Cubic graphs and related triangulations on orientable surfaces*

Wenjie Fang[†‡] Mihyun Kang[‡§]
Michael Moßhammer[§] Philipp Sprüssel[§]

Institute of Discrete Mathematics
Graz University of Technology
Graz, Austria

{fang,kang,mosshammer,spruessel}@math.tugraz.at

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Abstract

Let $\mathbb{S}_g$ be the orientable surface of genus g for a fixed non-negative integer g. We show that the number of vertex-labelled cubic multigraphs embeddable on $\mathbb{S}_g$ with $2n$ vertices is asymptotically $c_g n^{5/2(g-1)-1}\gamma^{2n}(2n)!$, where γ is an algebraic constant and c_g is a constant depending only on the genus g. We also derive an analogous result for simple cubic graphs and weighted cubic multigraphs. Additionally, for $g \geqslant 1$, we prove that a typical cubic multigraph embeddable on $\mathbb{S}_g$ has exactly one non-planar component.

Keywords: Cubic graphs; graphs on surfaces; triangulations; asymptotic enumeration; analytic combinatorics.

1 Introduction

Determining the numbers of maps and graphs *embeddable* on surfaces have been one of the main objectives of enumerative combinatorics for the last 50 years. Starting from the enumeration of *planar* maps by Tutte [34] various types of maps on the sphere were

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counted, e.g. planar *cubic* maps by Gao and Wormald [17]. Furthermore, Tutte's methods were generalised to enumerate maps on surfaces of higher genus [1, 2, 6].

An important subclass of maps are *triangulations*. Brown [9] determined the number of triangulations of a disc, and Tutte enumerated planar triangulations [33]. Triangulations on other surfaces have been considered as well. Gao enumerated 2-connected triangulations on the projective plane [13] as well as connected [14], 2-connected [15] and 3-connected [16] triangulations on surfaces of arbitrary genus.

Frieze [25] was arguably the first to ask about properties of random planar *graphs*. McDiarmid, Steger, and Welsh [26] showed the existence of an exponential growth constant for the number of vertex-labelled planar graphs with n vertices. This growth constant and the asymptotic number of planar graphs were determined by Giménez and Noy [18], while the corresponding results for the higher genus case were derived by Chapuy, Fusy, Giménez, Mohar and Noy [10] and independently by Bender and Gao [3]. Since then various other classes of planar graphs were counted [5, 7, 8, 21, 22, 24, 30].

An interesting subclass of planar graphs is the class of *cubic* planar graphs, which have been counted by Bodirsky, Kang, Löffler and McDiarmid [8]. Cubic planar graphs occur as substructures of sparse planar graphs and have thus been one of the essential ingredients in the study of sparse random planar graphs [22]. For surfaces of higher genus, the number of embeddable cubic graphs has not been studied.

Throughout the paper, let g be a fixed non-negative integer and let $\mathbb{S}_g$ be the orientable[1] surface of genus g. In this paper, we study cubic graphs embeddable on $\mathbb{S}_g$, in particular their asymptotic number. Similar to the case of planar graphs, cubic graphs embeddable on $\mathbb{S}_g$ appear as essential substructures of sparse graphs embeddable on $\mathbb{S}_g$. Therefore, the results of this paper pave the way to the study of sparse random graphs embeddable on $\mathbb{S}_g$ [23].

1.1 Main results

The main contributions of this paper are fourfold. We determine the asymptotic number of cubic multigraphs embeddable on $\mathbb{S}_g$. We also determine the asymptotic number of *weighted cubic multigraphs and cubic* simple graphs embeddable on $\mathbb{S}_g$. Finally we prove that almost all (multi)graphs from either of the three classes have exactly one non-planar component.

The first main result provides the exact asymptotic expression of the number of cubic multigraphs embeddable on $\mathbb{S}_g$.

Theorem 1.1. *The number $m_g(n)$ of vertex-labelled cubic multigraphs embeddable on $\mathbb{S}_g$ with $2n$ vertices is given by*

$$m_g(n) = \left(1 + O\left(n^{-1/4}\right)\right) d_g n^{5g/2-7/2} \gamma_1^{2n} (2n)!\,,$$

where γ_1 is an algebraic constant independent of the genus g and d_g is a constant depending only on g. The first digits of γ_1 are 3.986.

[1] We believe that one can also prove the main results for multigraphs embeddable on non-orientable surfaces, but with considerably more effort and case distinctions.

Our next main result concerns multigraphs weighted by the so-called *compensation factor* introduced by Janson, Knuth, Łuczak and Pittel [20]. This factor is defined as the number of ways to orient and order all edges of the multigraph divided by $2^r r!$, which is equal to the number of such oriented orderings if all edges were distinguishable. For example, a double edge results in a factor $\frac{1}{2}$ and simple graphs are the only multigraphs with compensation factor one.

Theorem 1.2. *The number $w_g(n)$ of vertex-labelled cubic multigraphs embeddable on $\mathbb{S}_g$ with $2n$ vertices* weighted by their compensation factor is given by

$$w_g(n) = \left(1 + O\left(n^{-1/4}\right)\right) e_g n^{5g/2-7/2} \gamma_2^{2n} (2n)!\,,$$

where $\gamma_2 = \frac{79^{3/4}}{54^{1/2}}$ and e_g is a constant depending only on the genus g. The first digits of γ_2 are 3.606

Theorem 1.2 can be used to derive the asymptotic number and structural properties of graphs embeddable on $\mathbb{S}_g$ [23]. *Planar* cubic multigraphs weighted by the compensation factor were counted by Kang and Łuczak [22]. The discrepancy to their exponential growth constant $\gamma \approx 3.38$ is due to incorrect initial conditions in [22], as pointed out by Noy, Ravelomanana and Rué [28]. While the explicit value of the correct exponential growth constant γ was not determined in [28], the implicit equations given there yield the same exponential growth constant γ_2 as in Theorem 1.2.

Our methods also allow us to count cubic *simple* graphs (graphs without loops and multi-edges) embeddable on $\mathbb{S}_g$.

Theorem 1.3. *The number $s_g(n)$ of vertex-labelled cubic* simple graphs embeddable on $\mathbb{S}_g$ with $2n$ vertices is given by

$$s_g(n) = \left(1 + O\left(n^{-1/4}\right)\right) f_g n^{5g/2-7/2} \gamma_3^{2n} (2n)!\,,$$

where γ_3 is an algebraic constant independent of the genus g and f_g is a constant depending only on g. The first digits of γ_3 are 3.133.

The exponential growth constant γ_3 coincides with the growth constant calculated for vertex-labelled cubic simple planar graphs by Bodirsky, Kang, Löffler and McDiarmid [8].

The final result describes the structure of cubic multigraphs embeddable on $\mathbb{S}_g$.

Theorem 1.4. *Let $g \geqslant 1$ and let G be a graph chosen uniformly at random from the class of vertex-labelled cubic multigraphs, cubic weighted multigraphs, or cubic simple graphs embeddable on $\mathbb{S}_g$ with n vertices, respectively. Then with probability $1 - O(n^{-2})$, G has one component that is embeddable on $\mathbb{S}_g$, but not on $\mathbb{S}_{g-1}$, while all other components of G are planar.*

1.2 Proof techniques

To derive our results we will use topological manipulations of surfaces called *surgeries*, constructive decomposition of graphs along connectivity, and singularity analysis of generating functions.

More precisely, in order to enumerate cubic multigraphs we apply constructive decompositions along connectivity. The basic building blocks in the decomposition are 3-connected cubic graphs, which we will then relate to their corresponding cubic *maps*. Note that, due to Whitney's Theorem [35], 3-connected *planar* graphs have a unique embedding on the sphere. Therefore, we can directly relate 3-connected planar graphs to the corresponding maps. For surfaces of positive genus, however, embeddings of 3-connected graphs are *not* unique. Following an idea from [10], we circumvent this problem by using the concept of the *facewidth* of a graph and by applying results of Robertson and Vitray [31] which relate 3-connected graphs and maps.

Counting 3-connected cubic maps on $\mathbb{S}_g$ is a challenging task. We shall use the dual of cubic maps, triangulations, in order to overcome this challenge. In fact, Gao [13, 15, 16] enumerated triangulations on $\mathbb{S}_g$ with various restrictions on the existence of loops and multi-edges. However, it turns out that the duals of 3-connected cubic maps on $\mathbb{S}_g$ have very specific constraints that have not been considered by Gao. In this paper we therefore investigate such triangulations by relating them to simple triangulations counted by Gao [15] (see Propositions 3.2 to 3.4). We strengthen Gao's result and derive very precise singular expansions of generating functions. These expansions are obtained from recursive formulas for the generating functions, which we derive by applying surgeries to the surfaces on which the respective triangulations are embedded. This enables us to apply singularity analysis to the generating functions of these triangulations, as well as to the generating functions of all other classes of maps and graphs considered in this paper.

This paper is organised as follows. In Section 2 we introduce some basic notions and notations. In Section 3 we enumerate the triangulations that are duals of 3-connected cubic maps and in Section 4 we prove the main results (Theorems 1.1 to 1.4) after giving a constructive decomposition along connectivity. Our strengthening of Gao's results and proofs, as well as other proofs for similar theorems from Section 3, are given in the appendix.

2 Preliminaries

A graph G is *simple* if it does not contain loops or multi-edges. If in a multigraph there are more than two edges connecting the same pair of vertices, we call each pair of those edges a *double edge*. Therefore, every multi-edge consisting of r edges between the same two vertices contains $\binom{r}{2}$ double edges. If e is a loop and incident to a vertex v, we say that v is the *base* of e. Similarly, we say that e is *based at* its base. An edge that is neither a loop nor part of a double edge is a *single edge*. An edge e of a connected multigraph G is called a *bridge* if deleting e disconnects G.

A multigraph is called *cubic* if each vertex has degree three. We adopt the convention that a loop counts as two in the degree of its base. By Φ we denote the cubic multigraph with two distinguished vertices u, v and three edges between u and v (i.e. a triple edge). Given a connected cubic multigraph G, let k and l denote the number of double edges and loops of G, respectively. We define the *weight* of G to be

$$W(G) = \begin{cases} \frac{1}{6} & \text{if } G = \Phi, \\ 2^{-(k+l)} & \text{otherwise.} \end{cases}$$

If G is not connected, we define $W(G)$ as the product of weights of its components. For cubic multigraphs, this weight coincides with the compensation factor introduced in [20]. Throughout this paper, when we refer to a *weighted cubic multigraph* G, the weight in consideration will always be $W(G)$.

Definition 2.1. An *embedding* of a multigraph G on $\mathbb{S}_g$ is a drawing of G on $\mathbb{S}_g$ without crossing edges. We consider G as a subset of $\mathbb{S}_g$, and therefore $\mathbb{S}_g \setminus G$ consists of connected components called *faces*. An embedding where additionally all faces are homeomorphic to open discs, or equivalently, where all faces are simply connected, is called a 2*-cell embedding*. Multigraphs that have an embedding are called *embeddable* on $\mathbb{S}_g$ and multigraphs that have a 2-cell embedding are called *strongly embeddable*.

A 2-cell embedding of a strongly embeddable multigraph is also called *map*. A *triangulation* is a map where each face is bounded by a triangle. These triangles might be degenerated, i.e., being three loops with the same base, or a double edge and a loop based at one of the end vertices of the double edge, or a loop and an edge from the base of the loop to a vertex of degree one.

If S is the disjoint union of $\mathbb{S}_{g_1}, \dots, \mathbb{S}_{g_r}$ for non-negative integers $g_1, \dots, g_r$ and M_i is a 2-cell embedding of a graph G_i on $\mathbb{S}_{g_i}$ for each $i = 1, \dots, r$, then the induced function $N : (G_1 \cup \cdots \cup G_n) \to S$ is called a *map* on S. Triangulations on S are defined analogously. We denote by $V(M)$, $E(M)$, and $F(M)$ the set of all vertices, edges, and faces of an embedding M, respectively.

We call a set $E' \subseteq E(M)$ *separating*, if the map $M' = (V(M), E')$ has at least two faces, i.e. if M' separates the surface.

From results of Mohar and Thomassen [27] we obtain some initial properties of embeddable graphs.

Proposition 2.2. *[27] Let G be a multigraph.*

(i) If G is connected and g is minimal such that G is embeddable on $\mathbb{S}_g$, then every embedding of G on $\mathbb{S}_g$ is a 2*-cell embedding. In particular, G is strongly embeddable on $\mathbb{S}_g$.*

(ii) G is embeddable on $\mathbb{S}_g$ if and only if each connected component C_i of G is strongly embeddable on a surface $\mathbb{S}_{g_i}$ such that $\sum_i g_i \leqslant g$.

Let M be a map on a surface $\mathbb{S}$. We construct the *dual map* of M by first putting a vertex in each face of M, then for each edge e in M, we draw an edge between the two (possibly coincident) vertices inside the faces on both side of e while crossing e exactly once. The newly drawn edges should only intersect at their end points. Note that the dual map has multi-edges if two faces of the original (*primal*) map have more than one edge in common. It is well known that the dual of a map is again a map, see e.g. [27].

For each vertex $v \in V(M)$ of a map M, the edges and faces incident to v have a canonical cyclic order $e_0, f_0, e_1, f_1, \dots, e_{d-1}, f_{d-1}$ by the way they are arranged around v (in counterclockwise direction). Note that faces can appear multiple times here and that a loop based at v will appear twice in this sequence. To avoid ambiguities, we distinguish the two ends of the loop in this sequence (e.g. by using half-edges or by orienting each loop). A triple $(v, e_i, e_{(i+1) \bmod d})$ of a vertex v and two consecutive edges $e_i, e_{i+1 \bmod d}$ in the cyclic sequence is called a *corner* (at v). We also say that $(v, e_i, e_{(i+1) \bmod d})$ is a corner *of the face* f_i. When we enumerate maps, we always work with maps with one distinguished corner, called the *root* of the map. If (v, e_i, e_{i+1}) is the root corner, we will call v the *root vertex*, e_i the *root edge*, and f_i the *root face*.

2.1 Generating functions and singularity analysis

We will use generating functions to enumerate the various classes of maps, graphs and multigraphs we consider. Unless stated otherwise, the formal variables x and y will always mark vertices and edges respectively. Generating functions for classes of *maps* will be *ordinary* unless stated otherwise. Generating functions for *multigraphs* will be *exponential* in x, because we always consider *vertex-labelled* multigraphs. If $\mathcal{A}$ is a class of maps, we write $\mathcal{A}(m)$ for the subclass of $\mathcal{A}$ containing all maps with exactly m edges. The generating function $\sum_m |\mathcal{A}(m)| y^m$ will be denoted by $A(y)$. If $\mathcal{B}$ is a class of multigraphs, we write $\mathcal{B}(n)$ for the subclass of $\mathcal{B}$ containing all multigraphs with exactly n vertices. The generating function $\sum_n \frac{|\mathcal{B}(n)|}{n!} x^n$ will be denoted by $B(x)$. For an ordinary generating function $F(z) = \sum_n f_n z^n$, we use the notation $[z^n]F(z) := f_n$. For an exponential generating function $G(z) = \sum \frac{g_n}{n!} z^n$, we write $[z^n]G(z) := \frac{g_n}{n!}$.

If two generating functions $F(z), G(z)$ satisfy $0 \leqslant [z^n]F(z) \leqslant [z^n]G(z)$ for all n, we say that F is *coefficient-wise smaller* than G, denoted by $F \preceq G$. The singularities of $F(z)$ with the smallest modulus are called *dominant singularities* of $F(z)$. Because every generating function we consider in this paper always has non-negative coefficients $[z^n]F(z)$, there is a dominant singularity located on the positive real axis by Pringsheim's Theorem [32, pp. 214 ff.]. We denote this dominant singularity by ρ_F. If an arbitrary function $F : \mathbb{C} \to \mathbb{C}$ has a unique singularity with smallest modulus and this singularity lies on the positive real axis, then we also denote it by ρ_F. The function F converges on the open disc of radius ρ_F and thus corresponds to a holomorphic function on this disc. In many cases, this function can be holomorphically extended to a larger domain. Given $\rho, R \in \mathbb{R}$ with $0 < \rho < R$ and $\theta \in (0, \pi/2)$,

$$\Delta(\rho, R, \theta) := \{z \in \mathbb{C} \mid |z| < R \wedge |\arg(z - \rho)| > \theta\}$$

is called a Δ-*domain*. Here, $\arg(z)$ denotes the *argument* of a complex number, that is $\arg(0) := 0$ and $\arg(re^{it}) := t$ for $r > 0$ and $t \in (-\pi, \pi]$. We say that F is Δ-*analytic* if it is holomorphically extendable to some Δ-domain $\Delta(\rho_F, R, \theta)$.

A function F is *subdominant* to a function G if either $\rho_F > \rho_G$ or $\rho_F = \rho_G$ and $\lim_{z\to\rho_G} \frac{F(z)}{G(z)} = 0$. In the latter case, if both F and G are Δ-analytic, then in the above limit, z is taken from some fixed Δ-domain to which both F and G are holomorphically extendable. If F is subdominant to G, we also write $F(z) = o(G(z))$. Analogously we write $F(z) = O(G(z))$ if either $\rho_F > \rho_G$ or $\rho_F = \rho_G$ and $\limsup_{z\to\rho_G} \frac{|F(z)|}{|G(z)|} < \infty$.

Given a function $F(z)$ with a dominant singularity ρ_F, we say that a function $G(z) = c\left(1-\rho_F^{-1}z\right)^{-\alpha}$ with $\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\leqslant 0}, c \in \mathbb{R}\setminus\{0\}$ or $G(z) = c\log\left(1-\rho_F^{-1}z\right)$ is the *dominant term* of F if there is a decomposition

$$F(z) = P(z) + G(z) + o(G(z)),$$

where $P(z)$ is a polynomial. The dominant term, if it exists, is uniquely defined and Δ-analytic. If $G(z) = c(1-\rho_F^{-1}z)^{-\alpha}$, the exponent $-\alpha$ is called the *dominant exponent* of F. If $G(z) = c\log\left(1-\rho_F^{-1}z\right)$, then we say that F has the dominant exponent 0.

The number of edges in cubic multigraphs and triangulations is always a multiple of three. In terms of generating functions, this is reflected by the existence of three different dominant singularities, all of which differ only by a third root of unity. The corresponding dominant terms will also be the same up to a third root of unity. Analogously, the number of vertices in cubic multigraphs is always even, resulting in two dominant singularities ρ_F and $-\rho_F$. Again, the dominant terms differ only by a factor of -1. In either case, the terms for the coefficients coming from the different dominant singularities will also differ only by the corresponding root of unity. Therefore, we will state our results only for the singularity ρ_F. With a slight abuse of notation, we will also refer to ρ_F as *the* dominant singularity.

Singularity analysis allows us to derive an asymptotic expression for the coefficients of a generating function $F(z)$ with help of the dominant singularity and the dominant term of $F(z)$. We state the well-known 'transfer theorem' for the specific cases we will need.

Theorem 2.3 ([11])**.** *Let $A(z)$ be a Δ-analytic generating function.*

(i) If

$$A(z) = P(z) + c\left(1-\rho_A^{-1}z\right)^{-\alpha} + O\left(\left(1-\rho_A^{-1}z\right)^{1/4-\alpha}\right)$$

with a polynomial $P(z)$ and constants $c \neq 0$, $\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\leqslant 0}$, then

$$[z^n]A(z) = \left(1+O\left(n^{-1/4}\right)\right)\frac{c}{\Gamma(\alpha)}n^{\alpha-1}\rho_A^{-n}.$$

Here, $\Gamma(\alpha) := \int_0^\infty z^{\alpha-1}e^{-z}\,\mathrm{d}z$ is the gamma function.

(ii) If

$$A(z) = P(z) + c\cdot\log\left(1-\rho_A^{-1}z\right) + O\left(\left(1-\rho_A^{-1}z\right)^{1/4}\right),$$

then

$$[z^n]A(z) = \left(1+O\left(n^{-1/4}\right)\right)(-c)n^{-1}\rho_A^{-n}.$$

We use the standard notation $\gamma_A = \rho_A^{-1}$ for the exponential growth constant of $[z^n]A(z)$. If we are counting rooted maps or multigraphs, the roots will be counted in the generating function unless stated otherwise. We will often mark vertices or edges of multigraphs or maps, which corresponds to applying the differential operator $z\frac{d}{dz}$ to the generating functions (with $z = x$ if vertices are marked and $z = y$ if edges are marked). To simplify notation we write δ_z for $z\frac{d}{dz}$ and δ_z^n for repeatedly applying n times the operator $z\frac{d}{dz}$, which corresponds to marking n vertices or edges, while *allowing multiple marks*. We use the notation $F'(z) = \frac{dF}{dz}$ for the standard differential operator.Vice versa, we say that F is a *primitive* of F'.

The dominant terms of derivatives and primitives of Δ-analytic functions can be determined using Theorems VI.8 and VI.9 from [12]. Again, we state these results in a slightly different way tailored for our specific needs.

Lemma 2.4 ([12])**.** *Let $A(z)$ be a Δ-analytic generating function with the dominant term $A_d(z)$. Suppose that there exists $\beta \in \mathbb{R}$ with*

$$A(z) = P(z) + A_d(z) + O\left(\left(1 - \rho_A^{-1} z\right)^{-\beta}\right),$$

where $P(z)$ is a polynomial and $\left(1 - \rho_A^{-1} z\right)^{-\beta} = o(A_d(z))$.

(i) We have

$$A'(z) = P'(z) + A_d'(z) + O\left(\left(1 - \rho_A^{-1} z\right)^{-\beta-1}\right).$$

(ii) If in addition $A_d(z) = c\left(1 - \rho_A^{-1} z\right)^{-\alpha}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant 0}$, then for any primitive $\mathbf{A}(z)$ of $A(z)$ there exists a primitive $\mathbf{P}(z)$ of $P(z)$ such that

$$\mathbf{A}(z) = \mathbf{P}(z) + \mathbf{A}_d(z) + O(R(z))$$

with

$$\mathbf{A}_d(z) = \begin{cases} \frac{c\rho_A}{\alpha-1}\left(1 - \rho_A^{-1} z\right)^{-\alpha+1} & \text{if } \alpha \neq 1, \\ -c\rho_A \log\left(1 - \rho_A^{-1} z\right) & \text{if } \alpha = 1, \end{cases}$$

and

$$R(z) = \begin{cases} \left(1 - \rho_A^{-1} z\right)^{-\beta+1} & \text{if } \beta \neq 1, \\ \log\left(1 - \rho_A^{-1} z\right) & \text{if } \beta = 1. \end{cases}$$

Theorem 2.3 and Lemma 2.4 are very helpful when the generating functions in question are Δ-analytic. However, for many of our generating functions we will not be able to guarantee Δ-analyticity. In order to utilise the results of this section also for generating functions that are *not necessarily* Δ-analytic, we introduce the following concept and notation.

Definition 2.5. Given a generating function $F(z)$ and Δ-analytic functions $A(z)$ and $B(z)$, we say that $F(z)$ is *congruent* to $A(z) + O(B(z))$ and write

$$F(z) \cong A(z) + O(B(z))$$

if there exist Δ-analytic functions $F^+(z), F^-(z)$ and polynomials $P^+(z), P^-(z)$ such that

- $F^- \preceq F \preceq F^+$;
- $F^+(z) = P^+(z) + A(z) + O(B(z))$;
- $F^-(z) = P^-(z) + A(z) + O(B(z))$.

Here we allow $A(z) \equiv 0$.

With Definition 2.5, we are able to apply the transfer theorem even if F itself is not Δ-analytic. The following lemma is an immediate consequence of Theorem 2.3 and the fact that $F^- \preceq F \preceq F^+$.

Lemma 2.6. *If $F(z) \cong A(z)$, where $A(z)$ is as in Theorem 2.3, then*

$$[x^n]F(z) = \left(1 + O\left(n^{-1/4}\right)\right)[z^n]A(z).$$

In this paper we will often encounter sums, products, differentials and integrals of generating functions. The following lemma states that these operations are compatible with the notion of congruence. We will frequently use this lemma without explicitly mentioning it.

Lemma 2.7. *Let A, A_1, A_2, B_1, B_2 be Δ-analytic functions with only finitely many negative coefficients. Let F_1, F_2 be generating functions such that*

$$F_1(z) \cong A_1(z) + O(B_1(z)) \qquad \text{and} \qquad F_2(z) \cong A_2(z) + O(B_2(z))$$

and let $F(z)$ be a generating function with

$$F(z) \cong A(z) + O\left(\left(1 - \rho_F^{-1}z\right)^{-\beta}\right),$$

where $\beta \in \mathbb{R}$ and the dominant term $A_d(z)$ of $A(z)$ satisfies $\left(1 - \rho_F^{-1}z\right)^{-\beta} = o(A_d(z))$. Then the following holds.

$$\begin{aligned}
F_1(z) \pm F_2(z) &\cong A_1(z) \pm A_2(z) + O\Big(B_1(z)) + O(B_2(z)\Big),\\
F_1(z)F_2(z) &\cong A_1(z)A_2(z) + O\Big(A_1(z)B_2(z) + B_1(z)A_2(z) + B_1(z)B_2(z)\Big),\\
F'(z) &\cong A'(z) + O\left(\left(1 - \rho_F^{-1}z\right)^{-\beta-1}\right).
\end{aligned}$$

Furthermore, if $A_d(z) = c\left(1 - \rho_F^{-1}z\right)^{-\alpha}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant 0}$, then for any primitive $\mathbf{F}(z)$ of $F(z)$ we have

$$\mathbf{F}(z) \cong \mathbf{A}_d(z) + O\left(R(z)\right),$$

where $\mathbf{A}_d$ and R are as in Lemma 2.4(ii).

Proof. The first congruence follows immediately from

$$F_1^- + F_2^- \preceq F_1 + F_2 \preceq F_1^+ + F_2^+ \qquad \text{and} \qquad F_1^- - F_2^+ \preceq F_1 - F_2 \preceq F_1^+ - F_2^-.$$

For the product $F_1(z) \cdot F_2(z)$, we may assume that F_1^- and F_2^- have nonnegative coefficients, since A_1 and A_2 have only finitely many negative coefficients. Hence

$$F_1^- \cdot F_2^- \preceq F_1 \cdot F_2 \preceq F_1^+ \cdot F_2^+$$

and the second congruence follows.

The last two congruences follow from Lemma 2.4 and the fact that

$$(F^-)' \preceq F' \preceq (F^+)' \qquad \text{and} \qquad \mathbf{F}^- \preceq \mathbf{F} \preceq \mathbf{F}^+,$$

where $\mathbf{F}^-, \mathbf{F}^+$ are primitives of F^-, F^+, respectively, with $\mathbf{F}^-(0) \leqslant \mathbf{F}(0) \leqslant \mathbf{F}^+(0)$. □

2.2 Maps with large facewidth

An *essential circle* on $\mathbb{S}_g$ is a circle that is not contractible to a point on $\mathbb{S}_g$. Let M be an embedding of a multigraph on $\mathbb{S}_g$. An *essential cycle* of M is a cycle of M that is an essential circle on the surface. The *facewidth* $\mathrm{fw}(M)$ of M is the minimal number of intersections of M with an essential circle on $\mathbb{S}_g$. The *edgewidth* $\mathrm{ew}(M)$ of M is defined as the minimal number of edges of an essential cycle of M. If $g = 0$, there are neither essential circles nor essential cycles and we use the convention $\mathrm{fw}(M) = \mathrm{ew}(M) = \infty$. Observe that if M is connected and *not* a 2-cell embedding, then $\mathrm{fw}(M) = 0$, as an essential circle can be found in any face that is not simply connected. The facewidth $\mathrm{fw}_g(G)$ of a multigraph G that is embeddable on $\mathbb{S}_g$ is defined as the *maximal* facewidth of all its embeddings on $\mathbb{S}_g$. If the genus is clear from the context, we omit it and write $\mathrm{fw}(G)$. When we count multigraphs with restrictions to their facewidth, we indicate the restriction by a superscript to the corresponding generating function, e.g. $G^{\mathrm{fw} \geqslant 2}(x)$ for the generating function of all multigraphs with facewidth at least two.

Having large facewidth proves to be a very helpful property, because it allows us to derive a constructive decomposition along connectivity as well as the existence of a unique embedding for 3-connected multigraphs. The following lemma was applied in a similar way in [10] as later in this paper.

Lemma 2.8. *[31] Let $g > 0$ and let M be an embedding of a connected multigraph G on $\mathbb{S}_g$.*

(i) M has facewidth $\mathrm{fw}(M) = k \geqslant 2$ if and only if M has a unique 2-connected component embedded on $\mathbb{S}_g$ with facewidth k and all other 2-connected components of M are planar.

(ii) If G is 2-connected, M has facewidth $\mathrm{fw}(M) = k \geqslant 3$ if and only if M has a unique 3-connected component embedded on $\mathbb{S}_g$ with facewidth k and all other 3-connected components of M are planar.

(iii) Let M_1, M_2 be embeddings of a 3-connected multigraph on $\mathbb{S}_g$ and suppose that $\mathrm{fw}(M_1) \geqslant 2g+3$. *Then there is a homeomorphism of $\mathbb{S}_g$ that maps M_1 to M_2.*

Lemma 2.8(iii) is a generalisation of Whitney's theorem [35] that all 3-connected planar multigraphs have a unique embedding up to orientation on the sphere. Because we will need Lemma 2.8 for multigraphs rather than for embeddings, we shall use the following easy corollary.

Corollary 2.9. *Let $g > 0$ and let G be a non-planar connected multigraph strongly embeddable on $\mathbb{S}_g$.*

(i) If $\mathrm{fw}_g(G) \geqslant 2$, then G has a unique 2-connected non-planar component strongly embeddable on $\mathbb{S}_g$ with facewidth $\mathrm{fw}_g(G)$ and all other 2-connected components are planar.

(ii) If G is 2-connected and has facewidth $\mathrm{fw}_g(G) \geqslant 3$, then G has a unique 3-connected non-planar component strongly embeddable on $\mathbb{S}_g$ with facewidth $\mathrm{fw}_g(G)$ and all other 3-connected components are planar.

(iii) If G is 3-connected and $\mathrm{fw}_g(G) \geqslant 2g+3$, then G has a unique 2-cell embedding on $\mathbb{S}_g$ up to orientation.

Proof. By Lemma 2.8(i), for any fixed embedding of G all but one components are planar. As G itself is not planar, that exceptional component has to be non-planar. As the component structure is the same for *all* embeddings, the non-planar component I is independent of the embedding. Therefore, I is the unique component described in (i). Part (ii) is proved analogously and (iii) is a direct consequence of Lemma 2.8(iii). □

3 Maps and triangulations

The goal of this section is to enumerate cubic 3-connected *maps* on $\mathbb{S}_g$. The duals of these maps are triangulations, which are characterised in the following proposition.

Proposition 3.1. *Let M be a 2-cell embedding of a cubic multigraph on $\mathbb{S}_g$ and let M^* be its dual map. Then M is 3-connected if and only if M^* is a triangulation with at least six edges and without separating loops, separating double edges, or separating pairs of loops.*

Proof. For cubic graphs with at least four vertices (and thus at least six edges), 3-connectivity and 3-edge-connectivity coincide. This can be seen by a simple case analysis. We thus use 3-edge-connectivity hereafter. Since a vertex in M corresponds to a face in M^*, deleting edges in the primal M has the same effect as cutting the surface along the dual edges of M^* (for a formal definition of “cutting” see Section A.1), and a set of edges is a separator in M if and only if cutting along the dual edges in M^* separates the surface. Thus, a bridge in M corresponds to a separating loop in M^*. A 2-edge-separator in M corresponds either to a separating double edge or a pair of loops in M^* which together separate the surface. □

In order to enumerate the triangulations described in Proposition 3.1, we will relate them to simple triangulations that have been studied by Bender and Canfield [1]. To this end we will use the following classes of triangulations.

Let $\mathcal{M}_g$ be the class of triangulations on $\mathbb{S}_g$ without separating loops, separating double edges, and separating pairs of loops and let $M_g(y)$ be its ordinary generating function. Note that these triangulations are either the duals of 3-connected cubic maps on $\mathbb{S}_g$ by Proposition 3.1 or a triangulation with exactly three edges. Furthermore, let $\mathcal{S}_g$ be the class of simple triangulations on $\mathbb{S}_g$ (i.e. without loops or double edges) and let $\hat{\mathcal{S}}_g$ be the class of triangulations on $\mathbb{S}_g$ without separating loops or separating double edges. Let $S_g(y)$ and $\hat{S}_g(y)$ be their generating functions, respectively.

The starting point in obtaining an asymptotic expansion for $M_g(y)$ will be results on simple triangulations which were obtained by Gao [14] and (in the planar case) by Tutte [33]. However, the results obtained by Gao are not strong enough in order to apply the theory of singularity analysis (developed in Section 2.1). We obtain more refined versions of their results by following the ideas of Bender and Canfield [1].

Proposition 3.2. *The dominant singularity of $S_g(y)$ is given by $\rho_S = \frac{3}{2^{8/3}}$. The generating function $S_0(y)$ is Δ-analytic and satisfies*

$$S_0(y) = \frac{1}{8} - \frac{9}{16}\left(1-\rho_S^{-1}y\right) + \frac{3}{2^{5/2}}\left(1-\rho_S^{-1}y\right)^{3/2} + O\left(\left(1-\rho_S^{-1}y\right)^2\right). \tag{1}$$

For $g \geqslant 1$ we have

$$S_g(y) \cong c_g\left(1-\rho_S^{-1}y\right)^{-5g/2+3/2}\left(1+O\left(\left(1-\rho_S^{-1}y\right)^{1/4}\right)\right), \tag{2}$$

where c_g is a constant depending only on g.

Furthermore, for $g \geqslant 0$, the asymptotic number of simple triangulations on $\mathbb{S}_g$ with m edges is given by

$$|\mathcal{S}_g(m)| = \left(1+O\left(m^{-1/4}\right)\right)\frac{c_g}{\Gamma(5(g-1)/2)}m^{5g/2-5/2}\rho_S^{-m},$$

where $c_0 = \frac{3}{2^{5/2}}$.

The exact values of c_g can be found in [16].

Along the same lines we obtain similar results for $\hat{S}_g(y)$.

Proposition 3.3. *We have $\hat{S}_0(y) = S_0(y)$ and for $g \geqslant 1$,*

$$\hat{S}_g(y) \cong c_g\left(1-\rho_S^{-1}y\right)^{-5g/2+3/2}\left(1+O\left(\left(1-\rho_S^{-1}y\right)^{1/4}\right)\right), \tag{3}$$

where c_g is the same constant depending only on g as in Proposition 3.2.

Furthermore, for $g \geqslant 0$, the asymptotic number of triangulations without separating loops or separating double edges on $\mathbb{S}_g$ with m edges is given by

$$\left|\hat{\mathcal{S}}_g(m)\right| = \left(1+O\left(m^{-1/4}\right)\right)\frac{c_g}{\Gamma(5g/2-5/2))}m^{5g/2-5/2}\rho_S^{-m}.$$

The proofs of Propositions 3.2 and 3.3 can be found in Appendix A.

From these two results and the fact that $\mathcal{S}_g \subseteq \mathcal{M}_g \subseteq \hat{\mathcal{S}}_g$ we obtain immediately our results for the number of triangulations in $\mathcal{M}_g$, i.e. triangulations on $\mathbb{S}_g$ without separating loops, separating double edges, and separating pairs of loops.

Proposition 3.4. *The dominant singularity of $M_g(y)$ is given by $\rho_M = \rho_S = \frac{3}{2^{8/3}}$. The generating function $M_0(y)$ is Δ-analytic and satisfies*

$$M_0(y) = \frac{1}{8} - \frac{9}{16}\left(1-\rho_M^{-1}y\right) + \frac{3}{2^{5/2}}\left(1-\rho_M^{-1}y\right)^{3/2} + O\left(\left(1-\rho_M^{-1}y\right)^2\right). \tag{4}$$

For $g \geqslant 1$ we have

$$M_g(y) \cong c_g\left(1-\rho_M^{-1}y\right)^{-5g/2+3/2}\left(1+O\left(\left(1-\rho_M^{-1}y\right)^{1/4}\right)\right), \tag{5}$$

where c_g is the same constant depending only on g as in Proposition 3.2.

Furthermore, for $g \geqslant 0$, the asymptotic number of triangulations in $\mathcal{M}_g(m)$ is given by

$$|\mathcal{M}_g(m)| = \left(1+O\left(m^{-1/4}\right)\right)\frac{c_g}{\Gamma(5g/2-5/2))}m^{5g/2-5/2}\rho_M^{-m}.$$

Observe that from Propositions 3.2 and 3.4 it follows immediately that the dual of a cubic map on $\mathbb{S}_g$ is simple with high probability, i.e. with probability tending to one as m tends to infinity.

4 Cubic graphs

Unless stated otherwise, graphs are unrooted. Recall that, in our generating functions, x marks vertices and y marks edges. Additionally, we will distinguish whether edges are single edges, double edges or loops because they will be treated differently when obtaining relations between graph classes. We will use the variable z to mark double edges and the variable w to mark loops. It is easy to see that 3-connected cubic graphs are simple and that 2-connected cubic multigraphs do not contain loops. The generating functions for these classes will only feature the variables of edges that can occur.

In order to derive asymptotic results we shall deal with univariate generating functions $F(v)$. As cubic (multi)graphs always have $2n$ vertices and $3n$ edges, where $n \in \mathbb{N}$, the coefficient $(2n)![v^n]F(v)$ will denote the number of graphs (or multigraphs or weighted multigraphs) in the corresponding class with $2n$ vertices and $3n$ edges. Such a univariate generating function can be obtained by the following substitution.

Definition 4.1. Let $\mathcal{F}$ be a class of connected cubic vertex-labelled multigraphs without triple edges and let

$$F(x,y,z,w) = \sum_{n,m,k,l \geqslant 0} \frac{f_{n,m,k,l}}{n!} x^n y^m z^k w^l$$

be its exponential generating function. We define functions $F(v)$, $F^u(v)$, and $F^s(v)$ as follows.

$$\begin{aligned} F(v) &:= F\left(v^{1/4}, v^{1/6}, \frac{v^{1/3}}{2}, \frac{v^{1/6}}{2}\right), \\ F^u(v) &:= F(v^{1/4}, v^{1/6}, v^{1/3}, v^{1/6}), \\ F^s(v) &:= F(v^{1/4}, v^{1/6}, 0, 0). \end{aligned}$$

If the generating function of $\mathcal{F}$ involves only two or three variables, we define $F(v)$, $F^u(v)$, and $F^s(v)$ analogously, only using the substitutions of those variables that occur.

We claim that $(2n)![v^n]F(v)$ is the number of weighted multigraphs in $\mathcal{F}(2n)$, i.e. the sum of $W(G)$ for all $G \in \mathcal{F}$ with $2n$ vertices (and thus with $3n$ edges). For, if $G \in \mathcal{F}(2n)$ has k double edges, l loops, and m single edges, then there are $2k+l+m = 3n$ edges in total and the substitution transforms the monomial $x^{2n}y^m z^k w^l$ into $2^{-(k+l)}v^{n/2+m/6+k/3+l/6} = W(G)v^n$. Similarly, $(2n)![v^n]F^u(v)$ is the number of (unweighted) multigraphs in $\mathcal{F}(2n)$. Finally, $(2n)![v^n]F^s(v)$ is the number of simple graphs in $\mathcal{F}(2n)$, since replacing z and w by 0 ensures that no graphs with double edges or loops are counted in $F^s(v)$.

4.1 From maps to graphs

Let $\mathcal{D}_g$ be the class of 3-connected cubic vertex-labelled graphs *strongly embeddable* on $\mathbb{S}_g$ and let $D_g(x,y)$ be its generating function. In this section we provide some necessary properties of $D_g(v)$. We will use the auxiliary classes $\overline{\mathcal{D}}_g$ of 3-connected cubic *edge-labelled* graphs strongly embeddable on $\mathbb{S}_g$, and $\overline{\mathcal{M}}_g$ of *edge-labelled, unrooted* triangulations where the triangulations are in $\mathcal{M}_g$.

Proposition 4.2. *The dominant singularity of $D_g(v)$ is $\rho_D = \rho_S^3 = \frac{27}{256}$ and we have the following congruences.*

$$\begin{aligned} D_0(v) &\cong c_0\left(1-\rho_D^{-1}v\right)^{5/2} + O\left(\left(1-\rho_D^{-1}v\right)^3\right), \\ D_1(v) &\cong c_1 \log\left(1-\rho_D^{-1}v\right) + O\left(\left(1-\rho_D^{-1}v\right)^{1/4}\right), \\ D_g(v) &\cong c_g\left(1-\rho_D^{-1}v\right)^{-5g/2+5/2} + O\left(\left(1-\rho_D^{-1}v\right)^{-5g/2+11/4}\right) \qquad \text{for } g \geqslant 2, \end{aligned}$$

where c_g is the same constant depending only on g as in Proposition 3.2.

Applying Theorem 2.3, we immediately obtain the coefficients of $D_g(v)$.

Corollary 4.3. *The coefficients of $D_g(v)$ satisfy*

$$[v^n]D_g(v) = \left(1+O\left(n^{-1/4}\right)\right)\overline{c}_g n^{5(g-1)/2-1}\rho_D^{-n},$$

where $\overline{c}_g$ is a constant depending only on g.

Proof of Proposition 4.2. First we compare $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$. For each triangulation $M \in \mathcal{M}_g$ with m edges, there are $m!$ possibilities of labelling its edges. Conversely, for a triangulation $\overline{M} \in \overline{\mathcal{M}}_g$, there are $2m$ possibilities of rooting. Therefore, the exponential generating function $\overline{M}_g(y)$ of $\overline{\mathcal{M}}_g$ satisfies

$$[y^m]M_g(y) = 2m[y^m]\overline{M}_g(y)$$

and thus

$$M_g(y) = 2\delta_y \overline{M}_g(y).$$

Every graph $G \in \overline{\mathcal{D}}_g$ has at least two (edge-labelled) 2-cell embeddings. By Proposition 3.1, the maps obtained in this way are precisely the duals of triangulations in $\overline{\mathcal{M}}_g$. As y marks the number of edges in $\overline{M}_g(v^{1/3})$ and v marks a third of the number of edges in $\overline{D}_g(v)$, we obtain

$$2\overline{D}_g(v) \preceq \overline{M}_g(v^{1/3}).$$

We claim that, for a cubic map M on $\mathbb{S}_g$, its facewidth $\mathrm{fw}(M)$ is exactly the edgewidth $\mathrm{ew}(M^*)$ of the triangulation M^* that is the dual of M. To see this, we observe that an essential cycle of M^* witnessing the edgewidth of M^* corresponds to an essential circle on $\mathbb{S}_g$ that meets M in $\mathrm{ew}(M^*)$ edges and no vertices, resulting in $\mathrm{fw}(M) \leqslant \mathrm{ew}(M^*)$. On the other hand, any two faces of M that share a vertex also share an edge, as M is cubic. Thus, there is an essential circle witnessing the facewidth of M that meets M only at edges. As this circle corresponds to an essential cycle of M^*, we have $\mathrm{fw}(M) \geqslant \mathrm{ew}(M^*)$.

Since by Lemma 2.8(iii) a 3-connected graph embeddable on $\mathbb{S}_g$ with facewidth at least $2g+3$ has exactly two embeddings, we have

$$2\overline{D}_g^{\mathrm{fw}\geqslant 2g+3}(v) = \overline{M}_g^{\mathrm{ew}\geqslant 2g+3}(v^{1/3}).$$

As obviously $\overline{D}_g^{\mathrm{fw}\geqslant 2g+3}(v) \preceq \overline{D}_g(v)$, we obtain the following relations:

$$\overline{M}_g^{\mathrm{ew}\geqslant 2g+3}(v^{1/3}) = 2\overline{D}_g^{\mathrm{fw}\geqslant 2g+3}(v) \preceq 2\overline{D}_g(v) \preceq \overline{M}_g(v^{1/3}). \tag{6}$$

Since there are no double edges in a 3-connected cubic graph, we know that the two generating functions $\overline{D}_g(v)$ and $D_g(v)$ are closely related. To be precise, $(2n)![v^n]D_g(v)$ is the number of vertex-labelled graphs in $\mathcal{D}_g(2n)$. Since every such graph has $3n$ edges, $(3n)!(2n)![v^n]D_g(v)$ is the number of 3-connected cubic graphs with $2n$ vertices embeddable on $\mathbb{S}_g$ with both vertices and edges labelled. As this number is equal to $(2n)!(3n)![v^n]\overline{D}_g(v)$ by an analogous argument, we have

$$[v^n]D_g(v) = [v^n]\overline{D}_g(v).$$

Therefore, we can replace $\overline{D}_g(v)$ by $D_g(v)$ in (6) to obtain

$$D_g(v) \preceq \frac{1}{2}\overline{M}_g(v^{1/3}) = \frac{1}{4}\int t^{-1}M_g(t)\,\mathrm{d}t\Big|_{t=v^{1/3}}.$$

By Lemma 2.4 we obtain an upper bound for $D_g(v)$ as claimed. To finish the proof we will show the following claim.

Claim 1. *The generating functions $M_g(y)^{\mathrm{ew}\geqslant 2g+3}$ and $M_g(y)$ have the same dominant singularity and*

$$M_g(y) - M_g^{\mathrm{ew}\geqslant 2g+3}(y) \cong O\left(\left(1-\rho_S^{-1}y\right)^{-5g/2+7/4}\right).$$

Before we prove the claim, let us note that Proposition 4.2 follows immediately from Claim 1, Lemma 2.7, and Proposition 3.4.

A statement more general than Claim 1 was proven in [4] for a variety of map classes. Although we believe that the proof in [4] generalises to $\mathcal{M}_g$, which was not considered in [4], we give a slightly different proof here for completeness.

The generating function of $\mathcal{M}_g \setminus \mathcal{S}_g$ is congruent to $O\left(\left(1-\rho_S^{-1}y\right)^{-5g/2+7/4}\right)$ by Propositions 3.2 to 3.4. It thus suffices to show that

$$S_g^{\mathrm{ew}\leqslant 2g+2} \cong O\left(\left(1-\rho_S^{-1}y\right)^{-5g/2+7/4}\right).$$

For $i \geqslant 3$, let $\mathcal{S}_g^{C=i}$ be the class of triangulations in $\mathcal{S}_g$ where one non-contractible cycle C of length i is marked, and we denote its generating function by $S_g^{C=i}(y)$. Clearly $S_g^{\mathrm{ew}=i}(y) \preceq S_g^{C=i}(y)$. Let $M \in \mathcal{S}_g^{C=i}$ and let C be the marked cycle of M. Consider the surface $\mathbb{S}_g$ on which M embeds. We cut $\mathbb{S}_g$ along the cycle C, and duplicate the vertices and edges of C so that the map structure in the neighbourhood on the two sides of C is preserved (for a precise definition of "cut", see Section A.1 in the appendix). We then close the two resulting holes by inserting disks to them in $\mathbb{S}_g$. This operation is also called "cutting along C on $\mathbb{S}_g$", and a more general and rigorous definition of such topological surgeries can be found in Appendix A. For each of the two disks, we then add a new vertex inside the disk and use it to triangulate the disk (see Figure 1). If C was separating, we mark one of the corners at the inserted vertex in the component that contains the original root face of M. For the other component, we choose one of the corners at the inserted vertex to be its root. If C was not separating, then we mark one corner at each of the two inserted vertices. In total we add $3i$ edges to the map. These operations result in

- two triangulations $M^{(1)}, M^{(2)}$, where $M^{(1)}$ contains the original root face of M and a marked corner;
- or one triangulation M^* with two marked corners.

All resulting triangulations are in $\mathcal{S}_{g'}$ for some $g' < g$, because the surgery does not create loops or double edges. Thus, disregarding markings, in the first case $M^{(1)} \in \mathcal{S}_{g_1}$ and $M^{(2)} \in \mathcal{S}_{g_2}$ with $g_1 + g_2 = g$ and $g_1, g_2 \geqslant 1$, and then in the second case $M^* \in \mathcal{S}_{g-1}$.

Since a corner (v_0, e, e') is uniquely defined once v_0 and e are given, marking a corner is equivalent to marking an edge and choosing one of its end vertices. In terms of generating functions, this corresponds to applying the operator $2\delta_y = 2y\frac{\partial}{\partial y}$. As in previous proofs, we will mark repeatedly, which will result in overcounting. Since we added $3i$ edges to M by our construction, we have to compensate by a factor of y^{3i}. Therefore, we obtain the relation

$$y^{3i} S_g^{C=i}(y) \preceq 4\delta_y^2\left(S_{g-1}(y)\right) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geqslant 1}} 2\delta_y\left(S_{g_1}(y)\right) S_{g_2}(y).$$

Figure 1: Cutting along C and triangulating the inserted disc.

By Proposition 3.2, we know that

$$4\delta_y^2\left(S_{g-1}(y)\right)+\sum_{\substack{g_1+g_2=g\\ g_j\geqslant 1}}2\delta_y\left(S_{g_1}(y)\right)S_{g_2}(y)\cong O\left(\left(1-\rho_S^{-1}y\right)^{-5g/2+7/4}\right).$$

Because

$$S_g^{\text{ew}\leqslant 2g+2}(y)=\sum_{i=3}^{2g+2}S_g^{\text{ew}=i}(y)\preceq\sum_{i=3}^{2g+2}S_g^{C=i}(y),$$

we complete the proof of the claim. □

4.2 From 3-connected graphs to connected multigraphs

In this section we derive dominance relations between different classes of cubic multigraphs. In the end we will relate connected cubic multigraphs via 2-connected cubic multigraphs to 3-connected cubic graphs enumerated in the previous section.

Denote by $\mathcal{D}_g$, $\mathcal{B}_g$, and $\mathcal{C}_g$ the classes of 3-connected, 2-connected, and connected vertex-labelled cubic multigraphs strongly embeddable on $\mathbb{S}_g$, respectively. Additionally, let $D_g(x,y)=\sum\frac{d_{n,m}}{n!}x^ny^m$, $B_g(x,y,z)=\sum\frac{b_{n,m,k}}{n!}x^ny^mz^k$, and $C_g(x,y,z,w)=\sum\frac{c_{n,m,k,l}}{n!}x^ny^mz^kw^l$, be the corresponding generating functions. In the generating function $C_g(x,y,z,w)$, the graph Φ consisting of two vertices connected by three edges will not be taken into account. This graph will be treated separately at the end.

First we give a relation between a subclass of $\mathcal{D}_g$ and a subclass of $\mathcal{B}_g$. To do this we need the class $\mathcal{N}$ of edge-rooted 2-connected vertex-labelled cubic planar multigraphs, called *networks*. In the exponential generating function $N(x,y,z)$ of $\mathcal{N}$ we mark the root always with y as a single edge, and with z marking double edges not including the root edge.

Lemma 4.4. *For $g\geqslant 1$, the generating functions of $\mathcal{D}_g$ and $\mathcal{B}_g$ satisfy*

$$D_g^{\text{fw}\geqslant 3}(x,\overline{y})-D_0(x,\overline{y})\preceq B_g^{\text{fw}\geqslant 3}(x,y,z)\preceq D_g^{\text{fw}\geqslant 3}(x,\overline{y}),\tag{7}$$

where $\overline{y}=y(1+N(x,y,z))$.

Proof. Let B be a multigraph in $\mathcal{B}_g^{\text{fw}\geqslant 3}$. We show that it is counted at least once on the right-hand side and at most once on the left-hand side of (7).

First, suppose that B is not planar. Then Corollary 2.9(ii) states that B has a unique 3-connected component T strongly embeddable on $\mathbb{S}_g$ with the same facewidth. T is in $\mathcal{D}_g^{\mathrm{fw}\geqslant 3}$ and therefore counted once in $D_g^{\mathrm{fw}\geqslant 3}(x,y)$. To get B from T, we have to attach 2-connected components along the edges (see Figure 2). That means, either we leave an edge as it is (obtaining a summand of y) or we replace it by two edges (obtaining a factor of y^2) and one multigraph in $\mathcal{N}$ without its root edge (obtaining a factor of $\frac{1}{y}N$). Thus, B is counted exactly once on the right-hand side of (7).

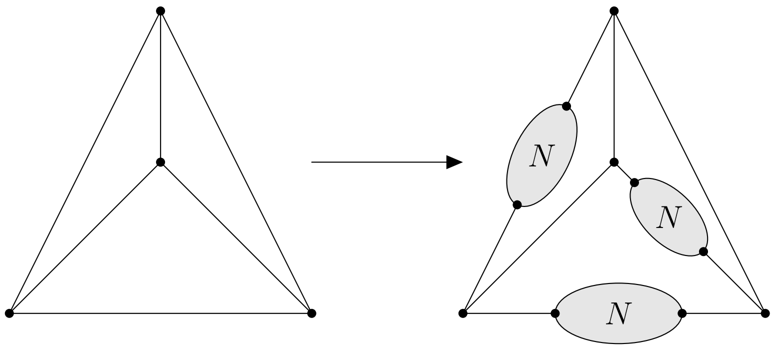

Figure 2: Obtaining 2-connected graphs from 3-connected graphs by substituting edges with networks.

If B is planar, then it might be counted more than once on the right-hand side. Indeed, in this case the 2-connected components carrying the facewidth might be different for different embeddings. Therefore, $D_g^{\mathrm{fw}\geqslant 3}(x, y + yN(x,y,z))$ is an upper bound. To get a lower bound we have to subtract all multigraphs we overcounted. This is achieved by subtracting $D_0(x, y + yN(x,y,z))$, as only planar multigraphs are overcounted and each such multigraph is subtracted once for each of its 3-connected components. $\square$

In the same spirit we can relate connected and 2-connected multigraphs, using the auxiliary class $\mathcal{Q}$ of all edge-rooted connected vertex-labelled cubic planar multigraphs whose root edge is a loop. To simplify the formulas later on, the root will be marked by y in the generating function $Q(x,y,z,w)$ and only non-root loops are marked by w.

Lemma 4.5. *For $g \geqslant 1$ the generating functions of $\mathcal{C}_g$ and $\mathcal{B}_g$ satisfy the following relation:*

$$B_g^{\mathrm{fw}\geqslant 2}(x,\overline{y},\overline{z}) - B_0(x,\overline{y},\overline{z}) \preceq C_g^{\mathrm{fw}\geqslant 2}(x,y,z,w) \preceq B_g^{\mathrm{fw}\geqslant 2}(x,\overline{y},\overline{z}), \tag{8}$$

where $\overline{y} = \frac{y}{1-Q(x,y,z,w)}$ and $\overline{z} = \frac{1}{2}(\frac{y}{1-Q(x,y,z,w)})^2 - \frac{y^2}{2} + z$.

Proof. Let $C \in \mathcal{C}_g^{\mathrm{fw}\geqslant 2}$. We shall show that C is counted at least once on the right-hand side and at most once on the left-hand side of (8).

First, suppose that C is not planar. Then Corollary 2.9(i) states that C has a unique 2-connected component B strongly embeddable on $\mathbb{S}_g$ with the same facewidth, i.e., $B \in \mathcal{B}_g^{\mathrm{fw}\geqslant 2}$. To construct C from B we have to replace each edge by a sequence of edges and multigraphs in $\mathcal{Q}$, which means we replace one edge by a sequence of alternating edges and multigraphs in $\mathcal{Q}$ without the root, starting and ending with an edge. Therefore, the replacement leads to the substitution $y \mapsto y\frac{1}{1-Q(x,y,z,w)}$ (see Figure 3).

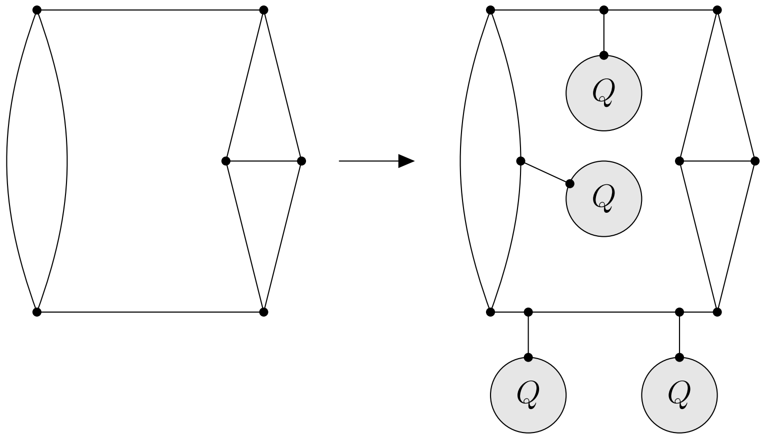

Figure 3: Obtaining connected graphs from 2-connected graphs by substituting edges with sequences of Q-graphs.

This results in a 1-to-1 correspondence between the generating functions $B_g^{\text{fw}\geqslant 2}(x,\overline{y},\overline{z})$ and $C_g^{\text{fw}\geqslant 2}(x,y,z,w)$ for non-planar multigraphs. The replacement for double edges results from replacing a set of two edges each as above, except when the two edges are left intact, then they should still be treated as a double edge instead of two simple edges. We thus have the correction term $-\frac{y^2}{2}+z$.

As in Lemma 4.4, if C is planar, the above argument does not necessarily result in a bijection. We thus have to subtract all corresponding planar multigraphs again in order to avoid overcounting on the left-hand side. Therefore, we get the claimed result analogously to Lemma 4.4. □

Combining Lemmas 4.4 and 4.5, we have the following upper and lower bounds for the generating function $C_g(x,y,z,w)$.

Corollary 4.6. *For $g \geqslant 1$, the generating function $C_g(x,y,z,w)$ satisfies*

$$\begin{aligned} &D_g^{\text{fw}\geqslant 3}\left(x, \overline{y}\left(1+N(x,\overline{y},\overline{z})\right)\right) - D_0\left(x, \overline{y}\left(1+N(x,\overline{y},\overline{z})\right)\right) \\ &\quad + B_g^{\text{fw}=2}(x,\overline{y},\overline{z}) - B_0(x,\overline{y},\overline{z}) + C_g^{\text{fw}=1}(x,y,z,w) \\ &\preceq C_g(x,y,z,w) \\ &\preceq D_g^{\text{fw}\geqslant 3}\left(x, \overline{y}\left(1+N(x,\overline{y},\overline{z})\right)\right) + B_g^{\text{fw}=2}(x,\overline{y},\overline{z}) + C_g^{\text{fw}=1}(x,y,z,w), \end{aligned} \tag{9}$$

where $\overline{y} = \frac{y}{1-Q(x,y,z,w)}$ and $\overline{z} = \frac{1}{2}\left(\frac{y}{1-Q(x,y,z,w)}\right)^2 - \frac{y^2}{2} + z$ as in Lemma 4.5.

We note that the upper and lower bounds of $C_g(x,y,z,w)$ differ only by terms involving generating functions of planar graphs. In Section 4.3 we will show that those generating functions are subdominant. Therefore, the two bounds match in asymptotics. We will also provide the asymptotic expressions for the other terms in Section 4.3. In order to do that, we first establish bounds on the generating functions for multigraph classes with fixed facewidth.

Lemma 4.7. *For* $g \geqslant 1$ *the following relations hold.*

$$B_g^{\mathrm{fw}=2}(x,y,z) \preceq 2\left(y+\frac{z}{y}\right)^2\left(\frac{1}{y}+\frac{y}{z}\right)^2\Bigg(\left(\delta_y+\delta_z\right)^2\left(B_{g-1}^{\mathrm{fw}\geqslant 2}(x,y,z)\right)$$
$$+\sum_{g'=1}^{g-1}\left(\delta_y+\delta_z\right)\left(B_{g'}^{\mathrm{fw}\geqslant 2}(x,y,z)\right)\left(\delta_y+\delta_z\right)\left(B_{g-g'}^{\mathrm{fw}\geqslant 2}(x,y,z)\right)\Bigg), \qquad (10)$$

$$C_g^{\mathrm{fw}=1}(x,y,z,w) \preceq (xyw)^2\left(\frac{1}{y}+\frac{y}{z}\right)\Bigg(\delta_w^2\left(C_{g-1}(x,y,z,w)\right)$$
$$+\sum_{g'=1}^{g-1}\delta_w\left(C_{g'}(x,y,z,w)\right)\delta_w\left(C_{g-g'}(x,y,z,w)\right)\Bigg). \qquad (11)$$

Proof. In order to show (10), let B be a multigraph in $\mathcal{B}_g^{\mathrm{fw}=2}$. Consider a fixed 2-cell embedding M of B on $\mathbb{S}_g$ with facewidth two and let $\{e_1 = \{v_1, w_1\}, e_2 = \{v_2, w_2\}\}$ be two edges such that there exists an essential circle C on $\mathbb{S}_g$ meeting M only in e_1 and e_2. Note that e_1, e_2 do not share vertices, because otherwise the facewidth would have been one. Then we delete e_1 and e_2, cut the surface along C and close both holes with a disk[2] (see Figure 4). By this surgery we either disconnect the surface or we reduce its genus by one.

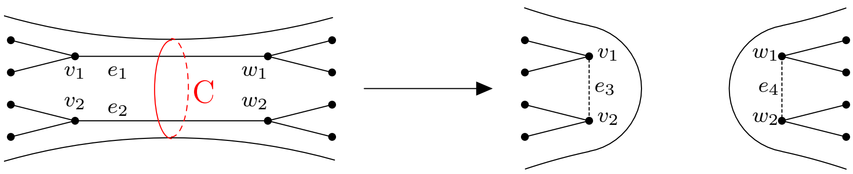

Figure 4: Surgery along an essential circle.

Case 1: Cutting along C disconnects the surface. As C was an essential loop, both components have genus at least one. Therefore, we obtain two multigraphs B_1^* and B_2^*, strongly embeddable on $\mathbb{S}_{g_1}$ and $\mathbb{S}_{g_2}$ respectively, with $g_1, g_2 \geqslant 1$ and $g_1+g_2 = g$. Without loss of generality, we can assume that $v_1, v_2 \in B_1^*$ and $w_1, w_2 \in B_2^*$. Furthermore, $\{e_1, e_2\}$ was a 2-edge-separator in B. Thus, B_1^* and B_2^* are connected as B is 2-edge-connected. Let B_1 be obtained from B_1^* by adding an edge $e_3 = \{v_1, v_2\}$ and marking e_3. Note that B_1 is also strongly embeddable on $\mathbb{S}_{g_1}$. We claim that B_1 is 2-connected. Indeed, any path in B between vertices in B_1 gives rise to a path in B_1 between the same vertices by replacing

[2]For a formal definition of this operation see Section A.1

any sub-path in $B \setminus B_1$ by the edge e_3. Thus, B_1 is 2-connected as B is. Analogously, we add the edge $e_4 = \{w_1, w_2\}$ to B_2^* to obtain a 2-connected multigraph B_2 strongly embeddable on $\mathbb{S}_{g_2}$. We also mark e_4. Furthermore, we claim that both B_1 and B_2 cannot have facewidth 1. Indeed, suppose that B_1 has facewidth 1 for a certain embedding M_1, then we can perform the reverse direction of the surgery to get an embedding of B. Since B is of facewidth at least 2, the only possibility is that the face containing the essential circle of length 1 in M_1 is one of those created in the surgery from B to B_1 and B_2, which is not possible by construction. Therefore, B_1 has facewidth at least 2, and analogously, B_2 has facewidth at least two as well. Therefore, in this case, we can conclude that a multigraph B can be constructed from a 2-connected multigraph embeddable on $\mathbb{S}_{g'}$ with one marked edge and a 2-connected multigraph embeddable on $\mathbb{S}_{g-g'}$ with one marked edge, with both multigraphs of facewidth at least 2, resulting in the term

$$\sum_{g'=1}^{g-1} (\delta_y + \delta_z) \left(B_{g'}^{\mathrm{fw}\geqslant 2}(x,y,z)\right) (\delta_y + \delta_z) \left(B_{g-g'}^{\mathrm{fw}\geqslant 2}(x,y,z)\right).$$

Note that e_3 and e_4 might be single edges or part of double edges. Therefore, differentiating with respect to both possibilities results in an upper bound. The factor $\left(\frac{1}{y} + \frac{y}{z}\right)^2$ accounts for the deletion of e_1 and e_2, each of which might have been a single edge or part of a double edge (hence deleting it turns a double edge into a single edge). The factor $\left(y + \frac{z}{y}\right)^2$ represents the insertion of e_3 and e_4, each of which either adds a single edge or turns a single edge into a double edge. Both factors contribute to the the upper bound as they overcount. Furthermore, we obtain a factor of two for the ways to obtain the original multigraph from B_1 and B_2.

Case 2: Cutting along C does not disconnect the surface. As the embedding after cutting is still a 2-cell embedding, $B \setminus \{e_1, e_2\}$ is connected. We can connect v_1, v_2, w_1, w_2 in $B \setminus \{e_1, e_2\}$ by two edges (without loss of generality $e_3 = \{v_1, v_2\}$ and $e_4 = \{w_1, w_2\}$) so as to obtain a multigraph B^*. The graph $\overline{B} = B \cup \{e_3, e_4\}$ has a 2-cell embedding $\overline{M}$ on $\mathbb{S}_g$ such that $e_1 \cup e_2 \cup e_3 \cup e_4$ bounds a face. Indeed, starting from M, e_3 and e_4 can be embedded so that they run close to e_1, e_2, and C. Let M^* be the embedding of B^* induced by $\overline{M}$. Suppose that B^* is not 2-connected, that is, it has a bridge e. Note that e cannot be e_3 or e_4 as $B^* \setminus \{e_3, e_4\} = B \setminus \{e_1, e_2\}$ is connected.

There is a (not necessarily essential) circle C' on $\mathbb{S}_g$ hitting M^* only in e. As e has not been a bridge in B, C' has to meet e_1 and e_2 as well. If it met neither e_1 nor e_2, it would either contradict B having facewidth two (if C' is essential) or the 2-connectivity of B (if C' is not essential). If it met only one of them, it would have to meet one of e_3, e_4, because e_1, e_2, e_3 and e_4 bound a disk in $\overline{M}$. This contradicts the fact that C' meets M^* only in e.

We now construct the circle C'' as follows. First, we follow C' from e to e_1 without traversing it. Then, we follow e_1 until reaching C and switch to C to reach e_2 without crossing e_1 and e_2. Finally, we return to C' along e_2 and then return to e (see Figure 5). C'' meets M only in e. Either C'' is an essential circle, contradicting the fact that B has facewidth two, or it is planar, contradicting the 2-connectivity of B. Similarly, we can also prove that B^* has facewidth at least 2. Thus we conclude that every multigraph B, where the surgery does not result in disconnecting the surface, can be constructed from a 2-connected multigraph embeddable on $\mathbb{S}_{g-1}$ with two marked edges and facewidth at least two, which results in the term

$$(\delta_y + \delta_z)^2 B_{g-1}(x, y, z).$$

The factor $2\left(y+\frac{z}{y}\right)^2\left(\frac{1}{y}+\frac{y}{z}\right)^2$ follows as in Case 1. We thus conclude (10).

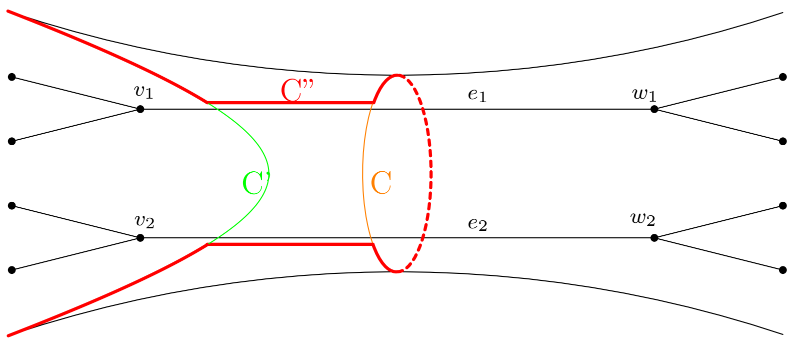

Figure 5: Finding an essential circle witnessing small facewidth

To prove (11), let G be a multigraph in $\mathcal{C}_g^{\mathrm{fw}=1}$. We fix a 2-cell embedding M of G on $\mathbb{S}_g$ of facewidth one and let $e_1 = \{v_1, v_2\}$ be an edge such that there exists an essential circle C on $\mathbb{S}_g$ meeting G only in e_1. Then we perform the following surgery: we delete e_1, cut the surface along C, close both holes with a disk, and attach an edge, an additional vertex and a loop to both v_1 and v_2. Remark that the edge deleted may be a single edge or part of a double edge. Thus, we have a factor of $(xyw)^2\left(\frac{1}{y}+\frac{y}{z}\right)$, overcounting all possibilities. The deleted edge cannot be a loop, since in cubic maps on orientable surfaces loops are always on the boundary of two different faces, and as such cannot be the only intersection of an embedding of a multigraph with an essential circle. This can easily be seen, as there is only one other edge at the base of the loop. Thus, the boundary of the face of the loop without this additional edge consists only of traversing the loop once. By this surgery, we either disconnect the surface or we reduce its genus by one.

If we separate the surface, we obtain two connected multigraphs each with one marked loop. These multigraphs are counted by $\delta_w(C_{g'})$ and by $\delta_w(C_{g-g'})$, as the genera of the two parts sum up to g and the embeddings resulting from the surgery are still 2-cell embeddings.

If the surface is not separated, the resulting embedding is a 2-cell embedding and hence the multigraph remains connected. Therefore, we obtain a multigraph counted by

$\delta_w^2(C_{g-1})$. The factor in front of the generating function is once again obtained by marking two loops. This proves (11). $\square$

In subsequent calculations, it will be more convenient to change the differential operators to operators with respect to x instead of y, z, and w.

Corollary 4.8. *For $g \geqslant 1$, we have*

$$B_g^{\mathrm{fw}=2}(x,y,z) \preceq \frac{9}{2}\left(y+\frac{z}{y}\right)^2\left(\frac{1}{y}+\frac{y}{z}\right)^2\left(\delta_x^2\left(B_{g-1}^{\mathrm{fw}\geqslant 2}(x,y,z)\right)\right.$$
$$\left.+\sum_{g'=1}^{g-1}\delta_x\left(B_{g'}^{\mathrm{fw}\geqslant 2}(x,y,z)\right)\delta_x\left(B_{g-g'}^{\mathrm{fw}\geqslant 2}(x,y,z)\right)\right), \tag{12}$$

and

$$C_g^{\mathrm{fw}=1}(x,y,z,w) \preceq (xyw)^2\left(\frac{1}{y}+\frac{y}{z}\right)\left(\delta_x^2\left(C_{g-1}(x,y,z,w)\right)\right.$$
$$\left.+\sum_{g'=1}^{g-1}\delta_x\left(C_{g'}(x,y,z,w)\right)\delta_x\left(C_{g-g'}(x,y,z,w)\right)\right). \tag{13}$$

Proof. Since the generating function $B_g^{\mathrm{fw}\geqslant 2}$ counts cubic graphs, it is the sum of monomials of the form $x^{2k}y^{3k-2\ell}z^{\ell}$ for some nonnegative integers k,ℓ with $2l \leqslant 3k$. We thus have

$$\delta_x x^{2k}y^{3k-2\ell}z^{\ell} = 2kx^{2k}y^{3k-2\ell}z^{\ell} \geqslant \frac{2}{3}(3k-\ell)x^{2k}y^{3k-2\ell}z^{\ell} = \frac{2}{3}\left(\delta_y+\delta_z\right)x^{2k}y^{3k-2\ell}z^{\ell}.$$

Therefore, we have $\frac{3}{2}\delta_x B_g^{\mathrm{fw}\geqslant 2} \geqslant \left(\delta_y+\delta_z\right)B_g^{\mathrm{fw}\geqslant 2}$. Combining this with (10) proves (12).

To show (13), we note that a cubic graph has at most as many loops as vertices, and thus replacing δ_w by δ_x increases each coefficient. Thus, we still have an upper bound when replacing δ_w by δ_x in (11). $\square$

Corollary 4.8 will be used to show that the number of multigraphs with small facewidth in Corollary 4.6 is negligible. Additionally, we need equations for the generating functions of the auxiliary classes $\mathcal{N}$ and $\mathcal{Q}$. Recall that $\mathcal{N}$ is the class of edge-rooted 2-connected vertex-labelled cubic planar multigraphs, and $\mathcal{Q}$ is the class of all edge-rooted connected vertex-labelled cubic planar multigraphs whose root edge is a loop.

Proposition 4.9. *The generating function $N(x,y,z)$ of $\mathcal{N}$ satisfies the system of equations*

$$\begin{aligned} N(x,y,z) &= \frac{u(1-2u)-x^2y(1+N(x,y,z))(y^2-2z)}{2}, \\ x^2y^3(1+N(x,y,z))^3 &= u(1-u)^3, \end{aligned} \tag{14}$$

and the generating function $Q(x,y,z,w)$ of $\mathcal{Q}$ satisfies

$$\begin{aligned} Q &= \frac{Q^2}{2} + \frac{x^2y^3}{2}\left(A + \overline{Q}\right) + x^2y^2w, \\ A &= Q + S + P + H, \\ S &= \frac{A^2}{A+1}, \\ P &= \frac{x^2y^3}{2}A^2 + x^2y^3A + x^2yz, \\ 2H(1+A) &= u(1-2u) - u(1-u)^3, \\ x^2y^3A^3 &= u(1-u)^3, \end{aligned} \tag{15}$$

where

$$\overline{Q} = \begin{cases} -Q & \text{for simple graphs,} \\ 0 & \text{for weighted multigraphs,} \\ Q & \text{for multigraphs.} \end{cases}$$

Proof. We obtain (15) by following the lines of Section 3 in [22] or Section 3 in [28]. The only difference is that we account for loops and double edges in the initial conditions. In order to derive the system for Q, one starts with an edge-rooted connected cubic planar graph and recursively decomposes it depending on the placement of the root. One of the following mutually exclusive cases occurs:

(i) the root is a loop;

(ii) the root is a bridge;

(iii) the root is part of a minimal separating edge set of size two;

(iv) the end vertices of the root separate the graph; or

(v) the root is part of a 3-connected component.

In Case (i) we obtain an equation for Q, while Case (ii) results in an equation that can immediately be eliminated from the system, Case (iii) in the equation for S, Case (iv) in the equation for P and Case (v) in the parametric equations for H in terms of u. It is shown in [22] that these cases are indeed exhaustive. For each of these cases, there is a decomposition of the graph resulting in the corresponding equation in the system. The difference for the three values of $\overline{Q}$ is due to the difference of how to deal with loops and double edges in the three different weightings. The only difference in the systems of all three weightings comes from Case (i), when the third edge at the root vertex is incident to a double edge (see Figure 6). While this case cannot happen for simple graphs (and thus it is not possible to obtain a loop as a root in this case), the difference regarding weighted and unweighted multigraphs is due to the weighting of $\frac{1}{2}$ of the double edge.

To obtain the equations for $N(x,y,z)$, we start with (15). Because $N(x,y,z)$ enumerates edge-rooted 2-connected planar cubic multigraphs, setting $w = 0$ and $Q(x,y,z,w) =$

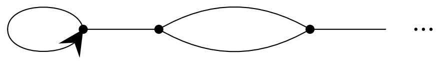

Figure 6: The exceptional case which has to be dealt with differently for simple graphs, weighted and unweighted multigraphs.

0 results in a system of equations for $N(x,y,z) = 1 + A$. The given equations follow by eliminating S, P, and H from the new system. □

4.3 Asymptotics

The goal of this section is to obtain asymptotics for $C_g(v)$ via Corollary 4.6. The analysis for $C_g^u(v)$ and $C_g^s(v)$ are analogous; we will point out the differences when they occur.

To use Corollary 4.6, we will prove asymptotic formulas for each of the occurring terms. In order to simplify notations, we define

$$\widetilde{D}_g(x,y,z,w) = D_g\left(x, \overline{y}\left(1 + N(x,\overline{y},\overline{z})\right)\right),$$

where

$$\overline{y} = \frac{y}{1 - Q(x,y,z,w)}, \quad \text{and} \quad \overline{z} = \frac{1}{2}\left(\frac{y}{1 - Q(x,y,z,w)}\right)^2 - \frac{y^2}{2} + z.$$

This change of variables comes from Lemma 4.5. Facewidth conditions can be added in the usual way. Additionally, we define

$$\widetilde{N}(x,y,z,w) = N\left(x,\overline{y},\overline{z}\right)$$
$$\widetilde{B}_g(x,y,z,w) = B_g\left(x,\overline{y},\overline{z}\right).$$

Furthermore, when writing v as the sole variable, we are always using the corresponding substitution from Definition 4.1.

In order to determine the dominant singularity of $\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v)$, one of the summands in Corollary 4.6, we first analyse the dominant singularity of $Q(v)$. The numerical values of the dominant singularities and other constants are different for $Q(v)$, $Q^u(v)$, and $Q^s(v)$, but the analysis works in exactly the same way.

Lemma 4.10. *The dominant singularity of $Q(v)$ is $\rho_Q = \frac{54}{79^{3/2}}$. Furthermore, $Q(v)$ is Δ-analytic and*

$$Q(v) \cong q_0(1 - \rho_Q^{-1}v)^{3/2} + O\left((1 - \rho_Q^{-1}v)^2\right),$$

where q_0 is a constant and we have $Q(\rho_Q) = 1 - \frac{17}{2\sqrt{79}}$.

Proof. Substituting v as in Definition 4.1 in (15) and eliminating S, P, H, u, and A in this order from (15), we get the following implicit equation for Q.

$$\begin{aligned} 0 =& 256Q^4 - 512Q^5 + 384Q^6 - 128Q^7 + 16Q^8 \\ &+ v(-320Q^3 - 224Q^4 + 2352Q^5 - 3304Q^6 + 2008Q^7 - 576Q^8 + 64Q^9) \\ &+ v^2(144Q^2 + 136Q^3 - 384Q^4 + 210Q^5 - 35Q^6) \\ &+ v^3(-28Q + 42Q^2 - 14Q^3) + 2v^4. \end{aligned} \tag{16}$$

Using standard methods for implicitly defined functions (see for example [12, VII.7.1]), we determine the dominant singularity to be at $\rho_Q = \frac{54}{79^{3/2}}$ and obtain the stated expression for $Q(v)$ and the value of $Q(v)$ at ρ_Q. □

This lemma is already strong enough to deal with the planar case. By unrooting the classes in Lemma 4.10, we obtain the asymptotic expansion of $C_0(v)$ as a corollary.

Corollary 4.11. *The dominant singularity of the generating function $C_0(v)$ of planar connected cubic vertex-labelled weighted multigraphs is $\rho_Q = \frac{54}{79^{3/2}}$. Furthermore, the generating function $C_0(v)$ is Δ-analytic and*

$$\begin{aligned} C_0(v) =& a_0 + a_1 \left(1 - \rho_C^{-1} v\right) + a_2 \left(1 - \rho_C^{-1} v\right)^2 \\ &+ c_0 \left(1 - \rho_C^{-1} v\right)^{5/2} + O\left(\left(1 - \rho_C^{-1} v\right)^3\right), \end{aligned}$$

where c_0, a_0, a_1, a_2 are constants.

Proof. The class of planar connected cubic multigraphs is given by unrooting the sum of the classes used in Lemma 4.10. It is easy to see that all those classes have the same dominant singularity as $Q(v)$, see [8, 22, 29] for more details. □

Similar results also hold for unweighted multigraphs and simple graphs with the same dominant singularities as $Q^u(v)$ and $Q^s(v)$, respectively.

Next we determine the asymptotic behaviour of networks in $\widetilde{N}(v)$. The only difference to the other two cases are the numerical values of n_0 and $\widetilde{N}(\rho_N)$.

Lemma 4.12. *The dominant singularity of $\widetilde{N}(v)$ occurs at $\rho_N = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore, $\widetilde{N}(v)$ is Δ-analytic, and*

$$\widetilde{N}(v) \cong n_0 \left((1 - \rho_N^{-1} v)^{3/2}\right) + O\left((1 - \rho_N^{-1} v)^2\right),$$

where n_0 is a constant and $\widetilde{N}(\rho_N) = 1/16$.

Proof. Starting from (14), we obtain a system of equations that is satisfied by $\widetilde{N}(v)$ by performing the appropriate substitutions on y and z in (14): first $y = \overline{y}$ and $z = \overline{z}$, then the substitutions from Definition 4.1. We thus obtain a system of two equations involving $\widetilde{N}(v)$, u, v and $Q(v)$. We then add (16), relating $Q(v)$ and v, and obtain a determined system. Eliminating $Q(v)$ and u from this system results in an implicit equation in $\widetilde{N}(v)$ and v.

Using standard methods (see for example [12, VII.7.1]) to deal with implicitly defined functions we determine the dominant singularity to be at $\rho_N = \rho_Q$ and derive the claimed properties of $\widetilde{N}(v)$. □

With the help of these two lemmas we obtain the singularity and singular expansion of the main term $\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v)$ in Corollary 4.6.

Lemma 4.13. *The generating function* $\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v)$ *has its dominant singularity at* $\rho_D = \rho_Q = \frac{54}{79^{3/2}}$. *Furthermore, we have*

$$\begin{aligned}
\widetilde{D}_0(v) &\cong c_0(1-\rho_Q)^{5/2} + O\left((1-\rho_Q)^3\right), \\
\widetilde{D}_1^{\mathrm{fw}\geqslant 3}(v) &\cong c_1 \log(1-\rho_Q^{-1}v) + O\left((1-\rho_Q^{-1}v)^{1/4}\right), \\
\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v) &\cong c_g(1-\rho_Q^{-1}v)^{-5g/2+5/2} + O\left((1-\rho_Q^{-1}v)^{-5g/2+11/4}\right) \quad \textit{for } g \geqslant 2.
\end{aligned}$$

Analogous results to Lemma 4.13 also hold for unweighted multigraphs and simple graphs. The only difference are the numerical values of the constants and the dominant singularities, where the latter coincide with the dominant singularities of $Q^u(v)$ and $Q^s(v)$, respectively.

Proof. The dominant singularity of $\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v)$ is given either by the singularity ρ_Q of $Q(v)$ and $\widetilde{N}(v)$, or by a solution of $\frac{v(1+\widetilde{N}(v))^3}{(1-Q(v))^3} = \rho_S^3$, where $\frac{v(1+\widetilde{N}(v))^3}{(1-Q(v))^3}$ is obtained by substituting v in $x^2(\overline{y}(1+N(x,\overline{y},\overline{z})))^3$, and ρ_S^3 is the dominant singularity of $D_g(v)$. By Proposition 3.2 and Lemma 4.12 we verify that $\frac{\rho_Q(1+\widetilde{N}(\rho_Q))^3}{(1-Q(\rho_Q))^3} = \rho_S^3$. This is the only solution of this equation, as $\frac{v(1+\widetilde{N}(v))^3}{(1-Q(v))^3}$ is a power series with positive coefficients, and thus monotone on the interval $[0,\rho_Q)$. Therefore, the dominant singularity of $\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v)$ is ρ_Q, and the composition is critical (in the sense of [12, pp. 411ff]). We thus conclude the proof by Proposition 4.2, and noting that $D_g^{\mathrm{fw}\geqslant 3}(v)$ has same asymptotic behaviour as $D_g(v)$. □

The next lemma shows the asymptotic behaviour of $\widetilde{B}_g^{\mathrm{fw}=2}(v)$, which is the next term occurring in the bounds of Corollary 4.6.

Lemma 4.14. *For* $g \geqslant 1$, *we have*

$$\widetilde{B}_g^{\mathrm{fw}=2}(v) \cong O\left(\left(1-\rho_N^{-1}v\right)^{-5g/2+11/4}\right). \tag{17}$$

Proof. First we observe that by Lemmas 4.4 and 4.13, the generating function $\widetilde{B}_g^{\mathrm{fw}\geqslant 3}(v)$ has its dominant singularity at $\rho_B = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore,

$$\begin{aligned}
\widetilde{B}_1^{\mathrm{fw}\geqslant 3}(v) &\cong c_1 \log(1-\rho_Q^{-1}v) + O\left((1-\rho_Q^{-1}v)^{1/4}\right), \\
\widetilde{B}_g^{\mathrm{fw}\geqslant 3}(v) &\cong c_g(1-\rho_Q^{-1}v)^{-5g/2+5/2} + O\left((1-\rho_Q^{-1}v)^{-5g/2+11/4}\right) \quad \text{for } g \geqslant 2.
\end{aligned}$$

We prove the claim by induction on g. Suppose that our claim is correct for all $g' < g$. By Corollary 4.8 and the fact that both $\overline{y}$ and $\overline{z}$ are formal power series with positive coefficients, we have

$$\begin{aligned}
B_g^{\mathrm{fw}=2}(x,\overline{y},\overline{z}) \preceq \frac{9}{2}\left(\overline{y}+\frac{\overline{z}}{\overline{y}}\right)^2 \left(\frac{1}{\overline{y}}+\frac{\overline{y}}{\overline{z}}\right)^2 \Big(&\delta_x^2 B_{g-1}^{\mathrm{fw}\geqslant 2}(x,\overline{y},\overline{z}) \\
&+ \sum_{g'=1}^{g-1} \delta_x B_{g'}^{\mathrm{fw}\geqslant 2}(x,\overline{y},\overline{z})\, \delta_x B_{g-g'}^{\mathrm{fw}\geqslant 2}(x,\overline{y},\overline{z}).
\end{aligned} \tag{18}$$

We now perform the substitutions as in Definition 4.1. Note that x is substituted by $v^{1/4}$, while $\overline{y}$ and $\overline{z}$ are formal power series in x, y, z, w, all substituted by positive powers of v. Therefore, we can replace δ_x by $4\delta_v$ while keeping an upper bound. We thus have

$$\widetilde{B}_g^{\mathrm{fw}=2}(v) \preceq 648 \left(\delta_v^2(\widetilde{B}_{g-1}^{\mathrm{fw}\geqslant 2}(v)) + \sum_{g'=1}^{g-1} \delta_v(\widetilde{B}_{g'}^{\mathrm{fw}\geqslant 2}(v)) \delta_v(\widetilde{B}_{g-g'}^{\mathrm{fw}\geqslant 2}(v)) \right). \tag{19}$$

The precise coefficient may change for unweighted multigraphs or simple graphs. By Lemma 2.4 and the fact that all generating functions on the right-hand side of (19) are for genus smaller than g, we deduce by induction that

$$\begin{aligned}
\delta_v^2(\widetilde{B}_{g-1}^{\mathrm{fw}\geqslant 2}(v)) &\cong O\left(\left(1 - \rho_B^{-1} v\right)^{-5g/2+3} \right), \\
\delta_v(\widetilde{B}_{g'}^{\mathrm{fw}\geqslant 2}(v)) &\cong O\left(\left(1 - \rho_B^{-1} v\right)^{-5g'/2+3/2} \right), \\
\delta_v(\widetilde{B}_{g-g'}^{\mathrm{fw}\geqslant 2}(v)) &\cong O\left(\left(1 - \rho_B^{-1} v\right)^{-5(g-g')/2+3/2} \right).
\end{aligned}$$

Substituting these congruences into (19) results in

$$\widetilde{B}_g^{\mathrm{fw}=2}(v) \preceq O\left(\left(1 - \rho_B^{-1} v\right)^{-5g/2+3} \right),$$

which immediately implies (17).

For the base case $g = 1$, the computation is the same, except that we have a term $\delta_v^2 \widetilde{B}_0(v)$, which is $\delta_v^2 C_0(v)$, since $B_0(x, \overline{y}, \overline{z}) = C_0(x, y, z, w)$. By Corollary 4.11, we have

$$\delta_v^2 \widetilde{B}_0(v) \cong a_2 + O\left((1 - \rho_B v)^{1/2} \right),$$

completing the proof. □

We use the asymptotic results in Corollary 4.11, Lemma 4.13, and Lemma 4.14 to examine the bounds in Corollary 4.6 and determine the dominant term of connected cubic multigraphs embeddable on $\mathbb{S}_g$.

Theorem 4.15. *For $g \geqslant 1$, the dominant singularity of the generating function $C_g(v)$ of connected cubic vertex-labelled weighted multigraphs that are strongly embeddable on $\mathbb{S}_g$ is $\rho_C = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore, we have*

$$\begin{aligned}
C_1(v) &\cong c_1 \log\left(1 - \rho_C^{-1} v\right) + O\left(\left(1 - \rho_C^{-1} v\right)^{1/4} \right), \\
C_g(v) &\cong c_g \left(1 - \rho_C^{-1} v\right)^{-5g/2+5/2} + O\left(\left(1 - \rho_C^{-1} v\right)^{-5g/2+11/4} \right) \quad \text{for } g \geqslant 2,
\end{aligned}$$

where c_g is a constant depending only on g.

Proof. We first do the substitution of Definition 4.1 in (9), which leads to

$$\begin{aligned}
&\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v) - \widetilde{D}_0(v) + \widetilde{B}_g^{\mathrm{fw}=2}(v) - \widetilde{B}_0(v) + C_g^{\mathrm{fw}=1}(v) \\
&\qquad \preceq C_g(v) \preceq \widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v) + \widetilde{B}_g^{\mathrm{fw}=2}(v) + C_g^{\mathrm{fw}=1}(v).
\end{aligned}$$

Comparing the terms other than $C_g^{\mathrm{fw}=1}(v)$ in these bounds, we obtain by Corollary 4.11, Lemma 4.13, and Lemma 4.14 that the dominant term is $\widetilde{D}_g^{\mathrm{fw}\geqslant 3}(v)$, which has the claimed singularity and decomposition.

To conclude the proof, it remains to show that

$$C_g^{\mathrm{fw}=1} \cong O\left(\left(1-\rho_C^{-1}v\right)^{-5g/2+11/4}\right). \tag{20}$$

By Corollary 4.8 and the fact that by replacing δ_x by $4\delta_v$, the coefficients do not decrease, we have the relation

$$C_g^{\mathrm{fw}=1}(v) \preceq \frac{27}{4}\delta_v^2(C_{g-1}(v)) + \frac{27}{4}\sum_{g'=1}^{g-1}\delta_v(C_{g'}(v))\delta_v(C_{g-g'}(v)). \tag{21}$$

By the fact that all generating functions on the right-hand side of (21) are for genus smaller than g, we can use induction on g as in the proof of Lemma 4.14 to deduce (20), concluding the proof. □

From Theorem 4.15 and Lemma 2.6, we can immediately evaluate the coefficients of $C_g(v)$.

Corollary 4.16. *The asymptotic number of connected cubic vertex-labelled multigraphs that are weighted by their compensation factor and are strongly embeddable on $\mathbb{S}_g$ is given by*

$$[v^n]C_g(v) = \left(1+O\left(n^{-1/4}\right)\right)c_g n^{5g/2-7/2}\rho_C^{-n}.$$

Here $\rho_C = \rho_Q = \frac{54}{79\sqrt{79}}$ and c_g is a constant only depending on g.

Again, both Theorem 4.15 and Corollary 4.16 work analogously for unweighted multigraphs and simple graphs with different constants and dominant singularities of $Q^u(v)$ and $Q^s(v)$, respectively.

4.4 Proof of Theorem 1.2

Using the results from Section 4.3 we can now prove Theorem 1.2. Recall that a cubic multigraph embeddable on $\mathbb{S}_g$ is given by a set of connected cubic multigraphs embeddable on $\mathbb{S}_{g_i}$ such that $\sum g_i \leqslant g$ (see Proposition 2.2). Therefore, we have the relation

$$G_g(v) \preceq \sum_{k=1}^{\infty}\sum_{\sum g_i\leqslant g}\frac{1}{k!}\prod_{i=1}^{k}\left(C_{g_i}(v)+\frac{v}{6}\right). \tag{22}$$

The summand $\frac{v}{6}$ accounts for the fact that a component might also be a triple edge, which is not taken into account in $C_{g_i}(v)$ (this additional summand will differ when proving Theorems 1.1 and 1.3). We only get an upper bound, because we overcount if a multigraph is strongly embeddable on surfaces of multiple genera. Later we will also obtain a lower bound with the same asymptotics to complete the proof.

If $g = 0$, the relation (22) simplifies to $G_0 = \exp(C_0 + \frac{v}{6})$, as there is no overcounting in this case. This coincides with Theorem 1 of [22] and therefore we can conclude our statement in this case. (Although it is not directly shown there, the same arguments can be used for unweighted planar cubic multigraphs. For simple graphs, see [8].)

Now suppose $g \geqslant 1$. The first step to obtain asymptotics from (22) is to rearrange the sum in such a way that all planar components are singled out. This results in

$$G_g(v) \preceq \sum_{k=0}^{g} \sum_{\substack{\sum g_i \leqslant g \\ g_i \geqslant 1}} \frac{1}{k!} \prod_{i=1}^{k} \left(C_{g_i}(v) + \frac{v}{6} \right) \sum_{j=0}^{\infty} \frac{k!}{(k+j)!} \left(C_0(v) + \frac{v}{6} \right)^j . \tag{23}$$

By the dominant term and the value of $C_0(v)$ at the singularity ρ_C from Corollary 4.11, we observe that the last sum contributes only a constant factor. Thus, it remains to derive the dominant term of $\frac{1}{k!} \prod \left(C_{g_i}(v) + \frac{v}{6} \right)$. As the first sum consists only of a constant number of summands, the dominant term of the right-hand side of (23) will be the (sum of the) dominant terms from $\frac{1}{k!} \prod \left(C_{g_i}(v) + \frac{v}{6} \right)$ up to the constant obtained from the planar components. That is, we shall compute the dominant term of

$$A(v) := \frac{1}{k!} \prod_{i=1}^{k} \left(C_{g_i}(v) + \frac{v}{6} \right),$$

where the g_i are positive and sum up to $g' \leqslant g$.

For $g = 1$, either $k = g' = 0$ or $k = g' = 1$. By Theorem 4.15, we have

$$A(v) = C_1(v) + \frac{v}{6} \cong c_1 \log(1 - \rho_C^{-1} v) + \frac{v}{6} + O\left(\left(1 - \rho_C^{-1} v\right)^{1/4} \right)$$

and thus

$$A(v) \preceq P_1(v) + c_1 \log(1 - \rho_C^{-1} v) + O\left(\left(1 - \rho_C^{-1} v\right)^{1/4} \right)$$

with $P_1(v)$ a polynomial and c_1 a constant.

Suppose now $g \geqslant 2$. Without loss of generality let $g_1, \ldots, g_l = 1$ and let $g_{l+1}, \ldots, g_k > 1$. Then

$$\begin{aligned} A(v) &\cong \left(1 + O\left(\left(1 - \rho_C^{-1} v\right)^{1/4} \right)\right) \frac{c_1^l}{k!} \left(\log\left(1 - \rho_C^{-1} v\right)\right)^l \prod_{i=l+1}^{k} c_{g_i} \left(1 - \rho_C^{-1} v\right)^{5(1-g_i)/2} \\ &\cong \left(c + O\left(\left(1 - \rho_C^{-1} v\right)^{1/4} \right)\right) \left(\log\left(1 - \rho_C^{-1} v\right)\right)^l \left(1 - \rho_C^{-1} v\right)^{-5g'/2 + 5k/2} . \end{aligned} \tag{24}$$

For $k = 1$ and $g' = g$ (and hence $l = 0$) we thus have

$$\frac{1}{1!} \prod_{g'=1}^{1} \left(C_{g_i} + \frac{v}{6} \right) \cong c_g (1 - \rho_C^{-1} v)^{-5g/2 + 5/2} + O\left((1 - \rho_C^{-1} v)^{-5g/2 + 11/4} \right). \tag{25}$$

For $k \geqslant 2$ or $g' < g$, (24) yields

$$\frac{1}{k!}\prod_{i=1}^{k} C_{g_i}(v) \cong O\left((1-\rho_C^{-1}v)^{-5g/2+5/2+2}\right) \tag{26}$$

and thus

$$G_g(v) \preceq c_g(1-\rho_C^{-1}v)^{-5g/2+5/2} + O\left((1-\rho_C^{-1}v)^{-5g/2+11/4}\right).$$

We derive a lower bound for $g \geqslant 1$ as follows. Let $\tilde{\mathcal{G}}_g$ be the class of graphs in $\mathcal{G}_g$ with one component of genus g and all other components planar. Then

$$\sum_{j=0}^{\infty} \frac{C_g(v)C_0^j(v)}{(j+1)!} \succeq \tilde{G}_g(v) \succeq \sum_{j=0}^{\infty} \frac{C_g(v)C_0^j(v)}{(j+1)!} - \sum_{j=0}^{\infty} \frac{(j+1)C_0^{j+1}(v)}{(j+1)!}.$$

Indeed, if the component of genus g is also planar, then the graph might be counted up to $j+1$ times (once for each component) on the left-hand side. Substituting the corresponding summands thus yields a lower bound of $G_g(v)$.

$$G_g(v) \succeq \tilde{G}_g(v) \succeq \sum_{j=0}^{\infty} \frac{1}{(j+1)!} C_g(v)C_0^j(v) - \sum_{j=1}^{\infty} \frac{j}{j!} C_0^j(v). \tag{27}$$

Applying Theorem 2.3 to the upper and lower bounds and setting $\gamma_2 = \rho_C^{-1}$ completes the proof. □

4.5 Proofs of Theorems 1.1, 1.3, and 1.4

Theorems 1.1 and 1.3 can be proven in a similar way as Theorem 1.2. We obtain $\rho_1 = \gamma_1^{-1}$ as the square root of the smallest positive solution of

$$0 = 46656 - 279936u - 7293760u^2 - 513216u^3 + 148716u^4 - 17469u^5 + 729u^6$$

and $\rho_3 = \gamma_3^{-1}$ as the square root of the smallest positive solution of

$$0 = 46656 + 279936u - 7293760u^2 + 513216u^3 + 148716u^4 + 17469u^5 + 729u^6.$$

Theorem 1.4 for the case of weighted multigraphs follows immediately from (25), (26), and Theorem 2.3. Indeed, (25) and Theorem 2.3 imply that the number of weighted multigraphs in $\mathcal{G}_g(n)$ that have a unique non-planar component that is not embeddable on $\mathbb{S}_{g-1}$ is

$$\left(1 + O\left(n^{-1/4}\right)\right) e_g n^{5g/2-5/2} \gamma_2^{2n} (2n)!\,.$$

On the other hand, (26) and Theorem 2.3 imply that the number of weighted multigraphs in $\mathcal{G}_g(n)$ that do not have such a component is

$$O\left(n^{5g/2-5/2-2} \gamma_2^{2n} (2n)!\right).$$

Thus, Theorem 1.4 follows. Observe that the probability $1-O(n^{-2})$ is not sharp. Indeed, the exponent in (26) could be improved to $-5g/2+5-\varepsilon$ for every $\varepsilon > 0$, which would yield a probability $1-O(n^{-5/2+\varepsilon})$. The statements of Theorem 1.4 for unweighted multigraphs and simple graphs are proved analogously. □

Remark 4.17. *Observe that the polynomials $p_1(u), p_3(u)$ whose smallest positive zeroes u_1 and u_3 give rise to the exponential growth constants γ_1 for cubic multigraphs and γ_3 for simple cubic graphs, respectively, satisfy the relation $p_1(u) = p_3(-u)$. It would be interesting to know whether this fact has a combinatorial meaning.*

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A Proof of Theorems 3.2 and 3.3

In this appendix, we compute the asymptotic numbers of triangulations in $\mathcal{S}_g$ and $\hat{\mathcal{S}}_g$, as stated in Propositions 3.2 and 3.3, respectively. Our proof follows the approach of [1].

A.1 Surgeries

When dealing with maps on $\mathbb{S}_g$ we will perform operations on the surfaces that are commonly known as *cutting* and *gluing*. In the course of these operations we will encounter *surfaces with holes*. A *surface with k holes* is a surface $\mathbb{S}_g$ from which the disjoint interiors $D_1, \ldots, D_k$ of k closed disks have been deleted. Each D_i is called a *hole*. Let S be the disjoint union of finitely many orientable surfaces, at least one of them with holes, and suppose that X and Y are homeomorphic subsets of the boundary of S. By *gluing S along X and Y* we mean the operation of identifying every point $x \in X$ with $f(x)$ for any fixed homeomorphism $f \colon X \to Y$. The identification of X and Y induces a surjection σ from S onto the resulting space $\tilde{S}$. We write $\tilde{X}$ for the subset $\sigma(X) = \sigma(Y)$ of $\tilde{S}$.

We will glue along subsets in two particular situations: when X and Y are

(i) disjoint boundaries of holes of S, or

(ii) sub-arcs of the boundary of the same hole that meet precisely in their endpoints.

For (ii), we shall additionally assume that the homeomorphism $f \colon X \to Y$ induces the identity on $X \cap Y$. In either case, the space $\tilde{S}$ resulting from gluing along X and Y is again the disjoint union of finitely many orientable surfaces with holes, with the number of components being either the same or one less than that for S. The subset $\tilde{X}$ of $\tilde{S}$ is a circle in Case (i) and homeomorphic to the closed unit interval in Case (ii). If S has k holes, then $\tilde{S}$ has $k-2$ holes in Case (i) and $k-1$ holes in Case (ii). A special case of (i) is when one of the components of S is a disk (i.e. a sphere with one hole) and Y is its boundary. In this case, we say that we *close the hole bounded by X by inserting a disk.*

If in addition we are given a map M on S, then we will glue along X and Y only if either both of them are contained in a face (not necessarily the same face for X and Y) or both are unions of the same number of vertices and the same number of edges of M. We also assume the homeomorphism $f\colon X \to Y$ to map vertices to vertices and edges to edges. Under these conditions, we obtain a map $\tilde{M}$ on $\tilde{S}$. The subset $\tilde{X}$ of $\tilde{S}$ is then either a subgraph of $\tilde{M}$ or a subset of a face of $\tilde{M}$. Observe that the surjection $\sigma\colon S \to \tilde{S}$ induces a bijection between the sets of corners of M and of corners of $\tilde{M}$. We will refer to this bijection by saying that every corner of M *corresponds* to a corner of $\tilde{M}$.

If $\tilde{S}$ is obtained from S by gluing along X and Y, we also say that vice versa, S is obtained from $\tilde{S}$ by *cutting along* $\tilde{X}$. The operation of cutting along a circle or interval is well defined in the sense that if $\tilde{S}$ and $\tilde{X}$ are given, then S, X, and Y are unique up to homeomorphism. If S has more components than $\tilde{S}$, we call $\tilde{X}$ *separating.* Cutting along a separating circle on $\mathbb{S}_g$ and closing the resulting holes by inserting disks will yield two surfaces $\mathbb{S}_{g_1}, \mathbb{S}_{g_2}$ with $g_1 + g_2 = g$. Cutting along a non-separating circle and closing the holes by inserting disks reduces the genus by one.

A combination of cutting and gluing surfaces along some subsets of their boundaries is called a *surgery.* Again, if a map $\tilde{M}$ results from performing surgeries on a map M, then every corner of $\tilde{M}$ corresponds to a corner of M.

A.2 Quasi-triangulations

We begin with some notations. The *valency* of a face f in a map is the number of corners of f. We call a rooted map M a *quasi-triangulation* if all faces except the root face f_r are bounded by triangles. Let $\mathcal{P}_g$ be the class of *simple* quasi-triangulations and $P_g(y,u)$ its generating function, where y marks the number of edges, and u marks the valency of f_r. Given an index set I and an injective function $h\colon I \to F(M) \setminus \{f_r\}$, we call M an *I-quasi-triangulation with respect to h* if all faces in $F(M) \setminus (h(I) \cup \{f_r\})$ are bounded by triangles. If in addition f_r is also bounded by a triangle, we say that M is an *I-triangulation* (with respect to h). Let $\mathcal{S}_{g,I}$ and $\mathcal{P}_{g,I}$ be the classes of simple I-triangulations and simple I-quasi-triangulations, respectively, with their generating functions denoted by $S_g(y, z_I)$ and $P_g(y, u, z_I)$, respectively. Here u again marks the valency of the root face f_r and $z_I = (z_i)_{i \in I}$ is a set of variables indexed by I, where z_i marks the valency of $h(i)$. Additionally, let $\hat{\mathcal{P}}_g$, $\hat{\mathcal{S}}_{g,I}$, and $\hat{\mathcal{P}}_{g,I}$ be the analogous classes for triangulations without separating loops or separating double edges and $\hat{P}_g(y,u)$, $\hat{S}_g(y, z_I)$, and $\hat{P}_g(y, u, z_I)$ their generating functions, respectively.

Note that $\mathcal{S}_g = \mathcal{S}_{g,\emptyset}$ and $\mathcal{P}_g = \mathcal{P}_{g,\emptyset}$. In the case $I = \emptyset$, we will therefore always use the generating functions $S_g(y)$ and $P_g(y,u)$ without mentioning variables z_i. To simplify notations, the one-vertex map is put into $\mathcal{P}_0$, although it is not a quasi-triangulation, since it does not have any corners and thus cannot be rooted. We say that a face f is *marked* if $f \in h(I)$ and that we are *marking a face* f if we add a new index i to the set I with $h(i) = f$. We use the same convention also for the corresponding classes and generating functions for triangulations without loops and double edges.

To prove Propositions 3.2 and 3.3, we first derive a recursive formula relating $\mathcal{P}_{g,I}$ (and

$\hat{\mathcal{P}}_{g,I}$) for different genera and different sizes of the set I. We then prove Propositions 3.2 and 3.3 by applying this formula inductively. In order to derive the recursive formula, we delete the root edge of a given quasi-triangulation and then perform surgeries that either separate the given surface or decrease its genus. One part of the reverse operation then consists of adding a new edge to a map. Let S be a map and $c_1 = (v_1, e_1^-, e_1^+)$ and $c_2 = (v_2, e_2^-, e_2^+)$ be two (not necessarily distinct) corners of the same face f of S. For T is a map with $V(T) = V(S)$ and $E(T) = E(S) \cup \{e_{\text{new}}\}$, we say that e_{new} is an *edge from* c_1 *to* c_2 if

- e_{new} is contained in f and its end vertices are v_1 and v_2;
- in the cyclic order of edges of T at v_1, e_{new} is the predecessor of e_1^+; and
- in the cyclic order of edges of T at v_2, e_{new} is the successor of e_2^-.

If $c_1 = c_2 =: c$, we also say that e_{new} is a *loop at* c.

A.3 The planar case

Before we derive the recursive formula, we first study the base case of *planar* quasi-triangulations.

Proposition A.1. *The generating function of planar quasi-triangulations satisfies*

$$P_0(y,u) = 1 + yu^2 P_0^2(y,u) + \frac{y(P_0(y,u) - 1)}{u} - y^2 u P_0(y,u) - S_0(y)(P_0(y,u) - 1). \quad (28)$$

Additionally, $\hat{P}_0(y,u) = P_0(y,u)$ *holds.*

Proof. As planar quasi-triangulations cannot have non-separating loops or double edges, $\hat{P}_0(y,u) = P_0(y,u)$ follows immediately.

The first summand in (28) corresponds to the one-vertex map. Let $S \in \mathcal{P}_0$ be a planar quasi-triangulation with at least one edge. We distinguish two cases.

First, suppose that the root edge e_r is a bridge; then the only face incident with e_r is the root face f_r. The union $f_r \cup e_r$ is not a disk and thus contains a non-contractible circle C. We delete e_r, cut along C, and close the two resulting holes by inserting disks. By this surgery, S is separated into two quasi-triangulations S_1, S_2. Let v_1 and v_2 be the end vertices of e_r in S_1 and S_2 respectively. One of these two vertices is the root vertex of S; by renaming S_1, S_2 we may assume that v_1 is the root vertex of S. In the cyclic order of the edges of S at v_1, let e_1^- and e_1^+ be the predecessor and successor of e_r respectively. Define e_2^- and e_2^+ analogously at v_2. We let (v_1, e_1^-, e_1^+) and (v_2, e_2^-, e_2^+) be the roots of S_1 and S_2 respectively. We thus have $S_1, S_2 \in \mathcal{P}_0$. Furthermore, S_1 and S_2 together have one edge less than S and the sum of the valencies of their root faces is two less than the valency of f_r. Thus, we obtain $yu^2P_0^2(x,u)$, the second term of the right-hand side of (28).

Now suppose that e_r is not a bridge. Then it lies on the boundary of the root face and of another face, which is bounded by a triangle. In the cyclic order of edges at the root vertex v_r, let e_r^- and e_r^+ be the predecessor and successor of e_r, respectively. We delete e_r and obtain a quasi triangulation S' that we root at $c_r' := (v_r, e_r^-, e_r^+)$. The valency of the root face of S' is larger by one than the valency of f_r. This is reflected by $\frac{y}{u}(P_0(y,u) - 1)$, the third term of the right-hand side of (28), because S' cannot be the quasi-triangulation consisting only of a single vertex. However, with this summand we have overcounted. Indeed, the reverse construction is as follows. Let f_r' be the root face of S'. Then the corners of f_r' can be ordered by walking along the boundary of f_r' in counterclockwise direction. In this order, starting from c_r', let $c' = (v, e, e')$ be the corner after the next; then S is obtained from S' by inserting an edge from c_r' to c'. If $v_r = v$ this results in a loop; if v_r and v are adjacent, we obtain a double edge (see Figure 7).

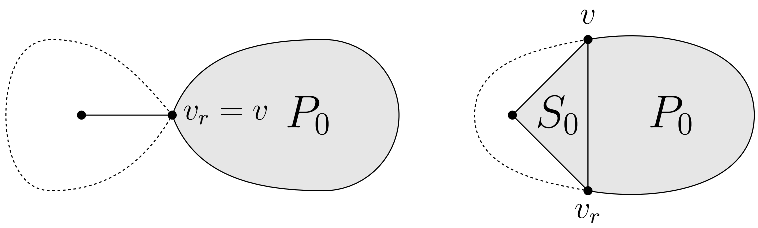

Figure 7: Obtaining a loop or a double edge by inserting an edge.

These cases have to be subtracted again in order to obtain a valid formula. First suppose that $v_r = v$. As we do not have double edges in S', this is only possible if the corner between c_r' and c' is at a vertex of degree one. We have to subtract $y^2uP_0(y,u)$, i.e. the fourth term of the right-hand side of (28), for this case (we add one vertex and two edges to a quasi-triangulation and increase the root face valency by one). Now suppose that v_r and v are adjacent, i.e. inserting an edge between them creates a double edge. In this case zipping the double edge separates the quasi-triangulation into two quasi-triangulations S_1, S_2. For one of them, without loss of generality for S_1, the root face valency is the same as for S, while the root face of S_2 has valency three. Thus, S_1 is in $\mathcal{P}_0$ but not the one-vertex map, while $S_2 \in \mathcal{S}_0$. Summing up we have to subtract $S_0(y)(P_0(y,u) - 1)$, the fifth term of the right-hand side of (28). □

We can use the quadratic method (see e.g. [19]) to obtain the main result for planar triangulations from Proposition A.1. Those were already obtained by Tutte [33] with slightly different parameters.

Lemma A.2. *It holds that* $\hat{S}_0(y) = S_0(y)$. *The dominant singularity of* $S_0(y)$ *is* $\rho_S = \frac{3}{2^{8/3}}$, $S_0(y)$ *is* Δ*-analytic and satisfies*

$$S_0(y) = \frac{1}{8} - \frac{9}{16}\left(1 - \rho_S^{-1} y\right) + \frac{3}{2^{5/2}}\left(1 - \rho_S^{-1} y\right)^{3/2} + O\left(\left(1 - \rho_S^{-1} y\right)^2\right). \qquad (29)$$

Furthermore, for $u = f(y)$ *with*

$$f(y) = \frac{t^{1/3}}{1+t} \quad \textit{and} \quad y = t^{1/3}(1-t),$$

the equations

$$P_0(y,f(y)) = \frac{5}{4} - \frac{3}{2^{5/2}}\left(1-\rho_S^{-1}y\right)^{1/2} + O\left(1-\rho_S^{-1}y\right),$$
$$\left(\frac{\partial}{\partial u}P_0(y,u)\right)\bigg|_{u=f(y)} = \frac{75}{2^{13/3}} - \frac{125\cdot 3^{3/4}}{2^{23/3}}\left(1-\rho_S^{-1}y\right)^{1/4} + O\left(\left(1-\rho_S^{-1}y\right)^{1/2}\right)$$

hold and $P_0(y,f(y))$ is Δ-analytic. Let $n \geqslant 2$ be an integer. Then

$$\left(\frac{\partial^n}{\partial u^n}P_0(y,u)\right)\bigg|_{u=f(y)} = c(n)\left(1-\rho_S^{-1}y\right)^{-n/2+3/4} + O\left(\left(1-\rho_S^{-1}y\right)^{-n/2+1}\right),$$

where $c(n)$ is a positive constant depending only on n.

Proof. As that planar quasi-triangulations cannot have non-separating loops or double edges, $\hat{S}_0(y) = S_0(y)$ follows immediately.

Multiplying (28) by $4yu^4$ and rearranging the terms yields

$$\left(2yu^3P_0(y,u) + q(y,u)\right)^2 = q(y,u)^2 + 4y^2u^3 - 4yu^4 - 4yu^4S_0(y), \tag{30}$$

where $q(x,u) = y - y^2u^2 - u - uS_0(y)$. Let

$$Q(y,u) = 2yu^3P_0(y,u) + q(y,u) \quad \text{and}$$
$$R(x,u) = q(y,u)^2 + 4y^2u^3 - 4yu^4 - 4yu^4S_0(y).$$

Then (30) reduces to $Q^2(y,u) = R(y,u)$. To obtain the claimed asymptotic behaviour one chooses $u = f(y)$ in such a way that $Q(y,f(y)) = 0$. This u is a double zero of $Q^2(y,u)$ and therefore both $R(y,u)$ and $\frac{\partial}{\partial u}R(y,u)$ are 0 at $u = f(y)$, giving

$$0 = q(y,u)^2 + 4y^2u^3 - 4yu^4 - 4yu^4S_0(y),$$
$$0 = 2q(y,u)(1 + S_0(y) + 2y^2u) + 16yu^3 + 16yu^3S_0(y) - 12y^2u^2.$$

By eliminating $f(y)$ from this system we obtain the implicit equation

$$S_0(y)^4 + 3S_0(y)^3 + S_0(y)^2(3+8y^3) + S_0(y)(1-20y^3) = (1-16y^3)y^3.$$

By standard methods for implicitly given functions (e.g. [12, VII.7.1]) we obtain the dominant singularity and the singular expansion of $S_0(y)$ as stated in (29).

Conversely, by eliminating $S_0(y)$ and substituting $y = t^{1/3}(1-t)$ we obtain $f(y) = \frac{t^{1/3}}{1+t} = \frac{y}{1-t^2}$ and $S_0(y) = t(1-2t)$. Since $\frac{1}{1-t^2}$ has only nonnegative coefficients in t and $t = t(y)$ has only nonnegative coefficients by Lagrange Inversion, $f(y)$ has only nonnegative coefficients as well. From the implicit equation for $f(y)$ we deduce that

$$f(y) = \frac{2^{4/3}}{5} - \frac{2^{11/6}}{25}\left(1-\rho_S^{-1}y\right)^{1/2} + O\left(1-\rho_S^{-1}y\right). \tag{31}$$

From $2yf(y)^3P_0(y,f(y))+q(y,f(y))=Q(y,f(y))=0$, (29), (31), and $y=\rho_S-\rho_S(1-\rho_S^{-1}y)$ we derive the claimed expression

$$P_0(y,f(y))=\frac{5}{4}-\frac{3}{2^{5/2}}\left(1-\rho_S^{-1}y\right)^{1/2}+O\left(1-\rho_S^{-1}y\right).$$

Given $n\in\mathbb{N}_0$, let us write $R^{(n)}(y,u)=\frac{\partial^n}{\partial u^n}R(y,u)$. By the choice of $f(y)$ we know that $R^{(0)}(y,f(y))=R^{(1)}(y,f(y))=0$. As $R(y,u)$ is a polynomial of degree four in u, we have $R^{(n)}(y,u)=0$ for all $n\geqslant 5$. For $n\in\{2,3,4\}$, we obtain the dominant term of $R^{(n)}(y,f(y))$ by first differentiating $R(y,u)$ with respect to u and then substituting $u=f(y)$, (29), (31), and $y=\rho_S-\rho_S(1-\rho_S^{-1}y)$. This yields

$$\begin{aligned}
R^{(2)}(y,f(y))&=\frac{27}{2^{7/2}}\left(1-\rho_S^{-1}y\right)^{1/2}+O\left(1-\rho_S^{-1}y\right),\\
R^{(3)}(y,f(y))&=-\frac{675}{2^{16/3}}+O\left(\left(1-\rho_S^{-1}y\right)^{1/2}\right),\\
R^{(4)}(y,f(y))&=-\frac{10125}{2^{23/3}}+O\left(\left(1-\rho_S^{-1}y\right)^{1/2}\right).
\end{aligned}$$

We define $Q^{(n)}(y,u)$ and $P_0^{(n)}(y,u)$ analogously to $R^{(n)}(y,u)$. From the facts that $Q(y,f(y))=0$ and $\frac{\partial^n}{\partial u^n}\left(Q^2(y,u)\right)=R^{(n)}(y,u)$ we deduce that

$$\begin{aligned}
2nQ^{(1)}(y,f(y))Q^{(n-1)}(y,f(y))&=R^{(n)}(y,f(y))\\
&\quad-\sum_{k=2}^{n-2}\binom{n}{k}Q^{(k)}(y,f(y))Q^{(n-k)}(y,f(y))
\end{aligned}\tag{32}$$

for every $n\in\mathbb{N}$. For $n=2$, this implies that

$$Q^{(1)}(y,f(y))=\overline{c}(1-\rho_S^{-1}y)^{1/4}+O\left((1-\rho_S^{-1}y)^{3/4}\right),$$

where $\overline{c}=\pm\frac{3^{3/2}}{2^{11/4}}$. By differentiating $Q(y,u)=2yu^3P_0(y,u)+q(y,u)$ with respect to u, we deduce that

$$P_0^{(1)}(y,f(y))=\frac{75}{2^{13/3}}+\overline{c}\frac{125}{2^{59/12}3^{3/4}}\left(1-\rho_S^{-1}y\right)^{1/4}+O\left(\left(1-\rho_S^{-1}y\right)^{1/2}\right).$$

Since $P_0^{(1)}(y,u)$ is a generating function of a combinatorial class, all of its coefficients $[y^ku^l]P_0^{(1)}(y,u)$ are nonnegative. As $f(y)$ has only nonnegative coefficients as well, all coefficients of $P_0^{(1)}(y,u)|_{u=f(y)}$ are nonnegative, implying that $\overline{c}=-\frac{3^{3/2}}{2^{11/4}}$.

For $n=3$, we deduce from (32) that

$$Q^{(2)}(y,f(y))=-\frac{675}{6\overline{c}2^{16/3}}(1-\rho_S^{-1}y)^{-1/4}+O\left((1-\rho_S^{-1}y)^{1/4}\right).$$

For $n\geqslant 4$, the term $R^{(n)}(y,f(y))$ is constant, while the sum on the right-hand side is nonempty. Since the sum only involves terms $Q^{(j)}(y,f(y))$ with $2\leqslant j\leqslant n-2$, we deduce by induction that

$$Q^{(n)}(y,f(y))=\overline{c}(n)\left(1-\rho_S^{-1}y\right)^{-n/2+3/4}+O\left(\left(1-\rho_S^{-1}y\right)^{-n/2+5/4}\right),\tag{33}$$

where $\overline{c}(n)$ is a constant depending only on n and $\overline{c}(n) > 0$ for $n \geqslant 2$.

The claimed expressions of $P_0^{(n)}(y, f(y))$ are now obtained by differentiating

$$Q(y,u) = 2y^2u^3P_0(y,u) + q(y,u)$$

n times and by (29), (31), (33), and induction.

As all generating functions in this proof are given by a system of algebraic equations, they are Δ-analytic. □

A.4 Recurrence for higher genus

Our next aim is to derive a recursion formula for $P_g(y, u, z_I)$ and $\hat{P}_g(y, u, z_I)$. Using the planar case in Lemma A.2 as the base case, inductively applying the recursion formula allows us to derive similar statements as Lemma A.2 for all g and I. In order to derive the recursion formula, we will perform different surgeries on the surface depending on the placement of the root. We distinguish four cases.

(A) The root edge e_r is only incident with the root face f_r and is a bridge;

(B) e_r is only incident with f_r and is not a bridge;

(C) e_r is incident with f_r and one marked face; and

(D) e_r is incident with f_r and one unmarked face.

The recursion formula is then of the form

$$P_g(y,u,z_I) = A_g(y,u,z_I) + B_g(y,u,z_I) + C_g(y,u,z_I) + D_g(y,u,z_I), \tag{34}$$

where $A_g(y, u, z_I)$, $B_g(y, u, z_I)$, $C_g(y, u, z_I)$, and $D_g(y, u, z_I)$ are the generating functions of the sub-classes $\mathcal{A}_{g,I}$, $\mathcal{B}_{g,I}$, $\mathcal{C}_{g,I}$, and $\mathcal{D}_{g,I}$ of $\mathcal{P}_{g,I}$ corresponding to the four cases (A), (B), (C), and (D) respectively. Each of the generating functions can be further decomposed as

$$\begin{aligned}
A_g(y,u,z_I) &= a(y,u)P_g(y,u,z_I) + M_A(g;y,u,z_I) - E_A(g;y,u,z_I),\\
B_g(y,u,z_I) &= b(y,u)P_g(y,u,z_I) + M_B(g;y,u,z_I) - E_B(g;y,u,z_I),\\
C_g(y,u,z_I) &= c(y,u)P_g(y,u,z_I) + M_C(g;y,u,z_I) - E_C(g;y,u,z_I),\\
D_g(y,u,z_I) &= d(y,u)P_g(y,u,z_I) + M_D(g;y,u,z_I) - E_D(g;y,u,z_I),
\end{aligned} \tag{35}$$

where $a(y, u)$, $b(y, u)$, $c(y, u)$, and $d(y, u)$ are functions only involving the generating functions P_0 and S_0 of the planar case, while the other functions involve terms of the type $P_{g'}(y, u, z_{I'})$ for $g' < g$ or $I' \subsetneq I$. This will enable us to use (34) to recursively determine the dominant terms of $P_g(y, u, z_I)$. In this recursion, the functions M_A, M_B, M_C, and M_D will contribute to the dominant term; the functions E_A, E_B, E_C, and E_D turn out to be of smaller order.

The classes $\hat{\mathcal{A}}_{g,I}$, $\hat{\mathcal{B}}_{g,I}$, $\hat{\mathcal{C}}_{g,I}$, and $\hat{\mathcal{D}}_{g,I}$ together with the functions $\hat{A}_g(y, u, z_I)$, $\hat{a}(y, u)$, $\hat{M}_A(g; y, u, z_I)$, and $\hat{E}_A(g; y, u, z_I)$ (and similarly for B, C, and D) are defined analogously.

We start by determining the functions for Case (A). In this case, after deleting the root edge we can split the map into two maps whose genera add up to g.

Lemma A.3. *The functions $a(y,u,z_I)$, $M_A(g;y,u,z_I)$, and $E_A(g;y,u,z_I)$ in (35) are given by*

$$\begin{aligned} a(y,u,z_I) &= 2yu^2 P_0(y,u), \\ M_A(g;y,u,z_I) &= yu^2 \sum_{t,J} P_t\left(y,u,z_J\right) P_{g-t}\left(y,u,z_{I\setminus J}\right), \\ E_A(g;y,u,z_I) &= 0. \end{aligned}$$

The sum is over $t = 0,\ldots,g$ and $J \subseteq I$ such that $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g,I)$.

Proof. Let S be an I-quasi-triangulation in $\mathcal{A}_{g,I}$, with respect to $h\colon I \to F(S)$, say. By (A), the union $f_r \cup e_r$ is not a disk and thus contains a non-contractible circle C. We delete e_r, cut along C, and close the two resulting holes by inserting disks. Since e_r was a bridge, this surgery results in two components S_1 and S_2. We define the roots of S_1 and S_2 like in Proposition A.1: let v_1 and v_2 be the end vertices of e_r in S_1 and S_2 respectively. Without loss of generality we may assume that v_1 is the root vertex of S. In the cyclic order of the edges of S at v_1, let e_1^- and e_1^+ be the predecessor and successor of e_r, respectively. Define e_2^- and e_2^+ analogously at v_2. We let (v_1,e_1^-,e_1^+) and (v_2,e_2^-,e_2^+) be the root of S_1 and S_2 respectively. Denote the root faces by f_1 and f_2 respectively. These are the faces of S_1 and S_2 into which the disks were inserted.

Since every face in $F(S)\setminus\{f_r\}$ corresponds to a face in $F(S_1)\setminus\{f_1\}$ or in $F(S_2)\setminus\{f_2\}$, h induces a function $\tilde{h}\colon I \to (F(S_1)\cup F(S_2))\setminus\{f_1,f_2\}$. If we write $J = \tilde{h}^{-1}(F(S_1))$, then S_1 is a J-quasi-triangulation on a surface of genus $t \leqslant g$; consequently, S_2 is an $(I\setminus J)$-quasi-triangulation on a surface of genus $g-t$. By deleting e_r, we decreased the number of corners of f_r by two; the surgery then distributed the remaining corners of f_r to f_1 and f_2. Therefore, the sum of valencies of f_1 and f_2 is smaller by two than the valency of f_r. On the other hand, we clearly have $|E(S_1)|+|E(S_2)| = |E(S)|-1$. The reverse operation of the surgery is to delete an open disk from each of f_1, f_2, glue the surfaces along the boundaries of these disks, and add an edge from the root corner of S_1 to the root corner of S_2. As this operation is uniquely defined, we deduce that

$$A_g(y,u,z_I) = yu^2 \sum_{t=0}^{g} \sum_{J\subseteq I} P_t\left(y,u,z_J\right) P_{g-t}\left(y,u,z_{i\setminus J}\right).$$

Extracting the terms for $(t,J) = (0,\emptyset)$ and $(t,J) = (g,I)$ finishes the proof. □

Remark A.4. *Analogously to Lemma A.3, we have*

$$\begin{aligned} \hat{a}(y,u,z_I) &= 2yu^2 \hat{P}_0(y,u), \\ \hat{M}_A(g;y,u,z_I) &= yu^2 \sum_{t,J} \hat{P}_t\left(y,u,z_J\right) \hat{P}_{g-t}\left(y,u,z_{I\setminus J}\right), \\ \hat{E}_A(g;y,u,z_I) &= 0, \end{aligned}$$

where the sum is over $t = 0,\ldots,g$ and $J \subseteq I$ such that $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g,I)$.

This follows by the same proof as for Lemma A.3, because no loops or double edges occur in that construction.

For Case (B), we will cut the surface along a circle contained in $f_r \cup e_r$ and close the holes by inserting disks. However, because the surface will not be separated by this surgery, one needs to keep track of where to cut and glue to reverse the surgery. To this end we have to mark faces. Therefore, the index set I will increase.

Lemma A.5. *The functions $b(y,u,z_I)$, $M_B(g;y,u,z_I)$, and $E_B(g;y,u,z_I)$ in* (35) *are given by*

$$\begin{aligned} b(y,u,z_I) &= 0, \\ M_B(g;y,u,z_I) &= yu^2 \delta_{z_{i_0}} \left(P_{g-1}(y,u,z_{I\cup\{i_0\}})\right) \big|_{z_{i_0}=u}, \\ 0 \preceq E_B(g;y,u,z_I) &\preceq \left(1+yu^2\right) \delta_u \left(P_{g-1}(y,u,z_I)\right). \end{aligned}$$

Proof. Let S be an I-quasi-triangulation in $\mathcal{B}_{g,I}$, with respect to $h\colon I \to F(S)$. We use the analogous surgery as in Lemma A.3, with the difference that S is not separated by cutting along the circle C. Therefore we only obtain one map T. One of the end vertices of e_r is the root vertex v_r of S. Let e_r^- and e_r^+ be the predecessor and successor of e_r in the cyclic order of edges of S at v_r, respectively. Then we define the root of T to be (v_r, e_r^-, e_r^+). Denote the root face of T by f_r'; this is one of the two faces into which we inserted disks to close the holes during our surgery. Denote the other such face by f_2. We mark f_2 by adding a new index i_0 to the index set I and extend the function h to $I \cup \{i_0\}$ by setting $h(i_0) := f_2$. Then T is an $(I\cup\{i_0\})$-quasi-triangulation on $\mathbb{S}_{g-1}$.

To reverse the surgery, we delete an open disk from each of f_r', f_2, glue the surface along the boundaries of these disks, add a new edge e_{new} from the root corner of T to a corner c_2 of f_2, and let $(v_r, e_{\text{new}}, e_r^+)$ be the new root corner. We thus have to mark a corner of f_2, which corresponds to applying the operator $\delta_{z_{i_0}}$ to the generating function. After gluing, the corners of f_2 become corners of the new root face; we thus have to remove i_0 from the index set and replace z_{i_0} by u in the generating function. Like in the previous cases, adding e_{new} increases the total number of edges by one and the valency of the root face by two, as e_{new} lies only on the boundary of the new root face. This results in the term $yu^2\delta_{z_{i_0}}\left(P_{g-1}(y,u,z_{I\cup\{i_0\}})\right)|_{z_{i_0}=u}$.

However, by this construction we have overcounted. If the vertex v of the corner c_2 is adjacent to v_r, then e_{new} will be part of a double edge; if $v = v_r$, e_{new} will be a loop. We want to subtract all resulting maps $\tilde{S}$ for which e_{new} is a loop or part of a double edge. Suppose first that e_{new} is part of a double edge. Since e_{new} lies only on the boundary of the root face of $\tilde{S}$, the double edge is not separating. Thus, zipping it results in an I-quasi-triangulation $\tilde{T}$ on $\mathbb{S}_{g-1}$. One of the two zipped edges is the root edge e_r' of $\tilde{T}$, denote the other zipped edge by e'. Both e_r' and e' lie on the boundary of the root face (see Figure 8). Let v_r' be the root vertex of $\tilde{T}$; then v_r' is one of the two copies of v_r. If we denote the other copy by v', then v' is an end vertex of e' and thus there is a corner $c' = (v', e'', e')$ of the root face of $\tilde{T}$. We can reconstruct $\tilde{S}$ from $\tilde{T}$ in the following way: cut along e_r' and e' and glue the surface along the boundaries of the resulting holes in the unique way that identifies v_r' and v'. Identifying the corner c' is bounded by marking an *arbitrary* corner of the root face of $\tilde{T}$. This corresponds to applying the operator δ_u to the generating function $P_{g-1}(y, u, z_I)$. As zipping a double edge does neither change the number of edges nor the valencies of faces, $\delta_u\left(P_{g-1}(y, u, z_I)\right)$ is an upper bound in this case.

Suppose now that e_{new} is a loop and recall that $(v_r, e_{\text{new}}, e_r^+)$ is the root corner of $\tilde{S}$. Since e_{new} lies only on the boundary of the root face of $\tilde{S}$, there is a unique edge $e_2 \neq e_r^+$ such that $(v_r, e_{\text{new}}, e_2)$ is a corner of the root face. We cut along e_{new}, close the two resulting holes by inserting disks, and delete the two copies of e_{new}. Again, cutting does not separate the surface. Thus, we obtain a map $\tilde{T}$ on $\mathbb{S}_{g-1}$ that does not have loops or double edges. Let v_r' be the copy of v_r in $\tilde{T}$ that is incident with e_r^+ and let v_2' be the other copy. Then the root of $\tilde{T}$ is (v_r', e', e_r^+) for some edge e'. Furthermore, the root face of $\tilde{T}$ has a corner (v_2', e_2', e_2). Now $\tilde{S}$ can be reconstructed from $\tilde{T}$ in the following way (see Figure 8).

(i) Add a loop at each of (v_r', e', e_r^+) and (v_2', e_2', e_2);

(ii) delete the resulting two faces of valency one;

(iii) identify the two loops.

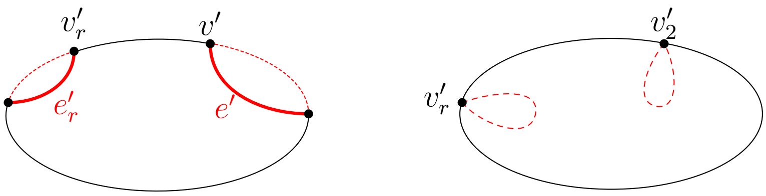

Figure 8: Deriving an upper bound for E_B.

In order to identify the corner (v_2', e_2', e_2), we mark an arbitrary corner of the root face, which is again overcounting. Since we have to add one edge to $\tilde{T}$ and increase the valency of the root face by two to reconstruct $\tilde{S}$, we have an additional factor of yu^2, resulting in the claimed upper bound for E_B. □

Similar arguments also show the corresponding result for $\hat{B}_g$.

Lemma A.6. *The functions $\hat{b}(y, u, z_I)$, $\hat{M}_B(g; y, u, z_I)$, and $\hat{E}_B(g; y, u, z_I)$ are given by*

$$\begin{aligned} \hat{b}(y, u, z_I) &= 0, \\ \hat{M}_B(g; y, u, z_I) &= yu^2 \delta_{z_{i_0}} \left(\hat{P}_{g-1}(y, u, z_{I \cup \{i_0\}})\right)\Big|_{z_{i_0}=u}, \\ \hat{E}_B(g; y, u, z_I) &= 0. \end{aligned}$$

The only difference to the proof of Lemma A.5 is that we do not need to compensate for overcounting in $\hat{E}_B$, as all loops and double edges occurring in the proof are non-separating and thus allowed.

In Case (C), the root edge is not a bridge. Therefore, we will not be able to find a circle C like in the previous two cases. On the other hand, deleting the root edge does not produce any faces that are not disks. Our construction in this case will thus start without cutting the surface.

Lemma A.7. *The functions $c(y,u,z_I)$, $M_C(g;y,u,z_I)$, and $E_C(g;y,u,z_I)$ in* (35) *are given by*

$$\begin{aligned} c(y,u,z_I) &= 0, \\ M_C(g;y,u,z_I) &= y\sum_{i\in I}\sum_{T\in\mathcal{P}_g(I\setminus\{i\})} y^{|E(T)|}\prod_{j\neq i} z_j^{\beta_j(T)}\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}, \\ 0\preceq E_C(g;y,u,z_I) &\preceq \sum_{i\in I}\Bigg((1+yuz_i)\,\delta_{z_i}\left(P_{g-1}(y,u,z_I)\right) \\ &\qquad + (1+yuz_i)\sum_{t=0}^{g}\sum_{J\subseteq I\setminus\{i\}} P_t(y,u,z_J)P_{g-t}(y,z_i,z_{I\setminus(J\cup\{i\})})\Bigg), \end{aligned}$$

where $\beta(T)$ and $\beta_j(T)$ denote the valencies of the root face of T and of the face with index j in T, respectively.

Note that the sum in M_C is over all $i\in I$ and all $I\setminus\{i\}$-quasi-triangulations. As such, M_C can be written as

$$M_C = y\sum_{i\in I}\frac{u^2 z_i P_g(y,u,z_{I\setminus\{i\}}) - u z_i^2 P_g(y,z_i,z_{I\setminus\{i\}})}{u-z_i}.$$

However, similarly to Lemma A.2, we shall replace u and z_i by $f(y)$ in order to derive the desired asymptotic formulas, which would result in a division by 0. For that reason we will use M_C as stated in Lemma A.7. We will derive a more convenient formulation in Proposition A.11.

Proof of Lemma A.7. Let S be an I-quasi-triangulation in $\mathcal{C}_{g,I}$ with respect to $h\colon I\to F(S)$. We delete the root edge e_r, thus obtaining a map T on $\mathbb{S}_g$. The root of T is defined as follows. Let e_r^- and e_r^+ be the predecessor and successor of e_r at v_r, respectively; then (v_r,e_r^-,e_r^+) is the root of T. By (C), e_r was incident with a marked face $h(i)$. The root face of T is $f_r' := f_r\cup e_r\cup h(i)$ and T is an $(I\setminus\{i\})$-quasi-triangulation with respect to $h|_{I\setminus\{i\}}$.

Let c be a corner of f_r' and let $\tilde{S}$ be obtained from T by adding an edge e_{new} from (v_r,e_r^-,e_r^+) to c and let the root of $\tilde{S}$ be

- $(v_r,e_{\text{new}},e_r^+)$ if $c\neq(v_r,e_r^-,e_r^+)$ and

- either $(v_r, e_{\text{new}}, e_r^+)$ or $(v_r, e_{\text{new}}, e_{\text{new}})$ otherwise.

These cases are illustrated in Figure 9. Adding e_{new} divides f_r' into two faces. One of these faces is the root face of $\tilde{S}$; we mark the other face with the index i and denote the corresponding function $I \to F(\tilde{S})$ by $\tilde{h}$. Clearly, there is a unique choice of $c \neq (v_r, e_r^-, e_r^+)$ such that $\tilde{S} = S$. If c is a corner at v_r (in particular if $c = (v_r, e_r^-, e_r^+)$), then e_{new} will be a loop. If c is a corner at a vertex adjacent to v_r, then e_{new} will be part of a double edge. In either case, $\tilde{S}$ will not be simple and thus not an I-quasi triangulation. Although the case $c = (v_r, e_r^-, e_r^+)$ is clearly one of the cases when $\tilde{S}$ is not simple, it is slightly easier to derive the formulas including this case.

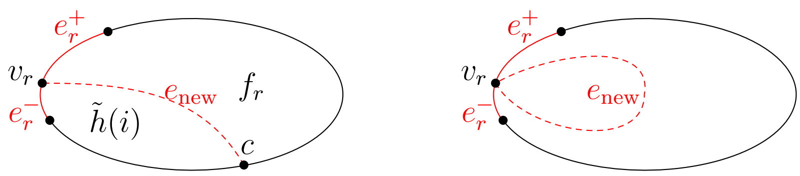

Figure 9: Adding the edge e_{new} from (v_r, e_r^-, e_r^+) to c to obtain $\tilde{S}$. If $c = (v_r, e_r^-, e_r^+)$, then each of the two faces can either be the root face or $\tilde{h}(i)$.

As f_r' has valency $\beta(T)$, there are $\beta(T)+1$ choices for $\tilde{S}$. The valency of the root face of $\tilde{S}$ is one if $c = (v_r, e_r^-, e_r^+)$ and the root face is $(v_r, e_{\text{new}}, e_{\text{new}})$. If $c = (v_r, e_r^-, e_r^+)$ and the root face is $(v_r, e_{\text{new}}, e_r^+)$, the valency is $\beta(T) + 1$. Depending on which corner is chosen as c, the valency of the root face can take any value k between 1 and $\beta(T) + 1$; the face $\tilde{h}(i)$ then has valency $\beta(T) + 2 - k$. The generating function of maps that can occur as $\tilde{S}$ from this particular $(I \setminus \{i\})$-quasi-triangulation T is thus given by

$$y^{|E(T)+1|} \left(\prod_{j \in I \setminus \{i\}} z_j^{\beta_j(T)} \right) \sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}.$$

This holds as the number of edges is increased by one and the valencies of all other marked faces do not change. After summing over all possible marked faces and all possible T, we obtain M_C.

As already mentioned, we overcount whenever the chosen corner c is at v_r or at a vertex adjacent to v_r, making e_{new} a loop or part of a double edge, respectively. Suppose first that e_{new} is part of a double edge. We zip the double edge. If it does not separate the surface, we have an upper bound $\delta_{z_i}(P_{g-1}(y, u, z_I))$ analogous to Lemma A.5. Indeed, the only difference to the corresponding case in Lemma A.5 is that we mark a corner of (the face corresponding to) $\tilde{h}(i)$ instead of a corner of the root face, because e_{new} was incident with both the root face and $\tilde{h}(i)$. If the double edge separates the surface, we obtain two maps T_1 on $\mathbb{S}_t$ for $0 \leqslant t \leqslant g$ and T_2 on $\mathbb{S}_{g-t}$. One of the two maps, without loss of generality T_2, contains (the face corresponding to) $\tilde{h}(i)$. As T_2 is rooted at a corner of that face and the root face is never marked, the number of marks decreases by one. Thus, T_1 is a J-quasi-triangulation on $\mathbb{S}_t$ and T_2 is a $(I \setminus (J \cup \{i\}))$-quasi-triangulation on $\mathbb{S}_{g-t}$, where $J \subseteq I \setminus \{i\}$. Going back, all corners of the root face of T_2 become corners of the face with index i, meaning that we have to replace u by z_i in $P_{g-t}(x, z_i, z_{I\setminus(J\cup\{i\})})$. This gives us an upper bound of

$$\sum_{t=0}^{g} \sum_{J\subseteq I\setminus\{i\}} P_t(x,u,J) P_{g-t}(x, z_i, z_{I\setminus(J\cup\{i\})}).$$

If e_{new} is a loop, then we proceed the same way as in Lemma A.5: we cut along e_{new}, close the two resulting holes by inserting disks, and delete the two copies of e_{new}. Like in Lemma A.5, the reverse construction yields the same bounds as in the case of e_{new} being part of a double edge; the additional factor yuz_i is due to the fact that we add one edge and increase the valencies of the root face and of $\tilde{h}(i)$ by one. □

Similar to Lemmas A.5 and A.6, the only difference when using triangulations without separating loops or separating double edges instead of simple triangulations is in the calculation of $\hat{E}_C$. The corresponding results are obtained by only keeping the terms in which separating loops or separating double edges are involved.

Lemma A.8. *The functions $\hat{c}(y,u,z_I)$, $\hat{M}_C(g;y,u,z_I)$, and $\hat{E}_C(g;y,u,z_I)$ are given by*

$$\begin{aligned}
\hat{c}(y,u,z_I) &= 0,\\
\hat{M}_C(g;y,u,z_I) &= y\sum_{i\in I}\sum_{T\in\hat{\mathcal{P}}_g(I\setminus\{i\})} y^{|E(T)|}\prod_{j\neq i} z_j^{\beta_j(T)}\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k},\\
0 \preceq \hat{E}_C(g;y,u,z_I) &\preceq \sum_{i\in I} 2\,(1+yuz_i)\sum_{t=0}^{g}\sum_{J\subseteq I\setminus\{i\}} \hat{P}_t(y,u,z_J)\hat{P}_{g-t}(y,z_i,z_{I\setminus(J\cup\{i\})}),
\end{aligned}$$

where $\beta(T)$ and $\beta_j(T)$ denote the valencies of the root face of T and of the face with index j in T, respectively.

The difference between Lemma A.7 and Lemma A.8 is that we do not need to compensate for the case where non-separating loops and double edges appear, since they are allowed in $\hat{\mathcal{P}}_{g,I}$.

The construction in Case (D) is similar to Case (C). The fact that the second face incident to e_r is not marked makes the analysis easier.

Lemma A.9. *The functions $d(y,u,z_I)$, $M_D(g;y,u,z_I)$, and $E_D(g;y,u,z_I)$ in (35) are given by*

$$\begin{aligned}
d(y,u,z_I) &= yu^{-1} - y^2u - S_0(y),\\
M_D(g;y,u,z_I) &= -S_g(y,z_I)P_0(y,u),\\
0 \preceq E_D(g;y,u,z_I) &\preceq 3P_{g-1}(y,u,z_{I\cup\{i_0\}})|_{z_{i_0}=u} + \sum_{t,J} S_t(y,J)P_{g-t}(y,u,z_{I\setminus J}),
\end{aligned}$$

where the sum is taken over all $0 \leqslant t \leqslant g$ and $J \subseteq I$ with $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g,I)$.

Proof. Let S be an I-quasi-triangulation in $\mathcal{D}_{g,I}$. We delete e_r and choose the root of the resulting map T to be $c'_r := (v_r, e_r^-, e_r^+)$ like in Lemma A.7. As the second face f incident with e_r is not marked and S is an I-quasi-triangulation, f is bounded by a triangle. Thus, T is also an I-quasi-triangulation and the valency of its root face f'_r is larger by one than the valency of f_r. For the reverse construction, consider the ordering of the corners of f'_r in clockwise direction along its boundary and let c be the corner after the next, starting from c'_r. We add an edge e_{new} from c'_r to c and let $(v_r, e_{\text{new}}, e_r^+)$ be the root of the resulting I-quasi-triangulation $\tilde{S}$. If T was obtained from S by deleting e_r, then $\tilde{S} = S$. However, if T is an *arbitrary* I-quasi-triangulation on $\mathbb{S}_g$, then e_{new} might be a loop or part of a double edge. Thus,

$$yu^{-1}P_g(y, u, z_I) \tag{36}$$

is only an upper bound for $D_g(y, u, z_I)$. Again, we have to subtract the cases when $\tilde{S}$ is not simple.

The case when e_{new} is a loop yields a term of

$$-y^2uP_g(x, u, z_I) \tag{37}$$

analogously to Proposition A.1. When e_{new} is part of a double edge, we need to distinguish whether this double edge separates the surface. If it does separate, we obtain

$$-\sum_{t=0}^{g} \sum_{J \subseteq I} S_t(y, J) P_{g-t}(y, u, z_{I \setminus J}) \tag{38}$$

by zipping the double edge, similar to Lemma A.3. The only differences are that the number of edges and the valencies of the faces do not change and that one of the two components is a J-triangulation, since its root face is f and thus has valency three. Finally, if the double edge does not separate, then after zipping it we have to mark f with a new index i_0 like in Lemma A.5. However, since the valency of f is three, we only have three possible ways to reverse the construction. As the number of edges and all valencies remain unchanged, we have a summand

$$-3P_{g-1}(y, u, z_{I \cup \{i_0\}})|_{z_{i_0}=u} . \tag{39}$$

Note that (39) is overcounting as the reverse construction can lead to additional loops or double edges.

Combining (36), (37), and the term from (38) with $(t, J) = (0, \emptyset)$, we deduce the claimed expression for $d(y, u, z_I)$. The term from (38) with $(t, J) = (g, I)$ results in $M_D(g; y, u, z_I)$; the remaining terms form the upper bound for $E_D(g; y, u, z_I)$. □

Throughout the proof of Lemma A.9, we do not encounter loops or multiple edges. Thus, the corresponding result for $\hat{D}$ follows immediately.

Lemma A.10. *The functions* $\hat{d}(y,u,z_I)$*,* $\hat{M}_D(g;y,u,z_I)$*, and* $\hat{E}_D(g;y,u,z_I)$ *are given by*

$$\begin{aligned}\hat{d}(y,u,z_I) &= yu^{-1} - y^2u - S_0(y),\\ \hat{M}_D(g;y,u,z_I) &= -\hat{S}_g(y,z_I)\hat{P}_0(y,u),\\ 0 \preceq \hat{E}_D(g;y,u,z_I) &\preceq 3\hat{P}_{g-1}(y,u,z_{I\cup\{i_0\}})|_{z_{i_0}=u} + \sum_{t,J}\hat{S}_t(y,J)\hat{P}_{g-t}(y,u,z_{I\setminus J}),\end{aligned}$$

where the sum is taken over all $0 \leqslant t \leqslant g$ *and* $J \subseteq I$ *with* $(t,J) \neq (0,\emptyset)$ *and* $(t,J) \neq (g,I)$*.*

A.5 Asymptotics

We now compute the asymptotics of all generating functions involved. Among the generating functions of all cases, the only one with a different structure than the others is M_C which cannot be easily expressed in terms of $P_{g'}(y,u,z_{I'})$ and $S_{g'}(y,I')$ with some genus g' and set I'. From M_B and E_B we observe that we need to calculate derivatives with respect to u and z_{i_0} and that we want to set $z_{i_0} = u$ in the end. We will be interested in the dominant term of $P_g(y,u,z_I)$ when we set $u = f(y)$ and $z_i = f(y)$ for all $i \in I$; we will abbreviate this by $u = z_I = f(y)$. Observe that setting $u = z_I = f(y)$ does not have any influence on the functions $S_g(y)$, as they only depend on the variable y.

The following proposition enables us to express arbitrary derivatives of M_C (and $\hat{M}_C$) at $u = z_I = f(y)$ in terms of derivatives of P_g (or $\hat{P}_g$).

Proposition A.11. *Let* $|y| < \rho_S$*,* $n \in \mathbb{N}_0$*, and* $\alpha_i \in \mathbb{N}_0$ *for all* $i \in I$*. Write* $|\alpha_I|$ *for* $\sum \alpha_i$*. Then*

$$\begin{aligned}&\left.\frac{\partial^{n+|\alpha_I|}}{\partial u^n \prod_{i\in I}\partial z_i^{\alpha_i}} M_C(g;y,u,z_I)\right|_{u=z_I=f(y)} \qquad (40)\\ &= y\left(\sum_{i\in I}\frac{n!\alpha_i!}{(n+\alpha_i+1)!}\frac{\partial^{n+1+|\alpha_I|}}{\partial u^{n+1+\alpha_i}\prod_{j\in I\setminus\{i\}}\partial z_j^{\alpha_j}}\left(u^3P_g(y,u,z_{I\setminus\{i\}})\right)\right)\Bigg|_{u=z_I=f(y)}.\end{aligned}$$

Proof. The generating function $yu^3P_g(y,u,z_{I\setminus\{i\}})$ is given by

$$yu^3P_g(y,u,z_{I\setminus\{i\}}) = y\sum_{T\in\mathcal{P}_g(I\setminus\{i\})} y^{|E(T)|}u^{\beta(T)+3}\prod_{j\in I\setminus\{i\}} z_j^{\beta_j(T)}.$$

By comparing this term with the summand in

$$M_C(g;y,u,z_I) = y\sum_{i\in I}\sum_{T\in\mathcal{P}_g(I\setminus\{i\})} y^{|E(T)|}\prod_{j\neq i} z_j^{\beta_j(T)}\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}$$

for a fixed index $i \in I$, one sees that the difference between them is that the factor $u^{\beta(T)+3}$ is replaced by $\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}$. Taking the derivatives with respect to u and z_i the given number of times and comparing the coefficients yields factors

$$\frac{(\beta(T)+3)!}{(\beta(T)+2-n-\alpha_i)!}u^{\beta(T)+2-n-\alpha_i} \qquad \text{and}$$

$$\sum_{k=n}^{\beta(T)+2-\alpha_i} \frac{k!(\beta(T)+2-k)!}{(k-n)!(\beta(T)+2-k-\alpha_i)!}u^{\beta(T)+2-n-\alpha_i},$$

respectively, when $n+\alpha_i+1 \leqslant \beta(T)+3$ and factors 0 otherwise. The quotient of these two coefficients equals $\frac{n!\alpha_i!}{(n+\alpha_i+1)!}$ by a binomial identity. Summing over $i \in I$ finishes the proof. □

Remark A.12. *With the same proof, the analogous result for $\hat{M}_C$ and $\hat{P}_g$ holds as well.*

The only other term where differentiating is not straight forward is M_B. By using the chain rule n times we obtain

$$\begin{aligned}
&\frac{\partial^{n+|\alpha_I|}}{\partial u^n \prod_{i\in I}\partial z_i^{\alpha_i}} M_B(g; y, u, z_I) \qquad (41)\\
&= \frac{\partial^{n+|\alpha_I|}}{\partial u^n \prod_{i\in I}\partial z_i^{\alpha_i}} \left(yu^2 \left(z_{i_0} \frac{\partial}{\partial z_{i_0}} P_{g-1}\left(y, u, z_{I\cup\{i_0\}}\right)\right)\Big|_{z_{i_0}=u}\right)\\
&= y \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\partial^{n-k+|\alpha_I|}}{\partial u^{n-k} \prod_{i\in I}\partial z_i^{\alpha_i}} \frac{\partial^{k+1}}{\partial z_{i_0}^{k+1}} \left(u^3 P_{g-1}\left(y, u, z_{I\cup\{i_0\}}\right)\right)\right)\Bigg|_{z_{i_0}=u}.
\end{aligned}$$

Using (40) and (41) we can now determine the dominant terms of the derivatives of S_g and P_g (and analogously the derivatives of $\hat{S}_g$ and $\hat{P}_g$).

Theorem A.13. *Let $\alpha_i \in \mathbb{N}_0$, $i \in I$, and $|\alpha_I| := \sum \alpha_i$. If $(g, I) \neq (0, \emptyset)$, then*

$$\frac{\partial^{|\alpha_I|}}{\prod \partial z_i^{\alpha_i}} S_g(y, z_I)\Big|_{z_I=f(y)} \cong a_0 + c_g\left(1-\rho_S^{-1}y\right)^{e_1} + O\left(\left(1-\rho_S^{-1}y\right)^{e_1+1/4}\right), \qquad (42)$$

where a_0 and $c_g = c_g(\alpha_i, i \in I)$ are positive constants and

$$e_1 = -\frac{5g}{2} - \frac{5|I|}{4} - \frac{|\alpha_I|}{2} + \frac{3}{2}.$$

$\frac{\partial^n}{\partial u^n} P_0(y, u)\big|_{u=f(y)}$ *is given as in Lemma A.2. If $(g, I, n) \neq (0, \emptyset, 0)$, then*

$$\frac{\partial^{n+|\alpha_I|}}{\partial u^n \prod \partial z_i^{\alpha_i}} P_g(y, u, z_I)\Big|_{u=z_I=f(y)} \cong c\left(1-\rho_S^{-1}y\right)^{e_2} + O\left(\left(1-\rho_S^{-1}y\right)^{e_2+1/4}\right), \qquad (43)$$

where $c = c(g, |I|, n, |\alpha_I|)$ is a positive constant and

$$e_2 = e_1 - \frac{n}{2} - \frac{3}{4}.$$

Proof. We show this by induction on $(g, |I|, n)$ in lexicographic order. Lemma A.2 shows that (43) is true for $(g, I) = (0, \emptyset)$ and $n > 0$. Note that $|\alpha_I| = 0$ for $I = \emptyset$.

Suppose now that (43) is true for all $(g, |I|, n) < (g_0, |I_0|, 0)$ and (42) is true for all $(g, |I|) < (g_0, |I_0|)$ with $(g, I) \neq (0, \emptyset)$. We first prove that (42) holds for $(g_0, |I_0|)$. By multiplying (34) by u and applying Lemmas A.3, A.5, A.7 and A.9 we obtain

$$u(1-a-d)P_{g_0}(y,u,z_{I_0}) = u(M_A + M_B + M_C) - uS_{g_0}(y, z_{I_0})P_0(y,u) - uE,$$

where $E = E_B + E_C + E_D$. The term

$$u(1-a-d) = u - 2yu^3P_0(y,u) - uS_0(y) - y + y^2u^2$$

is equal to $-Q(y,u)$ in (30) and thus

$$-Q(y,u)P_{g_0}(y,u,z_{I_0}) = u(M_A + M_B + M_C) - uS_{g_0}(y, z_{I_0})P_0(y,u) - uE. \tag{44}$$

Therefore, the left-hand side is zero when replacing u by $f(y)$. As this factor is independent of z_{I_0}, this does also hold when differentiating the equation α_i times with respect to z_i. Thus we obtain

$$uP_0(y,u)\frac{\partial^{|\alpha_{I_0}|}S_{g_0}(y,z_{I_0})}{\prod \partial z_i^{\alpha_i}}\bigg|_{u=z_{I_0}=f(y)} = u\frac{\partial^{|\alpha_{I_0}|}(M_A + M_B + M_C - E)}{\prod \partial z_i^{\alpha_i}}\bigg|_{u=z_{I_0}=f(y)}.$$

By inspecting the formulas for M_A to E_D in Lemmas A.3, A.5, A.7 and A.9, one sees that all occurring terms are lexicographically smaller than $(g_0, |I_0|, 0)$,and the induction hypothesis can thus be used. Inspection of the exponents of $(1 - \rho_S^{-1}y)$ in all those terms shows the following.

M_A: The summands have the form

$$\frac{\partial^{|\alpha_{I_0}|}}{\prod \partial z_i^{\alpha_i}}\left(P_t\left(y,u,z_J\right)P_{g_0-t}\left(y,u,z_{I_0\setminus J}\right)\right)$$

for $(t, J) \neq (0, \emptyset)$ and $(t, J) \neq (g_0, I_0)$. Thus, for $g_0 + |I_0| \leqslant 1$, we have $M_A = 0$. For all other values of $(g_0, |I_0|)$, each summand is of the form $c_A(1-\rho_S^{-1}y)^{m_A} + O\left((1-\rho_S^{-1}y)^{m_A+1/4}\right)$ with

$$m_A = -\frac{5t}{2} - \frac{5|J|}{4} - \frac{|\alpha_J|}{2} + \frac{3}{4} - \frac{5(g_0-t)}{2} - \frac{5|I_0 \setminus J|}{4} - \frac{|\alpha_{I_0\setminus J}|}{2} + \frac{3}{4} = e_1$$

by induction. Furthermore, all coefficients are positive by induction.

M_B: We have $M_B = yu^2\delta_{z_{i_0}}\left(P_{g_0-1}(y,u,z_{I\cup\{i_0\}})\right)|_{z_{i_0}=u}$. Thus, for $g_0 = 0$ we have $M_B = 0$ and otherwise $M_B = c_B(1-\rho_S^{-1}y)^{m_B} + O\left((1-\rho_S^{-1}y)^{m_B+1/4}\right)$ with

$$m_B = -\frac{5(g_0-1)}{2} - \frac{5|I_0 \cup \{i_0\}|}{4} - \frac{|\alpha_{I_0}|+1}{2} + \frac{3}{4} = e_1$$

by induction. Again, the coefficient is positive by induction.

M_C: We determine the expression for M_C by Proposition A.11. For $g_0 = 0$ and $|I_0| = 1$ we have $M_C \cong c_{C,1}(1-\rho_S^{-1}y)^{1/4} + O\left((1-\rho_S^{-1}y)^{1/2}\right)$, which is of the desired order, since $e_1 = 1/4$ in this case. For all other (g_0, I_0), induction yields that $M_C = c_{C,2}(1-\rho_S^{-1}y)^{m_C} + O\left((1-\rho_S^{-1}y)^{m_C+1/4}\right)$ with

$$m_C = -\frac{5g_0}{2} - \frac{5(|I_0|-1)}{4} - \frac{|\alpha_{I_0}|}{2} - \frac{1}{2} + \frac{3}{4} = e_1\,.$$

Like in the previous cases, the coefficient is positive by induction.

E_B: The function E_B is bounded from above by $(yu^2+1)\,\delta_u\left(P_{g_0-1}(y,u,z_{I_0})\right)$. For $g_0 = 0$, we thus have $E_B = 0$ and otherwise $E_B = O\left((1-\rho_S^{-1}y)^{e_B}\right)$ with

$$e_B = -\frac{5(g_0-1)}{2} - \frac{5|I_0|}{4} - \frac{|\alpha_{I_0}|}{2} - \frac{1}{2} + \frac{3}{4} \geqslant e_1 + \frac{1}{4}\,.$$

E_C: The first summand in the expression of E_C from Lemma A.7 is 0 if $g_0 = 0$ and otherwise $O((1-\rho_S^{-1}y)^{e_{C,1}})$ with

$$e_{C,1} = -\frac{5(g_0-1)}{2} - \frac{5|I_0|}{4} - \frac{|\alpha_{I_0}|+1}{2} + \frac{3}{4} \geqslant e_1 + \frac{1}{4}\,.$$

The second summand is $a_E + O\left((1-\rho_S^{-1}y)^{1/2}\right)$ if $g_0 = 0$ and $|I_0| = 1$. Suppose $(g_0, |I_0|) \neq (0,1)$. Then every term

$$(1+yuz_i)\,P_t(y,u,z_J)P_{g-t}(y,z_i,z_{I\setminus(J\cup\{i\})})$$

with $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g_0, I_0\setminus\{i\})$ is $O((1-\rho_S^{-1}y)^{e_{C,2}})$ with

$$\begin{aligned} e_{C,2} &= -\frac{5t}{2} - \frac{5|J|}{4} - \frac{|\alpha_J|}{2} + \frac{3}{4} - \frac{5(g_0-t)}{2} - \frac{5(|I_0\setminus J|-1)}{4} - \frac{|\alpha_{I_0\setminus J}|-\alpha_i}{2} + \frac{3}{4} \\ &\geqslant e_1 + \frac{1}{4} \end{aligned}$$

by induction. The corresponding terms for $(t,J) = (0,\emptyset)$ and $(t,J) = (g_0, I_0\setminus\{i\})$ are $O\left((1-\rho_S^{-1}y)^{e_{C,3}}\right)$ with

$$e_{C,3} = -\frac{5g_0}{2} - \frac{5(|I_0|-1)}{4} - \frac{|\alpha_{I_0}|-\alpha_i}{2} + \frac{3}{4} \geqslant e_1 + \frac{1}{4}\,.$$

In total, we have $E_C = O\left((1-\rho_S^{-1}y)^{e_1+1/4}\right)$.

E_D: The first summand in the expression of E_D from Lemma A.9 is 0 if $g_0 + |I_0| \leqslant 1$ and otherwise each of its summands is $O\left((1-\rho_S^{-1}y)^{e_{D,1}}\right)$ with

$$\begin{aligned} e_{D,1} &= -\frac{5t}{2} - \frac{5|J|}{4} - \frac{|\alpha_J|}{2} + \frac{3}{2} - \frac{5(g_0-t)}{2} - \frac{5|I_0\setminus J|}{4} - \frac{|\alpha_{I_0\setminus J}|}{2} + \frac{3}{4} \\ &\geqslant e_1 + \frac{1}{4} \end{aligned}$$

by induction. The second term is 0 for $g_0 = 0$ and $O\left((1-\rho_S^{-1}y)^{e_{D,2}}\right)$ otherwise with

$$e_{D,2} = -\frac{5(g_0-1)}{2} - \frac{5(|I_0|+1)}{4} - \frac{|\alpha_{I_0}|}{2} + \frac{3}{4} \geqslant e_1 + \frac{1}{4}\,.$$

Note that for $\hat{P}_g$ the only difference is in the formulas of $\hat{E}_B$ and $\hat{E}_C$, both of which satisfy the same inequalities as E_B and E_C above. Thus the following conclusions also hold for $\hat{P}_g$.

Combining these results, we have proved (42), where $a_0 = a_M - a_E$ for $g_0 = 0$ and $|I_0| = 1$. As this constant is the value of the generating function $S_0(y, z_{I_0})|_{z_{I_0}=f(y)}$ at its singularity ρ_S, a_0 is positive. For $|\alpha_{I_0}| > 0$ or $(g_0, |I_0|) \neq (0, 1)$, the exponent e_1 is negative and (42) is thus true with the same value for a_0. Finally, c_g is positive, since it is the sum of positive numbers.

To prove (43), recall that we assume that (43) is true for $(g, |I|, n) < (g_0, |I_0|, 0)$ and we have already shown that (42) is true for $(g, |I|) \leqslant (g_0, |I_0|)$. Let $n_0 \in \mathbb{N}_0$ and assume that (43) is also true for $(g_0, |I_0|, n)$ with $n < n_0$. Consider the derivative $\frac{\partial^{n+1}}{\partial u^{n+1}}$ of (44) and set $u = f(y)$; as $Q(y, f(y)) = 0$, this yields

$$- \sum_{k=0}^{n} \binom{n+1}{k} \frac{\partial^k}{\partial u^k} P_{g_0}(y, u, z_{I_0}) \frac{\partial^{n+1-k}}{\partial u^{n+1-k}} Q(y, u) \Big|_{u=z_{I_0}=f(y)}$$

for the left-hand side of (44). For the derivatives of M_A, M_B, and M_C, we obtain

$$\begin{aligned} M_A &= c_A(1 - \rho_S^{-1}y)^{m_A} + O\left((1 - \rho_S^{-1}y)^{m_A+1/4}\right), \\ M_B &= c_B(1 - \rho_S^{-1}y)^{m_B} + O\left((1 - \rho_S^{-1}y)^{m_B+1/4}\right), \\ M_C &= c_C(1 - \rho_S^{-1}y)^{m_C} + O\left((1 - \rho_S^{-1}y)^{m_C+1/4}\right) \end{aligned}$$

with positive constants c_A, c_B, c_C and $m_A = m_B = m_C = e_1 - \frac{n+1}{2} = e_2 + \frac{1}{4}$. For the derivatives of E_B, E_C, and E_D, the exponents $e_B, e_{C,1}, \dots$ in the considerations above reduces by $\frac{n+1}{2}$ as well. By (33) and the induction hypothesis, each term

$$\frac{\partial^k}{\partial u^k} P_{g_0}(y, u, z_{I_0}) \frac{\partial^{n+1-k}}{\partial u^{n+1-k}} Q(y, u) \Big|_{u=z_{I_0}=f(y)}$$

for $k < n$ is of the form $\bar{c}(k)(1-\rho_S^{-1}y)^{e_1-(n+1)/2} + O\left((1 - \rho_S^{-1}y)^{e_1-(n+1)/2+1/4}\right)$ with $\bar{c}(k) > 0$. Since $\frac{\partial}{\partial u} Q(y, u)|_{u=f(y)} = \bar{c}(1-\rho_S^{-1}y)^{1/4} + O\left((1 - \rho_S^{-1}y)^{1/2}\right)$ with $\bar{c} < 0$, (43) follows. □

An analogous proof yields the corresponding result for $\hat{\mathcal{S}}_g$ and $\hat{\mathcal{P}}_g$, with identical constants a_0, c_g, c, e_1, e_2. By Lemma A.2 and setting $I = \emptyset$ in (42) we deduce Proposition 3.2 and, from the corresponding result for $\hat{S}_g$, also Proposition 3.3.