A refined count of Coxeter element reflection factorizations

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Abstract

For well-generated complex reflection groups, Chapuy and Stump gave a simple product for a generating function counting reflection factorizations of a Coxeter element by their length. This is refined here to record the number of reflections used from each orbit of hyperplanes. The proof is case-by-case via the classification of well-generated groups. It implies a new expression for the Coxeter number, expressed via data coming from a hyperplane orbit; a case-free proof of this due to J. Michel is included.

Keywords: reflection group, Coxeter element, factorization, well-generated.

1 Introduction

A *complex reflection group* is a finite subgroup W of $GL(V)$, where $V = \mathbb{C}^n$, generated by the set of all *reflections* t in W, that is, the elements t whose fixed space $V^t := \ker(t-1)$ is a *hyperplane* H, meaning $\dim H = n-1$. Let $\mathcal{R}$ denote the set of all reflections in W,

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and $\mathcal{R}^*$ the collection of all reflecting hyperplanes. An important numerological role is played by the cardinalities of $\mathcal{R}, \mathcal{R}^*$, denoted N, N^*, respectively.

This paper focusses on the complex reflection groups W which act irreducibly on $V = \mathbb{C}^n$, and which are *well-generated* in the sense that they can be generated by n reflections. For such a group W, one can define the *Coxeter number* as $h := \frac{N+N^*}{n}$, and then the *Coxeter elements* c in W are the elements c which have at least one eigenvector v in $V^{\mathrm{reg}} := V \setminus \cup_{H \in \mathcal{R}^*} H$ with eigenvalue $\zeta_h := e^{\frac{2\pi i}{h}}$. It is known that there is only one conjugacy class of Coxeter elements c; see, for example, [2, 5].

Having fixed one Coxeter element c, one can ask for the number f_ℓ counting reflection factorizations of c having length ℓ, that is, sequences $(t_1, \ldots, t_\ell) \in \mathcal{R}^\ell$ with $c = t_1 t_2 \cdots t_\ell$. The main result of Chapuy and Stump [5] is the following amazingly simple product formula for the exponential generating function of the sequence $(f_\ell)_{\ell=0,1,2,\ldots}$:

$$\sum_{\ell \geqslant 0} f_\ell \frac{x^\ell}{\ell!} = \left(e^{\frac{N}{n}x} - e^{\frac{-N^*}{n}x} \right)^n. \tag{1}$$

In particular, this power series starts at x^n, as shortest factorizations of c have length n.

Our main result refines equation (1), accounting for how many reflections t_j appearing in $c = t_1 t_2 \cdots t_\ell$ have their reflecting hyperplane $\ker(t_j - 1)$ lying in the various W-orbits $\mathcal{R}_1^*, \ldots, \mathcal{R}_p^*$ that decompose $\mathcal{R}^* = \sqcup_{i=1}^p \mathcal{R}_i^*$. Stating it requires some numerology associated to each orbit $\mathcal{R}_i^*$ for $i = 1, 2, \ldots, p$. Let $\mathcal{R}_i$ denote the subset of reflections whose reflecting hyperplane lies in $\mathcal{R}_i^*$, so that $\mathcal{R} = \sqcup_{i=1}^p \mathcal{R}_i$. Define quantities N_i, N_i^*, n_i by

$$N_i := \#\mathcal{R}_i^*, \quad N_i^* := \#\mathcal{R}_i, \quad \text{and} \quad n_i := \# \left\{ \begin{array}{l} j \in \{1, 2, \ldots, n\} : t_j \in \mathcal{R}_i, \text{ for any} \\ \textit{shortest} \text{ factorization } c = t_1 t_2 \cdots t_n \end{array} \right\}$$

It is not obvious that these numbers n_i are well-defined, independent of the choice of a length n factorization for c, but this follows from work of Bessis [2, Prop. 7.6], who showed that any two such shortest factorizations can be connected by a sequence of *Hurwitz moves*

$$(t_1, t_2, \ldots, \ t_k, t_{k+1} \ , \ldots, t_{n-1}, t_n) \longmapsto (t_1, t_2, \ldots, \ t_k t_{k+1} t_k^{-1}, t_k \ , \ldots, t_{n-1}, t_n) \tag{2}$$

Let $f_{\ell_1, \ell_2, \ldots, \ell_p}$ be the number of tuples $(t_1, \ldots, t_\ell)$ factoring $c = t_1 t_2 \cdots t_\ell$ having the first ℓ_1 reflections $t_1, t_2, \ldots, t_{\ell_1}$ in $\mathcal{R}_1$, the next ℓ_2 reflections in $\mathcal{R}_2$, etc. (so $\ell = \sum_{i=1}^p \ell_i$). One can show using the Hurwitz moves above (or see Proposition 3 below), that $f_{\ell_1, \ell_2, \ldots, \ell_p}$ also counts factorizations in which the elements of $\mathcal{R}_i$ occur in *any* prescribed set of the ℓ_i positions, rather than all t_j in $\mathcal{R}_1$ first, then $\mathcal{R}_2$ second, etc.

Theorem 1. *For any irreducible, well-generated complex reflection group, and notation as above, one has*

$$\sum_{(\ell_1, \ldots, \ell_p) \in \mathbb{N}^p} f_{\ell_1, \ldots, \ell_p} \frac{x_1^{\ell_1} \cdots x_p^{\ell_p}}{\ell_1! \cdots \ell_p!} = \frac{1}{\#W} \prod_{i=1}^p \left(e^{\frac{N_i}{n_i} x_i} - e^{-\frac{N_i^*}{n_i} x_i} \right)^{n_i}.$$

Our proof is the same as Chapuy and Stump's proof of equation (1), via the classification[1] of irreducible, well-generated reflection groups, and Frobenius's character-theoretic technique for counting factorizations, reviewed in Section 2. Since there is little novelty in the methods, the proof in Section 4 is abbreviated as much as possible.

One **caveat**: The phrasing of Theorem 1, while convenient, may seem deceptively general, since the classification of irreducible complex reflection groups shows that $p = 1$ or 2 in every case. When $p = 1$, Theorem 1 is the same as equation (1), giving no further information. The remaining cases where $p = 2$ are listed in the table below, with the factorization in the theorem shown, using variables (x, y) instead of (x_1, x_2):

W	Coxeter-Shephard diagram	$\#W \cdot \sum_{(\ell_1,\ell_2)} f_{\ell_1,\ell_2} \frac{x^{\ell_1} y^{\ell_2}}{\ell_1! \ell_2!}$
$G(r,1,n)$ $r \geqslant 2$	(r) —4— (2) — (2) — ⋯ — (2)	$\left(e^{(r-1)nx} - e^{-nx}\right)\left(e^{\frac{nr}{2}y} - e^{-\frac{nr}{2}y}\right)^{n-1}$
$G(m,m,2)$ $m \geqslant 4$, even	(2) —m— (2)	$\left(e^{\frac{m}{2}x} - e^{-\frac{m}{2}x}\right)\left(e^{\frac{m}{2}y} - e^{-\frac{m}{2}y}\right)$
G_5	(3) —4— (3)	$\left(e^{8x} - e^{-4x}\right)\left(e^{8y} - e^{-4y}\right)$
G_6	(2) —6— (3)	$\left(e^{6x} - e^{-6x}\right)\left(e^{8y} - e^{-4y}\right)$
G_9	(2) —6— (4)	$\left(e^{12x} - e^{-12x}\right)\left(e^{18y} - e^{-6y}\right)$
G_{10}	(3) —4— (4)	$\left(e^{16x} - e^{-8x}\right)\left(e^{18y} - e^{-6y}\right)$
G_{14}	(3) —8— (2)	$\left(e^{16x} - e^{-8x}\right)\left(e^{12y} - e^{-12y}\right)$
G_{17}	(2) —6— (5)	$\left(e^{30x} - e^{-30x}\right)\left(e^{48y} - e^{-12y}\right)$
G_{18}	(3) —4— (5)	$\left(e^{40x} - e^{-20x}\right)\left(e^{48y} - e^{-12y}\right)$
G_{21}	(2) —10— (3)	$\left(e^{30x} - e^{-30x}\right)\left(e^{40y} - e^{-20y}\right)$
G_{26}	(3) — (3) —4— (2)	$\left(e^{12x} - e^{-6x}\right)^2\left(e^{9y} - e^{-9y}\right)$
G_{28}	(2) — (2) —4— (2) — (2)	$\left(e^{6x} - e^{-6x}\right)^2\left(e^{6y} - e^{-6y}\right)^2$

The second column of the table gives the *Coxeter-Shephard diagram* for these groups, reflecting the case-by-case observation that irreducible, well-generated groups W with $p = 2$ are all *Shephard groups*, that is, symmetry groups of *regular complex (or real) polytopes*. This implies (see [6]) that they have a *Shephard presentation*

$$W = \left\langle S = \{s_1, \ldots, s_n\} \quad \middle| \quad s_i^{p_i} = 1, \quad \underbrace{s_i s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} \right\rangle$$

[1] It should be noted that, at least for crystallographic real reflection groups (Weyl groups), J. Michel [12] has also produced a case-free derivation of (1), via properties of Deligne-Lusztig representations.

where here the integer $p_i \geqslant 2$ labels the node for s_i, and the integer $m_{ij} \geqslant 2$ labels the edge from s_i to s_j, with $m_{ij} = 2$ whenever $|i-j| \geqslant 2$ (and no edge from s_i to s_j is shown). It is known for Coxeter groups and Shephard groups that one can choose $s_1, \ldots, s_n$ in such a way that their product factors a Coxeter element $c = s_1 s_2 \cdots s_n$. The hyperplane orbits $\mathcal{R}_1^*, \mathcal{R}_2^*$ correspond to the connected components obtained when one erases the edges with even labels m_{ij} in the Coxeter-Shephard diagram, and in this case, n_1, n_2 may be re-interpreted as the number of nodes in the corresponding connected component.

We also explain (Proposition 5) how Theorem 1 necessarily specializes to recover the Chapuy-Stump formula (1). Comparing the two results then gives our first proof of the following seemingly new fact about the Coxeter number h.

Corollary 2. *For irreducible, well-generated complex reflection groups, and notation as above, each hyperplane orbit $\mathcal{R}_i^*$ for $i = 1, 2, \ldots, p$ satisfies*

$$h = \frac{N_i + N_i^*}{n_i}.$$

Because it uses both Theorem 1 and equation (1), this first proof of Corollary 2 relies on case-by-case checks. We also give a second proof which is case-free, but applies only to real reflection groups, and a third proof for the general case supplied by J. Michel, proving a more general assertion about regular elements (Theorem 6), which he has kindly allowed us to reproduce here.

2 Frobenius's method

Frobenius gave a method, using character theory, for counting factorizations of an element in any finite group W as a product of elements from specified conjugacy-closed subsets. Recall that (finite-dimensional, complex) representations $W \stackrel{\rho}{\to} GL(V)$ are determined up to equivalence by their character $\chi_\rho : W \to \mathbb{C}$ defined by $\chi_\rho(w) := \mathrm{Trace}(V \stackrel{\rho(w)}{\to} V)$. For subsets $A \subseteq W$, define $\chi(A) := \sum_{w \in A} \chi(w)$.

Proposition 3. *(Frobenius; see, e.g., [7, Thm A.1.9]) For $A_1, \ldots, A_\ell$ subsets of a finite group W, with each A_i closed under conjugation, and c in W, the number of factorizations $c = t_1 \cdots t_\ell$ with t_i in A_i equals*

$$\frac{1}{\#W} \sum_{\chi} \frac{\chi(c^{-1})\chi(A_1) \ldots \chi(A_\ell)}{\chi(1)^{\ell-1}}$$

where the sum is over all the characters χ of the inequivalent irreducible representations of G.

To apply this, recall that for Coxeter elements c in W a well-generated complex reflection group, we defined $f_{\ell_1,\ldots,\ell_p}$ as the number of sequences $(t_1, \ldots, t_\ell)$ factoring $c = t_1 \cdots t_\ell$ in which exactly ℓ_i of the factors t_j lie in $\mathcal{R}_i$, with the factors from $\mathcal{R}_1$ all coming first in the sequence, those from $\mathcal{R}_2$ coming next, etc.

Corollary 4. *With the above notations,*

$$f_{\ell_1,\ldots,\ell_p} = \frac{1}{\#W} \sum_{\chi} \frac{\chi(c^{-1})\chi(\mathcal{R}_1)^{\ell_1} \ldots \chi(\mathcal{R}_p)^{\ell_p}}{\chi(1)^{\ell-1}}.$$

3 Proofs of Corollary 2.

Before proving Theorem 1, we explain how it specializes to equation (1), and why this implies Corollary 2.

Note that in each summand on the right in Corollary 4, the order of the factors $\chi(\mathcal{R}_1)^{\ell_1} \ldots \chi(\mathcal{R}_p)^{\ell_p}$ does not matter. This explains an assertion from the Introduction: $f_{\ell_1,\ldots,\ell_p}$ also counts sequences $(t_1, \ldots, t_\ell)$ factoring $c = t_1 \cdots t_\ell$ in which exactly ℓ_i of the factors t_j lie in $\mathcal{R}_i$, but where one *fixes* any of the $\binom{\ell}{\ell_1,\ldots,\ell_p}$ choices of the positions in which the factors from $\mathcal{R}_1, \ldots, \mathcal{R}_p$ should occur. This then has the following implication.

Proposition 5. *The exponential generating functions in Theorem 1 and equation* (1) *are related by specialization:*

$$\sum_{\ell \geqslant 0} f_\ell \frac{x^\ell}{\ell!} = \left[\sum_{(\ell_1,\ldots,\ell_p) \in \mathbb{N}^p} f_{\ell_1,\ldots,\ell_p} \frac{x_1^{\ell_1} \cdots x_p^{\ell_p}}{\ell_1! \cdots \ell_p!} \right]_{x_i = x \text{ for } i=1,2,\ldots,p}.$$

Proof. The discussion of the preceding paragraph shows that

$$f_\ell = \sum_{\substack{(\ell_1,\ldots,\ell_p) \in \mathbb{N}^p: \\ \sum_i \ell_i = \ell}} \binom{\ell}{\ell_1, \ldots, \ell_p} f_{\ell_1,\ldots,\ell_p}$$

and the rest is simple manipulation of summations and factorials. □

From this one can now see why Theorem 1 and equation (1) imply Corollary 2.

First proof of Corollary 2. Plugging equation (1) into the left of Proposition 5 and plugging Theorem 1 into the right, gives this equality:

$$\left(e^{\frac{N}{n}x} - e^{-\frac{N^*}{n}x} \right)^n = \prod_{i=1}^{p} \left(e^{\frac{N_i}{n_i}x} - e^{-\frac{N_i^*}{n_i}x} \right)^{n_i}.$$

Factoring $e^{\frac{N}{n}x} - e^{-\frac{N^*}{n}x} = e^{-\frac{N^*}{n}x}(e^{\frac{N+N^*}{n}x} - 1) = e^{-\frac{N^*}{n}x}(e^{hx} - 1)$ on the left, and similarly on the right, gives

$$e^{-N^*x}(e^{hx} - 1)^n = e^{-x\sum_{i=1}^p N_i^*} \prod_{i=1}^{p} \left(e^{\frac{N_i + N_i^*}{n_i}x} - 1 \right)$$

On the other hand, by definition, $\sum_{i=1}^p N_i^* = N^*$, and hence

$$(e^{hx}-1)^n = \prod_{i=1}^p \left(e^{\frac{N_i+N_i^*}{n_i} x} - 1 \right).$$

Then the desired equality $\frac{N_i+N_i^*}{n_i} = h$ for $i = 1, 2, \ldots, p$ follows from this claim:

> **Claim:** Any power series of the form $P(x) = \prod_{i=1}^p (e^{a_i x} - 1)$ in $\mathbb{R}[[x]]$ uniquely determines the multiset $(a_1, \ldots, a_p)$.

One way to see this claim is to first write

$$P(x) = a_1 \cdots a_p \cdot x^p \prod_{i=1} \frac{e^{a_i x} - 1}{a_i x} = a_1 \cdots a_p x^p + o(x^{p+1})$$

where the last equality holds since $\frac{e^z-1}{z}$ in $\mathbb{R}[[z]]$ has constant term 1. Thus at least the product $a_1 \cdots a_p$ is determined by $P(x)$. Naming the coefficients c_k in the unique expansion $\log\left(\frac{e^z-1}{z}\right) = \sum_{k=0}^\infty c_k z^k$ in $\mathbb{R}[[z]]$ lets one read off from $P(x)$ all of the *power sums* $\{a_1^k + \cdots + a_p^k\}_{k=1,2,\ldots}$, via this calculation:

$$\log \frac{P(x)}{a_1 \cdots a_p x^p} = \sum_{i=1}^p \log\left(\frac{e^{a_i x}-1}{a_i x}\right) = \sum_{i=1}^p \sum_{k=0}^\infty c_k (a_i x)^k = \sum_{k=0}^\infty c_k x^k (a_1^k + \cdots + a_p^k).$$

But then these power sums uniquely determine the multiset $(a_1, \ldots, a_p)$. □

As mentioned in the Introduction, the above first proof of Corollary 2 relies on Theorem 1 and equation (1), both proven via case-by-case arguments. We therefore seek case-free proofs. The second proof will apply only when W is a *real* reflection group.

Second proof of Corollary 2, for real W, but case-free. Let W be an irreducible real reflection group, with simple reflections $S = \{s_1, \ldots, s_n\}$, root system Φ, and corresponding simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Then it is known that the Coxeter element $c = s_1 s_2 \cdots s_n$ generates a cyclic subgroup $\langle c \rangle$ of order h acting *freely* on the *root system*, decomposing $\Phi = \sqcup_{i=1}^n \Phi_i$ into n orbits Φ_i. Furthermore, one has $\langle c \rangle$-orbit representatives $\theta_j := s_n s_{n-1} \cdots s_{j+1}(\alpha_j)$, so that θ_j lies in the W-orbit of α_j; see[2] Bourbaki [1, Chap. VI, §11, Prop. 33]. The factorization $c = s_1 s_2 \cdots s_n$ then implies the first equality here

$$n_i = \#\{\alpha_1^\perp, \ldots, \alpha_n^\perp\} \cap \mathcal{R}_i^* = \#\{\theta_1^\perp, \ldots, \theta_n^\perp\} \cap \mathcal{R}_i^* = \frac{\#\Phi_i}{h} = \frac{2N_i}{h} = \frac{N_i + N_i^*}{h},$$

while the third equality comes from the fact that the θ_j represent the orbits for the free $\langle c \rangle$-action on Φ. □

[2]The results quoted here assume a *crystallographic* root system, but avoid the crystallographic hypothesis in their proof.

The promised third proof of Corollary 2, due to J. Michel, is case-free and even proves a more general assertion. Recall that a positive integer d is called a *regular number* for W if there is a regular element w in W (one with an eigenvector v in V^{reg}) having order d. Recall also that it is a consequence of a characterization of regular numbers (originally proven case-by-case by Lehrer and Springer [9], and later in a case-free fashion by Lehrer and Michel [8]) that the Coxeter number h is a regular number for every well-generated group.

Theorem 6. *(J. Michel) A complex reflection group W has every regular number d dividing $N_i + N_i^*$ for each $i = 1, 2, \ldots, p$. In particular, when W is irreducible, well-generated and $d = h$, one has $\frac{N_i+N_i^*}{h} = n_i$.*

The proof uses the theory of the *braid group* $B := \pi_1(V^{\text{reg}}/W)$ associated to a complex reflection group W; see Broué, Malle, and Rouquier [4], further developments by Bessis [2], and the exposition in Broué [3].

This theory emphasizes a certain generating set $\{s_H\}_{H\in\mathcal{R}^*}$ for W, where s_H is the *distinguished reflection* fixing H, the one having $\det(s_H) = \zeta_{\#W_H}$, where W_H is the cyclic subgroup pointwise fixing H.

Two surjections out of B play an important role here. First is the surjection $B \twoheadrightarrow W$ sending $b \mapsto w$, which arises because the quotient map $V^{\text{reg}} \to V^{\text{reg}}/W$ is a Galois covering with Galois group W; say that b *lifts* w in this situation. For each hyperplane H, there is an important family of lifts of s_H to elements $s_{H,\gamma}$ in B, called *braid reflections*; all of these braid reflection lifts $s_{H,\gamma}$ of s_H lie in the same B-conjugacy class.

Second is the *abelianization* map

$$\begin{array}{ccl} B & \twoheadrightarrow & B^{\text{ab}} = B/[B,B] \cong \mathbb{Z}^p \\ \gamma & \mapsto & \gamma^{\text{ab}}. \end{array}$$

The composite map $B \to \mathbb{Z}^p$ can be defined by the following property [4, Thm. 2.17]: if H lies in the W-orbit $\mathcal{R}_i^*$ inside $\mathcal{R}^*$, then each braid reflection $s_{H,\gamma}$ lifting s_H maps to the i^{th} standard basis vector of $\mathbb{Z}^p$.

Proof of Theorem 6. There is a special central element of B, denoted π in [4, Not. 2.3], and called the *full twist* τ in [2, Def. 6.12], with the following abelianized image [4, Lem. 2.22(2), Cor. 2.26]:

$$\pi^{\text{ab}} = (N_1 + N_1^*, \ldots, N_p + N_p^*).$$

When d is a regular number, there exists ρ in B with the property $\rho^d = \pi$, see [3, Prop. 5.24]. Consequently, any such ρ has ρ^{ab} satisfying

$$d \cdot \rho^{\text{ab}} = (\rho^d)^{\text{ab}} = \pi^{\text{ab}} = (N_1 + N_1^*, \ldots, N_p + N_p^*) \quad \text{in } \mathbb{Z}^p \tag{3}$$

proving the first assertion of the theorem, that d divides $N_i + N_i^*$ for $i = 1, 2, \ldots, p$.

In the special case where W is irreducible, well-generated, and $d = h$, Bessis defined [2, Def. 6.11] a certain element δ in B, which lifts a Coxeter element c in W (see [2, Lemmas 6.13, 7.3]), and which has a factorization $\delta = s_{H_1,\gamma_1} \cdots s_{H_n,\gamma_n}$ into n braid reflections; see

[2, Rmk. 6.10, Lem. 7.4]. Thus, by our earlier definition of n_i, and the aforementioned characterization of the abelianization map, one has

$$\delta^{\rm ab} = (s_{H_1,\gamma_1} \cdots s_{H_n,\gamma_n})^{\rm ab} = (n_1, \ldots, n_p).$$

Consequently, in this case (3) tells us that

$$h \cdot (n_1, \ldots, n_p) = h \cdot \delta^{\rm ab} = (N_1 + N_1^*, \ldots, N_p + N_p^*) \quad \text{in } \mathbb{Z}^d,$$

showing the desired equality $n_i = \frac{N_i + N_i^*}{h}$ for $i = 1, 2, \ldots, p$. □

4 Proof of Theorem 1

As explained in the caveat following Theorem 1, the number p of W-orbits of hyperplanes is either 1 or 2. When $p = 1$, the theorem is equivalent to equation (1), and so there is nothing further to prove. The irreducible, well-generated groups W having $p = 2$ appear in the table following the caveat, with only two infinite families $G(m, m, 2), G(r, 1, n)$, and several exceptional groups. Just as in [5], one can use Frobenius's Proposition 3 to verify the table entries– we give here the general calculations for the two infinite families in the next two subsections. The exceptional cases were handled via computer, accessing in `SAGE` (via the `Gap3` package `Chevie`, see [11]) the irreducible complex reflection groups and their character tables; we discuss the exceptional cases no further here.

4.1 The dihedral group $G(m, m, 2)$ for even m.

The group $G(m, m, 2)$ turns out to be the complexification of a real reflection group, the *dihedral group* of type $I_2(m)$ with Coxeter presentation

$$W = \langle s_1, s_2 : s_1^2 = s_2^2 = e = (st)^m \rangle.$$

Here the sets $\mathcal{R}^*, \mathcal{R}$ of reflecting hyperplanes (lines) and reflections both have size m. Both $\mathcal{R}^*, \mathcal{R}$ have a single W-orbit when m is odd, but when m is even, they decompose two orbits of size $N_1^* = N_1 = \frac{m}{2} = N_2 = N_2^*$, indexed here so that s_i lies in $\mathcal{R}_i$ for $i = 1, 2$. Furthermore, $c = s_1 s_2$, and $n_1 = n_2 = 1$.

Irreducible W-representations have dimension one or two, and for every 2-dimensional irreducible character χ, one has vanishing character values $\chi(s_1) = \chi(s_2) = 0$. Hence only one-dimensional characters χ contribute in the formula Corollary 4 for f_{ℓ_1,ℓ_2}. For m even, there are four such characters, namely $\{\mathbf{1},\ \chi_1,\ \chi_2,\ \chi_1\chi_2\}$, with values determined by $\chi_i(s_j) = -1$ if $i = j$ and $\chi_i(s_j) = +1$ if $i \neq j$, for $i, j \in \{1, 2\}$.

Using the $p = 2$ case of Corollary 4 then gives the following:

$$\begin{aligned}
\#W f_{\ell_1,\ell_2} &= \sum_{\chi} \frac{\chi(c^{-1}) \left(\frac{m}{2}\chi(s)\right)^{\ell_1} \left(\frac{m}{2}\chi(t)\right)^{\ell_2}}{\chi(1)^{\ell_1+\ell_2-1}} = \left(\frac{m}{2}\right)^{\ell_1+\ell_2} \sum_{\chi} \frac{\chi(c^{-1})\chi(s)^{\ell_1}\chi(t)^{\ell_2}}{\chi(1)^{\ell_1+\ell_2-1}} \\
&= \left(\frac{m}{2}\right)^{\ell_1+\ell_2} \left(1 + (-1)^{\ell_1-1} + (-1)^{\ell_2-1} + (-1)^{\ell_1+\ell_2-2}\right) \\
&= \left(\frac{m}{2}\right)^{\ell_1+\ell_2} \left(1 - (-1)^{\ell_1}\right) \left(1 - (-1)^{\ell_2}\right)
\end{aligned}$$

and hence, in agreement with Theorem 1, one calculates

$$\#W \sum_{(\ell_1,\ell_2)\in\mathbb{N}^2} f_{\ell_1,\ell_2}\frac{x^{\ell_1}y^{\ell_2}}{\ell_1!\ell_2!} = \sum_{(\ell_1,\ell_2)\in\mathbb{N}^2} \left(\frac{m}{2}\right)^{\ell_1+\ell_2}\left(1-(-1)^{\ell_1}\right)\left(1-(-1)^{\ell_2}\right)\frac{x^{\ell_1}y^{\ell_2}}{\ell_1!\ell_2!}$$
$$= \left(\sum_{\ell_1\in\mathbb{N}}\left(1-(-1)^{\ell_1}\right)\left(\frac{m}{2}\right)^{\ell_1}\frac{x^{\ell_1}}{\ell_1!}\right)\left(\sum_{\ell_1\in\mathbb{N}}\left(1-(-1)^{\ell_2}\right)\left(\frac{m}{2}\right)^{\ell_2}\frac{y^{\ell_2}}{\ell_2!}\right)$$
$$= \left(e^{\frac{m}{2}x}-e^{-\frac{m}{2}x}\right)\left(e^{\frac{m}{2}y}-e^{-\frac{m}{2}y}\right).$$

4.2 The monomial groups $G(r,1,n)$ for $r \geqslant 2$

The group $W = G(r,1,n)$ is the set of $n\times n$ matrices with one nonzero entry in each row and column, and that nonzero entry is an r^{th} root-of-unity in $\mathbb{C}$, a power of the primitive root $\zeta_r = e^{\frac{2\pi i}{r}}$. The reflecting hyperplane W-orbit decomposition is $\mathcal{R}^* = \mathcal{R}_1^* \sqcup \mathcal{R}_2^*$ where

$$\mathcal{R}_1^* = \{x_i = 0 : 1\leqslant i\leqslant n\}, \quad \text{so } N_1^* = n,$$
$$\mathcal{R}_2^* = \{x_i = \zeta_r^k x_j : 1\leqslant i<j\leqslant n, \text{ and } 0\leqslant k\leqslant r-1\}, \quad \text{so } N_2^* = r\binom{n}{2}.$$

The accompanying decomposition of the reflections $\mathcal{R} = \mathcal{R}_1\sqcup\mathcal{R}_2$ has $\mathcal{R}_1$ consisting of the $N_1 = (r-1)n$ reflections that scale one of the n coordinates by ζ_r^ℓ for some $1\leqslant\ell\leqslant r-1$, and fix all other coordinates, while $\mathcal{R}_2$ is the collection of $N_2 = N_2^* = r\binom{n}{2}$ order two reflections in each of the hyperplanes of $\mathcal{R}_2^*$.

To finish the computation, we use the character-theoretic analysis already detailed in [5, §5.3]. There the authors show that the only W-irreducible characters χ which do not vanish on c^{-1} form a two-parameter family denoted $\{\chi_{{}_q\mathfrak{h}_k^n}\}$ where $0\leqslant q\leqslant r-1$ and $0\leqslant k\leqslant n-1$, with these values:

$$\chi_{{}_q\mathfrak{h}_k^n}(1) = \binom{n-1}{k},$$
$$\chi_{{}_q\mathfrak{h}_k^n}(c^{-1}) = (-1)^k\zeta_r^{-q},$$
$$\frac{\chi_{{}_q\mathfrak{h}_k^n}(t)}{\chi_{{}_q\mathfrak{h}_k^n}(1)} = \begin{cases}\zeta_r^{q\ell} & \text{if } t\in\mathcal{R}_1 \text{ and } \det(t)=\zeta_r^\ell,\\ \frac{n-1-2k}{n-1} & \text{if } t\in\mathcal{R}_2.\end{cases}$$

Using the $p=2$ case of Corollary 4, one has

$$\#W\cdot f_{\ell_1,\ell_2}$$
$$= \sum_\chi \frac{\chi(c^{-1})\chi(\mathcal{R}_1)^{\ell_1}\chi(\mathcal{R}_2)^{\ell_2}}{\chi(1)^{\ell_1+\ell_2-1}} = \sum_\chi \chi(1)\cdot\chi(c^{-1})\cdot\left(\frac{\chi(\mathcal{R}_1)}{\chi(1)}\right)^{\ell_1}\left(\frac{\chi(\mathcal{R}_1)}{\chi(1)}\right)^{\ell_1}$$
$$= \sum_{k=0}^{n-1}\sum_{q=0}^{r-1}\binom{n-1}{k}\cdot(-1)^k\zeta_r^{-q}\cdot\left(n(\zeta_r^q+\zeta_r^{2q}+\cdots+\zeta_r^{(r-1)q})\right)^{\ell_1}\left(\frac{nr(n-1-2k)}{2}\right)^{\ell_2}$$
$$= \left(n^{\ell_1}\sum_{q=0}^{r-1}\zeta_r^{-q}\left(\sum_{\ell=1}^{r-1}\zeta_r^{q\ell}\right)^{\ell_1}\right)\left(\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\left(\frac{nr(n-1-2k)}{2}\right)^{\ell_2}\right)$$

Note that

$$\sum_{\ell=1}^{r-1} \zeta_r^{q\ell} = -1 + \sum_{\ell=0}^{r-1} \zeta_r^{q\ell} = \begin{cases} r-1 & \text{if } q = 0, \\ -1 & \text{if } q = 1, 2, \ldots, r-1, \end{cases}$$

and hence

$$\sum_{q=0}^{r-1} \zeta_r^{-q} \left(\sum_{\ell=1}^{r-1} \zeta_r^{q\ell} \right)^{\ell_1} = (r-1)^{\ell_1} - (-1)^{\ell_1}.$$

Therefore one can check agreement with Theorem 1 as follows:

$$\#W \sum_{(\ell_1,\ell_2)\in\mathbb{N}^2} f_{\ell_1,\ell_2} \frac{x^{\ell_1} y^{\ell_2}}{\ell_1! \ell_2!}$$
$$= \left(\sum_{\ell_1\in\mathbb{N}} \frac{x^{\ell_1}}{\ell_1!} n^{\ell_1} \left((r-1)^{\ell_1} - (-1)^{\ell_1} \right) \right) \left(\sum_{\ell_2\in\mathbb{N}} \frac{y^{\ell_2}}{\ell_2!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left(\frac{nr(n-1-2k)}{2} \right)^{\ell_2} \right)$$

in which the first factor on the right is

$$\sum_{\ell_1\in\mathbb{N}} \frac{x^{\ell_1}}{\ell_1!} n^{\ell_1} \left((r-1)^{\ell_1} - (-1)^{\ell_1} \right) = e^{(r-1)nx} - e^{-nx},$$

consistent with $n_1 = 1, N_1 = (r-1)n$ and $N_1^* = n$, while the second factor is

$$\sum_{\ell_2\in\mathbb{N}} \frac{y^{\ell_2}}{\ell_2!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left(\frac{nr(n-1-2k)}{2} \right)^{\ell_2}$$
$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \sum_{\ell_2\in\mathbb{N}} \frac{y^{\ell_2}}{\ell_2!} \left(\frac{nr(n-1-2k)}{2} \right)^{\ell_2}$$
$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{\frac{nr(n-1-2k)}{2} y}$$
$$= \left(e^{\frac{nr}{2} y} \right)^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} (-e^{-nry})^k = \left(e^{\frac{nr}{2} y} \right)^{n-1} (1 - e^{-nry})^{n-1} = \left(e^{\frac{nr}{2} y} - e^{\frac{-nr}{2} y} \right)^{n-1},$$

consistent with $n_2 = n-1$ and $N_2 = N_2^* = r\binom{n}{2} = (n-1)\frac{nr}{2}$.

This completes the proof for $W = G(r, 1, n)$, and the proof of Theorem 1.

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