Fixed Points of the Evacuation of Maximal Chains on Fuss Shapes

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Abstract

For a partition λ of an integer, we associate λ with a slender poset P the Hasse diagram of which resembles the Ferrers diagram of λ. Let X be the set of maximal chains of P. We consider Stanley's involution $\epsilon : X \to X$, which is extended from Schützenberger's evacuation on linear extensions of a finite poset. We present an explicit characterization of the fixed points of the map $\epsilon : X \to X$ when λ is a stretched staircase or a rectangular shape. Unexpectedly, the fixed points have a nice structure, i.e., a fixed point can be decomposed in half into two chains such that the first half and the second half are the evacuation of each other. As a consequence, we prove anew Stembridge's $q = -1$ phenomenon for the maximal chains of P under the involution ϵ for the restricted shapes.

Keywords: promotion; evacuation; slender posets; linear extensions; maximal chains; cyclic sieving phenomenon

1 Introduction

1.1 Schützenberger's evacuation

Promotion and evacuation are bijections on the set of linear extensions of a finite poset. It is well known that the RSK algorithm establishes a bijection between the permutations of $\{1, 2, \ldots, n\}$ and ordered pairs of n-cell standard Young tableaux of the same shape [10, pp. 320-321]. Evacuation was originally devised by Schützenberger to describe this bijection without involving the RSK algorithm [6]. Later Schützenberger extended the definition of evacuation to the linear extensions of a finite poset, described in terms of an operation called promotion [7]. One of the fundamental properties Schützenberger proved is that the evacuation is an involution.

Schützenbeger's work was simplified by Haiman [3], whose idea is to express linear extensions as *words* and then define the promotion and evacuation in terms of elementary operators on these words. For a finite poset P of p elements, a linear extension $f : P \to \{1, \ldots, p\}$ of P can be expressed as the word $u_1u_2 \ldots u_p$, where $u_i = f^{-1}(i) \in P$ for $1 \leqslant i \leqslant p$. Let $\mathcal{L}(P)$ be the set of linear extensions. For $1 \leqslant i \leqslant p-1$, define operators $\tau_i : \mathcal{L}(P) \to \mathcal{L}(P)$ by

$$\tau_i(u_1u_2 \ldots u_p) = \begin{cases} u_1u_2 \ldots u_p, & \text{if } u_i \text{ and } u_{i+1} \text{ are comparable in } P \\ u_1u_2 \ldots u_{i+1}u_i \ldots u_p, & \text{otherwise.} \end{cases}$$

Clearly, τ_i is a bijection and τ_i's satisfy the following relations.

$$\begin{aligned} \tau_i^2 &= 1, \quad 1 \leqslant i \leqslant p-1 \\ \tau_i\tau_j &= \tau_j\tau_i, \quad \text{if } |i-j| > 1. \end{aligned} \tag{1}$$

Then Schützenbeger's promotion is in fact the operator $\delta := \tau_1\tau_2 \ldots \tau_{p-1}$ and evacuation is the operator $\epsilon := \tau_1\tau_2 \ldots \tau_{p-1} \cdot \tau_1\tau_2 \ldots \tau_{p-2} \cdots \tau_1\tau_2 \cdot \tau_1$.

1.2 Stanley's point of view

Stanley noticed that the properties of promotion and evacuation depend only on the relations of τ_i's defined in Eq. (1) and hence the theory of promotion and evacuation can be extended to a more general context.

It is known that the set $J(P)$ of all order ideals of P, ordered by inclusion, is a finite distributive lattice of rank p and that there is a bijection between the maximal chains $\varnothing = I_0 \subset I_1 \subset \cdots \subset I_p = P$ of $J(P)$ and the linear extensions of P [9, §3.5], associated with this chain the linear extension $f : P \to \{1, \ldots, p\}$ defined by $f(t) = i$ if $t \in I_i - I_{i-1}$. Moreover, every interval of rank 2 of $J(P)$ contains either three or four elements. Stanley [8] described the promotion and evacuation on maximal chains of $J(P)$ by extending the definition of τ_i's as follows. For a maximal chain $C : \varnothing = I_0 \subset I_1 \subset \cdots \subset I_p = P$ of $J(P)$, either the interval $[I_{i-1}, I_{i+1}]$ contains the three elements I_{i-1}, I_i, I_{i+1} or there is exactly one other element I' in this interval, i.e., $I_{i-1} \subset I' \subset I_{i+1}$. In the former case define $C\tau_i = C$; in the latter case $C\tau_i$ is obtained from C by replacing I_i with I'.

As pointed out by Stanley, the same definition of τ_i works for any finite graded poset with a unique minimal element $\hat{0}$, unique maximal element $\hat{1}$ and the property that every interval of rank 2 contains either three or four elements, called *slender posets*. He also mentioned some examples of slender posets, such as intervals in the Bruhat order of Coxeter groups and face posets of regular CW-spheres.

1.3 Our work

In this paper we consider some families of slender posets the (tilted) Hasse diagrams of which resemble Ferrers diagrams of partitions of an integer, elaborate the properties of the evacuation of maximal chains of the posets and characterize the maximal chains fixed under evacuation.

For a partition $\lambda = (m_1, \dots, m_k)$ of n, denoted by $\lambda \vdash n$, we associate λ with a graded poset $(\mathcal{P}, \leqslant)$ on a $\mathcal{P}$ of lattice points in the plane $\mathbb{Z} \times \mathbb{Z}$ define as follows (sometimes we denote the relation by $\leqslant_P$ when there is a possibility of confusion).

(i) The minimum is $\hat{0} = (0, 0)$ and the maximum is $\hat{1} = (m_1, k)$.

(ii) Two points (x_1, y_1), $(x_2, y_2) \in \mathcal{P}$ are comparable $(x_1, y_1) \leqslant_P (x_2, y_2)$ if and only if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

(iii) The Hasse diagram of $(\mathcal{P}, \leqslant)$ comprises n unit squares in the form of Ferrers diagram of λ.

For example, the poset associated with $\lambda = (2, 1) \vdash 3$ is shown in Figure 1, with the set of points $\mathcal{P} = \{(x, y) : 0 \leqslant x, y \leqslant 2, y \geqslant x - 1\}$.

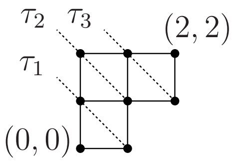


Figure 1: The poset associated with the partition $\lambda = (2, 1) \vdash 3$.

A maximal chain of $(\mathcal{P}, \leqslant)$ forms a lattice path from $\hat{0}$ to $\hat{1}$ using north step $(1, 0)$ and east step $(0, 1)$ staying within the Hasse diagram. Let N and E denote a north step and an east step, respectively. Let p be the rank of $(\mathcal{P}, \leqslant)$ and let $\mathcal{X}$ be the set of maximal chains of $(\mathcal{P}, \leqslant)$. For convenience, members of $\mathcal{X}$ are written as words on the alphabet $\{\mathsf{N}, \mathsf{E}\}$. For a maximal chain $C = z_1 \cdots z_p \in \mathcal{X}$ with $z_j \in \{\mathsf{N}, \mathsf{E}\}$ $(1 \leqslant j \leqslant p)$, the evacuation of C is another maximal chain in $\mathcal{X}$, denoted by $C\epsilon$. The elementary operators $\tau_i : \mathcal{X} \to \mathcal{X}$ $(1 \leqslant i \leqslant p - 1)$ that generate evacuation ϵ can be equivalently defined as follows. The chain $C\tau_i$ is obtained from C by interchanging the steps z_i and z_{i+1} if the resulting chain remains to be a member of $\mathcal{X}$; and $C\tau_i = C$ otherwise. For example, for the partition $\lambda = (2, 1) \vdash 3$, the associated poset $(\mathcal{P}, \leqslant)$ is of rank 4 with three elementary operators τ_1, τ_2, τ_3 (see Figure 1) and the operator $\epsilon = \tau_1\tau_2\tau_3 \cdot \tau_1\tau_2 \cdot \tau_1$. For the maximal chain $C = \mathsf{NEEN}$ of $(\mathcal{P}, \leqslant)$, the evacuation of C, $C\epsilon = \mathsf{NNEE}$, is obtained through the process shown in Figure 2.

By a fundamental property of evacuation obtained by Schützenberger [6, 7] (see also the proof given by Stanley [8, Lemma 2.2]), the operator ϵ establishes an involution on $\mathcal{X}$. The main result in this paper is that we obtain an explicit characterization of the fixed

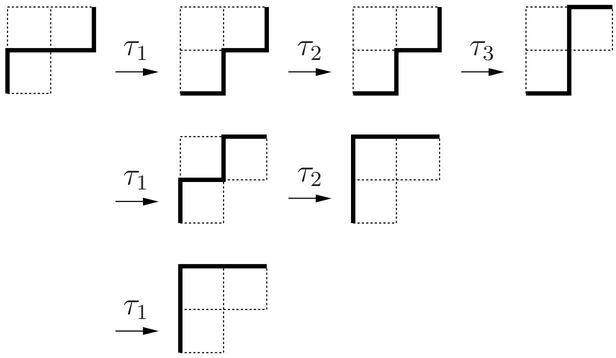


Figure 2: The process of evacuation of a maximal chain of the poset for $\lambda = (2, 1)$.

points of the map $\epsilon : \mathcal{X} \to \mathcal{X}$ when $\lambda = (n^s, (n-1)^s, \ldots, 1^s)$ is a stretched staircase for positive integers s, n or λ is a rectangular shape (Theorem 5.4 and Theorem 6.1). As a consequence, we prove anew Stembridge's $q = -1$ phenomenon for the maximal chains of $(\mathcal{P}, \leqslant)$ under the involution ϵ for the restricted shapes.

1.4 Cyclic sieving phenomenon

The cyclic sieving phenomenon is an enumerative property that the orbit structures of a cyclic action on a set X of combinatorial objects are encoded in an enumerator of the set X. More precisely, a triple $(X, X(q), \langle c \rangle)$ consisting of a finite set X, a polynomial $X(q) \in \mathbb{Z}[q]$, and a cyclic group $\langle c \rangle$, generated by an element c of order n, acting on X is said to exhibit the *cyclic sieving phenomenon* (CSP) if for all integers d, the number of elements fixed by c^d equals the evaluation $X(\zeta^d)$, where $\zeta = e^{\frac{2\pi i}{n}}$ is the root of unity of order n. The CSP was first defined by Reiner, Stanton and White [4]. The special case when $\langle c \rangle$ has order 2 was also known as Stembridge's $q = -1$ phenomenon.

Stanley presented an instance of CSP for the linear extensions of a finite poset P under evacuation. For a linear extension $\omega = u_1 \cdots u_p \in \mathcal{L}(P)$, the *descent set* $\mathsf{Des}(\omega)$ of ω is defined by $\mathsf{Des}(\omega) = \{i : u_i > u_{i+1}, 1 \leqslant i \leqslant p\}$. The CSP involves the enumerator $W(q)$ of $\mathcal{L}(P)$ respecting the *comajor index* $\mathsf{comaj}(\omega)$,

$$W(q) = \sum_{\omega \in \mathcal{L}(P)} q^{\mathsf{comaj}(\omega)}, \tag{2}$$

where $\mathsf{comaj}(\omega) = \sum_{i \in \mathsf{Des}(\omega)} (p - i)$. He proved that $W(-1)$ coincides with the number of self-evacuating linear extensions of P, i.e., $\omega\epsilon = \omega$, making use of another family of linear extensions called *domino linear extensions* as the intermediate stage [8, Theorem 3.1]. For the poset $(\mathcal{P}, \leqslant)$ associated with $\lambda = (n^s, (n-1)^s, \ldots, 1^s)$ or $\lambda = (n^{sn})$, we give an alternative proof of the CSP result in terms of the maximal chains of $(\mathcal{P}, \leqslant)$ under the action of evacuation $\langle \epsilon \rangle$.

As a q-polynomial for our CSP, we consider the enumerator of the maximal chains of $(\mathcal{P}, \leqslant)$ with respect to the statistic area, the number of unit squares above a maximal chain

$C \in \mathcal{X}$ within the Hasse diagram of $(\mathcal{P}, \leqslant)$. Let $X(q) = \sum_{C\in\mathcal{X}} q^{\mathsf{area}(C)}$. For example, with the partition $\lambda = (2,1) \vdash 3$, the associated poset contains five maximal chains, shown in Figure 3, with area-enumerator $X(q) = 1 + q + 2q^2 + q^3$. Note that $X(-1) = 1$ and there is exactly one maximal chain fixed by ϵ. Hence $(\mathcal{X}, X(q), \langle\epsilon\rangle)$ exhibits CSP. However, the map ϵ does not necessarily reverse the parity of the statistic area. As shown in Figure 3, the maximal chains in each orbit have area of the same parity.

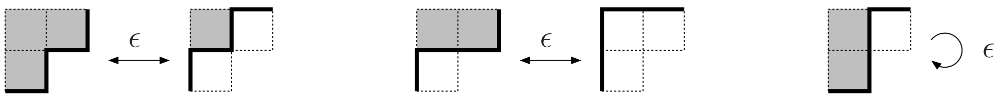


Figure 3: The three orbits of the maximal chains of the poset for $\lambda = (2,1)$ under evacuation.

With the area-enumerator $X(q)$, a partition λ of an integer is called a *good shape* if the triple $(\mathcal{X}, X(q), \langle\epsilon\rangle)$ of the poset associated with λ exhibits CSP. One can check that the partition $\lambda = (2,2,1) \vdash 5$ is not a good shape. A natural question is that what kind of partitions is a good shape? In the context of Coxeter combinatorics, there are two families of fundamental shapes, namely, Fuss shapes of type A and type B. A *Fuss shape of type A* is a stretched staircase defined by the partition $\lambda = (n^s, (n-1)^s, \dots, 1^s)$ for positive integers s, n, and a *Fuss shape of type B* a rectangular shape defined by the partition $\lambda = (n^{sn})$. As a consequence of the main result, Fuss shapes of type A and rectangular shapes, including Fuss shapes of type B, are good shapes.

1.5 The structure of this paper

The proof for Fuss shapes of type A occupies a large portion of this paper. In section 2, we evaluate $X(q)$ at $q = -1$. Since $X(q)$ has no closed form, the evaluation makes use of the generating function of an alternative expression of $X(q)$. Sections 3, 4 and 5 are devoted to characterize and enumerate the fixed points of the map $\epsilon : \mathcal{X} \to \mathcal{X}$. The characterization of the fixed points is quite neat but the proof is relatively sophisticated. Subject to a parity-condition, the maximal chains $C \in \mathcal{X}$ fixed by evacuation can be factorized in half as $C = C_1C_2$ such that C_2 is the evacuation of C_1 and vice versa. Some interesting and crucial points of the proof are listed below.

(i) We discover an interesting factor-swapping property of the evacuation of $C \in \mathcal{X}$ (see Proposition 3.7), which leads to a factorization of C into building blocks.

(ii) We come up with the notion of primitive factorization of $C \in \mathcal{X}$, which enables a characterization of $C\epsilon$ (see Theorem 4.5).

(iii) The characterization of the evacuation of primitive blocks in Proposition 4.3 is critical, which enables the determination of the primitive blocks fixed by evacuation (see Proposition 5.1) and the fixed points of the map $\epsilon : \mathcal{X} \to \mathcal{X}$ (see Theorem 5.4).

The proof for rectangular shapes is given in section 6. Concluding remarks and some problems for further studies are given in section 7.

2 Evaluation of $X(-1)$ for posets of stretched staircases

For positive integers s and n, the *Fuss-Catalan number*

$$\frac{1}{sn+1}\binom{sn+n}{n}$$

counts the number of lattice paths, called *s-Dyck paths* of *width* n, from the origin $(0,0)$ to the point (n, sn) using N and E steps staying weakly above the line $y = sx$. When $s = 1$ they are ordinary Dyck paths. Let $\mathcal{F}_n^{(s)}$ be the set of s-Dyck paths of width n. We consider the enumerator of the paths $\pi \in \mathcal{F}_n^{(s)}$ with respect to the number $\alpha(\pi)$ of unit squares enclosed by π and the line $y = sx$. Define

$$f_n^{(s)}(q) = \sum_{\pi\in\mathcal{F}_n^{(s)}} q^{\alpha(\pi)}. \tag{3}$$

The case $s = 1$, $f_n^{(1)}(q)$, was considered by Carlitz and Riordan [1], and Fürlinger and Hofbauer [2]. There is no known explicit form for $f_n^{(s)}(q)$.

For two integers $m < n$, let $[m, n] = \{m, m+1, \ldots, n\}$. For $\lambda = (n^s, (n-1)^s, \ldots, 1^s) \vdash s\binom{n+1}{2}$, let $(\mathcal{P}_n^{(s)}, \leqslant)$ denote the poset associated with λ defined on the set of points

$$\mathcal{P}_n^{(s)} = \{(x, y) : x \in [0, n], y \in [0, sn], y \geqslant sx - s\}.$$

Note that $(\mathcal{P}_n^{(s)}, \leqslant)$ is of rank $(s + 1)n$. A maximal chain of $(\mathcal{P}_n^{(s)}, \leqslant)$ forms a lattice path from the origin to the point (n, sn) using N and E steps staying weakly above the line $y = sx - s$. Let $\mathcal{X}_n^{(s)}$ denote the set of all maximal chains of $(\mathcal{P}_n^{(s)}, \leqslant)$. Note that $|\mathcal{X}_n^{(s)}| = |\mathcal{F}_{n+1}^{(s)}|$ and the area-enumerator of $\mathcal{X}_n^{(s)}$ is

$$X(q) = q^{s\binom{n+1}{2}} f_{n+1}^{(s)}(q^{-1}) \tag{4}$$

since a maximal chain of $(\mathcal{P}_n^{(s)}, \leqslant)$ is simply a s-Dyck path of width $n+1$ with the initial s steps and the terminal step removed. Sometimes members of $\mathcal{X}_n^{(s)}$ are also called *truncated s-Dyck paths* of *width n*.

Let $p = (s + 1)n$, the rank of $\mathcal{P}_n^{(s)}$. Let $\epsilon_n : \mathcal{X}_n^{(s)} \to \mathcal{X}_n^{(s)}$ denote the operator of evacuation on $\mathcal{X}_n^{(s)}$, which is defined as

$$\epsilon_n = \tau_1 \cdots \tau_{p-1} \cdot \tau_1 \cdots \tau_{p-2} \cdots \tau_1\tau_2 \cdot \tau_1. \tag{5}$$

Let $\langle\epsilon_n\rangle$ be the group of order 2 generated by ϵ_n. The CSP result is stated as follows.

Theorem 2.1. *For positive integers s and n, let $(\mathcal{P}_n^{(s)}, \leqslant)$ be the poset associated with $\lambda = (n^s, (n-1)^s, \ldots, 1^s)$. Let $\mathcal{X}_n^{(s)}$ be the set of maximal chains of the poset $(\mathcal{P}_n^{(s)}, \leqslant)$. Let $X(q)$ be the polynomial defined in Eq. (4). Let the group $\langle\epsilon_n\rangle$, generated by the operator ϵ_n of evacuation, act on $\mathcal{X}_n^{(s)}$. Then $(\mathcal{X}_n^{(s)}, X(q), \langle\epsilon_n\rangle)$ exhibits the cyclic sieving phenomenon.*

2.1 Evaluation of $X(-1)$

Define the generating function for $\{f_n^{(s)}(q)\}_{n\geqslant 0}$ as

$$F(x,q)=\sum_{n\geqslant 0} f_n^{(s)}(q)x^n.$$

Lemma 2.2. *The polynomial $F(x,q)$ satisfies the equation*

$$F(x,q)=1+xF(x,q)F(qx,q)\cdots F(q^s x,q).$$

Proof. For a s-Dyck path $\pi\in\mathcal{F}_n^{(s)}$, there is a standard factorization of π into s-Dyck paths $\pi_1,\cdots,\pi_{s+1}$, with respect to the first east step E returning to the line $y=sx$, as $\pi=N_1\pi_1\cdots N_s\pi_s E\pi_{s+1}$, where N_i is the last north step before E going from the line $y=sx+i-1$ to the line $y=sx+i$ for $1\leqslant i\leqslant s$. We observe that

$$f_n^{(s)}(q)=\sum_{k_1+\cdots+k_{s+1}=n-1} q^{k_1+2k_2+\cdots+sk_s} f_{k_1}^{(s)}(q)\cdots f_{k_s}^{(s)}(q) f_{k_{s+1}}^{(s)}(q),$$

with $f_0^{(s)}(q)=1$. The assertion follows from multiplying the equation by x^n and summing over $n\geqslant 0$. $\square$

A *s-ballot path* is a lattice path from the origin to some point above the line $y=sx$ using N and E steps staying weakly above the line $y=sx$. The enumeration of the following s-ballot paths will be useful for the evaluation $X(-1)$ and the enumeration of maximal chains of $(\mathcal{P}_n^{(s)},\leqslant)$ fixed by the operator ϵ_n.

Proposition 2.3. *For any nonnegative integer h, the number of s-ballot paths from the original to the point $(n,sn+h)$ is*

$$\frac{h+1}{sn+h+1}\binom{sn+n+h}{n}.$$

Proof. Let $G=G(x)=F(x,1)$, which is the generating function for the number of s-Dyck paths of width $n\geqslant 0$. By Lemma 2.2, G satisfies the equation

$$G=1+xG^{s+1}. \tag{6}$$

Let $r_{n;h}$ be the number of s-ballot paths from the original to the point $(n,sn+h)$. By a standard factorization, such a path can be factorized into s-Dyck paths $\pi_0,\ldots,\pi_h$ as $\pi_0N_1\pi_1\cdots N_h\pi_h$, where N_i is the last north step from the line $y=sx+i-1$ to the line $y=sx+i$ for $1\leqslant i\leqslant h$. By an argument similar to the proof of Lemma 2.2, we observe that the generating function for $\{r_{n;h}\}_{n\geqslant 0}$ is G^{h+1}. Setting $R=G-1$, we have $R = x(1+R)^{s+1}$. By Lagrange inversion formula [10, Corollary 5.4.3] with $H(u) = (1+u)^{h+1}$, for $n \geqslant 1$ we have

$$\begin{aligned}[x^n]G^{h+1} = [x^n](1+R)^{h+1} &= \frac{1}{n}[u^{n-1}]H'(u)(1+u)^{(s+1)n} \\ &= \frac{h+1}{n}[u^{n-1}](1+u)^{(s+1)n+h} \\ &= \frac{h+1}{n}\binom{(s+1)n+h}{n-1},\end{aligned}$$

as required. $\square$

Proposition 2.4. *The evaluation $X(-1)$ is given as follows.*

(i) *For s odd,*

$$X(-1) = \begin{cases} \dfrac{s+1}{sn+s+1}\dbinom{\frac{(s+1)n+s-1}{2}}{\frac{n}{2}} & \text{if } n \text{ is even} \\[2ex] 0 & \text{if } n \text{ is odd.} \end{cases}$$

(ii) *For s even,*

$$X(-1) = \begin{cases} \dfrac{s+2}{sn+s+2}\dbinom{\frac{(s+1)n+s}{2}}{\frac{n}{2}} & \text{if } n \text{ is even} \\[2ex] \dfrac{2}{s(n+1)+2}\dbinom{\frac{(s+1)(n+1)}{2}}{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, we evaluate $f_{n+1}^{(s)}(-1) = [x^{n+1}]F(x,-1)$. Let $P = F(x,-1)$ and $Q = F(-x,-1)$. We discuss the evaluation according to the parity of s.

Case I. For s odd, say $s = 2t+1$. By Lemma 2.2, we have $P = 1 + x(PQ)^{t+1}$ and $Q = 1 - x(PQ)^{t+1}$. Then $PQ = 1 - x^2(PQ)^{s+1}$, consisting only of the even degree terms. Comparing this equation with Eq. (6), we have

$$PQ = G(-x^2),$$

where $G(x) = F(x,1)$. Since $F(x,-1) = P = 1 + x(PQ)^{t+1} = 1 + x(PQ)^{\frac{s+1}{2}}$, $f_{n+1}^{(s)}(-1) = [x^n](PQ)^{\frac{s+1}{2}}$. Thus $f_{n+1}^{(s)}(-1) = 0$ if n is odd, and $f_{n+1}^{(s)}(-1) = [x^n]G(-x^2)^{\frac{s+1}{2}}$ otherwise. By the proof of Proposition 2.3, for n even we have

$$f_{n+1}^{(s)}(-1) = (-1)^{\frac{n}{2}}\frac{s+1}{sn+s+1}\binom{\frac{(s+1)n+s-1}{2}}{\frac{n}{2}}.$$

By Eq. (4), the assertion (i) follows from $X(-1) = (-1)^{\frac{sn(n+1)}{2}} f_{n+1}^{(s)}(-1)$.

Case II. For s even, say $s = 2t$. By Lemma 2.2, we have $P = 1 + xP^{t+1}Q^t$ and $Q = 1 - xP^tQ^{t+1}$. Then $P = (1 - x(PQ)^t)^{-1}$ and $Q = (1 + x(PQ)^t)^{-1}$. It follows that $PQ = 1 + x^2(PQ)^{s+1}$, consisting only of the even degree terms. Comparing this equation with Eq. (6), we have

$$PQ = G(x^2).$$

Moreover, multiplying both sides of $Q = 1 - xP^tQ^{t+1}$ by P, we have

$$F(x, -1) = P = PQ + x(PQ)^{\frac{s}{2}+1}.$$

Hence $f^{(s)}_{n+1}(-1) = [x^{n+1}]PQ = [x^{n+1}]G(x^2)$ if n is odd; and $f^{(s)}_{n+1}(-1) = [x^n](PQ)^{\frac{s}{2}+1} = [x^n]G(x^2)^{\frac{s}{2}+1}$ otherwise. By the proof of Proposition 2.3, we have

$$f^{(s)}_{n+1}(-1) = \begin{cases} \dfrac{s+2}{sn+s+2}\dbinom{\frac{(s+1)n+s}{2}}{\frac{n}{2}} & \text{if } n \text{ is even} \\[2ex] \dfrac{2}{s(n+1)+2}\dbinom{\frac{(s+1)(n+1)}{2}}{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Since s is even, by Eq. (4), $X(-1) = f^{(s)}_{n+1}(-1)$ and the assertion (ii) follows. ∎

3 Evacuation on maximal chains of $(\mathcal{P}^{(s)}_n, \leqslant)$

In this section we analyze the behavior of the maximal chains $C \in \mathcal{X}^{(s)}_n$ of $(\mathcal{P}^{(s)}_n, \leqslant)$ under evacuation. We found that the operator ϵ_n can be decomposed in a way depending on C such that the maximal chain $C\epsilon_n$ has a factor-swapping property (see Lemma 3.6 and Proposition 3.7).

Recall that the rank of $(\mathcal{P}^{(s)}_n, \leqslant)$ is $p = (s + 1)n$. We consider the following operators on the maximal chains of $(\mathcal{P}^{(s)}_n, \leqslant)$, generated by $\tau_1, \ldots, \tau_{p-1}$. For a positive integer t, define

$$\delta_t = \tau_1\tau_2\cdots\tau_t$$
$$\delta^*_t = \tau_t\tau_{t-1}\cdots\tau_1.$$

Note that $\delta^{-1}_t = \delta^*_t$. The operator ϵ_n of evacuation can be expressed in terms of $\delta_1, \ldots, \delta_{p-1}$ (resp. $\delta^*_1, \ldots, \delta^*_{p-1}$) as follows.

Lemma 3.1. *We have*

$$\epsilon_n = \delta_{p-1}\delta_{p-2}\cdots\delta_1 = \delta^*_1\delta^*_2\cdots\delta^*_{p-1}.$$

Proof. By the definition of ϵ_n in Eq. (5), $\epsilon_n = \delta_{p-1}\delta_{p-2}\cdots\delta_1$. Since ϵ_n is an involution, $\epsilon_n = \epsilon^{-1}_n = \delta^{-1}_1\delta^{-1}_2\cdots\delta^{-1}_{p-1} = \delta^*_1\delta^*_2\cdots\delta^*_{p-1}$. ∎

For convenience, members of $\mathcal{X}_n^{(s)}$ are also referred to as *paths*, with north steps and east steps. Given a path $C \in \mathcal{X}_n^{(s)}$, let $E_1, E_2, \dots, E_n$ be the east steps of C from left to right. The east step E_k is also said to be in the kth *column*. Let $y(E_k)$ denote the y-coordinate of the endpoint of E_k ($1 \leqslant k \leqslant n$). Note that $sk - s \leqslant y(E_k) \leqslant sn$. We encode the path C by a sequence $\Pi(C) = (d_1, d_2, \cdots, d_n)$, called the *depth-code* of C, defined by

$$d_k = sk - y(E_k)$$

for $1 \leqslant k \leqslant n$, which indicates the vertical depth of the endpoint of E_k from the line $y = sx$. Note that $0 \leqslant d_k \leqslant s$ (resp. $d_k < 0$) if the endpoint of E_k is weakly under (resp. strictly above) the line $y = sx$. For example, with $s = 3$ and $n = 3$, the first path shown in Figure 4 is encoded $(1, -1, 2)$.

We have the following observation about the operators $\delta_1, \dots, \delta_{p-1}$ applied to the paths in $\mathcal{X}_n^{(s)}$. Sometimes we write N^ℓ for a consecutive ℓ north steps.

Lemma 3.2. *Let $C = z_1 \cdots z_p \in \mathcal{X}_n^{(s)}$ be a path with depth-code $\Pi(C) = (d_1, \dots, d_n)$. Then the following results hold.*

(i) *If $z_1 = \mathsf{E}$ then $C\delta_{p-1} = z_2 \cdots z_p \mathsf{E}$ with $\Pi(C\delta_{p-1}) = (d_2 - s, d_3 - s, \dots, d_n - s, 0)$.*

(ii) *If $z_p = \mathsf{E}$ and $d_j \leqslant 0$ for all $j \in [1, n-1]$ then $C\delta^*_{p-1} = \mathsf{E} z_1 \cdots z_{p-1}$ with $\Pi(C\delta^*_{p-1}) = (0, d_1 + s, \dots, d_{n-1} + s)$.*

(iii) *For any positive integer $\ell \leqslant s$, if $d_j \leqslant s - \ell$ for all $j \in [1, n]$ then $C\delta_{p-1} \cdots \delta_{p-\ell} = z_{\ell+1} \cdots z_p \mathsf{N}^\ell$ with depth-code $\Pi(C\delta_{p-1} \cdots \delta_{p-\ell}) = (d_1 + \ell, \dots, d_n + \ell)$.*

(iv) *For any positive integer $\ell \leqslant s$, if $d_n \geqslant \ell$ then $C\delta^*_{p-\ell} \cdots \delta^*_{p-1} = \mathsf{N}^\ell z_1 \cdots z_{p-\ell}$ with depth-code $\Pi(C\delta^*_{p-\ell} \cdots \delta^*_{p-1}) = (d_1 - \ell, \dots, d_n - \ell)$.*

Proof. (i) Note that $\delta_{p-1} = \tau_1 \cdots \tau_{p-1}$ and τ_i interchanges the ith and $(i+1)$th steps of a path $C \in \mathcal{X}_n^{(s)}$. Since $z_1 = \mathsf{E}$, the path $C\delta_{p-1}$ is obtained from C by moving z_1 all the way to the end. As a result, the segment $z_2 \cdots z_p$ of C is moved to the left by one column and hence the code $\Pi(C\delta_{p-1})$ is obtained as asserted.

(ii) Note that $\delta^*_{p-1} = \tau_{p-1} \cdots \tau_1$ is the reverse operation of δ_{p-1}. Since $z_p = \mathsf{E}$, we observe that the last step can be moved all the way to the first position subject to the condition $d_j \leqslant 0$ for all $j \leqslant n - 1$. As a result, the segment $z_1 \cdots z_{p-1}$ of C is moved to the right by one column.

(iii) Since $d_1 \leqslant s - \ell$, the first ℓ steps of C are north steps. First, we compute $C\delta_{p-1}$. Note that $z_1 = \mathsf{N}$ can be moved all the way to end subject to the condition $d_j \leqslant s - 1$ for all $j \in [2, n]$. As a result, the segment $z_2 \cdots z_p$ of C is moved down one row and hence $\Pi(C\delta_{p-1}) = (d_1 + 1, \dots, d_n + 1)$. Next, apply δ_{p-2} to the segment $z_2 \cdots z_p$, leaving z_1 frozen in the last position. Continue in this way until $z_1, \dots, z_\ell$ are frozen in the back. The assertion follows.

(iv) Since $d_n \geqslant \ell$, the last ℓ steps of C are north steps. Note that $\delta^*_{p-\ell} \cdots \delta^*_{p-1}$ is the reverse operation of $\delta_{p-1} \cdots \delta_{p-\ell}$ and that the operator $\delta^*_{p-\ell}$ moves the step $z_{p-\ell+1} = \mathsf{N}$ all the way to the first position subject to no restriction on $\Pi(C)$. As a result, the segment

$z_1 \cdots z_{p-\ell}$ of C is moved up one row. The assertion follows from the similar operations of $\delta^*_{p-\ell+1}, \ldots, \delta^*_{p-1}$. ☐

We consider two restrictions on the depth-codes of the paths $C \in \mathcal{X}_n^{(s)}$. These restrictions will be used in the factorization of C with factor-swapping property. For $0 \leqslant \ell \leqslant s$, let $\mathcal{A}_n(\ell) \subseteq \mathcal{X}_n^{(s)}$ be the set of paths with code $(d_1, \ldots, d_n)$ such that $d_n \geqslant \ell$, and let $\mathcal{B}_n(\ell) \subseteq \mathcal{X}_n^{(s)}$ be the set of paths with code $(d_1, \ldots, d_n)$ such that $d_j \leqslant \ell$ for all $j \in [1, n]$. The following result is a property of the operator ϵ_n, carrying paths with one restriction to paths with the other.

Lemma 3.3. *The operator ϵ_n establishes a bijection between $\mathcal{A}_n(\ell)$ and $\mathcal{B}_n(s-\ell)$.*

Proof. The case $\ell = 0$ is trivial since $\mathcal{A}_n(0) = \mathcal{B}_n(s) = \mathcal{X}_n^{(s)}$. For $\ell > 0$ and a path $C \in \mathcal{A}_n(\ell)$, notice that the last ℓ steps of C are north and that they remain unaffected under the operations of $\delta^*_1, \cdots, \delta^*_{p-\ell-1}$. Applying $\delta^*_1, \cdots, \delta^*_{p-\ell-1}$ to C, let $C\delta^*_1 \cdots \delta^*_{p-\ell-1} = D\mathsf{N}^\ell$, where D is the resulting path of the first $p-\ell$ steps. Then applying $\delta^*_{p-\ell}, \ldots, \delta^*_{p-1}$ to $D\mathsf{N}^\ell$, by Lemma 3.2(iv) we have $(D\mathsf{N}^\ell)\delta^*_{p-\ell} \cdots \delta^*_{p-1} = \mathsf{N}^\ell D$, moving D up ℓ rows. Hence $C\epsilon_n = C\delta^*_1 \cdots \delta^*_{p-1} = \mathsf{N}^\ell D \in \mathcal{B}_n(s-\ell)$.

On the other hand, given a path $C \in \mathcal{B}_n(s-\ell)$, notice that $d_j \leqslant s-\ell$ for all $j \in [1, n]$. By Lemma 3.2(iii), we have $C\delta_{p-1} \cdots \delta_{p-\ell} = z_{\ell+1} \cdots z_p \mathsf{N}^\ell$. Then apply the operators $\delta_{p-\ell-1}, \cdots, \delta_1$ to $z_{\ell+1} \cdots z_p \mathsf{N}^\ell$, leaving the ℓ north steps in the back frozen. Hence $C\epsilon_n = (z_{\ell+1} \cdots z_p \mathsf{N}^\ell)\delta_{p-\ell-1} \cdots \delta_1 \in \mathcal{A}_n(\ell)$. ☐

For any positive integer $m < n$, if $d_m \geqslant 0$ then C can be factorized into two paths C_1C_2 with $C_1 \in \mathcal{X}_m^{(s)}$ and $C_2 \in \mathcal{X}_{n-m}^{(s)}$, where $C_1 = z_1 \cdots z_{(s+1)m}$ and $C_2 = z_{(s+1)m+1} \cdots z_{(s+1)n}$. We define an operator $\gamma_{n;m}$ that swaps C_1 with C_2, under certain restriction. Define

$$\gamma_{n;m} = (\tau_{(s+1)m} \cdots \tau_{p-1})(\tau_{(s+1)m-1} \cdots \tau_{p-2}) \cdots (\tau_1 \cdots \tau_{p-(s+1)m}).$$

In fact, $\gamma_{n;m}$ appears in a decomposition of the operator ϵ_n (see Lemma 3.6).

Lemma 3.4. *Let $C = C_1C_2$ with $C_1 \in \mathcal{X}_m^{(s)}$ and $C_2 \in \mathcal{X}_{n-m}^{(s)}$. If $\Pi(C) = (d_1, \ldots, d_n)$ satisfies the following conditions (i)-(iii) for some $\ell \leqslant s$, then $C\gamma_{n;m} = C_2C_1$.*

(i) $0 \leqslant d_m \leqslant s-\ell$,

(ii) $d_j \leqslant s-\ell$ *for all* $j \in [1, m-1]$,

(iii) $d_j \leqslant \ell$ *for all* $j \in [m+1, n]$.

Proof. Regarding the east steps $E_1, \ldots, E_m$ of C_1, we factorize C_1 as $\mathsf{N}^{t_0}E_1\mathsf{N}^{t_1} \cdots E_m\mathsf{N}^{t_m}$. Then the depth-code of C_1 can be expressed as

$$d_j = t_j + \cdots + t_m - s(m-j). \tag{7}$$

for $1 \leqslant j \leqslant m$. Let $\gamma_{n:m}$ be written as $\gamma_{n:m} = \rho_1 \cdots \rho_{(s+1)m}$, where $\rho_i = \tau_{(s+1)m-i+1} \cdots \tau_{p-i}$ for $1 \leqslant i \leqslant (s+1)m$. When $\gamma_{n;m}$ applies to C, by (i) and (iii) of Lemma 3.2, we observe that from $z_{(s+1)m}$ to z_1 the steps of C_1 are moved one by one to the back of C_2 by the operators $\rho_1, \ldots, \rho_{(s+1)m}$ accordingly and freeze in place afterwards, subject to the condition that the depth-code of C_2 meets the requirement in Lemma 3.2(iii) throughout the way. Notice that this is the case since $\Pi(C_1)$ satisfies conditions (i) and (ii) and $\Pi(C_2)$ satisfies condition (iii). $\square$

Example 3.5. Take $s = 3$, $n = 3$, $m = 1$ and $\ell = 2$. Then $p = (s + 1)n = 12$. Let $C = C_1C_2$ be the first path shown in Figure 4, where $C_1 = \mathsf{NNEN}$ and C_2 goes from Y to Z. Note that $C_2 \in \mathcal{B}_2(2)$, whose E's are in the shaded area. For $1 \leqslant i \leqslant 4$, let $\rho_i = \tau_{4-i+1} \cdots \tau_{12-i}$. Applying operator $\gamma_{3;1} = \rho_1\rho_2\rho_3\rho_4$ on C swaps C_1 with C_2, as shown in Figure 4.

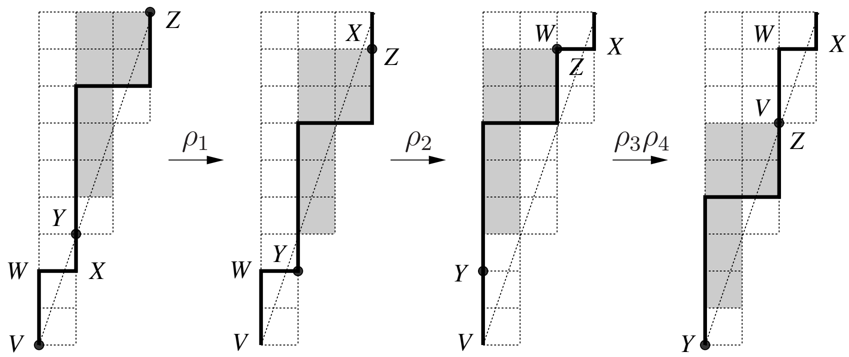


Figure 4: The operator $\gamma_{3;1}$ swaps the two factors of a path $C = C_1C_2 \in \mathcal{X}_3^{(3)}$.

Lemma 3.6. *For any positive integer $m < n$, the operator ϵ_n can be decomposed as*

$$\epsilon_n = \epsilon_m\gamma_{n;m}\epsilon_{n-m}.$$

Proof. For convenience, let $q := (s + 1)m$ and $r := p - q = (s + 1)(n - m)$. Following the relations (1), the operator ϵ_n is rearranged as follows. Note that $\epsilon_{n-m} = \delta_{r-1} \cdots \delta_1$. It suffices to consider the initial factor $\delta_{p-1} \cdots \delta_r$ of ϵ_n. Let δ_r be fixed. From right to left, move the τ_1 of δ_{r+1} to the right of the τ_2 of δ_{r+2}. Then move this τ_1, along with the factor $\tau_1\tau_2$ of δ_{r+2}, to the right of the τ_3 of δ_{r+3}. Repeat this process, moving the factor $(\tau_1 \cdots \tau_{j-1}) \cdots (\tau_1\tau_2)(\tau_1)$ to the right of the τ_j of δ_{r+j} for all $j \leqslant q - 1$. Now, we have the initial factor $(\tau_1 \cdots \tau_{q-1}) \cdots (\tau_1\tau_2)(\tau_1) = \delta_{q-1} \cdots \delta_2\delta_1 = \epsilon_m$. The stages of the operation is given below.

$$\begin{aligned}
\epsilon_n &= \delta_{p-1} \cdots \delta_r\epsilon_{n-m} \\
&= \delta_{p-1} \cdots \delta_{r+3}(\tau_1\tau_2)(\tau_1)(\tau_3 \cdots \tau_{r+2})(\tau_2 \cdots \tau_{r+1})\delta_r\epsilon_{n-m} \\
&= \delta_{p-1} \cdots \delta_{r+4}(\tau_1\tau_2\tau_3)(\tau_1\tau_2)(\tau_1)(\tau_4 \cdots \tau_{r+3})(\tau_3 \cdots \tau_{r+2})(\tau_2 \cdots \tau_{r+1})\delta_r\epsilon_{n-m} \\
&= (\tau_1 \cdots \tau_{q-1}) \cdots (\tau_1\tau_2)(\tau_1)(\tau_q \cdots \tau_{p-1}) \cdots (\tau_2 \cdots \tau_{r+1})(\tau_1 \cdots \tau_r)\epsilon_{n-m} \\
&= \epsilon_m\gamma_{n;m}\epsilon_{n-m},
\end{aligned}$$

as required. $\square$

On the basis of Lemmas 3.3, 3.4 and 3.6, we have the following fundamental property of evacuation. With this property, the evacuation of the maximal chains of $(\mathcal{P}_n^{(s)}, \leqslant)$ can be factorized into building blocks.

Proposition 3.7. *For all integers $m \in [1, n-1]$ and $\ell \in [0, s]$, let $C = z_1 \cdots z_p \in \mathcal{X}_n^{(s)}$ be a path whose depth-code $\Pi(C) = (d_1, \ldots, d_n)$ satisfies the condition $d_m \geqslant \ell$ and $d_j \leqslant \ell$ for all $j \in [m+1, n]$. Factorize C as C_1C_2, where $C_1 = z_1 \cdots z_{(s+1)m}$ and $C_2 = z_{(s+1)m+1} \cdots z_p$. Then the following properties hold.*

(i) *$C\epsilon_n = C_2'C_1'$, where $C_1' = C_1\epsilon_m$ and $C_2' = C_2\epsilon_{n-m}$.*

(ii) *If $\Pi(C\epsilon_n) = (d_1', \ldots, d_n')$ then $d_{n-m}' \geqslant s-\ell$ and $d_j' \leqslant s-\ell$ for all $j \in [n-m+1, n]$.*

Proof. Note that $C_1 \in \mathcal{A}_m(\ell)$ and $C_2 \in \mathcal{B}_{n-m}(\ell)$. We compute the evacuation $C\epsilon_n$ using the decomposition $\epsilon_n = \epsilon_m\gamma_{n;m}\epsilon_{n-m}$ of ϵ_n in Lemma 3.6. We observe that the operator ϵ_m applies to C_1, say $C_1' = C_1\epsilon_m$. By Lemma 3.3, we have $C_1' \in \mathcal{B}_m(s-\ell)$. Next, the operator $\gamma_{n:m}$ applies to $C_1'C_2$, leading to $C_1'C_2\gamma_{n;m} = C_2C_1'$. Then the operator ϵ_{n-m} applies to C_2, say $C_2' = C_2\epsilon_{n-m}$. By Lemma 3.3, we have $C_2' \in \mathcal{A}_m(s-\ell)$. The stages of the operation is given below.

$$\begin{aligned} C\epsilon_n &= C_1C_2\epsilon_m\gamma_{n;m}\epsilon_{n-m} \\ &= C_1'C_2\gamma_{n;m}\epsilon_{n-m} \\ &= C_2C_1'\epsilon_{n-m} \\ &= C_2'C_1'. \end{aligned}$$

The assertions (i) and (ii) follow. □

4 Primitive factorization of maximal chains of $(\mathcal{P}_n^{(s)}, \leqslant)$

In this section we characterize the evacuation of the maximal chains of $(\mathcal{P}_n^{(s)}, \leqslant)$ in terms of a specific factorization of the chains.

For any integer $\ell \in [0, s]$ and a path $B \in \mathcal{X}_n^{(s)}$ with $\Pi(B) = (d_1', \ldots, d_n')$, the path B is called a *primitive ℓ-block* of *width n* if $d_n' = \ell$ and $d_j' \leqslant \ell - 1$ for all $j \in [1, n-1]$. Every path $C \in \mathcal{X}_n^{(s)}$ can be uniquely factorized into primitive blocks of certain widths as follows. Let $\Pi(C) = (d_1, \ldots, d_n)$. Find the sequence $f_1 < f_2 < \cdots < f_b = n$ of integers such that $d_{f_1}, d_{f_2}, \ldots, d_{f_b}$ are the right-to-left weak maxima of $\Pi(C)$ for some integer b, namely

(i) $d_{f_1} \geqslant d_{f_2} \geqslant \ldots \geqslant d_{f_b} \geqslant 0$,

(ii) $d_i < d_{f_j}$ for $f_{j-1} < i < f_j$ and for all $j \in [1, b]$.

We assume $f_0 = 0$. For $1 \leqslant j \leqslant b$, let $e_j = f_j - f_{j-1}$. Then the path C can be factorized into primitive blocks $C = B_1B_2 \cdots B_b$, where B_j is a primitive d_{t_j}-block of width e_j for $1 \leqslant j \leqslant b$. This factorization is called the *primitive factorization* of C.

Example 4.1. For $s = 3$ and $n = 4$, let C be the path shown on the left hand side of Figure 5, with $\Pi(C) = (d_1, \dots, d_4) = (3, 2, -1, 2)$. The right-to-left weak maxima of $\Pi(C)$ are d_1, d_2, d_4. The primitive factorization $B_1B_2B_3$ of C consists of one primitive 3-block $B_1 = \mathsf{ENNN}$ and two primitive 2-blocks $B_2 = \mathsf{NENN}$ and $B_3 = \mathsf{NNNNEENN}$.

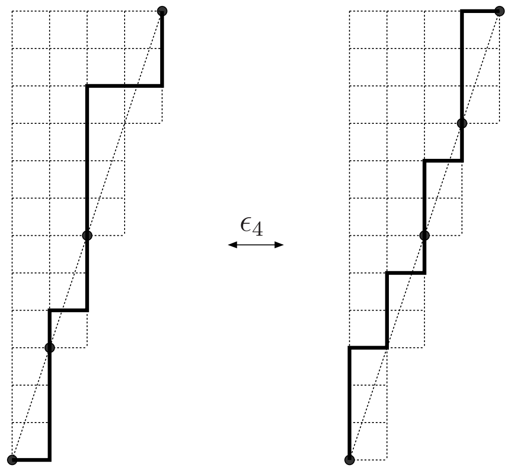


Figure 5: The primitive factorization of two paths in $\mathcal{X}_4^{(3)}$ in an orbit.

The evacuation of $C \in \mathcal{X}_n^{(s)}$ can be determined by the evacuation of individual primitive blocks of C (Theorem 4.5). The following decomposition of ϵ_n will be used in the proof of a characterization of the evacuation of primitive blocks (Proposition 4.3).

Lemma 4.2. *For any integer $\ell \in [0, s]$, the operator ϵ_n can be decomposed as*

$$\epsilon_n = (\delta_{p-1} \cdots \delta_{p-s+\ell})\epsilon_{n-1}\rho_1\rho_2\rho_3(\delta_\ell \cdots \delta_1),$$

where

$$\begin{aligned}
\rho_1 &= \tau_{p-s-1} \cdots \tau_{p-s+\ell-1}, \\
\rho_2 &= (\tau_{p-s-2} \cdots \tau_{p-s+\ell-3}) \cdots (\tau_2 \cdots \tau_{\ell+1})(\tau_1 \cdots \tau_\ell), \\
\rho_3 &= \tau_{p-s+\ell-2} \cdots \tau_{\ell+1}.
\end{aligned}$$

Proof. It suffices to consider the decomposition of the factor $\gamma := \delta_{p-s+\ell-1} \cdots \delta_{\ell+1}$ of ϵ_n. We describe the process of the decomposition.

(i) From right to left, move the τ_1 of $\delta_{\ell+2}$ to the right of the τ_2 of $\delta_{\ell+3}$ and then move this τ_1, along with the factor $\tau_1\tau_2$ of $\delta_{\ell+3}$, to the right of the τ_3 of $\delta_{\ell+4}$.

(ii) Repeat this process, moving the factor $(\tau_1 \cdots \tau_{j-1}) \cdots (\tau_1\tau_2)(\tau_1)$ to the right of the τ_j of $\delta_{\ell+j+1}$ for $2 \leqslant j \leqslant p - s - 1$ and let $\delta'_{\ell+j} = \tau_j \cdots \tau_{\ell+j}$ denote the operator obtained from $\delta_{\ell+j}$ with the factor $\tau_1 \cdots \tau_{j-1}$ removed. Now, the operator γ becomes $\epsilon_{n-1}\delta'_{p-s+\ell-1} \cdots \delta'_{\ell+2}\delta_{\ell+1}$.

(iii) Let $\rho_1 = \delta'_{p-s+\ell-1}$ be fixed. Factorize the remaining operator $\delta'_{p-s+\ell-2} \cdots \delta'_{\ell+2}\delta_{\ell+1}$ into $\rho_2\rho_3$ as follows. To form ρ_3, move the last element $\tau_{\ell+2}$ of $\delta'_{\ell+2}$ to the left of the $\tau_{\ell+1}$ of $\delta_{\ell+1}$ and then move the last element $\tau_{\ell+3}$ of $\delta'_{\ell+3}$ to the left of the factor $\tau_{\ell+2}\tau_{\ell+1}$ at the end.

(iv) Repeat this process, moving the last element $\tau_{\ell+j}$ of $\delta'_{\ell+j}$ to the left of the factor $\tau_{\ell+j-1} \cdots \tau_{\ell+1}$ at the end for $2 \leqslant j \leqslant p-s-2$.

The stages of the decomposition are given below.

$$\begin{aligned}
\gamma &:= \delta_{p-s+\ell-1} \cdots \delta_{\ell+1} \\
&= \delta_{p-s+\ell-1} \cdots \delta_{\ell+4}(\tau_1\tau_2)(\tau_1)(\tau_3 \cdots \tau_{\ell+3})(\tau_2 \cdots \tau_{\ell+2})\delta_{\ell+1} \\
&= (\tau_1 \cdots \tau_{p-s-2}) \cdots (\tau_1\tau_2)(\tau_1)\delta'_{p-s+\ell-1} \cdots \delta'_{\ell+2}\delta_{\ell+1} \\
&= \epsilon_{n-1}\delta'_{p-s+\ell-1} \cdots \delta'_{\ell+2}\delta_{\ell+1} \\
&= \epsilon_{n-1}\delta'_{p-s+\ell-1} \cdots \delta'_{\ell+3}(\tau_2 \cdots \tau_{\ell+1})(\tau_1 \cdots \tau_\ell)(\tau_{\ell+2}\tau_{\ell+1}) \\
&= \epsilon_{n-1}\delta'_{p-s+\ell-1}(\tau_{p-s-2} \cdots \tau_{p-s+\ell-3}) \cdots (\tau_2 \cdots \tau_{\ell+1})(\tau_1 \cdots \tau_\ell)(\tau_{p-s+\ell-2} \cdots \tau_{\ell+2}\tau_{\ell+1}) \\
&= \epsilon_{n-1}\rho_1\rho_2\rho_3,
\end{aligned}$$

as required. □

For $\ell \geqslant 1$ and a path $C \in \mathcal{A}_n(\ell)$ with $\Pi(C) = (d_1, \ldots, d_n)$, let $C^\perp = C\delta^*_{p-1}$ denote the path obtained from C by moving the last step, which is a north step, all the way to the beginning. As a result, the remaining part of C is moved up one row and hence $C^\perp \in \mathcal{A}_n(\ell-1)$ and $\Pi(C^\perp) = (d_1 - 1, \ldots, d_n - 1)$. The following result characterizes the evacuation of a primitive block, which leads to necessary conditions for a primitive block to be fixed by the operator ϵ_n; see Proposition 5.1.

Proposition 4.3. *Let $C = z_1 \cdots z_p \in \mathcal{X}^{(s)}_n$ be a primitive ℓ-block for some $\ell \in [0, s]$. Factorize C as $\mathsf{N}^{s-\ell}C^*\mathsf{E}\mathsf{N}^\ell$, where $C^* = z_{s-\ell+1} \cdots z_{p-\ell-1}$. Then $C\epsilon_n = \mathsf{N}^\ell(C^*\epsilon_{n-1})^\perp\mathsf{E}\mathsf{N}^{s-\ell}$ is a primitive $(s-\ell)$-block.*

Proof. Since C is a primitive ℓ-block, we observe that the segment C^* goes from the point $(0, s-\ell)$ to the point $(n-1, p-\ell)$ staying weakly above the line $y = sx - \ell$, which is a maximal chain of the subposet of $(\mathcal{P}^{(s)}_n, \leqslant)$ induced on the set of points

$$\mathcal{P}^{(s;\ell)}_{n-1} = \{(x, y) : x \in [0, n-1], y \in [s-\ell, sn-\ell], y \geqslant sx - \ell\}.$$

Let $\Pi(C\epsilon_n) = (d'_1, \ldots, d'_n)$. Making use of the decomposition of ϵ_n in Lemma 4.2, we have the evacuation

$$
\begin{aligned}
C\epsilon_n &= \mathsf{N}^{s-\ell}C^*\mathsf{E}\mathsf{N}^{\ell}(\delta_{p-1}\cdots\delta_{p-s+\ell})\epsilon_{n-1}\rho_1\rho_2\rho_3(\delta_\ell\cdots\delta_1)\\
&= C^*\mathsf{E}\mathsf{N}^{s}\epsilon_{n-1}\rho_1\rho_2\rho_3(\delta_\ell\cdots\delta_1)\\
&= (C^*\epsilon_{n-1})\mathsf{E}\mathsf{N}^{s}\rho_1\rho_2\rho_3(\delta_\ell\cdots\delta_1)\\
&= (C^*\epsilon_{n-1})\mathsf{N}^{\ell}\mathsf{E}\mathsf{N}^{s-\ell}\rho_2\rho_3(\delta_\ell\cdots\delta_1)\\
&= \mathsf{N}^{\ell}(C^*\epsilon_{n-1})\mathsf{E}\mathsf{N}^{s-\ell}\rho_3(\delta_\ell\cdots\delta_1)\\
&= \mathsf{N}^{\ell}(C^*\epsilon_{n-1})^{\perp}\mathsf{E}\mathsf{N}^{s-\ell}(\delta_\ell\cdots\delta_1)\\
&= \mathsf{N}^{\ell}(C^*\epsilon_{n-1})^{\perp}\mathsf{E}\mathsf{N}^{s-\ell}.
\end{aligned}
$$

The stages of the evacuation are described below.

(i) The initial $s-\ell$ steps are moved to the end by the operator $\delta_{p-1}\cdots\delta_{p-s+\ell}$. Now, $C^*\in\mathcal{B}_{n-1}(s-1)$.

(ii) Applying the operator ϵ_{n-1} to C^* leads to $C^*\epsilon_{n-1}\in\mathcal{A}_{n-1}(1)$.

(iii) By the operation of ρ_1, the east step E in the nth column is moved up ℓ rows and hence $d'_n = s-\ell$.

(iv) By the operation of ρ_2, the last step of $C^*\epsilon_{n-1}$, which is north, and $\ell-1$ north steps behind $C^*\epsilon_{n-1}$ are moved to the front. As an equivalent result, the path $C^*\epsilon_{n-1}$ is swapped with the ℓ north steps behind, and hence $d'_j\leqslant s-\ell$ for $1\leqslant j\leqslant n-1$.

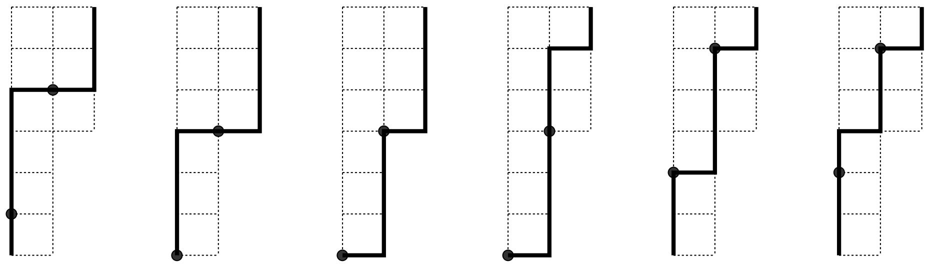
(v) By the operation of ρ_3, the path $C^*\epsilon_{n-1}$ becomes $(C^*\epsilon_{n-1})^{\perp}$, and hence $d'_j\leqslant s-\ell-1$ for $1\leqslant j\leqslant n-1$.

(vi) Since $d'_1\leqslant s-\ell-1$, the initial $\ell+1$ steps are north and remain unchanged under the operation of $\delta_\ell\cdots\delta_1$.

The proof is completed. $\square$

Example 4.4. Let $s=3$ and $n=2$. For the primitive 2-block C shown as (i) of Figure 6, the evacuation of C is obtained as follows. Factorize C as $\mathsf{N}C^*\mathsf{ENN}$, where C^* is the path from $(0,1)$ to $(1,4)$. By Lemma 4.2, the operator ϵ_2 is factorized as $\delta_7\epsilon_1\rho_1\rho_2\rho_3\delta_2\delta_1$. Figure 6 shows the stages of computing $C\epsilon_2$ given in the proof of Proposition 4.3. (i) The operator δ_7 moves the first step to the end. (ii) The operator ϵ_1 applies to C^*. (iii) By the operation of $\rho_1=\tau_4\tau_5\tau_6$, the east step in the second column is moved up 2 rows. (iv) By the operation of $\rho_2=(\tau_3\tau_4)(\tau_2\tau_3)(\tau_1\tau_2)$, as an equivalent result the path $C^*\epsilon_1$ is swapped with the two north steps behind. (v) By the operation of $\rho_3=\tau_5\tau_4\tau_3$, the path $C^*\epsilon_1$ becomes $(C^*\epsilon_1)^{\perp}$. (v) Applying the operator $\delta_2\delta_1$ to the initial three north steps leaves the path unchanged. The requested path $C\epsilon_2$ is a primitive 1-block, shown as (vi) of Figure 6.

Now, we characterize the evacuation of the paths in $\mathcal{X}_n^{(s)}$.



(i) (ii) (iii) (iv) (v) (vi)

Figure 6: The evacuation of a primitive 2-block in $\mathcal{X}_2^{(3)}$.

Theorem 4.5. *For any path $C \in \mathcal{X}_n^{(s)}$, let $C = B_1 B_2 \cdots B_b$ be the primitive factorization of C for some integer b, where B_j is a primitive ℓ_j-block of width e_j for $1 \leqslant j \leqslant b$. Then the primitive factorization of $C\epsilon_n$ is of the form $C\epsilon_n = B'_b B'_{b-1} \cdots B'_1$, where $B'_j = B_j \epsilon_{e_j}$ for $1 \leqslant j \leqslant b$.*

Proof. By Proposition 4.3, $B_j \epsilon_{e_j}$ is a primitive $(s - \ell_j)$-block for $1 \leqslant j \leqslant b$. We prove the assertion by induction on the number of blocks of the primitive factorization.

The case $b = 1$ follows from Proposition 4.3. For $b > 1$, by Lemma 3.6 the operator ϵ_n can be decomposed as $\epsilon_n = \epsilon_{e_1} \gamma_{n;e_1} \epsilon_{n-e_1}$. To find the evacuation of C, the operator ϵ_{e_1} applies to B_1, leading to a primitive $(s - \ell_1)$-block $B'_1 = B_1 \epsilon_{e_1}$. Next the operator $\gamma_{n;e_1}$ swaps B'_1 with $B_2 \cdots B_n$. Then the operator ϵ_{n-e_1} applies to $B_2 \cdots B_b$, leaving B'_1 frozen. By induction hypothesis, we have $B_2 \cdots B_b \epsilon_{n-e_1} = B'_b \cdots B'_2$, where $B'_j = B_j \epsilon_{e_j}$ for $2 \leqslant j \leqslant b$. The stages of operation are given below.

$$
\begin{aligned}
C\epsilon_n &= B_1 B_2 \cdots B_n \epsilon_{e_1} \gamma_{n;e_1} \epsilon_{n-e_1} \\
&= B'_1 B_2 \cdots B_b \gamma_{n;e_1} \epsilon_{n-e_1} \\
&= B_2 \cdots B_b B'_1 \epsilon_{n-e_1} \\
&= B'_b \cdots B'_2 B'_1,
\end{aligned}
$$

as required. ☐

Example 4.6. Let $s = 3$ and $n = 4$. Given the path C shown on the left hand side of Figure 5, let us construct the evacuation of C. As mentioned in Example 4.1, the primitive factorization of C consists of one primitive 3-block $B_1 = \mathsf{ENNN}$ and two primitive 2-blocks $B_2 = \mathsf{NENN}$ and $B_3 = \mathsf{NNNNEENN}$. Note that $B'_1 = \mathcal{B}_1 \epsilon_1 = \mathsf{NNNE}$ is a primitive 0-block and $B'_2 = \mathcal{B}_2 \epsilon_1 = \mathsf{NNEN}$ is a primitive 1-block. As shown in Example 4.4, $B'_3 = B_3 \epsilon_2 = \mathsf{NNNENNEN}$ is a primitive 1-block. By Theorem 4.5, we have the primitive factorization of $C\epsilon_4 = B'_3 B'_2 B'_1$, shown on the right hand side of Figure 5.

5 Enumeration of fixed points of the operator ϵ_n

In this section we characterize and enumerate the maximal chains of $\mathcal{P}_n^{(s)}$ fixed by the operator ϵ_n. First, we study the necessary conditions for a primitive block to be fixed under evacuation.

Proposition 5.1. *Let $C \in \mathcal{X}_n^{(s)}$ be a primitive ℓ-block for some $\ell \in [0, s]$. If $C = C\epsilon_n$ then the following properties hold.*

(i) *The integer s is even and $\ell = \frac{s}{2}$.*

(ii) *The integer n is odd and the path C passes the point $(\frac{n-1}{2}, \frac{s(n-1)}{2})$ and the point $(\frac{n-1}{2}, \frac{sn}{2})$.*

(iii) *If C is factorized as $\mathsf{N}^{\frac{s}{2}} D_1 D_2 \mathsf{E}\mathsf{N}^{\frac{s}{2}}$, where D_1 goes from $(0, \frac{s}{2})$ to $(\frac{n-1}{2}, \frac{sn}{2})$, then $D_2 = (D_1\epsilon_{\frac{n-1}{2}})^{\perp}$ and $D_1 = (D_2\epsilon_{\frac{n-1}{2}})^{\perp}$.*

Proof. (i) By Proposition 4.3, $C\epsilon_n$ is a primitive $s-\ell$ block. If $C = C\epsilon_n$ then $\ell = s - \ell$ and hence s is even and $\ell = \frac{s}{2}$.

(ii) For $n = 1$, $C = \mathsf{N}^{\frac{s}{2}}\mathsf{E}\mathsf{N}^{\frac{s}{2}}$ passes the point $(0, \frac{s}{2})$. For $n > 1$, factorize C as $\mathsf{N}^{\frac{s}{2}} C^* \mathsf{E}\mathsf{N}^{\frac{s}{2}}$. By Proposition 4.3, $C\epsilon_n = N^{\frac{s}{2}}(C^*\epsilon_{n-1})^{\perp}\mathsf{E}\mathsf{N}^{\frac{s}{2}}$. Note that C^* is a maximal chain of the subposet of $(\mathcal{P}_n^{(s)}, \leqslant)$ induced on the set of points

$$\mathcal{P}_{n-1}^{(s;\frac{s}{2})} = \{(x, y) : x \in [0, n-1], y \in [\tfrac{s}{2}, sn - \tfrac{s}{2}], y \geqslant sx - \tfrac{s}{2}\}.$$

Let $C^* = B_1 B_2 \cdots B_b$ be the primitive factorization of C^* with respect to $\mathcal{P}_{n-1}^{(s;\frac{s}{2})}$, where B_j a primitive ℓ_j-block of width e_j for $1 \leqslant j \leqslant b$ and $s - 1 \geqslant \ell_1 \geqslant \cdots \geqslant \ell_b \geqslant 0$. If $C = C\epsilon_n$ then $C^* = (C^*\epsilon_{n-1})^{\perp}$. Hence by Theorem 4.5, we have $\ell_j = s - \ell_{b-j+1} - 1$ and $e_j = e_{b-j+1}$ for $1 \leqslant j \leqslant b$.

If b is odd, say $b = 2a - 1$, then $\ell_a = s - \ell_a - 1$. It follows that $s = 2\ell_a + 1$ is against the parity of s in (i). Thus b is even, say $b = 2a$. Then $n - 1 = 2(e_1 + \cdots + e_a)$. Hence n is odd. Moreover, $\ell_a = s - \ell_{a+1} - 1 \geqslant s - \ell_a - 1$. Hence $\ell_a \geqslant \frac{s}{2}$. It follows that C^* passes the point $(\frac{n-1}{2}, \frac{s(n-1)}{2})$ and the point $(\frac{n-1}{2}, \frac{sn}{2})$, and so does C.

(iii) For $n = 1$, the path D_1D_2 is trivial. For $n > 1$ and the factorization $C^* = B_1 \cdots B_{2a}$, let $D_1 = B_1 \cdots B_a$ and $D_2 = B_{a+1} \cdots B_{2a}$. Note that $D_1, D_2 \in \mathcal{B}_{\frac{n-1}{2}}(s-1)$. By Lemma 3.4, $D_1D_2\epsilon_{n-1} = D_2'D_1'$, where $D_1' = D_1\epsilon_{\frac{n-1}{2}}$ and $D_2' = D_2\epsilon_{\frac{n-1}{2}}$ are members of $\mathcal{A}_{\frac{n-1}{2}}(1)$. Both of the last step of D_1' and D_2' are north. It follows from $C^* = (C^*\epsilon_{n-1})^{\perp}$ that $D_1D_2 = (D_2'D_1')^{\perp} = D_2'D_1'\delta^*_{p'-1}$, where $p' = (s+1)(n-1)$, and hence

$$\begin{aligned} D_1D_2 &= D_2'D_1'\tau_{p'-1}\cdots\tau_1 \\ &= D_2'(D_1')^{\perp}\tau_{\frac{p'}{2}}\cdots\tau_1 \\ &= D_2'(D_1')^{\perp}\tau_{\frac{p'}{2}-1}\cdots\tau_1 \\ &= (D_2')^{\perp}(D_1')^{\perp}. \end{aligned}$$

We describe the process as follows. The operator $\tau_{p'-1}\cdots\tau_{\frac{p'}{2}+1}$ moves the last step of D'_1 to the first position of D'_1. Next, $\tau_{\frac{p'}{2}}$ applies to the two north steps in the middle of $D'_2(D'_1)^{\perp}$, leaving the path unchanged. Then the operator $\tau_{\frac{p'}{2}-1}\cdots\tau_1$ moves the last step of D'_2 to the first position of D'_2. The assertion follows. $\square$

It turns out that any primitive $\frac{s}{2}$-block $C\in\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n is uniquely determined by the segment from the origin through $(\frac{n-1}{2},\frac{s(n-1)}{2})$ to $(\frac{n-1}{2},\frac{sn}{2})$.

Corollary 5.2. *Let D be a lattice path from the origin through $(\frac{n-1}{2},\frac{s(n-1)}{2})$ to $(\frac{n-1}{2},\frac{sn}{2})$ with $\Pi(D)=(d_1,\dots,d_{\frac{n-1}{2}})$. If $d_j\leqslant\frac{s}{2}-1$ for all $j\in[1,\frac{n-1}{2}]$ then D determines a unique primitive $\frac{s}{2}$-block in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n.*

Proof. For $n=1$, $D=\mathsf{N}^{\frac{s}{2}}$. The requested primitive $\frac{s}{2}$-block is $C=\mathsf{N}^{\frac{s}{2}}\mathsf{E}\mathsf{N}^{\frac{s}{2}}\in\mathcal{X}_1^{(s)}$. For $n>1$, factorize D as $\mathsf{N}^{\frac{s}{2}}D^*$. With respect to the poset $(\mathcal{P}_{n-1}^{(s;\frac{s}{2})},\leqslant)$, the segment D^* is a member of $\mathcal{B}_{\frac{n-1}{2}}(s-1)$ and hence the segment $D^*\epsilon_{\frac{n-1}{2}}$ is a member of $\mathcal{A}_{\frac{n-1}{2}}(1)$. Create a path $C=\mathsf{N}^{\frac{s}{2}}D^*(D^*\epsilon_{\frac{n-1}{2}})^{\perp}\mathsf{E}\mathsf{N}^{\frac{s}{2}}$. By Proposition 5.1(iii), the path C is the requested primitive $\frac{s}{2}$-block in $\mathcal{X}_n^{(s)}$ determined by D. $\square$

Example 5.3. Let $s=4$ and $n=3$. Given the lattice path D shown as Figure 7(i), let us construct the primitive 2-block fixed by ϵ_3 determined by D. Factorize $D=\mathsf{NN}D^*$. Then $D^*=\mathsf{NENNN}$ is a member of $\mathcal{B}_1(3)$ in the subposet $(\mathcal{P}_2^{(4;2)},\leqslant)$; see Figure 7(ii). Then the evacuation of D^* is $D^*\epsilon_1=\mathsf{NNNEN}$ and hence $(D^*\epsilon_1)^{\perp}=\mathsf{NNNNE}$. The path $\mathsf{NN}D^*(D^*\epsilon_1)^{\perp}$ is shown in Figure 7(iii). Finally, we obtain the requested primitive 2-block $\mathsf{NN}D^*(D^*\epsilon_1)^{\perp}\mathsf{ENN}\in\mathcal{X}_3^{(4)}$, shown as Figure 7(iv).

The following result characterizes the maximal chains of $(\mathcal{P}_n^{(s)},\leqslant)$ fixed by evacuation.

Theorem 5.4. *For any path $C\in\mathcal{X}_n^{(s)}$, the following results hold.*

(i) *For n even, if $C=C\epsilon_n$ then C passes the point $(\frac{n}{2},\frac{sn}{2}-\lceil\frac{s}{2}\rceil)$ and the point $(\frac{n}{2},\frac{sn}{2})$. Moreover, every path from the origin through $(\frac{n}{2},\frac{sn}{2}-\lceil\frac{s}{2}\rceil)$ to $(\frac{n}{2},\frac{sn}{2})$ staying weakly above the line $y=sx-s$ determines a unique path in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n.*

(ii) *For n odd, if $C=C\epsilon_n$ then the integer s is even and C passes the points $(\frac{n-1}{2},\frac{s(n-1)}{2})$ and $(\frac{n-1}{2},\frac{sn}{2})$. Moreover, every path from the origin through $(\frac{n-1}{2},\frac{s(n-1)}{2})$ to the point $(\frac{n-1}{2},\frac{sn}{2})$ staying weakly above the line $y=sx-s$ determines a unique path in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n.*

Proof. Let $C=B_1\cdots B_b$ be the primitive factorization of C, where B_j a primitive ℓ_j-block of width e_j for $1\leqslant j\leqslant b$. By Theorem 4.5, $C\epsilon_n=B'_b\cdots B'_1$, where B'_j is the evacuation of B_j for $1\leqslant j\leqslant b$. If $C=C\epsilon_n$ then $\ell_j=s-\ell_{b-j+1}$ and $e_j=e_{b-j+1}$ for $1\leqslant j\leqslant b$.

(i) For n even, if b is odd, say $b=2a-1$, then $B_a=B'_a=B_a\epsilon_{e_a}$, which is fixed under evacuation. By Proposition 5.1, e_a is odd. It follows that $n=2(e_1+\cdots+e_{a-1})+e_a$ is

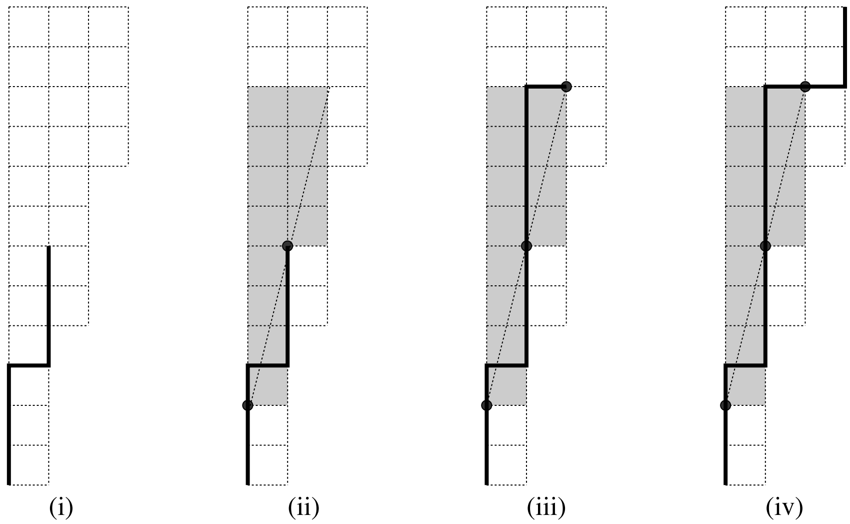


Figure 7: A primitive 2-block in $\mathcal{X}_3^{(4)}$ determined by a path NNNENNN.

against the parity of n. Hence b is even, say $b = 2a$. Then $\ell_a = s - \ell_{a+1} \geqslant s - \ell_a$ and hence $\ell_a \geqslant \frac{s}{2}$. It follows that C passes the point $(\frac{n}{2}, \frac{sn}{2} - \lceil \frac{s}{2} \rceil)$ and the point $(\frac{n}{2}, \frac{sn}{2})$.

On the other hand, suppose D is a path from the origin through $(\frac{n}{2}, \frac{sn}{2} - \lceil \frac{s}{2} \rceil)$ to the point $(\frac{n}{2}, \frac{sn}{2})$. Then D has a unique primitive factorization, say with abuse of notation $D = B_1 \cdots B_{a'}$ for some a', where B_j is a primitive ℓ_j-block of width e_j and $\ell_1 \geqslant \cdots \geqslant \ell_{a'} \geqslant \lceil \frac{s}{2} \rceil$. Then the evacuation of D is $D\epsilon_{\frac{n}{2}} = B'_{a'} \cdots B'_1$, where $B'_j = B_j \epsilon_{e_j}$ is a primitive $(s - \ell_j)$-block for $1 \leqslant j \leqslant a'$. Create a path $C = D(D\epsilon_{\frac{n}{2}})$. Then C is the requested path in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n.

(ii) For n odd, if b is even, say $b = 2a$ then $n = 2(e_1 + \cdots + e_a)$ is against the parity of n. Hence b is odd, say $b = 2a - 1$. Then $B_a = B_a \epsilon_{e_a}$ is fixed under evacuation. By Proposition 5.1, it follows that the path B_a passes the points $(\frac{n-1}{2}, \frac{s(n-1)}{2})$ and $(\frac{n-1}{2}, \frac{sn}{2})$, and so does C.

On the other hand, suppose D is a path from the origin through $(\frac{n-1}{2}, \frac{s(n-1)}{2})$ to the point $(\frac{n-1}{2}, \frac{sn}{2})$. Then D has a unique factorization $D = B_1 B_2 \cdots B_{a'}$, for some a', satisfying the following conditions.

- B_j is a primitive ℓ_j-block of width e_j for $1 \leqslant j \leqslant a' - 1$ and $\ell_1 \geqslant \cdots \geqslant \ell_{a'-1} \geqslant \frac{s}{2}$.
- $B_{a'}$ is either $\mathsf{N}^{\frac{s}{2}}$ or a path of width $e_{a'}$ with $\Pi(B_{a'}) = (d_1, \ldots, d_{e_{a'}})$ such that $d_i \leqslant \frac{s}{2} - 1$ for $1 \leqslant i \leqslant e_{a'}$.

For $1 \leqslant j \leqslant a' - 1$, create a path $B'_j = B_j \epsilon_{e_j}$. By Corollary 5.2, $B_{a'}$ determines a unique primitive $\frac{s}{2}$-block $B'_{a'}$ of width $2e_{a'} + 1$. Then the path $C = B_1 \ldots B_{a'-1} B'_{a'} B'_{a'} B'_{a'-1} \cdots B'_1$ is the requested path in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n. $\square$

Now, we enumerate the fixed points of the map $\epsilon_n : \mathcal{X}_n^{(s)} \to \mathcal{X}_n^{(s)}$.

Corollary 5.5. *The following results hold.*

(i) *For n even, the number of paths in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n is*

$$\begin{cases} \dfrac{s+2}{sn+s+2}\dbinom{\frac{(s+1)n+s}{2}}{\frac{n}{2}} & \text{if } s \text{ is even} \\[2ex] \dfrac{s+1}{sn+s+1}\dbinom{\frac{(s+1)n+s-1}{2}}{\frac{n}{2}} & \text{if } s \text{ is odd.} \end{cases}$$

(ii) *For n odd, the number of paths in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n is*

$$\begin{cases} \dfrac{2s+2}{sn+s+2}\dbinom{\frac{(s+1)n+s-1}{2}}{\frac{n-1}{2}} & \text{if } s \text{ is even} \\[2ex] 0 & \text{if } s \text{ is odd.} \end{cases}$$

Proof. (i) For n even, let $U_n^{(s)}$ be the set of lattice paths from the origin to the point $(\frac{n}{2}, \frac{sn}{2} - \lceil \frac{s}{2} \rceil)$ staying weakly above the line $y = sx - s$. By Theorem 5.4(i), the number of paths in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n is $|U_n^{(s)}|$. For every path in $U_n^{(s)}$, add a prefix of s north steps and move up s rows. This establishes a one-to-one correspondence between $U_n^{(s)}$ and the set of lattice paths from the origin to the point $(\frac{n}{2}, \frac{sn}{2} + \lfloor \frac{s}{2} \rfloor)$ staying weakly above the line $y = sx$. By Proposition 2.3, we have

$$|U_n^{(s)}| = \frac{\lfloor \frac{s}{2} \rfloor + 1}{\frac{sn}{2} + \lfloor \frac{s}{2} \rfloor + 1}\binom{\frac{sn+n}{2} + \lfloor \frac{s}{2} \rfloor}{\frac{n}{2}},$$

as require.

(ii) For n odd, let $V_n^{(s)}$ be the set of lattice paths from the origin to the point $(\frac{n-1}{2}, \frac{s(n-1)}{2})$ staying weakly above the line $y = sx - s$. By Theorem 5.4(ii), the number of paths in $\mathcal{X}_n^{(s)}$ fixed by the operator ϵ_n is $|V_n^{(s)}|$ if s is even, and 0 otherwise. For every path in $V_n^{(s)}$, add a prefix of s north steps and move up s rows. This establishes a one-to-one correspondence between $V_n^{(s)}$ and the set of lattice paths from the origin to the point $(\frac{n-1}{2}, \frac{s(n-1)}{2} + s)$ staying weakly above the line $y = sx$. By Proposition 2.3, for s even we have

$$|V_n^{(s)}| = \frac{2s+2}{sn+s+2}\binom{\frac{(s+1)n+s-1}{2}}{\frac{n-1}{2}},$$

as required. □

Since the results in Corollary 5.5 agree with that in Proposition 2.4, the proof of Theorem 2.1 is completed.

6 CSP for the posets of a rectangular shape

For positive integers m, n, consider the lattice paths from the origin to (n, m) using N and E steps staying within the $m \times n$ rectangle. Recall that the q-binomial coefficients are polynomials defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

where $[n]!_q = [1]_q[2]_q \cdots [n]_q$ and $[i]_q = 1 + q + \cdots + q^{i-1}$. By a known result [9, p. 30], the area-enumerator $X(q)$ of the lattice paths within the $m \times n$ rectangle is given by

$$X(q) = \begin{bmatrix} m+n \\ n \end{bmatrix}_q. \tag{8}$$

In this section we shall prove the following result.

Theorem 6.1. *For positive integers m and n, let $(\mathcal{P}, \leqslant)$ be the poset associated with $\lambda = (n^m)$. Let $\mathcal{X}$ be the set of maximal chains of $(\mathcal{P}, \leqslant)$. Let $X(q)$ be the polynomial defined in Eq. (8). Let the group $\langle \epsilon \rangle$, generated by the operator ϵ of evacuation, act on $\mathcal{X}$. Then $(\mathcal{X}, X(q), \langle \epsilon \rangle)$ exhibits the cyclic sieving phenomenon.*

Proof. We shall prove that the evaluation $X(-1)$ coincides with the number of maximal chains in $\mathcal{X}$ fixed by the operator ϵ.

For positive integers k_1, k_2 and k, we have the following facts. (i) $[k]_{q=-1} = 0$ if and only if k is even. (ii) If k_1, k_2 have the same parity, then

$$\lim_{q \to -1} \frac{[k_1]_q}{[k_2]_q} = \begin{cases} \frac{k_1}{k_2} & \text{if } k_1, k_2 \text{ are even} \\ 1 & \text{if } k_1, k_2 \text{ are odd.} \end{cases}$$

Making use of the facts (i) and (ii), we observe that

$$X(-1) = \lim_{q \to -1} \frac{[m+1]_q [m+2]_q \cdots [m+n]_q}{[1]_q [2]_q \cdots [n]_q} = \begin{cases} 0 & \text{if } m, n \text{ are odd} \\ \binom{\lfloor \frac{m+n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} & \text{otherwise.} \end{cases}$$

On the other hand, for a path $C = z_1 \cdots z_{mn} \in \mathcal{X}$, note that since λ is a rectangular shape, interchanging any two consecutive steps of C remains to be a path in $\mathcal{X}$. So applying the elementary operator τ_i on C always interchanges z_i and z_{i+1} $(1 \leqslant i \leqslant mn-1)$. Then the evacuation of C is obtained from C by reversing the order of the steps, i.e., $C\epsilon = z_{mn} \cdots z_1$. Hence $C\epsilon = C$ if and only if C is centrally symmetric, i.e., has 2-fold rotational symmetry. Note that the number of centrally symmetric paths in the $m \times n$ rectangle is 0 if m, n are odd, and $\binom{\lfloor \frac{m+n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}$ otherwise. The proof is completed. $\square$

7 Concluding Remarks

Note that the poset $(\mathcal{P}_n^{(s)}, \leqslant)$ associated with the partition $\lambda = (n^s, (n-1)^s, \ldots, 1)$ is the poset $J(P)$ constructed from the order ideals of a poset P. Stanley's result [8, Theorem 3.1] gives an alternative CSP for Fuss shapes of type A, which involves the q-polynomial $W(q)$ in Eq. (2), the enumerator of linear extensions of P respecting the comajor index. For example, the poset associated with $\lambda = (2,1) \vdash 3$ can be constructed from the order ideals of the poset shown on the left hand side in Figure 8, with $W(q) = 1+q+q^2+q^3+q^4$ independent of the labeling of its elements (see Figure 9).

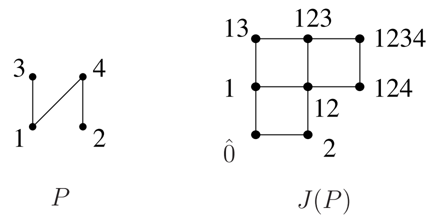


Figure 8: The paths in $\mathcal{X}_4$ fixed by evacuation.

However, the evacuation ϵ does not necessarily reverse the parity of the comajor index of linear extensions of P; see Figure 9 (sometimes self-evacuating linear extensions have an odd comajor index). He proved that the evaluation $W(-1)$ coincides with the number of domino linear extensions of P, i.e., the linear extensions $\omega \in \mathcal{L}(P)$ with the property $\omega\tau_{p-1}\tau_{p-3}\tau_{p-5}\cdots\tau_h = \omega$, where $h = 1$ if p is even, and $h = 2$ otherwise. To determine the self-evacuating linear extensions of P, he established a bijection $\omega \to \widetilde{\omega}$ between the domino linear extensions ω and the self-evacuating linear extensions $\widetilde{\omega}$ of P by

$$\widetilde{\omega} = \omega\tau_1 \cdot \tau_3\tau_2\tau_1 \cdot \tau_5\tau_4\tau_3\tau_2\tau_1 \cdots \tau_g\tau_{g-1}\cdots\tau_1,$$

where $g = p-1$ if p is even, and $g = p-2$ otherwise. For the poset P shown on the left hand side in Figure 8, the only domino linear extension ω corresponds to the maximal chain NNEE of $J(P)$ and the self-evacuating linear extension $\omega\tau_1 \cdot \tau_3\tau_2\tau_1$ corresponds to the maximal chain ENNE of $J(P)$. However, it is still unclear how to describe self-evacuating linear extension of a poset P explicitly. We contribute a neat characterization of the maximal chains of $J(P)$ fixed under evacuation for $J(P) = (\mathcal{P}_n^{(s)}, \leqslant)$.

As mentioned earlier, the map ϵ_n does not necessarily reverse the parity of the statistic area of maximal chains of $(\mathcal{P}_n^{(s)}, \leqslant)$. This suggests the following problem.

Problem 1. *Find a statistic of s-Dyck paths (linear extensions of a poset, respectively) equidistributed with* area *(*comaj*, respectively) so that the evacuation is parity-reversing.*

Among various cyclic sieving results on Catalan objects (e.g. [4, Theorem 7.1], [5, Theorem 8]), the case $s = 1$ in Theorem 2.1 gives an instance of CSP on a Catalan object, the triple $(\mathcal{X}_n, X(q), \langle\epsilon_n\rangle)$ of the poset associated with the partition $\lambda = (n, n-1, \ldots, 1)$. Note that $|\mathcal{X}_n| = c_{n+1}$ is the number of truncated Dyck paths of width n, where $c_n =$

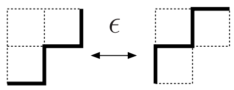
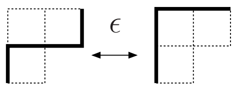
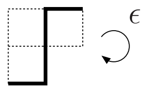
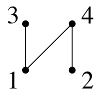
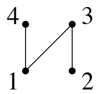
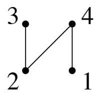
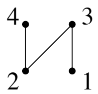
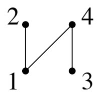
					
	q^4	1	q	q^2	q^3
	q^3	q	1	q^2	q^4
	q	q^3	q^4	q^2	1
	1	q^4	q^3	q^2	q
	q^4	q^2	q	1	q^3

Figure 9: The comaj-enumerator $W(q)$ of linear extensions of the poset shown on the left hand side of Figure 8, regarding labeling of its elements.

$\frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. It is worth mentioning that to our knowledge this result is the first instance using the area-enumerator $X(q)$ as the q-polynomial while other known results using the q-analogue of Catalan number $\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q$.

By Theorem 5.4, for n even the paths $C \in \mathcal{X}_n$ fixed by evacuation can be factorized as $C = C_1C_2$, where C_1 goes from the origin through $(\frac{n}{2}, \frac{n}{2} - 1)$ to $(\frac{n}{2}, \frac{n}{2})$. Moreover, $C\epsilon_n = C_2'C_1'$, where $C_1' = C_1\epsilon_{\frac{n}{2}}$ and $C_2' = C_2\epsilon_{\frac{n}{2}}$. For example, inspecting the orbits of $\mathcal{X}_2$ under evacuation shown in Figure 3, one can predict the two paths in $\mathcal{X}_4$ fixed by evacuation, as shown in Figure 10.

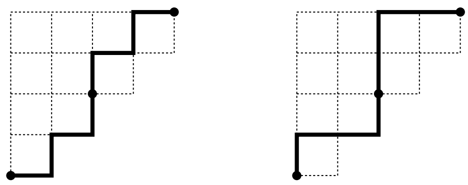


Figure 10: The paths in $\mathcal{X}_4$ fixed by evacuation.

Recall that not all partitions λ of an integer n are good shapes, i.e., the triple $(\mathcal{X}, X(q), \langle\epsilon\rangle)$ of the poset associated with λ exhibits CSP. Let g_n be the number of good shapes $\lambda \vdash n$. We obtain the initial terms of the sequence $\{g_n\}_{n\geqslant 0}$ by computer

$$1, 1, 2, 3, 5, 6, 11, 13, 21, 24, 40, 45, 71, 78, 122, 135, 202.$$

A question might arise.

Problem 2. *Determine g_n and characterize good shapes $\lambda \vdash n$, with an explicit characterization of the fixed points under evacuation.*

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