ we obtain by Lemma 12 (2,3)

$$\begin{aligned}\mathbf{D}_i(D_\sigma) &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} \circ \mathbf{D}_i\right)(D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l}\right)\left((\mathbf{D}_{i+1} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} + \mathbf{D}_i - \mathbf{D}_{i+1})(D_{\sigma'})\right).\end{aligned}$$

By the induction hypothesis $((\mathbf{D}_{i+1} \circ \mathbf{D}_i \circ \mathbf{D}_{i+1} + \mathbf{D}_i - \mathbf{D}_{i+1})(D_{\sigma'}))$ is a linear combination of D_τ's with $\tau \leqslant \hat{\sigma}$ where $\hat{\sigma}$ is σ' with a box added on the i-th and $i+1$-th diagonal. Using again the induction hypothesis for the D_τ's with $\tau \leqslant \hat{\sigma}$ proofs the claim.

(d) Let $a = 0, b = 1$ and let $\hat{\sigma}$ be the noncrossing matching whose Young diagram consists of $\lambda(\sigma')$ and the boxes labelled with α_i for $1 \leqslant i \leqslant A$. Lemma 12 (2,3) implies

$$\begin{aligned}\mathbf{D}_i(D_\sigma) &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ \mathbf{D}_i \circ \mathbf{D}_{i-1} \circ \mathbf{D}_i \circ \prod_{l=1}^{A} \mathbf{D}_{\alpha_l}\right)(D_{\sigma'}) \\ &= \left(\prod_{l=1}^{B} \mathbf{D}_{\beta_l} \circ (\mathbf{D}_{i-1} \circ \mathbf{D}_i \circ \mathbf{D}_{i-1} + \mathbf{D}_i - \mathbf{D}_{i-1})\right)(D_{\hat{\sigma}}).\end{aligned}$$

We finish the proof by using the induction hypothesis analogously to the above case. □

Let $\pi \in \mathrm{NC}_n$ be a noncrossing matching given by $\pi = \pi_1\pi_2$ where π_i is a noncrossing matching of size n_i for $i = 1, 2$. We want to generalise D_π and Theorem 13 in the sense that we can write $\Psi_\pi = \Psi_{\pi_1\pi_2}$ as a linear combination of D_{τ_1,τ_2} with $\tau_i \leqslant \pi_i$ for $i = 1, 2$. This will not be possible for Ψ_π but for $\Psi_{\rho^{n_2}\pi}$. Let the Young diagram corresponding to π_2 be given as $\lambda(\pi_2) = (\lambda_1, \ldots, \lambda_l)$. The wheel polynomial D_{π_1,π_2} is then defined by the following algorithm. First we write in every box of $\lambda(\pi_2)$ the number of the diagonal the box lies on. The wheel polynomial $D_{\pi_1\pi_2}$ is then constructed recursively by "reading" in the Young diagram $\lambda(\pi_2)$ the rows from top to bottom and in the rows all boxes from left to right and apply $\mathbf{D}_{\textit{number in the box}-n}$ to the previous wheel polynomial, starting with $D_{(\pi_1)_{n_2}}$, which is defined in (13). Remember that we have extended the definition of $\mathbf{D}_k$ to all integers via $\mathbf{D}_k = \mathbf{D}_{k+2n}$. We can express D_{π_1,π_2} also by the following formula

$$D_{\pi_1,\pi_2} := \left(\prod_{i=1}^{l} \prod_{j=1}^{\lambda_{l+1-i}} \mathbf{D}_{(i-l)+(\lambda_{l+1-i}-j)}\right)(D_{(\pi_1)_{n_2}}).$$

For π_2 as in Figure 12 we obtain

$$D_{\pi_1,\pi_2} = (\mathbf{D}_{-3} \circ \mathbf{D}_{-2} \circ \mathbf{D}_0 \circ \mathbf{D}_{-1} \circ \mathbf{D}_3 \circ \mathbf{D}_2 \circ \mathbf{D}_1 \circ \mathbf{D}_0)\,(D_{(\pi_1)_{n_2}}).$$

Theorem 15. *Let $\pi = \pi_1\pi_2, \pi_1, \pi_2$ be noncrossing matching of size n, n_1 or n_2 respectively and set $q = e^{\frac{2\pi i}{3}}$. The wheel polynomial*

$$\Psi_{\rho^{n_2}(\pi_1\pi_2)}(z_1, \ldots, z_{2n}) = \Psi_{\pi_1\pi_2}(z_{2n+1-n_2}, \ldots, z_{2n}, z_1, \ldots, z_{2n-n_2})$$

can be expressed as a linear combination of D_{τ_1,τ_2}'s where $\tau_i \leqslant \pi_i$ and the coefficient of D_{π_1,π_2} is 1.

Proof. We calculate $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ in three steps:

1. $\Psi_{(\pi_1)_{n_2}}$ is by Theorem 13 a linear combination of $D_{(\tau_1)_{n_2}}$'s with $\tau_1 \leqslant \pi_1$ and the coefficient of $D_{(\pi_1)_{n_2}}$ is 1.
2. Theorem 10 implies
$$\Psi_{\pi_1()_{n_2}} = \Psi_{\rho^{-n_2}((\pi_1)_{n_2})} = \Psi_{(\pi_1)_{n_2}}(z_{n_2+1}, \ldots, z_{2n}, z_1, \ldots, z_{n_2}).$$
3. Use the recursion (10) of Theorem 10 to obtain $\Psi_{\pi_1\pi_2}$ starting from $\Psi_{\pi_1()_{n_2}}$. By Lemma 2 the τ appearing in the sum in (10) are of the form $\pi_1\tau_2$ with $\tau_2 \leqslant \pi_2$.

The algorithm for calculating $\Psi_{(\pi_2)_{n_1}}$ and the third step of calculating $\Psi_{\pi_1\pi_2}$ differ by the initial condition – in the first case $\Psi_{()_n}$, in the second $\Psi_{\pi_1()_{n_2}}$ – and each $\mathbf{D}_i$ of the first algorithm is replaced by $\mathbf{D}_{i+n_2}$. Hence we can use Theorem 13 to express $\Psi_{\pi_1\pi_2}$ as a linear combination of $\hat{D}_{\tau_2}$ with $\tau_2 \leqslant \pi_2$, where $\hat{D}_{(\tau_2)_{n_1}}$ is obtained by taking $D_{(\tau_2)_{n_1}}$ and changing every $\mathbf{D}_i$ to a $\mathbf{D}_{i+n_2}$ and $\Psi_{()_n}$ is replaced by $\Psi_{\pi_1()_{n_2}}$. Together with the first two parts of the algorithm this implies that $\Psi_{\rho^{n_2}(\pi_1\pi_2)}$ is a linear combination of D_{τ_1,τ_2}'s with $\tau_i \leqslant \pi_i$ and the coefficient of D_{π_1,π_2} is 1. □

Remark 16. Let $\Psi_{\pi_i} = \sum_{\tau_i \leqslant \pi_i} \alpha_{\tau_i} D_{\tau_i}$ for $i = 1, 2$. The above proof implies

$$\Psi_{\rho^{n_2}(\pi_1\pi_2)} = \sum_{\tau_1 \leqslant \pi_1, \tau_2 \leqslant \pi_2} \alpha_{\tau_1}\alpha_{\tau_2} D_{\tau_1,\tau_2}.$$

Hence gaining knowledge about

$$A_{\pi_1\pi_2} = \Psi_{\pi_1\pi_2}|_{z_1=\ldots=z_{2n}=1, q^3=1} = \Psi_{\rho^{n_2}(\pi_1\pi_2)}|_{z_1=\ldots=z_{2n}=1, q^3=1}$$

could be achieved by understanding the coefficients α_{τ_i} and the behaviour of D_{τ_1,τ_2} for $\tau_i \leqslant \pi_i$. However this seems to be very difficult.

4 Fully packed loops with a set of nested arches

In order to prove Theorem 1 we will need to calculate D_{π_1,π_2} at $z_1 = \ldots = z_{2(n_1+n_2)} = 1$ for two noncrossing matchings π_1, π_2. The following notations will simplify this task. We define

$$f(i,j) := \frac{qz_i - q^{-1}z_j}{q - q^{-1}}, \quad g(i) := \frac{q - q^{-1}z_i}{q - q^{-1}}, \quad h(i) := \frac{qz_i - q^{-1}}{q - q^{-1}},$$

for $1 \leqslant i \neq j \leqslant 2n$. Using this notations we obtain

$$\Psi_{()_n} = \prod_{1 \leqslant i < j \leqslant n} f(i,j) f(n+i, n+j).$$

One verifies the following lemma by simple calculation.

Lemma 17. *For $1 \leqslant i,j,k \leqslant 2n$ and $i \neq j$ one has*

1. $$\mathbf{D}_k(f(i,j)) = \begin{cases} (q+q^{-1}) f(k,k+1) & (i,j) = (k,k+1), \\ -(q+q^{-1}) f(k,k+1) & (i,j) = (k+1,k), \\ q f(k,k+1) & i = k;\ j \neq k+1, \\ -q f(k,k+1) & i = k+1;\ j \neq k, \\ -q^{-1} f(k,k+1) & j = k;\ i \neq k+1, \\ q^{-1} f(k,k+1) & j = k+1;\ i \neq k, \\ 0 & \{i,j\} \cap \{k,k+1\} = \emptyset, \end{cases}$$

2. $$\mathbf{D}_k(g(i)) = \begin{cases} -q^{-1} f(k,k+1) & i = k, \\ q^{-1} f(k,k+1) & i = k+1, \\ 0 & \textit{otherwise}, \end{cases}$$

3. $$\mathbf{D}_k(h(i)) = \begin{cases} q f(k,k+1) & i = k, \\ -q f(k,k+1) & i = k+1, \\ 0 & \textit{otherwise}. \end{cases}$$

4. *Let m be a positive integer, then the following holds*

$$\mathbf{D}_k(f(i,j)^m) = \mathbf{D}_k(f(i,j)) \sum_{l=0}^{m-1} f(i,j)^l \mathbf{S}_k(f(i,j)^{m-1-l}),$$

$$\mathbf{D}_k(g(i)^m) = \mathbf{D}_k(g(i)) \sum_{l=0}^{m-1} g(i)^l \mathbf{S}_k(g(i)^{m-1-l}),$$

$$\mathbf{D}_k(h(i)^m) = \mathbf{D}_k(h(i)) \sum_{l=0}^{m-1} h(i)^l \mathbf{S}_k(h(i)^{m-1-l}).$$

We further introduce the abbreviation

$$P(\alpha_{i,j}|\beta_i|\gamma_i) := \prod_{1\leqslant i\neq j\leqslant 2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} h(i)^{\gamma_i},$$

where $\alpha_{i,j}, \beta_i, \gamma_i$ are nonnegative integers for $1 \leqslant i \neq j \leqslant 2n$. Our goal is to obtain a useful expression for $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P(\alpha_{i,j}|\beta_i|\gamma_i))|_{z_1=\ldots=z_{2n}=1}$ for special values of $\alpha_{i,j}, \beta_i$ and γ_i. By using the previous lemma it is very easy to see that $\mathbf{D}_{i_1}\circ\cdots\circ\mathbf{D}_{i_m}(P(\alpha_{i,j}|\beta_i|\gamma_i))$ is a sum of products of the form $P(\alpha'_{i,j}|\beta'_i|\gamma'_i)$. The explicit form of this sum is easy to understand when only one $\mathbf{D}$-operator is applied but gets very complicated for more. However it turns out that $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P(\alpha_{i,j}|\beta_i|\gamma_i))|_{z_1=\cdots=z_{2n}=1}$ is a polynomial in $\alpha_{i,j}, \beta_i$ and γ_i, which is stated in Lemma 19. The next example hints at the basic idea behind this fact.

Example 18. Let $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$ and $n = 1$. We calculate $\mathbf{D}_1(P)_{z_1=z_2=1}$ explicitly. By using Lemma 9 and Lemma 17 we obtain for $\mathbf{D}_1(P)$ the expression.

$$\begin{aligned}
\mathbf{D}_1(P) =&\mathbf{D}_1\left(f(1,2)^{\alpha_{1,2}}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1}g(2)^{\beta_2}h(1)^{\gamma_1}h(2)^{\gamma_2}\right)\\
=&(q+q^{-1})\sum_{t=0}^{\alpha_{1,2}-1} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t}f(2,1)^{t}g(1)^{\beta_2}g(2)^{\beta_1}h(1)^{\gamma_2}h(2)^{\gamma_1}+\\
&-(q+q^{-1})\sum_{t=0}^{\alpha_{2,1}-1} f(1,2)^{\alpha_{1,2}+\alpha_{2,1}-t}f(2,1)^{t}g(1)^{\beta_2}g(2)^{\beta_1}h(1)^{\gamma_2}h(2)^{\gamma_1}+\\
&-q^{-1}\sum_{t=0}^{\beta_1-1} f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1+\beta_2-t-1}g(2)^{t}h(1)^{\gamma_2}h(2)^{\gamma_1}+\\
&+q^{-1}\sum_{t=0}^{\beta_2-1} f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1+\beta_2-t-1}g(2)^{t}h(1)^{\gamma_2}h(2)^{\gamma_1}+\\
&+q\sum_{t=0}^{\gamma_1-1} f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1}g(2)^{\beta_2}h(1)^{\gamma_1+\gamma_2-t-1}h(2)^{t}+\\
&-q\sum_{t=0}^{\gamma_2-1} f(1,2)^{\alpha_{1,2}+1}f(2,1)^{\alpha_{2,1}}g(1)^{\beta_1}g(2)^{\beta_2}h(1)^{\gamma_1+\gamma_2-t-1}h(2)^{t}.
\end{aligned}$$

By evaluating this at $z_1 = z_2 = 1$ we obtain:

$$\mathbf{D}_1(P)|_{z_1=z_2=1} = (q+q^{-1})(\alpha_{1,2}-\alpha_{2,1}) + q^{-1}(\beta_2-\beta_1) + q(\gamma_1-\gamma_2),$$

which is a polynomial in the $\alpha_{i,j}, \beta_i, \gamma_i$.

The proof of Theorem 1 is achieved by using two main ingredients. First Theorem 15 allows us to express the wheel polynomial $\Psi_{(\pi_1)_m\pi_2}$ in a suitable basis and second Lemma 19 tells us what we have to expect when evaluating the basis at $z_1 = \cdots = z_{2N} = 1$.

Proof of Theorem 1. In the following we show that the number $A_{(\pi_1)_m\pi_2}$ of FPLs with link pattern $(\pi_1)_m\pi_2$ is a polynomial in m. Together with [4, Theorem 6.7], which states that $A_{(\pi_1)_m\pi_2}$ is a polynomial in m with requested degree and leading coefficient for large values of m, this proves Theorem 1.

Set $N = m + n_1 + n_2$ and $q = e^{\frac{2\pi i}{3}}$. By Theorem 11, Theorem 5 and Theorem 10 one has

$$A_{(\pi_1)_m\pi_2} = \Psi_{(\pi_1)_m\pi_2}|_{z_1=\cdots=z_{2N}=1} = \Psi_{\rho^{n_2}((\pi_1)_m\pi_2)}|_{z_1=\cdots=z_{2N}=1}.$$

Theorem 15 implies that $\Psi_{\rho^{n_2}((\pi_1)_m\pi_2)}$ is a linear combination of $D_{(\tau_1)_m,\tau_2}$ with $\tau_i \leqslant \pi_i$ for $i = 1, 2$. By definition $D_{(\tau_1)_m,\tau_2}$ is of the form $\prod_{j=1}^{k} \mathbf{D}_{i_j}(\Psi_{()_N})$ with $k \leqslant |\lambda(\pi_1)| + |\lambda(\pi_2)|$ and $i_j \in \{1, \ldots, n_2 - 2, N - n_1 + 2, \ldots, N + n_1 - 2, 2N - n_2 + 2, \ldots, 2N\}$ for $1 \leqslant j \leqslant k$. The operator $\mathbf{D}_{i_j}$ acts for $1 \leqslant j \leqslant k$ trivially on z_i with $i \in I := \{n_2 + 1, \ldots, N - n_1, N + n_1 + 1, \ldots, 2N - n_2\}$. Hence one has

$$\left(\prod_{j=1}^{k} \mathbf{D}_{i_j}(\Psi_{()_N})\right)\Bigg|_{z_1=\cdots=z_{2N}=1} = \left(\prod_{j=1}^{k} \mathbf{D}_{i_j}(\Psi_{()_N|\forall i\in I: z_i=1})\right)\Bigg|_{\forall i\in\{1,\ldots,2N\}\setminus I: z_i=1}.$$

The polynomial $\Psi_{()_N}\big|_{z_i=1\forall i\in I}$ is a polynomial in the $2(n_1 + n_2)$ variables z_i, where i is an element of $\{1, \ldots, 2N\} \setminus I$. For simplicity we substitute these remaining variables with $z_1, \ldots, z_{2(n_1+n_2)}$ whereby we keep the same order on the indices. Hence $\Psi_{()_N}\big|_{z_i=1\forall i\in I}$ can be written in the form $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$ with

$$\alpha_{i,j} = \begin{cases} 1 & i < j \text{ and } (j \leqslant n_1 + n_2 \text{ or } i > n_1 + n_2)\,, \\ 0 & \text{otherwise,} \end{cases}$$
$$\beta_i = \begin{cases} m & i \in \{n_2 + 1, \ldots, n_1 + n_2, 2n_1 + n_2 + 1, \ldots, 2(n_1 + n_2)\}, \\ 0 & \text{otherwise,} \end{cases}$$
$$\gamma_i = \begin{cases} m & i \in \{1, \ldots, n_2, n_1 + n_2 + 1, \ldots, 2n_1 + n_2\}, \\ 0 & \text{otherwise,} \end{cases}$$

whereas all the z_i in $f(i, j), g(i)$ and $h(i)$ are replaced by $\hat{z}_i$. Lemma 19 implies that $\prod_{j=1}^{k} \mathbf{D}_{i_j}(P)$ is a polynomial in m of degree at most $k \leqslant |\lambda(\pi_1)| + |\lambda(\pi_2)|$ which proves the statement. □

We conclude the proof of Theorem 1 by the following Lemma.

Lemma 19. *Let $P = P(\alpha_{i,j}|\beta_i|\gamma_i)$, m an integer and $i_1, \ldots, i_m \in \{1, \ldots, 2n\}$. There exists a polynomial $Q \in \mathbb{Q}(q)[y_1, \ldots, y_{2n(2n+1)}]$ with total degree at most m such that*

$$\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P)\big|_{z_1=\cdots=z_{2n}=1} = Q((\alpha_{i,j}), (\beta_i), (\gamma_i)).$$

Proof. We prove the theorem by induction on m. The statement is trivial for $m = 0$, hence let $m > 0$ and set $k = i_m$. We can express $\mathbf{D}_k(P)$ as

$$\mathbf{D}_k P = \sum_{s \in S} a_s P_s, \tag{15}$$

for a finite set S of indices, $a_s \in \{\pm q, \pm q^{-1}, \pm(q + q^{-1})\}$ and $P_s = P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i})$ for all $s \in S$. Indeed we can use iteratively the product rule for the operator $\mathbf{D}_k$, stated in Lemma 9, to split $\mathbf{D}_k(P)$ into a sum. Since this splitting depends on the order of the factors, we fix it to be

$$P = \prod_{i=1}^{2n} \prod_{\substack{j=1,\\ j \neq i}}^{2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} \prod_{i=1}^{2n} h(i)^{\gamma_i}.$$

Lemma 17 implies that every summand is of the form $P_s = P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i})$ and the coefficients a_s are as stated above, which verifies (15).

We express $\mathbf{D}_k(P)$ more explicitly by using the above defined ordering of the factors and Lemma 9

$$\begin{aligned}
\mathbf{D}_k(P) &= \mathbf{D}_k\left(\prod_{1 \leqslant i \neq j \leqslant 2n} f(i,j)^{\alpha_{i,j}} \prod_{i=1}^{2n} g(i)^{\beta_i} h(i)^{\gamma_i} \right) \\
&= \sum_{1 \leqslant i \neq j \leqslant 2n} \prod_{\substack{1 \leqslant i' \neq j' \leqslant 2n \\ (i'<i) \vee (i'=i, j'<j)}} f(i',j')^{\alpha_{i',j'}} \times \mathbf{D}_k(f(i,j)^{\alpha_{i,j}}) \\
&\times \mathbf{S}_k \left(\prod_{\substack{1 \leqslant i' \neq j' \leqslant 2n \\ (i'>i) \vee (i'=i, j'>j)}} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{2n} g(i')^{\beta_{i'}} h(i')^{\gamma_{i'}} \right) && \text{(16a)} \\
&+ \sum_{i=1}^{2n} \prod_{1 \leqslant i' \neq j' \leqslant 2n} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{i-1} g(i')^{\beta_{i'}} \times \mathbf{D}_k(g(i)^{\beta_i}) \\
&\times \mathbf{S}_k \left(\prod_{i'=i+1}^{2n} g(i')^{\beta_{i'}} \prod_{i'=1}^{2n} h(i')^{\gamma_{i'}} \right) && \text{(16b)} \\
&+ \sum_{i=1}^{2n} \prod_{1 \leqslant i' \neq j' \leqslant 2n} f(i',j')^{\alpha_{i',j'}} \prod_{i'=1}^{2n} g(i')^{\beta_{i'}} \prod_{i'=1}^{i-1} h(i')^{\gamma_{i'}} \times \mathbf{D}_k(h(i)^{\gamma_i}) \\
&\times \mathbf{S}_k \left(\prod_{i'=i+1}^{2n} h(i')^{\gamma_{i'}} \right). && \text{(16c)}
\end{aligned}$$

Using Lemma 17 we split every summand in (16a) up into a sum of P_s with $s \in S$ and say that these P_s originate from this very summand. We define $A_{i,j}$ for $1 \leqslant i \neq j \leqslant 2n$ to be the set consisting of all $s \in S$ such that P_s originates from the summand in (16a) with control variables i, j. analogously we define for $1 \leqslant i \leqslant 2n$ the sets B_i and C_i to consist of all $s \in S$ such that P_s originates from the summand with control variable i in (16b) or (16c) respectively. Hence we can write the set S as the disjoint union

$$S = \left(\bigcup_{1 \leqslant i \neq j \leqslant 2n} A_{i,j} \right) \cup \left(\bigcup_{1 \leqslant i \leqslant 2n} B_i \right) \cup \left(\bigcup_{1 \leqslant i \leqslant 2n} C_i \right).$$

Lemma 17 implies $\mathbf{D}_k(f(i,j)) = 0$ for $\{i,j\} \cap \{k, k+1\} = \emptyset$ and $\mathbf{D}_k(g(i)) = \mathbf{D}_k(h(i)) = 0$ for $i \notin \{k, k+1\}$. Therefore the sets $A_{i,j}, B_i, C_i$ are empty in these cases.

Let $1 \leqslant i \neq j \leqslant 2n$ be fixed with $\{i,j\} \cap \{k, k+1\} \neq \emptyset$ and let $\sigma \in \mathfrak{S}_{2n}$ be the permutation $\sigma = (k, k+1)$. Set $\Lambda_{i,j} = \{(i', j') : 1 \leqslant i' \neq j' \leqslant 2n, (i' < i) \vee (i' = i, j' < j)\}$. The definition of $A_{i,j}$ and Lemma 17 imply for all $(i', j') \notin \{(i,j), (\sigma(i), \sigma(j)), (k, k+1)\}$ and all $s \in A_{i,j}$:

$$\alpha_{s;i',j'} = \begin{cases} \alpha_{i',j'} & \{i',j'\} \cap \{k,k+1\} = \emptyset \text{ or } ((i',j'),(\sigma(i'),\sigma(j')) \in \Lambda_{i,j}), \\ \alpha_{i',j'} + \alpha_{\sigma(i'),\sigma(j')} & \{i',j'\} \cap \{k,k+1\} \neq \emptyset, (i',j') \in \Lambda_{i,j}, (\sigma(i'),\sigma(j')) \notin \Lambda_{i,j}, \\ 0 & \{i',j'\} \cap \{k,k+1\} \neq \emptyset, (i',j') \notin \Lambda_{i,j}, (\sigma(i'),\sigma(j')) \in \Lambda_{i,j}, \\ \alpha_{\sigma(i'),\sigma(j')} & \{i',j'\} \cap \{k,k+1\} \neq \emptyset, (i',j'), (\sigma(i'),\sigma(j')) \notin \Lambda_{i,j}. \end{cases}$$

If $(k, k+1) \notin \{(i,j), (\sigma(i), \sigma(j))\}$, the parameter $\alpha_{s;k,k+1}$ is given as the adequate value of the above case analysis added by 1. Further we obtain $\beta_{s;i'} = \beta_{\sigma(i')}$ and $\gamma_{s;i'} = \gamma_{\sigma(i')}$ for all $1 \leqslant i' \leqslant 2n$ and $s \in A_{i,j}$. By Lemma 17 the constant a_s is for all $s \in A_{i,j}$ determined by the corresponding constant of $\mathbf{D}_k(f(i,j))$ and hence not depending on s. The last statement of Lemma 17 implies that we can list the elements of $A_{i,j} = \{s_1, \ldots, s_{\alpha_{i,j}}\}$ such that we have the following description for the remaining parameters $\alpha_{s;i,j}$ and $\alpha_{s;\sigma(i),\sigma(j)}$:

$$\alpha_{s_t;i,j} = \begin{cases} \alpha_{i,j} + \alpha_{j,i} + 1 - t & i = k, j = k+1, \\ \alpha_{i,j} - t & i = k+1, j = k, \\ \alpha_{i,j} + \alpha_{\sigma(i),\sigma(j)} - t & \{i,j\} \cap \{k,k+1\} = \{k\}, \\ \alpha_{i,j} - t & \{i,j\} \cap \{k,k+1\} = \{k+1\}, \end{cases}$$

$$\alpha_{s_t;\sigma(i),\sigma(j)} = \begin{cases} \alpha_{i,j} + \alpha_{j,i} - \alpha_{s_t;i,j} & \{i,j\} = \{k,k+1\}, \\ \alpha_{i,j} + \alpha_{\sigma(i),\sigma(j)} - \alpha_{s_t;i,j} - 1 & \text{otherwise}, \end{cases}$$

with $1 \leqslant t \leqslant \alpha_{i,j}$. If $k = 2n$ the first two and last two cases in the description of $\alpha_{s_t;i,j}$ switch places, which is due to the fact that we identify $k+1$ with 1 for $k = 2n$.

There exists an analogue description for the sets B_i, C_i and $i \in \{k, k+1\}$ as above, whereas the only parameters that change are given in the case of B_i by

$$\beta_{s_t;k} = \beta_k + \beta_{k+1} - t, \qquad \beta_{s_t;k+1} = t - 1,$$

with $1 \leqslant t \leqslant \beta_i$ and in the case of C_i by

$$\gamma_{s_t;k} = \gamma_k + \gamma_{k+1} - t, \qquad \gamma_{s_t;k+1} = t - 1,$$

with $1 \leqslant t \leqslant \gamma_i$. For $k = 2n$ the description of $\beta_{s_t;k}, \beta_{s_t;k+1}$ and $\gamma_{s_t;k}, \gamma_{s_t;k}$ are interchanged.

We know by induction that $\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_{m-1}} \left(P(a_{i,j}|b_i|c_i)\right)|_{z_1=\cdots=z_{2n}=1}$ is a polynomial Q' of degree at most $m-1$ in $(a_{i,j}), (b_i)$ and (c_i). Since the operators $\mathbf{D}_i$ are linear we can write

$$\begin{aligned}\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P)|_{z_1=\cdots=z_{2n}=1} \\ &= \mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_{m-1}} \left(\sum_{s \in S} a_s P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i}) \right)\Bigg|_{z_1=\cdots=z_{2n}=1} \\ &= \sum_{s \in S} a_s \mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_{m-1}} \left(P(\alpha_{s;i,j}|\beta_{s;i}|\gamma_{s;i})\right)|_{z_1=\cdots=z_{2n}=1} \\ &= \sum_{s \in S} a_s Q((\alpha_{s;i,j}), (\beta_{s;i}), (\gamma_{s;i})). \end{aligned} \tag{17}$$

The description above implies that if we restrict ourselves to $s \in A_{i,j}$, $s \in B_i$ or $s \in C_i$ respectively, a_s is independent of s, the parameters $\alpha_{s;i',j'}, \beta_{s;i'}, \gamma_{s;i'}$ are constant for $(i', j') \neq (i, j), (\sigma(i), \sigma(j))$ or $i' \neq k, k+1$ respectively and otherwise depending linearly on a parameter t which runs from 1 up to the cardinality of the set $A_{i,j}$, B_i or C_i respectively. The fact, that for a polynomial $p(t)$ of degree d the sum $\sum_{x \leqslant t \leqslant y} p(t)$ is a polynomial in x and y of degree at most $d+1$, together with the previous statement imply that the sum

$$\sum_{s \in A_{i,j}} a_s Q((\alpha_{s;i,j}), (\beta_{s;i}), (\gamma_{s;i})),$$

and the analogous sums for $s \in B_i$ or $s \in C_i$ respectively are polynomials in $(\alpha_{i,j}), (\beta_i), (\gamma_i)$ of degree at most m for all $1 \leqslant i \neq j \leqslant 2n$. Therefore

$$\mathbf{D}_{i_1} \circ \cdots \circ \mathbf{D}_{i_m}(P)|_{z_1=\cdots=z_{2n}=1} = \sum_{s \in S} a_s Q((\alpha_{s;i,j}), (\beta_{s;i}), (\gamma_{s;i})),$$

is a polynomial in $(\alpha_{i,j}), (\beta_i), (\gamma_i)$ of degree at most m. □

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