

An Arithmetic Property of Moments of the β -Hermite Ensemble and Certain Map Enumerators

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Abstract

Moments of the β -Hermite ensemble are known to be related to the enumerative theory of topological maps. When $\beta \in \{1, 2\}$, asymptotic information about these moments has been used to deduce asymptotics on the number of maps of given genus, and arithmetic information about these moments can sometimes be explained by underlying group actions on the set of maps. In this paper we establish a new arithmetic property about the $2q$ -th moment of the β -Hermite ensemble, for any prime $q \geq 3$ and real number $\beta > 0$, that has a combinatorial interpretation in terms of maps but no known combinatorial explanation. In the process, we derive several additional results that might be of independent interest, including a general integrality statement and an efficient algorithm for evaluating expectations of multi-part elementary symmetric polynomials of bounded length.

Keywords: Topological maps; β -Hermite ensemble; elementary symmetric functions

1 Introduction

Since the work of Tutte [28, 29] over fifty years ago, the study of map enumeration has been a popular area of research among combinatorialists. One basic question in this field asks for the number of maps with V vertices and F faces that exist on a given surface of genus g , for fixed integers V , F , and g . In most cases, closed form expressions for these numbers are unavailable, so researchers have instead asked for properties that these numbers satisfy.

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For instance, questions asking for asymptotic properties of these numbers date back at least to 1986 with the work of Bender and Canfield [3]. Since then, there has been significant effort in trying to find more asymptotic results in map enumeration; see, for instance, [4, 7, 14, 15, 27, 31]. Earlier work in this area, such as [14], established asymptotics in entirely combinatorial ways. However, many more recent works, such as [7, 15, 27, 31], take advantage of a connection between random matrices and map enumeration; by studying the asymptotics of random matrix integrals, one can sometimes obtain asymptotics on map enumerators.

Instead of trying to find asymptotic properties of matrix integrals, our goal in this paper is find their arithmetic properties; these can then be interpreted as arithmetic properties of map enumerators. As we will see later (in Section 1.2), results of this type can sometimes be combinatorially understood through an underlying group action on the set of maps on a given surface.

Before discussing this further, let us begin by introducing some of the notions of random matrix theory.

1.1 Matrix Integrals

We first need some notation about partitions and symmetric functions; we will follow the conventions used by Macdonald in [24]. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is a finite, nondecreasing sequence of positive integers. The *size* of λ is equal to the sum $\sum_{i=1}^r \lambda_i$ and will be denoted by $|\lambda|$. The *length* of λ is r and will be denoted by $\ell(\lambda)$. For any positive integer i , let $m_i(\lambda)$ denote the number of indices j for which $\lambda_j = i$; observe that $\ell(\lambda) = \sum_{i=1}^{|\lambda|} m_i(\lambda)$. For any positive integer $i \leq r$, let $\lambda \setminus \lambda_i$ denote the partition $(\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_r)$ obtained from removing one occurrence of λ_i from λ . For any positive integer j , let (j, λ) denote the partition $\lambda \cup \{j\}$ obtained from adjoining j to λ .

Now, let N be a positive integer and $X = \{x_1, x_2, \dots, x_N\}$ be a set of N real variables. For each nonnegative integer k , let $p_k(X) = \sum_{i=1}^N x_i^k$ denote the k th *power sum*. Moreover, let

$$e_k(X) = \sum_{\substack{S \subseteq X \\ |S|=k}} \prod_{s \in S} s,$$

denote the k th *elementary symmetric polynomial*, where the sum ranges over all unordered k -element subsets $S \subseteq X$. For any partition λ , let $p_\lambda(X) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(X)$ denote the *multi-part power sum* and let $e_\lambda(X) = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}(X)$ denote the *multi-part elementary symmetric polynomial*.

In this paper we will be interested in expectations of multi-part power sums and multi-part elementary symmetric polynomials against the β -Hermite ensemble (also called the Gaussian β Ensemble or β -Ensemble), which is a probability distribution that generalizes the joint eigenvalue distributions of particular Gaussian ensembles of random matrices. Let us define these random matrices, following the notation used by Mehta in [25].

A *random matrix* is a matrix whose entries are random variables. Examples of such matrices are given by the *Gaussian Orthogonal Ensemble* (GOE), which is a collection of symmetric random matrices $\mathbf{GOE}_n = \{g_{i,j}\}_{1 \leq i,j \leq n}$ whose upper triangular entries $\{g_{i,j}\}_{1 \leq i < j \leq n}$ are independently distributed, normalized, real Gaussian random variables; whose lower triangular entries $\{g_{i,j}\}_{1 \leq j < i \leq n}$ are symmetrically determined by the upper triangular entries through the equality $g_{i,j} = g_{j,i}$; and whose diagonal entries are independently distributed real Gaussian variables with variance 2. The *Gaussian Unitary Ensemble* (GUE) and *Gaussian Symplectic Ensemble* (GSE) are complex and quaternionic analogs of the GOE, respectively.

The joint eigenvalue distributions of the GOE, GUE, and GSE are part of a one-parameter family of probability measures $\mathbb{P}_\beta = \mathbb{P}_{\beta,N}$, which are defined by

$$d\mathbb{P}_\beta(X) = c(\beta) \exp\left(\frac{-\beta}{4} \sum_{i=1}^N x_i^2\right) \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N dx_i, \quad (1.1)$$

for $\beta = 1, 2, 4$, respectively, where

$$c(\beta) = 2^{N(\beta - \beta N - 4)/4} \pi^{-N/2} \beta^{N(\beta N - \beta + 2)/4} \prod_{i=1}^N \Gamma\left(1 + \frac{\beta}{2}\right) \Gamma\left(1 + \frac{i\beta}{2}\right)^{-1}$$

is a normalizing function in β .

Although \mathbb{P}_β is defined above for $\beta \in \{1, 2, 4\}$, it is a probability distribution for all $\beta > 0$. This distribution is called the β -*Hermite ensemble*. Dense matrix models (such as the GOE, GUE, and GSE) are not known to exist for the β -Hermite ensemble if $\beta \notin \{1, 2, 4\}$. However, Dumitriu and Edelman [9] showed that \mathbb{P}_β has a tridiagonal matrix model for all positive β ; we will discuss this further in Section 2.2.

For any function $f(X)$ of N variables $X = \{x_1, x_2, \dots, x_N\}$, let

$$\langle f(X) \rangle_\beta = \int_{\mathbb{R}^N} f(X) d\mathbb{P}_\beta(X)$$

denote the β -*expectation* of f . For instance, if $f(X) = p_k(X)$ for some positive integer k , then $\langle p_k \rangle_2 = \mathbb{E}[\text{Tr } \mathbf{GUE}_N^k]$, where \mathbb{E} denotes expectation. By symmetry, this is equal to 0 if k is odd.

An intriguing result of Harer and Zagier [20] states that $\langle p_\lambda(X) \rangle_2$ is a nonnegative integer polynomial in N , for all partitions λ . When $\lambda = (2s)$ consists of one part, Harer and Zagier expressed $\langle p_{2s}(X) \rangle_2$ explicitly through the identity

$$\langle p_{2s}(X) \rangle_2 = \frac{(2s)!}{2^s s!} \sum_{j=0}^s 2^j \binom{s}{j} \binom{N}{j+1}, \quad (1.2)$$

which holds for all integers $s \geq 1$ (see also [19] and [23] for combinatorial proofs of this fact).

This identity implies the following arithmetic property about the integer polynomial $\langle p_{2q}(X) \rangle_2$ when $q \geq 3$ is a prime.

Proposition 1.1. *We have that $\langle p_{2q}(X) \rangle_2 \equiv 2N^{q+1} - 2N^2 \pmod q$ for any prime $q \geq 3$.*

Proof. The equality (1.2) may be rewritten as

$$2^q \left(\langle p_{2q}(X) \rangle_2 - \frac{1}{q+1} \binom{2q}{q} \prod_{i=0}^q (N-i) \right) = \frac{1}{q+1} \binom{2q}{q} \sum_{j=0}^{q-1} 2^j \binom{q}{j} \frac{(q+1)!}{(j+1)!} \prod_{i=0}^j (N-i). \quad (1.3)$$

Now, $\binom{2q}{q}/(q+1)$ is the q -th Catalan number and thus is integral. Furthermore, q divides $\binom{q}{j}$ for all nonnegative integers $j < q$, unless $j = 0$, in which case q divides $(q+1)!/(j+1)!$. Therefore, the right side of (1.3) is divisible by q . Since $q \neq 2$ and $\langle p_{2q}(X) \rangle_2$ is an integer polynomial in N , we have that

$$\langle p_{2q}(X) \rangle_2 - \frac{1}{q+1} \binom{2q}{q} \prod_{i=0}^q (N-i) \equiv 0 \pmod q.$$

The proposition now follows from the facts that

$$\frac{1}{q+1} \binom{2q}{q} \equiv 2 \pmod q; \quad \prod_{i=0}^q (N-i) \equiv N^{q+1} - N^2 \pmod q. \quad (1.4)$$

□

Remark 1.2. From (1.2), we find that $\langle p_4(X) \rangle_2 = 2N^3 + N$. Thus, Proposition 1.1 does not hold for $q = 2$.

Similar results for $\langle p_\lambda(X) \rangle_1$ were found by Goulden and Jackson in [18]. They showed that $\langle p_\lambda(X) \rangle_1$ is a nonnegative integer polynomial in N , for all partitions λ . When $\lambda = (2s)$ consists of one part, Goulden and Jackson established the identity

$$\langle p_{2s}(X) \rangle_1 = s! \sum_{j=0}^s 2^{2s-j} \sum_{i=0}^s \binom{s-\frac{1}{2}}{s-i} \binom{i+j-1}{i} \binom{(N-1)/2}{i} + \frac{(2s)!}{2^s s!} \sum_{j=0}^s 2^j \binom{s}{j} \binom{N}{j+1}, \quad (1.5)$$

for all positive integers s . The following result is analogous to Proposition 1.1 and can be verified similarly; we omit its proof.

Proposition 1.3. *We have that $\langle p_{2q}(X) \rangle_1 \equiv 2N^{q+1} + N^q - 2N^2 - N \pmod q$ for any prime q .*

Proposition 1.1 and Proposition 1.3 might initially seem arbitrary in this matrix-theoretic context, but in the next section we will discuss their combinatorial significance; we will show how these two results can be deduced in a different way, using combinatorial interpretations of the coefficients of the polynomials $\langle p_{2q}(X) \rangle_2$ and $\langle p_{2q}(X) \rangle_1$ as enumerators of topological maps.

1.2 Maps

In this section, we will describe how to establish Proposition 1.1 and Proposition 1.3 in a more combinatorial way, using the enumerative theory of topological maps. Let us begin by introducing some of the terminology in this field, following the notation given in Chapter 2.2 of [22].

Let G be a topological connected graph (possibly with multiple edges and loops), and let \mathcal{M} be a surface. A *map* $m : G \rightarrow \mathcal{M}$ is a two-cell embedding of G into \mathcal{M} . The *edges* and *vertices* of m are the edges and vertices of G , respectively; the *faces* of m are the connected components of $\mathcal{M} \setminus m(G)$. The map m is *orientable* if the surface \mathcal{M} is orientable and is otherwise *non-orientable*; following Goulden and Jackson [18], we call all maps *locally orientable*. We call m *planar* if \mathcal{M} has genus 0.

A map $m' : G' \rightarrow \mathcal{M}'$ is *equivalent* to m if there is an orientation-preserving homeomorphism from \mathcal{M} to \mathcal{M}' that induces a graph isomorphism from G to G' . A map is *rooted* by distinguishing an edge $e \in G$ and a *flag* f , which is a side-end position of e ; each edge has two sides and two ends, so four flags. In what follows, all maps will be rooted.

In his original paper [28], Tutte was interested in the number $n(F, E)$ of planar rooted maps, up to equivalence, with a specified number of faces F and edges E . The number of vertices V is determined by the equality $V + F = E + 2$ because the map is planar. More generally, we would have that

$$V + F = E + 2 - g, \tag{1.6}$$

where g is the Eulerian genus of \mathcal{M} , which we will also refer to as the *genus* of the map m . In [29], Tutte combinatorially deduced recursive relations for $n(F, E)$ and used them to solve for a generating function of the form $\sum_{F=0}^{\infty} \sum_{E=0}^{\infty} n(F, E)x^F y^E$.

Remark 1.4. Euler's equality describing the standard genus g' of an orientable map states that $V + F = E + 2 - 2g'$. We use the Eulerian genus, which is twice the standard genus, in order to discuss the genera of orientable and non-orientable surfaces simultaneously.

For the remainder of this section, we will restrict ourselves to maps with one face ($F = 1$), although analogs of the following results are known for maps with multiple faces. For any nonnegative integers E and g , let $\tilde{k}_g(E)$ be the number of genus- g orientable rooted maps with one face and E edges, and let $k_v(E)$ denote the number of orientable rooted maps with one face, v vertices, and E edges. Observe that $n(1, E) = \tilde{k}_0(E)$; furthermore, (1.6) implies that $k_{E+1-g}(E) = \tilde{k}_g(E)$. Similarly, let $\tilde{K}_g(E)$ denote the number of genus- g locally orientable rooted maps with one face and E edges, and let $K_v(E)$ denote the number locally orientable rooted maps with one face, v vertices, and E edges.

Extending on work of 't Hooft [21] and Brézin-Itzykson-Parisi-Zuber [6], Harer and Zagier showed that a generating function for the orientable map enumerators $k_v(E)$ could be found from moments of \mathbb{P}_2 [20]. Specifically, they established that

$$\langle p_{2E}(X) \rangle_2 = \sum_{v=1}^{E+1} k_v(E) N^v, \tag{1.7}$$

for any positive integer E ; we refer to [20] or Section 5 of the survey [31] for a proof. Now, Proposition 1.1 may be restated as follows.

Proposition 1.5. *Let $q \geq 3$ be any prime number. For any nonnegative integer $v \notin \{2, q + 1\}$, we have that $k_v(2q) \equiv 0 \pmod q$. Furthermore, $k_{q+1}(2q) \equiv 2 \pmod q$ and $k_2(2q) \equiv -2 \pmod q$.*

In the previous section, we deduced this fact from (1.2), which follows from matrix theoretic methods. However, as stated, Proposition 1.5 also has a direct combinatorial proof.

Proof of Proposition 1.5. For each nonnegative integer g , $\tilde{k}_g(2q)$ counts the number of genus- g rooted maps with one face of degree $2q$. This is equal to the number of *gluings* (that is, the number of ways to identify pairs of edges; see Section 5.1 of [31]) of a labelled regular $2q$ -gon \mathcal{P} to form a surface of Eulerian genus g . Label the edges of \mathcal{P} with the integers $1, 2, \dots, 2q$ in clockwise order; any gluing may be represented by a collection of g pairs $\{(a_i, b_i)\}$ of integers between 1 and $2q$, specifying which edges to identify. For instance, when $q = 3$, the collection $\{(1, 4), (2, 5), (3, 6)\}$ glues opposite sides of the hexagon \mathcal{P} to form a surface of Eulerian genus 2.

The cyclic group C_q of order q acts on the set of genus- g gluings; a generator of this group acts on a gluing by rotating \mathcal{P} clockwise by an angle of $2\pi/q$ around its center. Specifically, this generator sends any gluing identifying the pairs $\{(a_i, b_i)\}_{i \in [1, q]}$ to the gluing identifying the pairs $\{a_i + 2, b_i + 2\}_{i \in [1, q]}$ (here, the $a_i + 2$ and $b_i + 2$ are taken modulo $2q$). Any gluing not fixed by the action of this group has an orbit of size q ; therefore, the orbits of such gluings do not contribute to the residue of $\tilde{k}_g(2q)$ modulo q .

The cyclic group C_q fixes q gluings, which may be represented by collections of the form $\{(1, k), (3, k+2), \dots, (2q-1, k+2q-2)\}$, where k is a positive even integer less than $2q+1$. The two gluings $\{(1, 2), (3, 4), \dots, (2q-1, 2q)\}$ and $\{(1, 2q), (3, 2), \dots, (2q-1, 2q-2)\}$ have $q+1$ vertices; the other $q-2$ gluings fixed by C_q have 2 vertices. Therefore, $k_{q+1}(2q) \equiv 2 \pmod q$; $k_2(2q) \equiv -2 \pmod q$; and $k_v(2q) \equiv 0 \pmod q$ if $v \notin \{2, q + 1\}$. \square

Remark 1.6. The proof of Proposition 1.5 uses the fact that the cyclic group C_q acts on the set of genus- g gluings by rotations. Rotating a genus- g gluing is equivalent to re-rooting its corresponding genus- g map. Therefore, the above proof implicitly uses the fact that re-rooting a map does not change its genus or orientability. As we will mention in Section 1.3, this is not true in the general β setting.

The analog of (1.7) for locally orientable maps was established by Goulden and Jackson in [18]. Specifically, they showed that

$$\langle p_{2E}(X) \rangle_1 = \sum_{v=1}^{E+1} K_v(E) N^v, \tag{1.8}$$

for any positive integer E . Now, Proposition 1.3 can be rewritten as follows.

Proposition 1.7. *Let q be any prime number. For any nonnegative integer $v \notin \{1, 2, q, q+1\}$, we have that $K_v(2q) \equiv 0 \pmod{q}$. Furthermore, $K_{q+1}(2q) \equiv 2 \pmod{q}$; $K_q(2q) \equiv 1 \pmod{q}$; $K_2(2q) \equiv -2 \pmod{q}$; and $K_1(2q) \equiv -1 \pmod{q}$.*

Similar to Proposition 1.5, Proposition 1.7 might be shown using the action of C_q on the set of oriented gluings; we omit this proof.

Thus, while both Proposition 1.1 and Proposition 1.3 may initially appear as statements about matrix integrals, they can also be interpreted as arithmetic properties of the $k_v(2q)$ and $K_v(2q)$ that can be combinatorially explained through an underlying group action on the set of orientable or locally orientable genus- g topological maps; the nonzero residues in these statements result from the fixed points of this group action. In Section 1.3, we will present our main result, which is a generalization of both Proposition 1.1 and Proposition 1.3 that has also has a combinatorial interpretation but no known combinatorial explanation.

1.3 Results

The extension of (1.7) and (1.8) to the general β -Hermite ensemble was established by La Croix [22]. He showed as Lemma 4.13 of Chapter 4 of [22] that moments of the β -Hermite ensemble satisfy the recursive relations

$$\langle p_{j+2,\lambda}(X) \rangle_\beta = b(j+1) \langle p_{j,\lambda}(X) \rangle_\beta + (b+1) \sum_{i=1}^{\ell(\lambda)} \lambda_i \langle p_{\lambda_i+j,\lambda \setminus \lambda_i}(X) \rangle_\beta + \sum_{i=0}^j \langle p_{i,j-i,\lambda}(X) \rangle_\beta, \quad (1.9)$$

for all integers $j \geq -1$ (when $j = -1$, only the second term on the right side of (1.9) should be viewed as nonzero), positive reals $\beta > 0$, and partitions λ . In (1.9), $b = 2/\beta - 1$ denotes the *shifted Jack parameter*. Later, we will also use the notation $\alpha = 2/\beta = b + 1$ to denote the *Jack parameter*.

Since $\langle p_0(X) \rangle_\beta = N$, (1.9) and induction on $|\lambda|$ imply that $\langle p_\lambda(X) \rangle_\beta$ is a polynomial in N and b with nonnegative integer coefficients, for any partition λ . Restricting to the case when $\lambda = (2E)$ consists of one even part (again, analogs exist for $\ell(\lambda) > 1$, but we will not discuss them here), this implies that there exist nonnegative integers $K_{v,\eta}(E)$ such that

$$\langle p_{2E}(X) \rangle_\beta = \sum_{v=1}^{E+1} N^v \sum_{\eta=0}^E K_{v,\eta}(E) b^\eta. \quad (1.10)$$

Inserting $b = 0$ into (1.10) and applying (1.7) yields $k_v(E) = K_{v,0}(E)$; in this case, only orientable maps are counted in (1.10). Similarly, inserting $b = 1$ into (1.10) and applying (1.8) yields $K_v(E) = \sum_{\eta=0}^E K_{v,\eta}(E)$; in this case, all maps are counted in (1.10). Therefore, it might be reasonable to guess that $K_{v,\eta}(E)$ enumerates the number of rooted maps with v vertices, one face, E edges, and some “degree of non-orientability” that

increases with the integer η . This was conjectured implicitly by Harer, Goulden, and Jackson as Conjecture 3.5 of [17], and later in its above form in Section 5 of [16].

Through a suitable combinatorial interpretation of the recurrence in (1.9), LaCroix showed this conjecture to be true; the parameter η is known as a *b-invariant*. One may refer to Definition 4.1 of [22] for a precise definition of the *b-invariant* η .

Since the $K_{v,\eta}(\lambda)$ have a combinatorial interpretation as “partially orientable” map enumerators, one might expect the $K_{v,\eta}(2q)$ to have arithmetic properties similar to $k_v(2q)$ and $K_v(2q)$ for all odd primes q . The following result, which appears to be a new, simultaneous generalization of Proposition 1.5 ($b = 0$) and Proposition 1.7 ($b = 1$), shows this to be true. We will establish this theorem (in fact its equivalent Theorem 1.9) in Section 2.5.

Theorem 1.8. *Let $q \geq 3$ be any prime number. We have that $K_{q+1,0} \equiv 2 \pmod q$; $K_{q,1} \equiv 1 \pmod q$; $K_{2,0} \equiv -2 \pmod q$; $K_{2,1} \equiv 1 \pmod q$; and $K_{1,1} \equiv -2 \pmod q$. For all integers $\eta \in [2, q - 1]$, we have that $K_{2,\eta} \equiv (-1)^{\eta+1} \pmod q$ and $K_{1,\eta} \equiv (-1)^\eta \pmod q$. For all other pairs of nonnegative integers (v, η) , we have that $K_{v,\eta}(2q) \equiv 0 \pmod q$.*

Similar to Proposition 1.5 and Proposition 1.7, Theorem 1.8 suggests the existence of an underlying group action on the set of maps with one face of degree $2q$, a fixed number of vertices, and a fixed *b-invariant*. As before, the nonzero residues appearing in Theorem 1.8 might then correspond to fixed points of this group action. However, Goulden and Jackson [17] (see also Theorem 3.30 of [22]) showed that the *b-invariant* η of a map depends on the map’s rooting. Therefore, due to Remark 1.6, the cyclic group C_q does not necessarily act on the set of maps with a fixed *b-invariant*.

This leads us to ask whether there is a combinatorial proof of Theorem 1.8, perhaps through a different group action or through a different combinatorial interpretation of the integers $K_{v,\eta}(\lambda)$. We remark that a different arithmetic property with no known combinatorial explanation (unless $b \in \{0, 1\}$, in which case it follows from re-rooting) was given by La Croix as Corollary 4.17 of [22].

Our proof of Theorem 1.8 is matrix theoretic. Due to (1.10), we can rewrite Theorem 1.8 as follows.

Theorem 1.9. *For each prime number $q \geq 3$, we have that*

$$\langle p_{2q} \rangle_\beta \equiv 2N^{q+1} + bN^q + \left(-\sum_{i=1}^{q-1} (-b)^i - 2 \right) N^2 + \left(\sum_{i=2}^{q-1} (-b)^i - 2b \right) N \pmod q.$$

Unlike the case $\beta \in \{1, 2\}$, there is no known analog of (1.2) and (1.5) that explicitly evaluates $\langle p_{2s}(X) \rangle_\beta$ for general $\beta > 0$. Instead, we know of five ways to find exact expressions for $\langle p_{2s}(X) \rangle_\beta$. The first is by evaluating traces of powers of tridiagonal matrix models, given in Chapter 5.4 and Chapter 6.2 of [8] by Dumitriu; the second is through a change of basis from the Jack polynomials to power sums, given in Section 4 of [10] by Dumitriu, Edelman, and Shuman; the third is through the recursive equation (1.9), given in Chapter 4.4 of [22] by La Croix; and the fourth uses loop equations for the resolvent,

given in Sections 2 and 3 of [12] by Forrester and Witte (see also the references therein). We do not know of a way to deduce arithmetic properties about $\langle p_{2q}(X) \rangle_\beta$ through any of these four existing methods.

Therefore, we introduce the fifth method by expressing $\langle p_{2q}(X) \rangle_\beta$ in terms of expectations $\langle e_\lambda \rangle_\beta$ of multi-part elementary symmetric polynomials; through an integrality result about these expectations (see Proposition 2.5), we will be able to establish Theorem 1.9 in Section 2.

In the process, we will derive several results about expectations of symmetric polynomials that might be of independent interest. The first is an integrality result (Corollary 2.15) about $\langle f(X) \rangle_\beta$ that holds for arbitrary symmetric polynomials f with integer coefficients; as we will discuss in Remark 2.17, special cases of this result have appeared implicitly before, but not in full generality until now. The second is a recursion (Theorem 3.1) that explicitly evaluates expectations of multi-part elementary symmetric functions. This gives rise to an algorithm (Corollary 3.10) that finds $\langle e_\lambda(X) \rangle_\beta$ efficiently for partitions λ of small length; as we will discuss in Section 3, this complements the algorithm given by Dumitriu, Edelman, and Shuman in [10].

2 Proof of the Arithmetic Property

In this section, we will prove Theorem 1.9. Let us begin by outlining the proof.

2.1 Outline of Proof of Theorem 1.9

Instead of attempting to evaluate the power-sum expectation $\langle p_{2q}(X) \rangle_\beta$ directly, we will first express the $2q$ -th power sum as a rational linear combination of multi-part elementary symmetric functions through the Newton-Girard identity. This yields

$$\langle p_{2q}(X) \rangle = 2q \sum_{|\lambda|=2q} c_\lambda \langle e_\lambda(X) \rangle, \quad (2.1)$$

where the c_λ are explicit rational numbers, and we simplify notation by using $\langle f(X) \rangle$ in place of $\langle f(X) \rangle_\beta$ for any function f of N variables. The prefactor of $2q$ in (2.1) might initially appear useful for reduction modulo q , but there seem to be two issues.

First, the coefficients c_λ are rational, and their denominators might be divisible by q . Thus, $2qc_\lambda$ might not be divisible by q , so there might be many summands on the right side of (2.1) that are nonzero modulo q . For instance, if $\lambda = 2^q = (2, 2, \dots, 2)$ (with q twos), then $c_\lambda = -q^{-1}$.

Second, as seen from the following proposition (originally due Aotomo [2]), $\langle e_\lambda(X) \rangle$ is not always an integer polynomial in N and b .

Proposition 2.1 ([2]). *If k is a nonnegative even integer, then*

$$\langle e_k(X) \rangle = (-1)^{k/2} \binom{N}{k} \prod_{j=1}^{k/2-1} (2j+1).$$

The term $\langle e_{2q}(X) \rangle$ appears as a summand in (2.1) with coefficient $c_{(2q)} = -1$, and the denominator of $\langle e_{2q}(X) \rangle$ is divisible by q due to Proposition 2.1. Therefore, the contribution from the single-part partition $\lambda = (2q)$ to (2.1) is nonzero modulo q , even though $2qc_\lambda = -2q$ is a multiple of q . Many other partitions also contribute nonzero residues to (2.1).

Remark 2.2. Proposition 2.1 implies that $\langle e_k(X) \rangle$ is independent of β . It can be viewed as a restatement of the fact that the expectation of the characteristic polynomial of a random matrix from the Gaussian β Ensemble is the Hermite polynomial, for each $\beta \in \mathbb{R}_{>0}$.

Remark 2.3. In fact, Aomoto established a more general result than Proposition 2.1; he found the expectations of single-part elementary symmetric polynomials against the β -Jacobi ensemble, which exhibits a limit degeneration to the β -Hermite ensemble.

The first issue can be quickly resolved through the following lemma, which we will establish in Section 2.5. In what follows, 1^{2q} denotes the partition $(1, 1, \dots, 1)$ (with $2q$ ones).

Lemma 2.4. *For any partition $\lambda \notin \{1^{2q}, 2^q\}$ of size $2q$, there exists an integer k_λ , not divisible by q , such that $k_\lambda c_\lambda \in \mathbb{Z}$; equivalently, the denominator of c_λ is not divisible by q unless $\lambda = 1^{2q}$ or $\lambda = 2^q$. Furthermore, if $\lambda = 1^{2q}$, then $2qc_\lambda = 1$; if $\lambda = 2^q$, then $2qc_\lambda = -2$.*

Lemma 2.4 shows that all but two of the coefficients c_λ are well-behaved with respect to q ; it also evaluates the other two coefficients explicitly. Using Lemma 2.4, we may rewrite (2.1) as

$$\langle p_{2q}(X) \rangle = \langle e_1(X)^{2q} \rangle - 2\langle e_2(X)^q \rangle + \sum_{\substack{|\lambda|=2q \\ \lambda \notin \{1^{2q}, 2^q\}}} \tilde{c}_\lambda \langle e_\lambda(X) \rangle, \quad (2.2)$$

where the \tilde{c}_λ are rational numbers whose numerators are multiples of q . The polynomials $\langle e_1(X)^{2q} \rangle$ and $\langle e_2(X)^q \rangle$ will be addressed separately in Section 2.4; the coefficients of all other polynomials on the right side of (2.2) are divisible by q .

Now, let us turn to the second issue. Although $\langle e_\lambda(X) \rangle$ is not always an integer polynomial, Proposition 2.1 indicates that $\langle e_k(X) \rangle$ is in the $\mathbb{Z}[b]$ -span of the binomial coefficients $\binom{N}{0}, \binom{N}{1}, \dots$. The following lemma shows that this is also true for β -expectations of arbitrary multi-part elementary symmetric polynomials.

Lemma 2.5. *For each partition λ , there exist integer polynomials $a_{\lambda,0}, a_{\lambda,1}, \dots, a_{\lambda,|\lambda|} \in \mathbb{Z}[b]$ such that $\langle e_\lambda(X) \rangle = \sum_{i=0}^{|\lambda|} a_{\lambda,i} \binom{N}{i}$.*

Remark 2.6. Lemma 2.5 does not seem to be immediate from integrating $e_\lambda(X)$ against (1.1). In Section 2.3, we will present a combinatorial proof of this result using a tridiagonal matrix model.

In view of Lemma 2.5, we can rewrite (2.2) as

$$\langle p_{2q}(X) \rangle = \sum_{i=0}^{2q} a_{1^{2q},i} \binom{N}{i} - 2 \sum_{i=0}^{2q} a_{2^q,i} \binom{N}{i} + \sum_{\substack{|\lambda|=2q \\ \lambda \notin \{1^{2q}, 2^q\}}} \sum_{i=0}^{2q} \tilde{c}_\lambda a_{\lambda,i} \binom{N}{i}. \quad (2.3)$$

Since the index i in (2.3) runs from 0 to $2q$, the issue remains that q appears in the denominators of many summands on the right side of (2.3). Moreover, the fact that i can equal $2q$ in (2.3) seems to suggest that $\langle p_{2q}(X) \rangle$ is of degree $2q$ in N . However, we will see in Section 2.5 that it is in fact a polynomial of degree $q+1$. Thus, there is cancellation among the high-degree terms in (2.3). Specifically, we may rewrite (2.3) as

$$\langle p_{2q}(X) \rangle = \sum_{i=0}^{q+1} a_{1^{2q},i} \binom{N}{i} - 2 \sum_{i=0}^{q+1} a_{2^q,i} \binom{N}{i} + \sum_{\substack{|\lambda|=2q \\ \lambda \notin \{1^{2q}, 2^q\}}} \sum_{i=0}^{q+1} \tilde{c}_\lambda a_{\lambda,i} \binom{N}{i}, \quad (2.4)$$

where now the index i does not exceed $q+1$.

Furthermore, we will see from a result of Forrester and Witte (which will be more precisely stated in Section 2.5; see Proposition 2.24) that the degree- $(q+1)$ and degree- q terms on the right side of (2.4) are explicit. Therefore, (2.4) may be rewritten as

$$\begin{aligned} \langle p_{2q}(X) \rangle &= s_q(q+1)! \binom{N}{q+1} + t_q q! \binom{N}{q} + \sum_{\substack{|\lambda|=2q \\ \lambda \notin \{1^{2q}, 2^q\}}} \sum_{i=0}^{q-1} \tilde{c}_\lambda a_{\lambda,i} \binom{N}{i} \\ &\quad + \sum_{i=0}^{q-1} a_{1^{2q},i} \binom{N}{i} - 2 \sum_{i=0}^{q-1} a_{2^q,i} \binom{N}{i}, \end{aligned} \quad (2.5)$$

for explicit constants s_q and t_q .

Now, the first two terms on the right side of (2.5) are integer polynomials that can be evaluated modulo q directly, since s_q and t_q are explicit. The third term (which contributes the most summands to (2.5)) is zero modulo q since q divides \tilde{c}_λ and does not divide the denominator of $\binom{N}{i}$ for any integer $i < q$. The remaining two sums in (2.5) will be evaluated modulo q through an examination of the polynomials $\langle e_1(X)^{2q} \rangle$ and $\langle e_2(X)^q \rangle$ in Section 2.4. Thus, the right side of (2.5) can be evaluated modulo q termwise, from which we will be able to deduce Theorem 1.9.

The organization for the remainder of this section is as follows. In Section 2.2, we will introduce the tridiagonal matrix model for the β -Hermite ensemble. In Section 2.3, we will use this matrix model to establish Lemma 2.5. In Section 2.4, we will understand the modulo q properties of $\langle e_1(X)^{2q} \rangle$ and $\langle e_2(X)^q \rangle$. In Section 2.5, we will prove Lemma 2.4 and Theorem 1.9.

2.2 The Tridiagonal Matrix Model

Our proof of Lemma 2.5 will use the *tridiagonal matrix model* for the β -Hermite ensemble, introduced by Dumitriu and Edelman in [9]. In this section, we will describe this matrix model.

$$\mathbf{T}_3 = \frac{1}{\sqrt{\beta}} \begin{bmatrix} G_1\sqrt{2} & X_\beta & & \\ X_\beta & G_2\sqrt{2} & X_{2\beta} & \\ & X_{2\beta} & G_3\sqrt{2} & \\ & & & \end{bmatrix}; \quad \mathbf{T}_4 = \frac{1}{\sqrt{\beta}} \begin{bmatrix} G_1\sqrt{2} & X_\beta & & & \\ X_\beta & G_2\sqrt{2} & X_{2\beta} & & \\ & X_{2\beta} & G_3\sqrt{2} & X_{3\beta} & \\ & & X_{3\beta} & G_4\sqrt{2} & \\ & & & & \end{bmatrix}$$

Figure 1: The tridiagonal matrix models \mathbf{T}_3 and \mathbf{T}_4 are depicted above.

For any real number $\beta > 0$ and positive integer N , let $\mathbf{T}_N = \mathbf{T}_{N,\beta}$ denote the $N \times N$ tridiagonal symmetric random matrix whose (i, j) entry $t_{i,j}$ is distributed as follows. If $|i - j| \geq 2$, then $t_{i,j} = 0$; if $i = j$, then $t_{i,j} = G_i\sqrt{2/\beta}$, where G_i denotes a normalized, real Gaussian random variable; if $j = i + 1$, then $t_{i,j} = X_{i\beta}/\sqrt{\beta}$, where $X_{i\beta}$ denotes a $\chi_{i\beta}$ distributed random variable; and if $i = j + 1$, then $t_{i,j}$ is determined through the symmetric equality $t_{i,j} = t_{j,i}$. Here, the upper diagonal entries are mutually independent. Examples of \mathbf{T}_N for $N \in \{3, 4\}$ are shown in Figure 2.

The following result of Dumitriu and Edelman appears as Theorem 2.12 in [9].

Proposition 2.7 ([9, Theorem 2.12]). *For any positive integer N , the joint eigenvalue distribution of \mathbf{T}_N is $\mathbb{P}_{\beta,N}$.*

Remark 2.8. The tridiagonal model introduced above is slightly different from the tridiagonal model given by Dumitriu and Edelman. Specifically, they use the tridiagonal model $\mathbf{T}_N\sqrt{\beta/2}$ (meaning that all entries in \mathbf{T}_N are multiplied by $\sqrt{\beta/2}$). This is due to a difference in normalization; they define a β -Hermite ensemble \mathbb{P}'_β through

$$d\mathbb{P}'_\beta(X) = c'(\beta) \exp\left(\frac{-1}{2} \sum_{i=1}^N x_i^2\right) \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N dx_i, \quad (2.6)$$

where $c'(\beta) = (2/\beta)^{n(\beta n - \beta + 2)/4} c(\beta)$. Their normalization is useful for understanding how the β -Hermite distribution behaves as β tends to ∞ ; however, our normalization is useful for understanding the connection between the β -Hermite ensemble and map enumeration (see Section 1.2 and Section 1.3). Observe that our normalization (1.1) is related to the normalization (2.6) through the identity $\mathbb{P}'_\beta(X) = \mathbb{P}_\beta(X\sqrt{2/\beta})$, where $Xc = \{cx_1, cx_2, \dots, cx_N\}$ for any $c \in \mathbb{R}$.

For any nonnegative integer k and $N \times N$ matrix \mathbf{M} , let $P_k(\mathbf{M})$ denote the sum of the $\binom{N}{k}$ principal minors of \mathbf{M} . The following result shows how to obtain β -expectations of multi-part elementary symmetric polynomials from \mathbf{T}_N .

Corollary 2.9. *Let N be a positive integer, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition. Then, $\langle e_\lambda(X) \rangle = \mathbb{E}[\prod_{i=1}^r P_{\lambda_i}(\mathbf{T}_N)]$.*

Proof. For any nonnegative integer k and $N \times N$ matrix \mathbf{M} with eigenvalues $\mu_1, \mu_2, \dots, \mu_N$, we have that $e_k(\mu_1, \mu_2, \dots, \mu_N) = P_k(\mathbf{M})$. Therefore, $e_\lambda(\mu_1, \mu_2, \dots, \mu_N) = \prod_{i=1}^r P_{\lambda_i}(\mathbf{M})$. Letting $\mathbf{M} = \mathbf{T}_N$, taking expectations, and applying Proposition 2.7 yields the corollary. \square

When \mathbf{M} is a tridiagonal matrix, we can evaluate $P_k(\mathbf{M})$ explicitly in terms of the entries of \mathbf{M} . We begin with the following recursion.

Proposition 2.10. *Let $u_1, u_2, \dots, u_{N-1}; v_1, v_2, \dots, v_N$; and w_1, w_2, \dots, w_{N-1} be real numbers. Suppose that \mathbf{M} is $N \times N$ tridiagonal matrix whose (i, j) entry is v_i if $i = j$; is u_j if $i = j + 1$; is w_i if $j = i + 1$; and is 0 otherwise. Let \mathbf{M}' denote the $(N - 1) \times (N - 1)$ matrix obtained from removing the bottommost row and rightmost column of \mathbf{M} , and let \mathbf{M}'' denote the $(N - 2) \times (N - 2)$ matrix obtained from removing the bottommost row and rightmost column of \mathbf{M}' . For each integer k , we have that*

$$P_k(\mathbf{M}) = P_k(\mathbf{M}') + v_N P_{k-1}(\mathbf{M}') - u_{N-1} w_{N-1} P_{k-2}(\mathbf{M}'').$$

Proof. Any principal $k \times k$ minor of \mathbf{M} either contains the entry v_N (in which case we call it a *type 1 minor*) or does not contain the entry v_N . The sum of all type 1 minors is $v_N P_{k-1}(\mathbf{M}')$. Now, any $k \times k$ principal minor that is not type 1 is either contained in \mathbf{M}' (in which case we say that it is a *type 2 minor*) or contains the entry u_{N-1} (in which case we say it is a *type 3 minor*). The sum of all type 2 minors is $P_k(\mathbf{M}')$. Any nonzero type 3 minor also contains the entry w_{N-1} , so the sum of all type 3 minors is $-u_{N-1} w_{N-1} P_{k-2}(\mathbf{M}'')$. Summing over all types of minors yields the proposition. \square

The following corollary now expresses $P_k(\mathbf{M})$ using the entries of \mathbf{M} .

Corollary 2.11. *Suppose that the conditions of Proposition 2.10 hold. For any nonnegative integer k , we have that*

$$P_k(\mathbf{M}) = \sum_{|S_1|+2|S_2|=k} (-1)^{|S_2|} \prod_{i \in S_1} v_i \prod_{i \in S_2} u_i w_i, \quad (2.7)$$

where the sum is over all subsets $S_1 \subset [1, N]$ and $S_2 \subset [1, N - 1]$ such that $|S_1| + 2|S_2| = k$ and such that the following holds. For any $i \in S_2$, we have that $i, i + 1 \notin S_1$ and, for any $i \in S_1$, we have that $i - 1, i \notin S_2$.

Proof. One can verify the cases $k \in \{0, 1\}$ and $N \in \{1, 2\}$ directly. Now, let the right side of (2.7) be $\widetilde{P}_k(\mathbf{M})$. Then, the \widetilde{P}_k satisfy the recursion

$$\widetilde{P}_k(\mathbf{M}) = \widetilde{P}_k(\mathbf{M}') + v_N \widetilde{P}_{k-1}(\mathbf{M}') - u_{N-1} w_{N-1} \widetilde{P}_{k-2}(\mathbf{M}'').$$

The corollary now follows by induction on k and Proposition 2.10. \square

2.3 Proof of Lemma 2.5

In this section, we will use Corollary 2.9 and Corollary 2.11, to prove Lemma 2.5. However, it might be useful to first give an alternate proof of Proposition 2.1, in order to indicate the types of methods that we will use. As mentioned in Remark 2.3, our proof of Proposition 2.1 will complement the one given by Aomoto in [2]. Instead of being analytic, it will be combinatorial, along the lines of the one given by Ullah [30] in the case $\beta = 1$.

Proposition 2.1 ([2]). *If k is a nonnegative even integer, then*

$$\langle e_k(X) \rangle = (-1)^{k/2} \binom{N}{k} \prod_{j=1}^{k/2-1} (2j+1).$$

Proof. For any positive integer N and nonnegative integer k , let $\mathbb{E}[P_k(\mathbf{T}_N)] = P_k(N)$; observe that $P_k(N) = \langle e_k(X) \rangle$ by Corollary 2.9. Due to Proposition 2.10, we have that

$$P_k(\mathbf{T}_N) = P_k(\mathbf{T}_{N-1}) + G_N \sqrt{\frac{2}{\beta}} P_{k-1}(\mathbf{T}_{N-1}) - \beta^{-1} X_{(N-1)\beta}^2 P_{k-2}(\mathbf{T}_{N-2}),$$

where G_N is a normalized, real Gaussian random variable that is independent from all entries in \mathbf{T}_{N-1} , and $X_{(N-1)\beta}$ is a $\chi_{(N-1)\beta}$ random variable that is independent from all entries in \mathbf{T}_{N-2} . Taking expectations, we obtain that

$$\begin{aligned} \mathbb{E}[P_k(\mathbf{T}_N)] &= \mathbb{E}[P_k(\mathbf{T}_{N-1})] + \mathbb{E}\left[G_N \sqrt{\frac{2}{\beta}}\right] \mathbb{E}[P_{k-1}(\mathbf{T}_{N-1})] \\ &\quad - \mathbb{E}[\beta^{-1} X_{(N-1)\beta}^2] \mathbb{E}[P_{k-2}(\mathbf{T}_{N-2})]. \end{aligned}$$

Using the facts that $\mathbb{E}[G_N \sqrt{2/\beta}] = 0$ and $\mathbb{E}[\beta^{-1} X_{(N-1)\beta}^2] = N - 1$, we deduce that

$$P_k(N) = P_k(N-1) - (N-1)P_{k-2}(N-2).$$

Since $P_0(N) = 1$, induction on k yields that $P_k(N) = (-1)^{k/2} \binom{N}{k} \prod_{j=1}^{k/2-1} (2j+1)$; this implies the proposition. \square

Now, let us turn to Lemma 2.5. In order for this lemma to be true, we should have that $\langle e_\lambda(X) \rangle \in \mathbb{Z}[b]$ for any fixed positive integer N and partition λ . The following proposition establishes this fact.

Proposition 2.12. *For any fixed positive integer N and partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, we have that $\langle e_\lambda(X) \rangle \in \mathbb{Z}[b]$.*

Proof. By Corollary 2.11, we have that

$$\prod_{i=1}^r P_{\lambda_i}(\mathbf{T}_N) = \prod_{i=1}^r \sum_{|S_{1,i}|+2|S_{2,i}|=\lambda_i} \left(\frac{2}{\beta}\right)^{|S_{1,i}|/2} (-1)^{|S_{2,i}|} \prod_{j \in S_{1,i}} G_j \prod_{j \in S_{2,i}} \frac{X_{j\beta}^2}{\beta},$$

where the sum is over all subsets $S_{1,i} \subset [1, N]$ and $S_{2,i} \subset [1, N-1]$ such that $|S_{1,i}|+2|S_{2,i}| = k$ and such that the following holds. For any $j \in S_{2,i}$, we have that $j, j+1 \notin S_{1,i}$ and, for any $j \in S_{1,i}$, we have that $j-1, j \notin S_{2,i}$. Here, the G_j are normalized, real Gaussian random variables, and the $X_{j\beta}$ are $\chi_{j\beta}$ distributed random variables.

Taking expectations, applying Corollary 2.9, and recalling that $\alpha = 2/\beta$, we deduce that

$$\langle e_\lambda(X) \rangle = \mathbb{E} \left[\prod_{i=1}^r \sum_{|S_{1,i}|+2|S_{2,i}|=\lambda_i} \alpha^{|S_{1,i}|/2} (-1)^{|S_{2,i}|} \prod_{j \in S_{1,i}} G_j \prod_{j \in S_{2,i}} \frac{X_{j\beta}^2}{\beta} \right],$$

where the sums are as above. Expanding the product yields that $\langle e_\lambda(X) \rangle$ is equal to

$$\sum_{|S_{1,1}|+2|S_{2,1}|=\lambda_1} \cdots \sum_{|S_{1,r}|+2|S_{2,r}|=\lambda_r} \alpha^{\sum_{i=1}^r |S_{1,i}|/2} (-1)^{\sum_{i=1}^r |S_{2,i}|} \prod_{j=1}^N \mathbb{E}[G_j^{n_{1,j}}] \prod_{j=1}^{N-1} \mathbb{E} \left[\frac{X_{j\beta}^{2n_{2,j}}}{\beta^{n_{2,j}}} \right], \tag{2.8}$$

where the $S_{1,i}$ and $S_{2,i}$ are summed as above; $n_{1,j}$ denotes the number of integers $i \in [1, r]$ for which $j \in S_{1,i}$; and $n_{2,j}$ denotes the number of integers $i \in [1, r]$ for which $j \in S_{2,i}$.

If $|\lambda|$ is odd, then $\langle e_\lambda(X) \rangle = 0$ due to symmetry of (1.1). If $|\lambda|$ is even, then $\sum_{i=1}^r |S_{1,i}|$ is even because $\sum_{i=1}^r |S_{1,i}| + 2 \sum_{i=1}^r |S_{2,i}| = |\lambda|$. Therefore, the exponent of α in (2.8) is integral. Furthermore, for all nonnegative integers $n_{1,j}$ and $n_{2,j}$, we have that

$$\mathbb{E}[G_j^{n_{1,j}}] \in \mathbb{Z}; \quad \mathbb{E} \left[\frac{X_{j\beta}^{2n_{2,j}}}{\beta^{n_{2,j}}} \right] = \prod_{i=0}^{n_{2,j}-1} (i + j\alpha) \in \mathbb{Z}[\alpha].$$

Thus, (2.8) is the sum of polynomials in α with integral coefficients, so $\langle e_\lambda(X) \rangle \in \mathbb{Z}[\alpha]$. Since $b = \alpha - 1$, we conclude that $\langle e_\lambda(X) \rangle \in \mathbb{Z}[b]$. \square

Using Proposition 2.12, we may now establish Lemma 2.5.

Lemma 2.5. *For each partition λ , there exist integer polynomials $a_{\lambda,0}, a_{\lambda,1}, \dots, a_{\lambda,|\lambda|} \in \mathbb{Z}[b]$ such that $\langle e_\lambda(X) \rangle = \sum_{i=0}^{|\lambda|} a_{\lambda,i} \binom{N}{i}$.*

Proof. By (1.9) and induction on $|\lambda|$, we find that $\langle p_\lambda(X) \rangle \in \mathbb{Q}[N, b]$ is a rational polynomial whose degree in N is at most equal to $|\lambda|$. Since the multi-part elementary symmetric polynomials are in the \mathbb{Q} -span of the multi-part power sums, we deduce that $\langle e_\lambda(X) \rangle \in \mathbb{Q}[N, b]$ is a rational polynomial whose degree in N at most equal to $|\lambda|$. Thus, there are rational polynomials $R_{\lambda,i}(N) \in \mathbb{Q}[N]$ such that $\langle e_\lambda(X) \rangle = \sum_{i=0}^{d_\lambda} b^i R_{\lambda,i}(N)$, for some nonnegative integer d_λ .

From Proposition 2.12, $R_{\lambda,i}(N) \in \mathbb{Z}$ for each fixed positive integer N , partition λ , and integer $i \in [0, d_\lambda]$. Therefore, the $R_{\lambda,i}$ are integer linear combinations of the binomials $\binom{N}{0}, \binom{N}{1}, \dots$. Applying this to the equality $\langle e_\lambda(X) \rangle = \sum_{i=0}^{d_\lambda} b^i R_{\lambda,i}(N)$ and using the fact that $\langle e_\lambda(X) \rangle$ is of degree at most $|\lambda|$ in N , we deduce the lemma. \square

Remark 2.13. In the proof of Lemma 2.5, we used the fact that $\langle p_\lambda(X) \rangle \in \mathbb{Q}[N, b]$, which was shown by LaCroix through analytic methods (in fact, he shows the stronger statement that $\langle p_\lambda(X) \rangle \in \mathbb{Z}[N, b]$). It is also possible to give a combinatorial proof of this fact using the tridiagonal matrix model reviewed in Section 2.2, but we will not pursue this here.

Remark 2.14. Lemma 2.5 does not explicitly evaluate $\langle e_\lambda(X) \rangle$. We will investigate the question of how to explicitly evaluate this expectation in Section 3.

The following corollary of Lemma 2.5 will not be used but may be of independent interest.

Corollary 2.15. *For any symmetric polynomial f with integer coefficients, there are integers d_f and $h_{i,j}^{(f)}$ such that*

$$\langle f(X) \rangle = \sum_{i=0}^{d_f} \sum_{j=0}^{\deg f} h_{i,j}^{(f)} b^i \binom{N}{j}.$$

Proof. This follows from Lemma 2.5 and the fact that f is in the \mathbb{Z} -span of the multi-part elementary symmetric polynomials e_λ . \square

Remark 2.16. One can show that $d_f \leq \deg f/2$ using the fact that $\langle p_\lambda(X) \rangle$ is of degree at most $\deg f/2$ in b (which follows by (1.9) and induction on $|\lambda|$).

Remark 2.17. Corollary 2.15 may be of independent interest, since it has implicitly appeared previously in multiple articles in random matrix theory but has not been stated or proven in full generality until now. For instance, (1.10) (and thus its special cases (1.7) and (1.8)) exhibit the phenomenon stated in Corollary 2.15. Furthermore, in response to Conjecture 3.4 of Goulden and Jackson in [18], Okounkov [26] (and later Dumitriu in Theorem 8.5.1 in [8]) explicitly evaluated the β -expectation of the Jack polynomial $J^{(2/\beta)}(X)$; one may also verify that this expectation satisfies the statement of Corollary 2.15.

Remark 2.18. The proofs of the facts mentioned in Remark 2.17 are analytic [8, 22, 26]. However, the proof of Corollary 2.15 is mainly elementary and combinatorial, as it used the tridiagonal matrix model for the β -Hermite ensemble instead of explicit integration against the distribution (1.1).

2.4 The Polynomials $\langle e_1(X)^{2q} \rangle$ and $\langle e_2(X)^q \rangle$

In this section, we will discuss the modulo q properties of the polynomials $\langle e_1(X)^{2q} \rangle$ and $\langle e_2(X)^q \rangle$. Let us begin with the following result that evaluates $\langle p_1(X)^{2s} \rangle$ and $\langle p_2(X)^s \rangle$ explicitly.

Proposition 2.19. *For any positive integer s , we have that*

$$\langle p_1(X)^{2s} \rangle = N^s (b+1)^s \prod_{i=1}^s (2i-1) \tag{2.9}$$

and

$$\langle p_2(X)^s \rangle = \prod_{i=0}^{s-1} (N^2 + bN + 2i(b+1)). \quad (2.10)$$

Proof. Inserting $j = -1$ and $\lambda = 1^{2s-1}$ into (1.9) yields

$$\langle p_1(X)^{2s} \rangle = (b+1)(2s-1) \langle p_0(X)p_1(X)^{2s-2} \rangle.$$

Since $p_0(X) = N$, (2.9) follows by induction on s .

Inserting $j = 0$ and $\lambda = 2^{s-1}$ into (1.9) yields

$$\langle p_2(X)^s \rangle = b \langle p_0(X)p_2(X)^{s-1} \rangle + 2(s-1)(b+1) \langle p_2(X)^{s-1} \rangle + \langle p_0(X)^2 p_2(X)^{s-1} \rangle.$$

Again using the fact that $p_0(X) = N$, we deduce (2.10) by induction on s . □

Remark 2.20. An alternative proof of (2.9) can be obtained by changing variables from x_i to $x_i + t$ in (1.1), integrating over all x_i , and differentiating with respect to t . Similarly, an alternative proof of (2.10) can be deduced from changing variables from x_i to tx_i in (1.1), and then integrating and differentiating. The author is grateful to Peter Forrester for mentioning this.

Next, we transfer from power sums to elementary symmetric polynomials to understand the polynomials $\langle e_1(X)^{2q} \rangle$ and $\langle e_2(X)^q \rangle$ modulo q .

Corollary 2.21. *For any prime $q \geq 3$, we have that $\langle e_1(X)^{2q} \rangle$ and $2^q \langle e_2(X)^q \rangle$ are integer polynomials in N and b . Furthermore, $\langle e_1(X)^{2q} \rangle \equiv 0 \pmod{q}$ and*

$$\begin{aligned} 2^q \langle e_2(X)^q \rangle &\equiv ((b+1)^{q-1} - 1)N^2 + (b(b+1)^{q-1} - b^q)N \\ &\quad - (2q)! \binom{N}{2q} - 2(q+1)! \binom{N}{q+1} - b^q q! \binom{N}{q} \pmod{q}. \end{aligned}$$

Proof. The first statement follows from the facts that $e_1(X) = p_1(X)$, $2e_2(X) = p_1(X)^2 - p_1(X)$, and $\langle p_\lambda(X) \rangle \in \mathbb{Z}[N, b]$ for all partitions λ (the last fact holds by (1.9) and induction on $|\lambda|$). The second statement follows from (2.9) since q divides $\prod_{i=1}^q (2i-1)$.

For the third statement, observe that

$$\begin{aligned} 2^q \langle e_2(X)^q \rangle &\equiv \langle (p_1(X)^2 - p_1(X))^q \rangle \pmod{q} \\ &\equiv p_1(X)^{2q} - p_1(X)^q \pmod{q}, \end{aligned}$$

since $\langle p_\lambda(X) \rangle \in \mathbb{Z}[N, b]$ for each partition λ . Using the fact that $\langle p_1(X)^{2q} \rangle \equiv 0 \pmod{q}$ and applying (2.10), we deduce that

$$2^q \langle e_2(X)^q \rangle \equiv - \prod_{i=0}^{q-1} (N^2 + bN + 2i(b+1)) \pmod{q}.$$

Using the known fact that $\prod_{i=0}^{q-1} (Y + 2iZ) \equiv Y^q - YZ^{q-1} \pmod q$ for any integer polynomials $Y, Z \in \mathbb{Z}[N, b]$, we deduce (by letting $Y = N^2 + bN$ and $Z = b + 1$) that

$$\begin{aligned} 2^q \langle e_2(X)^q \rangle &\equiv (b + 1)^{q-1} (N^2 + bN) - (N^2 + bN)^q \\ &\equiv (b + 1)^{q-1} (N^2 + bN) - N^{2q} - b^q N^q \pmod q. \end{aligned} \quad (2.11)$$

Now, applying (2.11) and using the facts that

$$\begin{aligned} (2q)! \binom{N}{2q} &\equiv \prod_{i=0}^{2q-1} (N - i) \equiv N^2 (N^{q-1} - 1)^2 \equiv N^{2q} - 2N^{q+1} + N^2 \pmod q; \\ (q + 1)! \binom{N}{q + 1} &\equiv N^{q+1} - N^2 \pmod q; \quad q! \binom{N}{q} \equiv N^q - N \pmod q, \end{aligned}$$

we deduce the corollary. □

Remark 2.22. In the proof of Corollary 2.21, we explicitly evaluated $\langle e_1(X)^{2q} \rangle$, but did not explicitly evaluate $\langle e_2(X)^q \rangle$. Instead, we found the residue of this term modulo q through the expectation $\langle p_2(X)^q \rangle$ of a multi-part power sum. This leads to the question of whether we can find $\langle e_2(X)^q \rangle$ explicitly; Proposition 3.5 in Section 3 will lead to a partial answer.

We can now establish the following lemma.

Lemma 2.23. *For any prime number $q \geq 3$, we have that*

$$\sum_{i=0}^{q-1} a_{1^{2q},i} \binom{N}{i} - 2^q \sum_{i=0}^{q-1} a_{2^q,i} \binom{N}{i} \equiv (N - N^2) \sum_{i=1}^{q-1} (-b)^i \pmod q,$$

where the $a_{\lambda,i}$ are the polynomials defined by Lemma 2.5.

Proof. From Corollary 2.21, we obtain

$$\begin{aligned} \sum_{i=0}^{2q} a_{1^{2q},i} \binom{N}{i} - 2^q \sum_{i=0}^{2q} a_{2^q,i} \binom{N}{i} &\equiv \langle e_1(X)^{2q} \rangle - 2^q \langle e_2(X)^q \rangle \\ &\equiv (2q)! \binom{N}{2q} + 2(q + 1)! \binom{N}{q + 1} + b^q q! \binom{N}{q} \\ &\quad - ((b + 1)^{q-1} - 1) N^2 - (b(b + 1)^{q-1} - b^q) N \pmod q. \end{aligned}$$

Therefore, we deduce that

$$\sum_{i=0}^{q-1} a_{1^{2q},i} \binom{N}{i} - 2 \sum_{i=0}^{q-1} a_{2^q,i} \binom{N}{i} \equiv (1 - (b + 1)^{q-1}) N^2 - (b(b + 1)^{q-1} - b^q) N \pmod q.$$

Now the lemma follows from the fact that

$$1 - (b + 1)^{q-1} \equiv b(b + 1)^{q-1} - b^q \equiv - \sum_{i=1}^{q-1} (-b)^i \pmod q. \quad \square$$

2.5 Proof of Theorem 1.9

In this section, we will establish Theorem 1.9. However, we first require a proof of Lemma 2.4. Throughout, we will follow the notation used in Section 2.1.

Lemma 2.4. *For any partition $\lambda \notin \{1^{2q}, 2^q\}$ of size $2q$, there exists an integer k_λ , not divisible by q , such that $k_\lambda c_\lambda \in \mathbb{Z}$; equivalently, the denominator of c_λ is not divisible by q unless $\lambda = 1^{2q}$ or $\lambda = 2^q$. Furthermore, if $\lambda = 1^{2q}$, then $2qc_\lambda = 1$; if $\lambda = 2^q$, then $2qc_\lambda = -2$.*

Proof. The relationship between power sums and multi-part elementary symmetric polynomials is given by the Newton-Girard identity, which states that

$$p_{2q}(X) = 2q \sum_{|\lambda|=2q} \left(\frac{(\ell(\lambda) - 1)!}{\prod_{i=1}^{2q} (m_i(\lambda))!} \prod_{i=1}^{\lfloor |\lambda|/2 \rfloor} (-1)^{m_{2i}(\lambda)} \right) e_\lambda(X).$$

Therefore, the coefficients c_λ are defined by

$$c_\lambda = \frac{(\ell(\lambda) - 1)!}{\prod_{i=1}^{2q} (m_i(\lambda))!} \prod_{i=1}^{\lfloor |\lambda|/2 \rfloor} (-1)^{m_{2i}(\lambda)}, \quad (2.12)$$

from which we deduce that

$$c_\lambda \ell(\lambda) = \pm \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \dots, m_{2q}(\lambda)} \quad (2.13)$$

is a (possibly negative) multinomial coefficient and thus integral. Hence the denominator of c_λ can be divisible by q only if q divides $\ell(\lambda)$.

Since $\ell(\lambda) \leq |\lambda| = 2q$, this implies that $\ell(\lambda) \in \{q, 2q\}$. If $\ell(\lambda) = 2q$, then $\lambda = 1^{2q} = (1, 1, \dots, 1)$ (with $2q$ ones). If $\ell(\lambda) = q$, then the right side of (2.13) is divisible by q unless $m_i(\lambda) = q = \ell(\lambda)$ for some integer $i \in [1, 2q]$. This implies that $i = 2$ and $\lambda = 2^q$.

From (2.12), we deduce that $c_\lambda = (2q)^{-1}$ when $\lambda = 1^{2q}$ and $c_\lambda = -q^{-1}$ when $\lambda = 2^q$. \square

Next, we will need the following proposition of Forrester and Witte [12], which states that $\langle p_{2q}(X) \rangle$ is a polynomial of degree $q + 1$ whose high-degree coefficients are explicit.

Proposition 2.24 ([12, Theorem 3]). *For each integer $s \geq 1$, we have that*

$$\langle p_{2s}(X) \rangle = \frac{1}{s+1} \binom{2s}{s} N^{s+1} + \left(2^{2s-1} - \binom{2s-1}{s} \right) b N^s + r(N, b),$$

where r is a polynomial of degree $s - 1$ in N .

Remark 2.25. Forrester and Witte prove Proposition 2.24 as Theorem 3 in [12] by finding loop equations for the resolvent of the β -Hermite ensemble; in fact, they find the first six leading order terms of $\langle p_{2s}(X) \rangle$. Though interesting, their methods seem unrelated to ours; therefore, we refer the reader to the original paper [12] for the proof of this proposition.

Now, we may establish Theorem 1.9.

Theorem 1.9. *For each prime number $q \geq 3$, we have that*

$$\langle p_{2q} \rangle_\beta \equiv 2N^{q+1} + bN^q + \left(-\sum_{i=1}^{q-1} (-b)^i - 2 \right) N^2 + \left(\sum_{i=2}^{q-1} (-b)^i - 2b \right) N \pmod{q}.$$

Proof. Due to the fact that

$$\begin{aligned} & \frac{1}{s+1} \binom{2s}{s} N^{s+1} + \left(2^{2s-1} - \binom{2s-1}{s} \right) bN^s - \frac{(2s)!}{s!} \binom{N}{s+1} \\ & - b \left(2^{2s-1} + (s-1) \binom{2s-1}{s} \right) s! \binom{N}{s} \end{aligned}$$

is a polynomial of degree at most $s-1$ in N , Proposition 2.24 states that there exists a polynomial $\tilde{r}(N, b)$ of degree $q-1$ in N such that

$$\langle p_{2s}(X) \rangle = \frac{(2q)!}{q!} \binom{N}{q+1} + b \left(2^{2q-1} + (q-1) \binom{2q-1}{q} \right) q! \binom{N}{q} + \tilde{r}(N, b).$$

Thus, (2.3) may be rewritten as

$$\begin{aligned} \langle p_{2q}(X) \rangle &= \frac{(2q)!}{q!} \binom{N}{q+1} + b \left(2^{2q-1} + (q-1) \binom{2q-1}{q} \right) q! \binom{N}{q} \\ &+ \sum_{\substack{|\lambda|=2q \\ \lambda \notin \{1^{2q}, 2^q\}}} \sum_{i=0}^{q-1} \tilde{c}_\lambda a_{\lambda, i} \binom{N}{i} + \sum_{i=0}^{q-1} a_{1^{2q}, i} \binom{N}{i} - 2 \sum_{i=0}^{q-1} a_{2^q, i} \binom{N}{i}. \end{aligned} \quad (2.14)$$

Now let us clear denominators in (2.14). Let $K = (q-1)! \prod_\lambda k_\lambda$, where λ ranges over all partitions, not equal to 1^{2q} or 2^q , of size $2q$, and the k_λ are defined by Lemma 2.4. Let $\tilde{K} = 2^{q-1}K$; then, q does not divide \tilde{K} .

Multiplying (2.14) by \tilde{K} , we obtain that

$$\begin{aligned} \tilde{K} \langle p_{2q}(X) \rangle &= \frac{\tilde{K}(2q)!}{q!} \binom{N}{q+1} + \tilde{K}b \left(2^{2q-1} + (q-1) \binom{2q-1}{q} \right) q! \binom{N}{q} \\ &+ K \left(2^{q-1} \sum_{i=0}^{q-1} a_{1^{2q}, i} \binom{N}{i} - 2^q \sum_{i=0}^{q-1} a_{2^q, i} \binom{N}{i} \right) \\ &+ \tilde{K} \sum_{\substack{|\lambda|=2q \\ \lambda \notin \{1^{2q}, 2^q\}}} \sum_{i=0}^{q-1} \tilde{c}_\lambda a_{\lambda, i} \binom{N}{i}. \end{aligned} \quad (2.15)$$

The first and second terms in (2.15) can be evaluated directly in terms of \tilde{K} . Indeed, from (1.4), we deduce that

$$\frac{(2q)!}{q!} \binom{N}{q+1} \equiv \frac{1}{q+1} \binom{2q}{q} \prod_{i=0}^q (N-i) \equiv 2N^{q+1} - 2N^2 \pmod{q}. \quad (2.16)$$

Moreover, since

$$2^{2q-1} \equiv 2 \pmod{q}; \quad (q-1) \binom{2q-1}{q} \equiv -1 \pmod{q}; \quad \prod_{i=0}^{q-1} (N-i) \equiv N^q - N \pmod{q},$$

we have that

$$b \left(2^{2q-1} + (q-1) \binom{2q-1}{q} \right) q! \binom{N}{q} \equiv bN^q - bN \pmod{q}. \quad (2.17)$$

The third term on the right side of (2.15) can be evaluated in terms of K using Lemma 2.23 since $2^{q-1} \equiv 1 \pmod{q}$, and the fourth term is zero modulo q due to Lemma 2.4. Thus, we obtain that

$$\tilde{K} \langle p_{2q}(X) \rangle \equiv \tilde{K} (2N^{q+1} + bN^q - 2N^2 - bN) + K \left(\sum_{i=1}^{q-1} (-b)^i \right) (N - N^2) \pmod{q}. \quad (2.18)$$

Since $2^{q-1} \equiv 1 \pmod{q}$, we have that $\tilde{K} \equiv K \pmod{q}$. Thus, the theorem follows after multiplying (2.18) by $\tilde{K}^{-1} \pmod{q}$. \square

3 Expectations of Elementary Symmetric Polynomials

Several times in the proof of Theorem 1.9, we asked the question of how to explicitly evaluate the β -expectation $\langle e_\lambda(X) \rangle$ of a multi-part elementary symmetric polynomial (see, for instance, Proposition 2.1, Remark 2.14, and Remark 2.22). In this section, we will show one way of doing this through a recurrence that parallels the power-sum recursion (1.9).

3.1 Historical Context

In general, the question of how to explicitly evaluate the expectations of symmetric polynomials against the β -Hermite ensemble has been one of both probabilistic and combinatorial interest since the mid 1980s. Let us review some of the known results in this direction.

In 1986, Harer and Zagier [20] found an explicit form for the 2-expectation $\langle p_{2k}(X) \rangle_2$ of a single-part power sum and gave a combinatorial interpretation to the 2-expectation $\langle p_\lambda(X) \rangle_2$ of arbitrary multi-part power sums. The analogs of these results for $\beta = 1$

were found by Goulden and Jackson [18] in 1997. In connection with map enumeration, Goulden and Jackson conjectured an explicit form for the β -expectation of the Jack polynomial $J^{(2/\beta)}(X)$ in 1997 [18]. This conjecture was first proven in [26] by Okounkov and then in a different way in Chapter 8.5 of [8] by Dumitriu. The extension of Harer and Zagier's interpretation to the general β -Hermite ensemble was established by La Croix [22] in 2009. Moments of the β -Hermite ensemble have also been understood through loop equations for the resolvent, studied recently by Forrester and Witte in [12] (see also references therein and [13] for related work).

Also in 1986, Ullah [30] found a closed form for the 1-expectation $\langle e_k(X) \rangle_1$ of a single-part elementary symmetric polynomial (this now follows as a special case of Proposition 2.1). Aomoto later found a different, more analytic, proof of Proposition 2.1 that generalizes to yield expectations of one-part elementary symmetric polynomials against the β -Jacobi ensemble [2]. In Chapter 17.8 of his book [25], Mehta uses Aomoto's method to explicitly find the β -expectations $\langle m_\lambda(X) \rangle_\beta$ of monomial symmetric polynomials $m_\lambda(X)$, in which all parts of the partition λ are less than or equal to 3. These statements may be extended to more general partitions λ but with more complex results as the parts of λ increase.

Relatedly, Andrews, Goulden, and Jackson explicitly evaluated the expectations of moments of determinants for the GOE and GUE in 2003 [1]. Although an explicit form for the expectation $\langle e_N(X)^k \rangle_\beta$ of the k -th moment of the determinant of the general β -Hermite ensemble is not currently known, Dumitriu found the second moment explicitly and found recursive equations for the third and fourth moments (see Chapter 8.5 of [8]). Recently, Edelman and La Croix [11] used a singular value decomposition to find the law of the absolute value of the determinant of the GUE; the analog for the GOE was later found by Bornemann and La Croix in [5].

Thus, there has been a significant effort over the past 30 years in explicitly evaluating $\langle f(X) \rangle_\beta$, for various symmetric functions f . We will be interested in the case when $f = e_\lambda$ is a multi-part elementary symmetric polynomial. In Section 3.2, we will present a recurrence relation that evaluates $\langle e_\lambda(X) \rangle_\beta$ explicitly; we will also show how it can be applied in several cases. As a related application, we will show in Section 3.3 how this recursion can be used to give an algorithm that finds $\langle e_\lambda(X) \rangle_\beta$ efficiently for all partitions λ of small length.

3.2 The Recursion

The recursion for expectations of elementary symmetric polynomials may be stated as follows.

Theorem 3.1. *For any partition $\lambda = (\lambda_1, \dots, \lambda_r)$ and integer $k \geq \max_{i \in [1, r]} \lambda_i$, we have*

that

$$k\langle e_{k,\lambda}(X)\rangle_\beta = (N - k + 1)\alpha \sum_{i=1}^r \langle e_{k-1,\lambda_i-1,\lambda\setminus\lambda_i}(X)\rangle_\beta - (N - k + 1)(N - k + 2)\langle e_{k-2,\lambda}(X)\rangle_\beta \\ - \alpha \sum_{i=1}^r \sum_{j=1}^{\lambda_i-1} (k - \lambda_i + 2j)\langle e_{k+j-1,\lambda_i-j-1,\lambda\setminus\lambda_i}(X)\rangle_\beta.$$

Remark 3.2. From Theorem 3.1, we deduce that $\langle e_{k,\lambda}(X)\rangle_\beta$ is a linear combination of $\langle e_\mu(X)\rangle_\beta$, where μ is either of the form $(k, \lambda_i - 1, \lambda \setminus \lambda_i)$ for some $i \in [1, r]$; of the form $(k + j - 1, \lambda_i - j - 1, \lambda \setminus \lambda_i)$ for some $i \in [1, r]$ and $j \in [1, \lambda_i - 1]$; or equal to $(k - 2, \lambda)$. In each of these cases, $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for each $i \in [2, r]$. This property, which is not shared by the recurrence (1.9) given by LaCroix, will later allow for an efficient algorithm that finds $\langle e_\lambda(X)\rangle_\beta$ for partitions λ of small length (see Corollary 3.10).

We will give the proof of Theorem 3.1 in Section 3.4, but let us first show how it can be used to explicitly evaluate $\langle e_\lambda(X)\rangle_\beta$ for several classes of partitions λ ; some of our results are summarized in the table in Figure 3. For instance, we can obtain another proof of Proposition 2.1.

Proposition 2.1 ([2]). *If k is a nonnegative even integer, then*

$$\langle e_k(X)\rangle = (-1)^{k/2} \binom{N}{k} \prod_{j=1}^{k/2-1} (2j + 1).$$

Proof. Observe that $\langle e_0(X)\rangle_\beta = 1$ since \mathbb{P}_β is a probability measure. Applying Theorem 3.1 with empty λ yields $k\langle e_k(X)\rangle_\beta = (N - k + 1)(k - 2 - N)\langle e_{k-2}(X)\rangle$ for each integer $k \geq 2$. The proposition then follows by induction on k . \square

Using Theorem 3.1 can also establish the following results.

Corollary 3.3. *If k is odd and positive, then*

$$\langle e_{k,1}(X)\rangle_\beta = (-1)^{(k-1)/2} \binom{N}{k} \alpha \prod_{j=1}^{(k-1)/2} (2j + 1).$$

Proof. Applying Theorem 3.1 with the single-part partition $\lambda = (1)$, we obtain that

$$\langle e_{k,1}(X)\rangle_\beta = (N - k + 1)\alpha \langle e_{k-1}(X)\rangle_\beta - (N - k + 1)(N - k + 2)\langle e_{k-2,1}(X)\rangle_\beta. \quad (3.1)$$

If $k = 1$, then (3.1) implies that $\langle e_{1,1}(X)\rangle_\beta = N\alpha$. If $k \geq 3$, then the result follows from Proposition 2.1, (3.1), and induction on k . \square

Proposition 3.4. *If k is an even nonnegative integer, then*

$$\langle e_{k,2}(X)\rangle_\beta = (-1)^{k/2-1} \binom{N}{k} \left(k \binom{\alpha + 1}{2} + \binom{N}{2} \right) \prod_{j=1}^{k/2-1} (2j + 1).$$

Proof. If $k = 0$, then the result follows from Proposition 2.1, so suppose that $k \geq 2$. Inserting the single-part partition $\lambda = (2)$ into Theorem 3.1 yields

$$\begin{aligned} k\langle e_{k,2}(X) \rangle_\beta &= (N - k + 1)\alpha\langle e_{k-1,1}(X) \rangle_\beta - (N - k + 1)(N - k + 2)\langle e_{k-2,2}(X) \rangle_\beta - \alpha k\langle e_k(X) \rangle_\beta \\ &= (-1)^{k/2-1}k(\alpha + 1)\alpha \binom{N}{k} \prod_{j=1}^{k/2-1} (2j + 1) - (N - k + 1)(N - k + 2)\langle e_{k-2,2}(X) \rangle_\beta. \end{aligned}$$

The proposition now follows by induction on k . □

Proposition 3.5. *If s is a nonnegative integer, then $\langle e_1(X)^{2s} \rangle_\beta = (N\alpha)^s \prod_{j=1}^{s-1} (2j + 1)$. Moreover, for any positive integers s and r , we have that*

$$\begin{aligned} 2\langle e_1(X)^{2s} e_2(X)^r \rangle_\beta &= 2s(N - 1)\alpha\langle e_1(X)^{2s} e_2(X)^{r-1} \rangle_\beta \\ &\quad + (N - 1)(r - 1)\alpha\langle e_1(X)^{2s+2} e_2(X)^{r-2} \rangle_\beta \\ &\quad - N(N - 1)\langle e_1(X)^{2s} e_2(X)^{r-1} \rangle_\beta - 2\alpha(r - 1)\langle e_1(X)^{2s} e_2(X)^{r-1} \rangle_\beta. \end{aligned}$$

Proof. Inserting $k = 1$ and $\lambda = 1^{2s-1}$ into Theorem 3.1 yields $\langle e_1(X)^{2s} \rangle_\beta = N\alpha(2s - 1)\langle e_1(X)^{2s-2} \rangle_\beta$. We therefore deduce the first part of the proposition by induction on k . The second statement of the proposition follows from inserting $k = 2$ and $\lambda = 1^s 2^{r-1}$ (with s ones and $r - 1$ twos) into Theorem 3.1. □

Remark 3.6. The first statement of Proposition 3.5 coincides with (2.9) since $\alpha = b + 1$. The second statement of Proposition 3.5 allows one to evaluate $\langle e_2(X)^s \rangle_\beta$ through an efficient recurrence, thereby partially answering the question asked in Remark 2.22. We do not know of a compact, explicit expression for $\langle e_2(X)^s \rangle_\beta$, but Proposition 3.5 might be sufficient to provide an alternative derivation of the arithmetic statement Corollary 2.21.

Proposition 3.7. *For any partition λ , $\langle e_\lambda(X) \rangle_\beta \in \mathbb{Q}[N, \alpha]$ is a polynomial whose degree in N is at most $|\lambda|$. Furthermore, $|\lambda|!\langle e_\lambda(X) \rangle_\beta \in \mathbb{Z}[N, \alpha]$.*

Proof. We have that $\langle e_0(X) \rangle = 1$ and $\langle e_1(X) \rangle = 0$; therefore, the statement holds when $|\lambda| \in \{0, 1\}$. Now the proposition follows from Theorem 3.1 and induction on $|\lambda|$. □

Remark 3.8. Observe that Proposition 3.7 also follows from Lemma 2.5. We do not know how to establish the latter result directly from Theorem 3.1.

3.3 Algorithm

Due to the interest and applications of the β -Hermite ensemble, Dumitriu, Edelman, and Shuman introduced an algorithm that evaluates $\langle f(X) \rangle_\beta$ explicitly in terms of N and $\alpha = 2/\beta$, for any symmetric polynomial f of N variables [10]. Though effective, their algorithm involves a change-of-basis to the Jack symmetric polynomials, which is a high-complexity process. For instance, in order to evaluate $\langle e_{2k}(X) \rangle_\beta$, this algorithm requires

Expectations of Multi-part Elementary Symmetric Polynomials	
$F(X)$	$\langle F(X) \rangle_\beta$
$e_{2k}(X)$	$(-1)^k \binom{N}{2k} \prod_{j=1}^{k-1} (2j+1)$
$e_{2k+1,1}(X)$	$(-1)^k \alpha \binom{N}{2k+1} \prod_{j=1}^k (2j+1)$
$e_{2k+2,2}(X)$	$(-1)^k \binom{N}{2k+2} \left(k \binom{\alpha+1}{2} + \binom{N}{2} \right) \prod_{j=1}^k (2j+1)$
$e_1(X)^{2s}$	$(N\alpha)^s \prod_{j=1}^{s-1} (2j+1)$
$e_1(X)^{2s} e_2(X)$	$\frac{1}{2} (2s\alpha - N)(N-1)(N\alpha)^s \prod_{j=1}^{s-1} (2j+1)$

Figure 2: Some applications of Theorem 3.1 are listed in the above table.

complexity $\exp(\Theta(\sqrt{k}))$ (see Section 3 of [10]), even though $\langle e_{2k}(X) \rangle_\beta$ is explicit by Proposition 2.1.

One may observe that LaCroix's recursive relation (1.9) also yields an algorithm that finds β -expectations of multi-part power sums $\langle p_\lambda(X) \rangle_\beta$ explicitly in terms of N and b , for any partition λ . A change-of-basis (which, depending on f , might be a high-complexity process) would then let one evaluate $\langle f(X) \rangle_\beta$, for any symmetric polynomial f , as above. However, one may verify that this algorithm also requires complexity $\exp(\Theta(\sqrt{k}))$ in order to evaluate $\langle e_{2k}(X) \rangle_\beta$.

In general, we do not know if there is an algorithm that evaluates β -expectations of general multi-part elementary symmetric polynomials, power sums, or monomials efficiently. More specifically, it is unknown whether there exists an integer d and an algorithm that finds $\langle e_\lambda(X) \rangle_\beta$ (or $\langle p_\lambda(X) \rangle_\beta$ or $\langle m_\lambda(X) \rangle_\beta$) in $O(|\lambda|^d)$ space and time, for any partition λ . However, in this section, we will provide a partial result in this direction by presenting an algorithm \mathcal{A}_λ that finds $\langle e_\lambda(X) \rangle_\beta$, as a polynomial in N and α , with complexity $O(|\lambda|^d(d^2 + |\lambda|))$ for all partitions λ satisfying $\ell(\lambda) \leq d-1$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition, and let S_λ denote the set of partitions $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ such that either $|\mu| \leq |\lambda| - 2$, $s \leq r$, and $\mu_i \leq \lambda_i$ for each integer $i \in [2, s]$ or such that $\mu = \lambda$. Our algorithm \mathcal{A}_λ will evaluate $\langle e_\mu(X) \rangle_\beta$ for all $\mu \in S_\lambda$ in $O(|\lambda||S_\lambda|)$ time and $O(\ell(\lambda)^2|S_\lambda|)$ space. We will have \mathcal{A}_λ perform the following steps.

1. Select an ordering $\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(|S_\lambda|)}$ of the partitions in S_λ such that $|\mu^{(i)}| \leq |\mu^{(i+1)}|$ for each integer $i \in [1, |S_\lambda| - 1]$.
2. Create a list of variables $\{e_\mu\}_{\mu \in S_\lambda}$ indexed by S_λ , and initialize $e_0 = 1$ and $e_1 = 0$.

3. Let i be the minimum integer such that $e_{\mu^{(i)}}$ has not been defined. Let $\mu^{(i)} = (k, \mu)$, where k is the largest part of $\mu^{(i)}$.
4. Define $e_{k,\mu}$ through the equality

$$\begin{aligned}
ke_{k,\mu} &= (N - k + 1)\alpha \sum_{i=1}^r e_{k-1,\mu_i-1,\mu \setminus \mu_i} - (N - k + 1)(N - k + 2)e_{k-2,\mu} \\
&\quad - \alpha \sum_{i=1}^r \sum_{j=1}^{\mu_i-1} (k - \mu_i + 2j)e_{k+j-1,\mu_i-j-1,\mu \setminus \mu_i}.
\end{aligned} \tag{3.2}$$

5. Repeat Steps 3 and 4 until e_λ is defined; then output e_λ , and stop.

By Remark 3.2, the recursion given by Theorem 3.1 expresses $\langle e_\lambda(X) \rangle_\beta$ as a linear combination of $|\lambda| + 1$ terms of the form $\langle e_\mu(X) \rangle_\beta$ for $\mu \in S_\lambda$. Furthermore, since $S_\mu \subseteq S_\lambda$ for any partition $\mu \in S_\lambda$, we deduce that each e_ν on the right side of (3.2) has been previously defined by \mathcal{A}_λ (due to the minimality of $|(k, \mu)|$). Therefore, Theorem 3.1 ensures that \mathcal{A}_λ outputs $\langle e_\lambda(X) \rangle_\beta$, since the e_λ and the $\langle e_\lambda(X) \rangle_\beta$ satisfy the same recurrence relation with the same initial conditions.

It remains to analyze the complexity of \mathcal{A} .

Proposition 3.9. *The algorithm \mathcal{A}_λ runs in time $O(|\lambda||S_\lambda|)$ and space $O(\ell(\lambda)^2|S_\lambda|)$.*

Proof. The fact that \mathcal{A}_λ requires space $O(\ell(\lambda)^2|S_\lambda|)$ is due to the fact that the data stored by \mathcal{A}_λ consists of the variable set $\{e_\mu\}_{\mu \in S_\lambda}$, and each variable is defined by a polynomial in N and α of degree at most $\ell(\lambda)$ in each variable. Now let us turn to the run time of \mathcal{A} . Since $|\mu| \in [0, |\lambda|]$, for each $\mu \in S_\lambda$, the run time of Step 1 is $O(|\lambda||S_\lambda|)$. The run time of Step 2 is $O(|S_\lambda|)$. The run time of Step 3 is $O(1)$, and the run time of Step 4 is $O(|\lambda|)$, since there are $|\lambda| + 1$ terms on the right side of (3.2). Step 5 repeats Steps 3 and 4 $O(|S_\lambda|)$ times, so the run time of Step 5 is \mathcal{A} is $O(|\lambda||S_\lambda|)$. Summing over the steps of \mathcal{A} , we deduce that the run time of \mathcal{A} is $O(|\lambda||S_\lambda|)$. \square

Corollary 3.10. *Suppose that λ is a partition satisfying $\ell(\lambda) \leq d$. Then, \mathcal{A} evaluates $\langle e_\lambda(X) \rangle_\beta$ in space $O(d^2|\lambda|^d)$ and time $O(|\lambda|^{d+1})$.*

Proof. This follows from Proposition 3.9 and the fact that $|S_\lambda| \leq |\lambda|^{\ell(\lambda)}$. \square

Remark 3.11. Generally, $\ell(\lambda)$ is not bounded, which means that \mathcal{A} does not provide a polynomial-time and polynomial-space algorithm that evaluates $\langle e_\lambda(X) \rangle_\beta$. However, $\ell(\lambda)$ is bounded for some classes of partitions λ ; in such cases, \mathcal{A} is efficient. For instance, \mathcal{A} is efficient when evaluating small moments of determinants of the β -Hermite ensemble.

3.4 Proof of Theorem 3.1

Our proof of Theorem 3.1 will be similar to Aomoto's proof [2] of Proposition 2.1. In particular, we will use his “de-symmetrizing” idea, which involves transferring between expressions of the form $\langle e_k(X) \rangle$ and expressions of the form $\langle e_j(X \setminus \{x_1\}) \rangle$. As in Section 2, we write $\langle f(X) \rangle$ in place of $\langle f(X) \rangle_\beta$ for any symmetric polynomial f .

The following lemma, whose proof will be deferred Section 3.5, will facilitate this transferal.

Lemma 3.12. *Let $a \geq b \geq 0$ be integers, λ be a partition, and $X = (x_1, x_2, \dots, x_N)$ be a set of real variables. Then,*

$$N \langle e_{a,b}(X \setminus \{x_1\}) e_\lambda(X) \rangle = (N - a) \langle e_{a,b}(X) e_\lambda(X) \rangle - \sum_{i=1}^b (a - b + 2i) \langle e_{a+i,b-i}(X) e_\lambda(X) \rangle.$$

Using this lemma, we can prove Theorem 3.1.

Theorem 3.1. *For any partition $\lambda = (\lambda_1, \dots, \lambda_r)$ and integer $k \geq \max_{i \in [1,r]} \lambda_i$, we have that*

$$\begin{aligned} k \langle e_{k,\lambda}(X) \rangle_\beta &= (N - k + 1) \alpha \sum_{i=1}^r \langle e_{k-1,\lambda_i-1,\lambda \setminus \lambda_i}(X) \rangle_\beta - (N - k + 1)(N - k + 2) \langle e_{k-2,\lambda}(X) \rangle_\beta \\ &\quad - \alpha \sum_{i=1}^r \sum_{j=1}^{\lambda_i-1} (k - \lambda_i + 2j) \langle e_{k+j-1,\lambda_i-j-1,\lambda \setminus \lambda_i}(X) \rangle_\beta. \end{aligned}$$

Proof. Let

$$\Omega_\beta(X) = c(\beta) \exp\left(\frac{-\beta}{4} \sum_{i=1}^N x_i^2\right) \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta$$

denote the density function of the β -Hermite ensemble (1.1). Observe that

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\partial}{\partial x_1} (e_{k-1}(X \setminus \{x_1\}) e_\lambda(X) \Omega_\beta(X)) dx_1 \\ &= \lim_{z \rightarrow \infty} e_{k-1}(X \setminus \{x_1\}) e_\lambda(z, X \setminus \{x_1\}) \Omega_\beta(z, X \setminus \{x_1\}) \\ &\quad - \lim_{z \rightarrow -\infty} e_{k-1}(X \setminus \{x_1\}) e_\lambda(z, X \setminus \{x_1\}) \Omega_\beta(z, X \setminus \{x_1\}) = 0, \end{aligned} \quad (3.3)$$

where the second equality holds because both limits are equal to 0 (due to the exponential term in $\Omega_\beta(X)$). The derivative on the left side of (3.3) is equal to

$$\begin{aligned} &\frac{\partial}{\partial x_1} (e_{k-1}(X \setminus \{x_1\}) e_\lambda(X) \Omega_\beta(X)) \\ &= \Omega_\beta(X) \left(\sum_{i=1}^{\ell(\lambda)} e_{k-1}(X \setminus \{x_1\}) e_{\lambda_i-1}(X \setminus \{x_1\}) e_{\lambda \setminus \lambda_i}(X) \right. \\ &\quad \left. + \beta e_\lambda(X) \sum_{i=1}^N \frac{e_{k-1}(X \setminus \{x_1\})}{x_1 - x_i} - \frac{\beta x_1 e_{k-1}(X \setminus \{x_1\}) e_\lambda(X)}{2} \right). \end{aligned}$$

Integrating first over x_1 , then integrating over $X \setminus \{x_1\}$, and applying (3.3) yields

$$\sum_{i=1}^{\ell(\lambda)} \left\langle e_{k-1}(X \setminus \{x_1\}) e_{\lambda_i-1}(X \setminus \{x_1\}) e_{\lambda \setminus \lambda_i}(X) \right\rangle + \beta \left\langle e_{\lambda}(X) \sum_{i=1}^N \frac{e_{k-1}(X \setminus \{x_1\})}{x_1 - x_i} \right\rangle - \frac{\beta \langle x_1 e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X) \rangle}{2} = 0.$$

Multiplying by $2N/\beta = N\alpha$, and rearranging terms yields

$$\begin{aligned} N \langle x_1 e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X) \rangle &= 2N \sum_{i=1}^N \left\langle \frac{e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X)}{x_1 - x_i} \right\rangle \\ &\quad + N\alpha \sum_{i=1}^{\ell(\lambda)} \langle e_{k-1}(X \setminus \{x_1\}) e_{\lambda_i-1}(X \setminus \{x_1\}) e_{\lambda \setminus \lambda_i}(X) \rangle. \end{aligned} \tag{3.4}$$

By symmetry, $N \langle x_1 e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X) \rangle = k \langle e_k(X) e_{\lambda}(X) \rangle$. Using this fact and applying Lemma 3.12 to the second summand on the right side of (3.4) yields

$$\begin{aligned} k \langle e_k(X) e_{\lambda}(X) \rangle &= 2N \sum_{i=1}^N \left\langle \frac{e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X)}{x_1 - x_i} \right\rangle \\ &\quad + \alpha(N - k + 1) \sum_{i=1}^{\ell(\lambda)} \langle e_{k-1, \lambda_i-1}(X) e_{\lambda \setminus \lambda_i}(X) \rangle \\ &\quad - \alpha \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i-1} (k - \lambda_i + 2j) \langle e_{k+j-1, \lambda_i-j-1}(X) e_{\lambda^{(i)}}(X) \rangle. \end{aligned} \tag{3.5}$$

By symmetry, the first term on the left side of (3.5) is equal to

$$\begin{aligned} 2N \sum_{i=1}^N \left\langle \frac{e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X)}{x_1 - x_i} \right\rangle &= N \sum_{i=1}^N \left\langle \frac{e_{k-1}(X \setminus \{x_1\}) e_{\lambda}(X)}{x_1 - x_i} - \frac{e_{k-1}(X \setminus \{x_i\}) e_{\lambda}(X)}{x_1 - x_i} \right\rangle \\ &= -N \sum_{i=1}^N \langle e_{k-2}(X \setminus \{x_1, x_i\}) e_{\lambda}(X) \rangle \\ &= -N(N-1) \langle e_{k-2}(X \setminus \{x_1, x_2\}) e_{\lambda}(X) \rangle \\ &= (k - N - 1)(N - k + 2) \langle e_{k-2}(X) e_{\lambda}(X) \rangle. \end{aligned}$$

Inserting this into (3.5) yields the theorem. □

3.5 Proof of Lemma 3.12

Here, we will give a proof of Lemma 3.12.

Lemma 3.12. Let $a \geq b \geq 0$ be integers, λ be a partition, and $X = (x_1, x_2, \dots, x_N)$ be a set of real variables. Then,

$$N \langle e_{a,b}(X \setminus \{x_1\}) e_\lambda(X) \rangle = (N - a) \langle e_{a,b}(X) e_\lambda(X) \rangle - \sum_{i=1}^b (a - b + 2i) \langle e_{a+i,b-i}(X) e_\lambda(X) \rangle.$$

Proof. For any nonnegative integers t and s , let

$$d_{t,s}(X) = \sum_{\substack{|I|=t \\ |J|=s}} \prod_{\substack{i \in I \\ j \in J}} x_i x_j^2,$$

where the sum ranges over all pairs (I, J) of disjoint subset $I, J \subseteq \{1, 2, \dots, n\}$ such that $|I| = t$ and $|J| = s$. Observe that

$$e_{a,b}(X) = e_a(X) e_b(X) = \sum_{i=0}^b d_{a-b+2i,b-i}(X) \binom{a-b+2i}{i}. \quad (3.6)$$

Furthermore, by symmetry, we have that

$$N \langle g(X \setminus \{x_1\}) f(X) \rangle = \sum_{i=1}^N \langle g(X \setminus \{x_i\}) f(X) \rangle \quad (3.7)$$

and

$$(N - t - s) \langle d_{t,s}(X \setminus \{x_i\}) f(X) \rangle = \langle d_{t,s}(X) f(X) \rangle, \quad (3.8)$$

for any symmetric polynomials f and g in N variables. Applying (3.6) and (3.8) to (3.7) (with $f(X) = e_\lambda(X)$ and $g(X) = e_{a,b}(X)$) yields

$$\begin{aligned} N \langle e_{a,b}(X \setminus \{x_1\}) e_\lambda(X) \rangle &= \left\langle e_\lambda(X) \sum_{i=1}^N e_{a,b}(X \setminus \{x_i\}) \right\rangle \\ &= \left\langle e_\lambda(X) \sum_{i=0}^b \binom{a-b+2i}{i} d_{a-b+2i,b-i}(X \setminus \{x_1\}) \right\rangle \\ &= \left\langle e_\lambda(X) \sum_{i=0}^b \binom{a-b+2i}{i} (N - a - i) d_{a-b+2i,b-i}(X) \right\rangle. \end{aligned} \quad (3.9)$$

Multiplying (3.6) by $N - a$ and subtracting (3.9) yields

$$\begin{aligned} (N - a) \langle e_{a,b}(X) e_\lambda(X) \rangle - N \langle e_{a,b}(X \setminus \{x_1\}) e_\lambda(X) \rangle &= \left\langle e_\lambda(X) \sum_{i=1}^b i \binom{a-b+2i}{i} d_{a-b+2i,b-i}(X) \right\rangle \\ &= \left\langle e_\lambda(X) \sum_{i=0}^{b-1} (i+1) \binom{a-b+2i+2}{i+1} d_{a-b+2i+2,b-i-1}(X) \right\rangle. \end{aligned} \quad (3.10)$$

Now, observe that

$$(i+1) \binom{a-b+2i+2}{i+1} = (a-b+2) \binom{a-b+2i+2}{i} + i \binom{a-b+2i+2}{i}, \quad (3.11)$$

for each integer $i \in [0, b-1]$. Multiplying (3.11) by $d_{a-b+2i+2, b-i-1}$, summing i between 0 and $b-1$, and applying (3.6) (with the pair (a, b) replaced by $(a+1, b-1)$), we deduce that

$$\begin{aligned} & \sum_{i=0}^{b-1} (i+1) \binom{a-b+2i+2}{i+1} d_{a-b+2i+2, b-i-1}(X) \\ &= (a-b+2)e_{a+1, b-1}(X) + \sum_{i=1}^{b-1} i \binom{a-b+2i+2}{i} d_{a-b+2i+2, b-i-1}(X) \\ &= (a-b+2)e_{a+1, b-1}(X) + \sum_{i=0}^{b-2} (i+1) \binom{a-b+2i+4}{i+1} d_{a-b+2i+4, b-i-2}(X). \end{aligned} \quad (3.12)$$

Applying (3.12) with (a, b) replaced with $(a+1, b-1)$ yields that

$$\begin{aligned} & \sum_{i=0}^{b-2} (i+1) \binom{a-b+2i+4}{i+1} d_{a-b+2i+4, b-i-2}(X) \\ &= (a-b+4)e_{a+2, b-2}(X) + \sum_{i=0}^{b-3} (i+1) \binom{a-b+2i+6}{i+1} d_{a-b+2i+6, b-i-3}(X). \end{aligned} \quad (3.13)$$

Inserting (3.13) into the right side of (3.12), we deduce that

$$\begin{aligned} & \sum_{i=0}^{b-1} (i+1) \binom{a-b+2i+2}{i+1} d_{a-b+2i+2, b-i-1}(X) \\ &= \sum_{i=1}^2 (a-b+2i)e_{a+i, b-i}(X) + \sum_{i=0}^{b-3} (i+1) \binom{a-b+2i+6}{i+1} d_{a-b+2i+6, b-i-3}(X). \end{aligned}$$

Repeating this procedure $b-2$ more times yields

$$\sum_{i=0}^{b-1} (i+1) \binom{a-b+2i+2}{i+1} d_{a-b+2i+2, b-i-1}(X) = \sum_{i=1}^b (a-b+2i)e_{a+i, b-i}(X). \quad (3.14)$$

Multiplying (3.14) by $e_\lambda(X)$, taking expectations, and inserting the resulting equation into (3.10) yields

$$(N-a) \langle e_{a,b}(X) e_\lambda(X) \rangle - N \langle e_{a,b}(X \setminus \{x_1\}) e_\lambda(X) \rangle = \sum_{i=1}^b (a-b+2i) \langle e_{a+i, b-i}(X) e_\lambda(X) \rangle,$$

from which we deduce the lemma. \square

4 Further Directions

In this paper we established an arithmetic property of moments of the β -Hermite ensemble, and interpreted this result combinatorially. However, instead of combinatorial, our methods were matrix theoretic, which led to us to come across several intermediate results that might be of independent interest to combinatorialists and random matrix theorists alike.

Still, our study was not exhaustive. We would like to conclude this discussion by listing several related directions that might be interesting to pursue.

1. Is there a combinatorial proof of Theorem 1.8?
2. Our methods seem specific to evaluating $\langle p_{2q}(X) \rangle_\beta$ modulo q . What modulo q properties do $\langle p_\lambda(X) \rangle_\beta$ satisfy for other partitions λ ? For instance, what happens when $\lambda = (4q)$?
3. The result (1.10) shows that the coefficients $h_{i,j}^{(f)}$ from Corollary 2.15 are nonnegative and have combinatorial interpretations when f is a power sum. For what other symmetric polynomials f are these coefficients nonnegative, and when do they have a combinatorial interpretation?
4. In Section 3.3, we will present an algorithm that finds $\langle e_\lambda(X) \rangle_\beta$ in $O(|\lambda|^{\ell(\lambda)+2})$ complexity; this is of polynomial time and space when λ has small length. Does there exist an algorithm that evaluates $\langle e_\lambda(X) \rangle_\beta$ in polynomial time and space, for all partitions λ ?

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