q-Stirling Identities Revisited

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Abstract

We give combinatorial proofs of q-Stirling identities using restricted growth words. This includes a poset theoretic proof of Carlitz's identity, a new proof of the q-Frobenius identity of Garsia and Remmel and of Ehrenborg's Hankel q-Stirling determinantal identity. We also develop a two parameter generalization to unify identities of Mercier and include a symmetric function version.

Keywords: q-analogues, q-Stirling numbers, restricted growth words, poset decomposition

1 Introduction

The classical Stirling number of the second kind $S(n,k)$ is the number of set partitions of n elements into k blocks. The Stirling numbers of the second kind first appeared in work of Stirling in 1730, where he gave the Newton expansion of the functions $f(z) = z^n$ in terms of the falling factorial basis [31, Page 8]. Kaplansky and Riordan [19] found the

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combinatorial interpretation that the Stirling number $S(n,k)$ enumerates the number of ways to place $n-k$ non-attacking rooks on a triangular board of side length $n-1$. From later work of Garsia and Remmel, this is equivalent to the number of set partitions of n elements into k blocks [14].

The q-Stirling numbers of the second kind arose from Carlitz's development of a q-analogue of the Bernoulli numbers and is predated by a problem of his involving abelian groups [3, 4]. There is a long history of studying set partitions [6, 14, 21, 28], Stirling numbers of the second kind and their q-analogues [3, 12, 15, 25, 34, 35].

In the literature there are many identities involving Stirling and q-Stirling numbers of the second kind. Stirling identities which appear in Jordan's text [18] have been transformed to q-identities by Ernst [13] using the theory of Hahn–Cigler–Carlitz–Johnson, Carlitz–Gould and the Jackson q-derivative. Verde-Star uses the divided difference operator [32] and the complete homogeneous symmetric polynomials in the indeterminates $x_k = 1 + q + \cdots + q^k$ in his work [33].

The goal of this paper is to give bijective proofs of many of these q-Stirling identities as well as a number of new identities. Underlying these proofs is the theory of restricted growth words which we review in the next section. In Section 3 we discuss recurrence structured q-Stirling identities, while in Section 4 we focus on Gould's ordinary generating function for the q-Stirling number. A poset theoretic proof of Carlitz's identity is given in Section 5. Section 6 contains combinatorial proofs of the de Médicis–Leroux q-Vandermonde convolutions. We provide new proofs of Garsia and Remmel's q-analogue of the Frobenius identity and of Ehrenborg's Hankel q-Stirling determinantal identity in Sections 7 and 8. In Section 9 we prove two identities of Carlitz each using a sign-reversing involution on RG-words. In Section 10 we prove two parameter q-Stirling identities, generalizing identities of Mercier and include a symmetric function reformulation. We end with concluding remarks.

2 Preliminaries

A word $w = w_1 w_2 \cdots w_n$ of length n where the entries are positive integers is called a *restricted growth word*, or *RG-word* for short, if w_i is bounded above by $\max(0, w_1, w_2, \ldots, w_{i-1})+1$ for all indices i. This class of words was introduced by Milne in the papers [24, 25]. The set of RG-words of length n where the largest entry is k is in bijective correspondence with set partitions of the set $\{1, 2, \ldots, n\}$ into k blocks. Namely, if $w_i = w_j$, place the elements i and j in the same block of the partition. To describe the inverse of this bijection, write the partition $\pi = B_1/B_2/\cdots/B_k$ in standard form, that is, with $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. The associated RG-word is given by $w = w_1 \cdots w_n$ where $w_i = j$ if the entry i appears in the jth block B_j of π.

One way to obtain a q-analogue of Stirling numbers of the second kind is to introduce a weight on RG-words. Let $RG(n,k)$ denote the set of all RG-words of length n with maximal entry k. Observe that $RG(n,k)$ is the empty set if $n < k$. The set $RG(n,0)$ is also empty for $n > 0$ but the set $RG(0,0)$ is the singleton set consisting of the empty word ϵ. Define the weight of $w = w_1 w_2 \cdots w_n \in RG(n,k)$ by

$$\mathrm{wt}(w) = q^{\sum_{i=1}^{n}(w_i - 1) - \binom{k}{2}}. \tag{2.1}$$

The *q-Stirling numbers of the second kind* are given by

$$S_q[n,k] = \sum_{w \in RG(n,k)} \mathrm{wt}(w). \tag{2.2}$$

See Cai and Readdy [2, Sections 2 and 3].

This definition satisfies the recurrence definition for q-Stirling numbers of the second kind originally due to Carlitz; see [3, pages 128–129] and [4, Section 3]:

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q \cdot S_q[n-1,k] \quad \text{for } 1 \leqslant k \leqslant n,$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$. To see this, consider a word $w \in RG(n,k)$. If the last letter is a left-to-right maxima then the word w is of the form $w = v \cdot k$ where $v \in RG(n-1,k-1)$, yielding the first term of the recurrence. Otherwise w is of the form $w = v \cdot i$ where $v \in RG(n-1,k)$ and $1 \leqslant i \leqslant k$, which yields the second term of the recurrence. The boundary conditions $S_q[n,0] = \delta_{n,0}$ and $S_q[0,k] = \delta_{0,k}$ also follow from the interpretation (2.2). For other weightings of RG-words which generate the q-Stirling numbers of the second kind, see [26] and [34].

For a word $w = w_1 w_2 \cdots w_n$ define the *length* of w to be $\ell(w) = n$. Similarly, define the ls-*weight* of w to be $\mathrm{ls}(w) = q^{\sum_{i=1}^{n}(w_i - 1)}$. This is a generalization of the ls-statistic of RG-words [34, Section 2]. The concatenation of two words u and v is denoted by $u \cdot v$. The word v is a *factor* of the word w if one can write $w = u_1 \cdot v \cdot v_2$. A word $v = v_1 v_2 \cdots v_k$ is a *subword* of w if there is a sequence $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $w_{i_j} = v_j$ for all $1 \leqslant j \leqslant k$. In other words, a factor of w is a subword consisting of consecutive entries. For S a set of positive integers, let S^k denote the set of all words of length k with entries in S. Furthermore, let S^* denote the union $S^* = \dot{\bigcup}_{k \geqslant 0} S^k$, that is, the set of all words with entries in S. Observe that when S is the empty set then S^* consists only of the empty word ϵ. Let $[j,k]$ denote the interval $[j,k] = \{i \in \mathbb{P} \,:\, j \leqslant i \leqslant k\}$.

Recall the q-Stirling numbers of the second kind are specializations of the homogeneous symmetric function h_{n-k}:

$$S_q[n,k] = h_{n-k}([1]_q, [2]_q, \ldots, [k]_q). \tag{2.3}$$

See for instance [22, Chapter I, Section 2, Example 11]. This follows directly by observing that a word $w \in RG(n,k)$ has a unique expansion of the form

$$w = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdots k \cdot u_k, \tag{2.4}$$

where u_i is a word in $[1,i]^*$. By summing over all words u_i for $i = 1, \ldots, k$ such that the sum of their lengths is $\ell(u_1) + \ell(u_2) + \cdots + \ell(u_k) = n - k$, equation (2.3) follows.

3 Recurrence related identities

In this section we focus on recurrence structured identities for the q-Stirling numbers of the second kind. The proofs we provide here use the combinatorics of RG-words.

We begin with Mercier's identity [23, Theorem 3]. This is a q-analogue of Jordan [18, equation 9, page 187]. Mercier's original proof of Theorem 3.1 was by induction. Later a combinatorial proof using 0-1 tableaux was given by de Médicis and Leroux [7]. In the same paper, de Médicis and Leroux proved Theorems 3.3 and 3.4 using 0-1 tableaux.

Theorem 3.1 (Mercier, 1990)**.** *For nonnegative integers n and k, the following identity holds:*

$$S_q[n+1,k+1] = \sum_{m=k}^{n} \binom{n}{m} \cdot q^{m-k} \cdot S_q[m,k]. \tag{3.1}$$

Proof. When $n < k$ there is nothing to prove. For any word $w \in RG(n+1,k+1)$, suppose there are m entries in w that are not equal to one. Remove the $n+1-m$ entries equal to one in w and then subtract one from each of the remaining m entries to obtain a new word u. Observe $u \in RG(m,k)$ and $\mathrm{wt}(w) = q^{m-k} \cdot \mathrm{wt}(u)$. Conversely, given a word $u \in RG(m,k)$, one can first increase each of the m entries by one and then insert $n+1-m$ ones into the word to obtain an RG-word $w \in RG(n+1,k+1)$. There are $\binom{n}{n-m} = \binom{n}{m}$ ways to insert the $n+1-m$ ones since the first entry in an RG-word must be one. In other words, for any $u \in RG(m,k)$ we can obtain $\binom{n}{m}$ new RG-words in $RG(n+1,k+1)$ under the map described above, which gives the desired identity. $\square$

Using similar ideas we also prove the following q-identity. It is a q-analogue of a result due to Jordan [18, equation 7, page 187].

Theorem 3.2. *For two non-negative integers n and m, the following identity holds:*

$$q^{n-m} \cdot S_q[n,m] = \sum_{k=m}^{n} (-1)^{n-k} \cdot \binom{n}{k} \cdot S_q[k+1,m+1]. \tag{3.2}$$

Proof. For a subset $A \subseteq \{2,3,\ldots,n+1\}$ observe that the sum over the weights of RG-words in the set $RG(n+1,m+1)$ with ones in the set of positions containing the set A is given by $S_q[n+1-|A|,m+1]$. Hence by inclusion-exclusion the right-hand side of equation (3.2) is the sum of the weights of all words in $RG(n+1,m+1)$ where the element 1 only occurs in first position. This set of RG-words is also obtained by taking a word in $RG(n,m)$, adding one to each entry, which multiplies the weight by q^{n-m}, and concatenating it with a one on the left. $\square$

Theorems 3.3 and 3.4 appear in [7, Propositions 2.3 and 2.5]. We now give straightforward proofs of each result using RG-words.

Theorem 3.3 (de Médicis–Leroux, 1993)**.** *For nonnegative integers n and k, the following identity holds:*

$$S_q[n+1,k+1] = \sum_{j=k}^{n} [k+1]_q^{n-j} \cdot S_q[j,k]. \tag{3.3}$$

Proof. Factor a word $w \in RG(n+1, k+1)$ as $w = x \cdot (k+1) \cdot y$ where $x \in RG(j, k)$ for some $j \geqslant k$ and y belongs to $[1, k+1]^*$. The factor y has length $n - j$. The sum of the weights of these words is $[k+1]_q^{n-j} \cdot S_q[j, k]$. The result follows by summing over all possible integers j. $\square$

Theorem 3.4 (de Médicis–Leroux, 1993). *For nonnegative integers n and k, the following identity holds:*

$$(n-k) \cdot S_q[n, k] = \sum_{j=1}^{n-k} S_q[n-j, k] \cdot ([1]_q^j + [2]_q^j + \cdots + [k]_q^j). \tag{3.4}$$

Proof. For a word $w \in RG(n, k)$ consider factorizations $w = x \cdot y \cdot z$ with the following two properties: (1) the rightmost letter of the factor x, call this letter i, is a left-to-right maxima of x, and (2) the word y is non-empty and all letters of y are at most i.

We claim that the number of such factorizations of w is $n - k$. Let s_i be the number of letters between the first occurrence of i and the first occurrence of $i + 1$, and let s_k be the number of letters after the first occurrence of k. For a particular i, we have s_i choices for the word y. But $\sum_{i=1}^{k} s_i = n - k$ since there are $n - k$ repeated letters in w. This completes the claim.

Fix integers $1 \leqslant j \leqslant n - k$ and $1 \leqslant i \leqslant k$. Given a word $u \in RG(n - j, k)$, we can factor it uniquely as $x \cdot z$, where the last letter of x is the first occurrence of i in the word u. Pick y to be any word of length j with letters at most i. Finally, let $w = x \cdot y \cdot z$. Observe that this is a factorization satisfying the conditions from the previous paragraph. Furthermore, we have $\mathrm{wt}(w) = \mathrm{wt}(u) \cdot \mathrm{ls}(y)$. Summing over all words $u \in RG(n, k)$ and words $y \in [1, i]^j$ yields $S_q[n-j, k] \cdot [i]_q^j$. Lastly, summing over all i and j gives the desired equality. $\square$

4 Gould's generating function

Gould [15, equation (3.4)] gave an analytic proof for the ordinary generating function of the q-Stirling numbers of the second kind. Later Ernst [13, Theorem 3.22] gave a proof using the orthogonality of the q-Stirling numbers of the first and second kinds. Wachs and White [34] stated a p, q-version of this generating function without proof. Here we prove Gould's q-generating function using RG-words.

Theorem 4.1 (Gould, 1961). *The q-Stirling numbers of the second kind $S_q[n, k]$ have the generating function*

$$\sum_{n \geqslant k} S_q[n, k] \cdot t^n = \frac{t^k}{\prod_{i=1}^{k} (1 - [i]_q \cdot t)}. \tag{4.1}$$

Proof. The left-hand side of (4.1) is the sum of over all RG-words w of length at least k with largest letter k where each term is $\mathrm{wt}(w) \cdot t^{\ell(w)}$. Using the expansion in equation (2.4), that is, $w = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdots k \cdot u_k$ where u_i is a word in $[1, i]^*$, observe the weight of w factors as $\mathrm{wt}(w) = \mathrm{ls}(u_1) \cdot \mathrm{ls}(u_2) \cdots \mathrm{ls}(u_k)$ whereas the term $t^{\ell(w)}$ factors as $t^k \cdot t^{\ell(u_1)} \cdot t^{\ell(u_2)} \cdots t^{\ell(u_k)}$.

Since there are no restrictions on the length of ith word u_i, all the words u_i for $i = 1, \ldots, k$ together contribute the factor $1 + [i]_q \cdot t + [i]_q^2 \cdot t^2 + \cdots = 1/(1 - [i]_q \cdot t)$. By multiplying together all the contributions from all of the words u_i, the identity follows. $\square$

5 A poset proof of Carlitz's identity

In this section we state a poset decomposition theorem for the Cartesian product of chains. This decomposition implies Carlitz's identity. For basic poset terminology and background, we refer the reader to Stanley's treatise [29, Chapter 3].

Let C_m denote the chain on m elements. Recall that $\mathbb{P}^n$ is the set of all words of length n having positive integer entries. We make this set into a poset, in fact, a lattice, by entrywise comparison, where the partial order relation is given by $v_1 v_2 \cdots v_n \leqslant w_1 w_2 \cdots w_n$ if and only if $v_i \leqslant w_i$ for all indices $1 \leqslant i \leqslant n$. Note that $[1, m]^n$ is the subposet consisting of all words of length n where the entries are at most m.

For a word v in $RG(n, k)$, factor v according to equation (2.4), that is, write v as the product $v = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdots u_{k-1} \cdot k \cdot u_k$, where each factor u_i belongs to $[1, i]^*$. For $m \geqslant n$ define the word $\omega_m(v) = m \cdot u_1 \cdot m \cdot u_2 \cdots u_{k-1} \cdot m \cdot u_k$. Effectively, each left-to-right maxima is replaced by the integer m. Directly it is clear that the interval $[v, \omega_m(v)]$ in $\mathbb{P}^n$ is isomorphic to a product of chains, that is,

$$[v, \omega_m(v)] \cong C_m \times C_{m-1} \times \cdots \times C_{m-k+1}.$$

Theorem 5.1. *The n-fold Cartesian product of the m-chain has the decomposition*

$$[1, m]^n = \dot{\bigcup_{0 \leqslant k \leqslant \min(m,n)}} \dot{\bigcup_{v \in RG(n,k)}} [v, \omega_m(v)].$$

Proof. Define a map $f : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ as follows. Let $w = w_1 w_2 \cdots w_n$ be a word in $\mathbb{P}^n$. If w is an RG-word, let $f(w) = w$. Otherwise let i be the smallest index in w reading from left to right that makes w fail to be an RG-word. In other words, i is the smallest index such that $\max(0, w_1, w_2, \ldots, w_{i-1}) + 1 < w_i$. Let $f(w)$ be the new word formed by replacing the ith entry of w with $\max(0, w_1, w_2, \ldots, w_{i-1}) + 1$. Observe that for all words w we obtain the poset inequality $f(w) \leqslant w$.

Since the word w only has n entries, we know that the $(n + 1)$st iteration of f is equal to the nth iteration of f, that is, $f^{n+1}(w) = f^n(w)$. Furthermore, $f^n(w)$ is an RG-word. Finally, define $\varphi : \mathbb{P}^n \longrightarrow \dot{\bigcup}_{0 \leqslant k \leqslant n} RG(n, k)$ to be the map f^n. Observe that φ is a surjection since every RG-word is a fixed point. Furthermore, for all words w the inequality $\varphi(w) \leqslant w$ holds in the poset $\mathbb{P}^n$.

Let v be a word in $RG(n, k)$. Use the expansion (2.4) to write v in the form $v = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdots u_{k-1} \cdot k \cdot u_k$, where $u_i \in [1, i]^*$. It is straightforward to check that the fiber $\varphi^{-1}(v)$ is given by

$$\varphi^{-1}(v) = \{j_1 \cdot u_1 \cdot j_2 \cdot u_2 \cdots u_{k-1} \cdot j_k \cdot u_k \ : \ i \leqslant j_i \text{ for } i = 1, 2, \ldots, k\}. \tag{5.1}$$

Observe that as a poset this fiber is isomorphic to $\mathbb{P}^k$. When we restrict to $[1,m]^n$ we obtain that the intersection $\varphi^{-1}(v) \cap [1,m]^n$ is the interval $[v, \omega_m(v)]$. Taking the disjoint union over all RG-words v, the decomposition follows. $\square$

By considering the rank generating function of Theorem 5.1, we can obtain a poset theoretic proof of Carlitz's identity [4, Section 3]. Other proofs are due to Milne using finite operator techniques on restricted growth functions [25], de Médicis and Leroux via interpreting the identity as counting products of matrices over the finite field $\mathrm{GF}(q)$ having non-zero columns [7], and Ehrenborg and Readdy using the theory of juggling sequences [12].

Corollary 5.2 (Carlitz, 1948)**.** *The following q-identity holds:*

$$[m]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \cdot S_q[n,k] \cdot [k]_q! \cdot \begin{bmatrix} m \\ k \end{bmatrix}_q . \tag{5.2}$$

Proof. The cases when $n = 0$ or $m = 0$ are straightforward. The rank generating function of the left-hand side of Theorem 5.1 is $[m]_q^n$. The rank generating function of the interval $[v, \omega_m(v)]$ is $q^{\binom{k}{2}} \cdot \mathrm{wt}(v) \cdot [m]_q \cdot [m-1]_q \cdots [m-k+1]_q$. By summing over all RG-words, the result follows. $\square$

Remark 5.3. *Similar poset techniques used to prove Theorem 5.1 and Corollary 5.2 can be applied to obtain the identities in Section 3. The proofs are omitted.*

The map φ that appears in the proof of Theorem 5.1 has interesting properties.

Proposition 5.4. *The map $\varphi : \mathbb{P}^n \longrightarrow \dot{\bigcup}_{0 \leqslant k \leqslant n} RG(n,k)$ is the dual of a closure operator, that is, it satisfies the following three properties:*

(i) $\varphi(w) \leqslant w$,

(ii) $\varphi^2(w) = \varphi(w)$ *and*

(iii) $v \leqslant w$ *implies that* $\varphi(v) \leqslant \varphi(w)$.

Proof. Properties (i) and (ii) are direct from the construction of the map φ. To prove property (iii) assume that we have $v \leqslant w$ but $\varphi(v) \not\leqslant \varphi(w)$. Let i be the smallest index such that $\varphi(w)_i < \varphi(v)_i$. Especially for $j < i$ we have $\varphi(w)_j \geqslant \varphi(v)_j$. Since $\varphi(w)_i < \varphi(v)_i \leqslant v_i \leqslant w_i$, we know that the ith coordinate is changed when computing $\varphi(w)$. Hence $\varphi(w)_i = \max(0, \varphi(w)_1, \ldots, \varphi(w)_{i-1}) + 1 \geqslant \max(0, \varphi(v)_1, \ldots, \varphi(v)_{i-1}) + 1 \geqslant \varphi(v)_i$, a contradiction. $\square$

As a consequence of the closure property, if v and w are RG-words then so is their join $v \vee w = u$ where the ith entry is given by $u_i = \max(v_i, w_i)$. This can also be proven directly. Note however that the set of RG-words is not closed under the meet operation; see for instance the two RG-words 1123 and 1213.

6 q-Vandermonde convolutions

Verde-Star gave Vandermonde convolution identities for Stirling numbers of the second kind [32, equations (6.24), (6.25)]. Chen gave a grammatical proof for the first of these identities [5, Proposition 4.1]. For q-analogues of both identities, de Médicis and Leroux used $0,1$-tableaux for their argument [8, equations (1.12), (1.14)]. In this section we present combinatorial proofs of the de Médicis–Leroux results using RG-words.

As a remark, Theorem 3.1 is the special case of $n = 1$ in Theorem 6.1.

Theorem 6.1 (de Médicis–Leroux, 1995)**.** *The following q-Vandermonde convolution holds for q-Stirling numbers of the second kind:*

$$S_q[m+n,k] = \sum_{i+j \geqslant k} \binom{m}{j} \cdot q^{i\cdot(i+j-k)} \cdot [i]_q^{m-j} \cdot S_q[n,i] \cdot S_q[j,k-i]. \tag{6.1}$$

Proof. Given a word $w \in RG(m+n,k)$, factor it as $w = u \cdot z$ where u has length n and i is the largest entry in u. By assumption, $u \in RG(n,i)$.

The second factor $z = w_{n+1} \cdot w_{n+2} \cdots w_{n+m}$ has length m and its maximal entry is at most k. In particular, if $i < k$ then the maximal entry for z is exactly k. Suppose there are j entries in z that are strictly larger than i. These j entries from z form a subword v. Denote by $v^{(-i)}$ the shift of v by subtracting i from each entry in v. It is straightforward to check that $v^{(-i)} \in RG(j,k-i)$.

For any word $w \in RG(m+n,k)$, we can decompose it as described above. In such a decomposition, the first segment u contributes to a factor of $S_q[n,i]$. The subsequence v of the second segment z contributes to a factor of $S_q[j,k-i] \cdot q^{i\cdot(j-(k-i))}$ since the shift $v^{(-i)}$ causes a weight loss of q^i from each of the $j-(k-i)$ repeated entries in v. Finally, the remaining entries in z that are less than or equal to i range from 1 to i. Each will contribute to a factor of $[i]_q$. These $m-j$ entries can be assigned at any position in z, which gives $\binom{m}{j}$ choices. Multiplying all these weights, we obtain the desired identity. □

Note that Theorem 3.3 is a special case of Theorem 6.2 when one takes $r = 0$.

Theorem 6.2 (de Médicis–Leroux, 1995)**.** *The following q-Vandermonde convolution holds for q-Stirling numbers of the second kind:*

$$S_q[n+1,k+r+1] = \sum_{i=0}^{n} \sum_{j=r}^{i} \binom{i}{j} \cdot q^{(k+1)\cdot(j-r)} \cdot [k+1]_q^{i-j} \cdot S_q[j,r] \cdot S_q[n-i,k]. \tag{6.2}$$

Proof. This result is proved in a similar fashion as Theorem 6.1. For any $w \in RG(n+1,k+r+1)$, suppose w is of the form $x \cdot (k+1) \cdot y$ where $x \in RG(n-i,k)$ for some i. Consider the remaining word $y = w_{n-i+2} \cdots w_{n+1}$ of length i. The maximal entry of y is $k+r+1$. Suppose there are j entries in y that are at least $k+2$. These j entries form a subword v, and $v^{(-k-1)}$, obtained by subtracting $k+1$ from each entry in v, is an RG-word in $RG(j,r)$, giving a total weight of $S_q[j,r]$. The weight loss from the shift is $q^{(k+1)\cdot(j-r)}$ since there are $j-r$ repeated entries. The remaining $i-j$ entries in y can be any value from the interval $[1, k+1]$. Each such entry contributes to a factor of $[k+1]_q$. Finally, there are $\binom{i}{j}$ ways to place the j entries back into u. This proves identity (6.2). $\square$

Remark 6.3. *Theorem 3.2 can be viewed as an inversion of Theorem 3.1. Furthermore, Theorem 3.1 is a special case of Theorem 6.1. Is there any sort of natural inversion analogue to Theorem 6.1?*

7 A q-analogue of the Frobenius identity

We now prove a q-analogue of the Frobenius identity by Garsia and Remmel [14, equation I.1].

Theorem 7.1 (Garsia–Remmel, 1986)**.** *The following q-Frobenius identity holds:*

$$\sum_{m \geqslant 0} [m]_q^n \cdot x^m = \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} \cdot S_q[n,k] \cdot [k]_q! \cdot x^k}{(1-x)\cdot(1-qx)\cdots(1-q^k x)}. \tag{7.1}$$

Proof. When $n = 0$ the result is direct. We concentrate on the case $n > 0$. For a word w in $\mathbb{P}^n$ let $\max(w)$ denote its maximal entry. Hence the left-hand side of equation (7.1) is given by

$$\sum_{m \geqslant 0} [m]_q^n \cdot x^m = \sum_{m \geqslant 0} \sum_{\substack{w \in \mathbb{P}^n \\ \max(w) \leqslant m}} \mathrm{ls}(w) \cdot x^m.$$

Recall the poset map $\varphi : \mathbb{P}^n \longrightarrow \bigcup_{0 \leqslant k \leqslant n} RG(n,k)$ appearing in the proof of Theorem 5.1. Let $v \in RG(n,k)$. The fiber $\varphi^{-1}(v)$ is given in equation (5.1). The sum over this fiber appears in the proof of Corollary 5.2, that is,

$$\sum_{\substack{w \in \varphi^{-1}(v) \\ \max(w) \leqslant m}} \mathrm{ls}(w) = \mathrm{wt}(v) \cdot q^{\binom{k}{2}} \cdot [m]_q \cdot [m-1]_q \cdots [m-k+1]_q.$$

Multiplying the above by x^m and summing over all $m \geqslant 0$ yields

$$\begin{aligned} \sum_{m \geqslant 0} \sum_{\substack{w \in \varphi^{-1}(v) \\ \max(w) \leqslant m}} \mathrm{ls}(w) \cdot x^m &= \mathrm{wt}(v) \cdot q^{\binom{k}{2}} \cdot [k]_q! \cdot \sum_{m \geqslant 0} \begin{bmatrix} m \\ k \end{bmatrix}_q \cdot x^m \\ &= \frac{\mathrm{wt}(v) \cdot q^{\binom{k}{2}} \cdot [k]_q! \cdot x^k}{(1-x)\cdot(1-qx)\cdots(1-q^k x)}. \end{aligned}$$

The result now follows by summing over all RG-words of length n. $\square$

8 A determinantal identity

The following identity was first stated by Ehrenborg [10, Theorem 3.1] who proved it using juggling patterns. We now present a proof using RG-words.

Theorem 8.1 (Ehrenborg, 2003)**.** *Let n and s be non-negative integers. Then the following identity holds:*

$$\det(S_q[s+i+j, s+j])_{0\leqslant i,j\leqslant n} = [s]_q^0 \cdot [s+1]_q^1 \cdots [s+n]_q^n.$$

Proof. Let T be the set of all $(n+2)$-tuples $(\sigma, w(0), w(1), \ldots, w(n))$ where σ is a permutation of the $n+1$ elements $\{0, 1, \ldots, n\}$, and $w(i)$ is a word in $RG(s+i+\sigma(i), s+\sigma(i))$ for all $0 \leqslant i \leqslant n$. The determinant expands as the sum

$$\det(S_q[s+i+j, s+j])_{0\leqslant i,j\leqslant n} = \sum_{(\sigma,w(0),\ldots,w(n))\in T} (-1)^{\sigma} \cdot \mathrm{wt}(w(0)) \cdot \mathrm{wt}(w(1)) \cdots \mathrm{wt}(w(n)).$$

Factor the word $w(i) \in RG(s+i+\sigma(i), s+\sigma(i))$ as $w(i) = u(i) \cdot v(i)$ where the lengths are given by $\ell(u(i)) = s+i$ and $\ell(v(i)) = \sigma(i)$. Furthermore, let a_i denote the number of repeated entries in the RG-word $w(i)$ that appear in the factor $v(i)$, that is,

$$a_i = |\{j \,:\, j > s+i, w(i)_j = w(i)_r \text{ for some } r < j\}|.$$

There are $\sigma(i) - a_i$ left-to-right maxima of $w(i)$ that appear in $v(i)$. Since $w(i)$ has $s+\sigma(i)$ left-to-right maxima, we obtain that the first factor $u(i)$ has $(s+\sigma(i)) - (\sigma(i) - a_i) = s + a_i$ left-to-right maxima, that is, the factor $u(i)$ belongs to the set $RG(s+i, s+a_i)$. To be explicit, the left-to-right maxima of $u(i)$ are given by $1, 2, \ldots, s+a_i$. Lastly, observe that there are i repeated entries in any word $w(i) \in RG(s+i+\sigma(i), s+\sigma(i))$ and $\sigma(i)$ is the length of $v(i)$, yielding the bound $a_i \leqslant \min(i, \sigma(i))$ for all i.

Let $T_1 \subseteq T$ consist of all tuples $(\sigma, w(0), \ldots, w(n))$ where the sequence of a_i's for $i = 0, \ldots, n$ are distinct. This implies $a_i = i = \sigma(i)$, that is, σ is the identity permutation. Furthermore, the first factor $u(i)$ is equal to $12\cdots(s+i)$ and the second factor $v(i)$ can be any word of length i with the entries from the interval $[1, s+i]$. Thus $\mathrm{wt}(w(i)) = \mathrm{ls}(v(i))$ and the sum over all such words $v(i)$ gives a total weight of $[s+i]_q^i$. Thus we have

$$\sum_{(\sigma,w(0),\ldots,w(n))\in T_1} (-1)^{\sigma} \cdot \mathrm{wt}(w(0)) \cdot \mathrm{wt}(w(1)) \cdots \mathrm{wt}(w(n)) = \prod_{i=0}^{n} [s+i]_q^i. \tag{8.1}$$

Let $T_2 = T - T_1$ be the complement of T_1. Define a sign-reversing involution φ on T_2 as follows. For $t = (\sigma, w(0), \ldots, w(n)) \in T_2$ there exists indices i_1 and i_2 such that $a_{i_1} = a_{i_2}$. Let (j, k) be the least such pair of indices in the lexicographic order. First let $\sigma' = \sigma \circ (j, k)$ where (j, k) denotes the transposition. Second, let $w(i)' = w(i)$ for $i \neq j, k$. Finally, define $w(j)'$ and $w(k)'$ by switching the second factors in the factorizations, that is, $w(j)' = u(j)v(k)$ and $w(k)' = u(k)v(j)$. Overall, the function is given by $\varphi(t) = (\sigma', w(0)', \ldots, w(n)')$

Since $u(j)$ and $u(k)$ have the same number of left-to-right maxima, it is straightforward to check that $w(j)' = u(j)v(k)$ belongs to $RG(s+j+\sigma(k), s+\sigma(k)) = RG(s+j+\sigma'(j), s+\sigma'(j))$. Hence it follows that $\varphi(t) \in T_2$. Let a_i' be the number repeated entries in $w(i)'$ that occur beyond position $s+i$. Directly, we have $a_i' = a_i$ and we obtain that φ is an involution. Finally, we have $(-1)^{\sigma'} = -(-1)^{\sigma}$ implying φ is a sign-reversing involution.

Finally, it is direct to see that $\mathrm{wt}(w(j)) \cdot \mathrm{wt}(w(k)) = \mathrm{wt}(w(j)') \cdot \mathrm{wt}(w(k)')$ using the observation that $u(j)$ and $u(k)$ have the same number of left-to-right maxima. Hence the map φ is a sign-reversing involution on T_2 which preserves the weight $\mathrm{wt}(w(0)) \cdots \mathrm{wt}(w(n))$. Thus the determinant is given by equation (8.1). $\square$

9 A pair of identities of Carlitz

In this section we turn our attention to two identities of Carlitz. Observe that setting $q = 1$ in Theorems 9.2 and 9.3 in this section does not yield any information about the Stirling number $S(n, k)$ of the second kind.

We first prove a theorem from which Carlitz's Theorem 9.2 will follow.

Theorem 9.1. *For two non-negative integers n and k not both equal to 0, the following identity holds:*

$$(1-q)^{n-k} \cdot S_q[n, k] = \sum_{j=0}^{n-k} (-q)^j \cdot \binom{n-1}{n-k-j} \cdot \begin{bmatrix} j+k-1 \\ j \end{bmatrix}_q . \tag{9.1}$$

Proof. For a word $u = u_1 u_2 \cdots u_n$ in $RG(n, k)$ let $\mathrm{NLRM}(u)$ be the set of all positions r such that the letter u_r is not a left-to-right-maxima of the word u, that is, $u_r \leqslant \max(u_1, u_2, \ldots, u_{r-1})$. Furthermore, for a position $r \in NLRM(u)$ define the bound $b(r)$ to be $\max(u_1, u_2, \ldots, u_{r-1})$. Note that $b(r)$ is the largest possible value we could change u_r to be so that the resulting word remains an RG-word in $RG(n, k)$.

To describe the left-hand side of (9.1), consider the set of pairs (u, P) where $u \in RG(n, k)$ and $P \subseteq \mathrm{NLRM}(u)$. Define the weight of such a pair (u, P) to be $(-q)^{|P|} \cdot \mathrm{wt}(u)$. It is clear that the sum of the weight over all such pairs (u, P) is given by the left-hand side of (9.1).

Define a sign-reversing involution as follows. For the pair (u, P) pick the smallest position r in $\mathrm{NLRM}(u)$ such that either $r \in P$ and $u_r \leqslant b(r) - 1$ or $r \notin P$ and $2 \leqslant u_r$. In the first case, send P to $P - \{r\}$ and replace the rth letter u_r with $u_r + 1$. In the second case, send P to $P \cup \{r\}$ and replace the rth letter u_r with $u_r - 1$. This is a sign-reversing involution which pairs terms having the same weight, but opposite signs. Furthermore, the remaining pairs (u, P) satisfy for all positions $r \in \mathrm{NLRM}(u)$ either $r \in P$ and $u_r = b(r)$ or $r \notin P$ and $u_r = 1$.

We now sum the weight of these remaining pairs (u, P). First select the cardinality j of the set P. Note that $0 \leqslant j \leqslant n - k$ and that it yields a factor of $(-q)^j$. Second, select the positions r of the non-left-right-maxima such that r will not be in the set P and $u_r = 1$. There will be $n - k - j$ such positions and they can be anywhere in the interval $[2, n]$, yielding $\binom{n-1}{n-k-j}$ possibilities. Third, select a weakly increasing word z of length j with letters from the set $[k]$. This will be the letters corresponding to the non-left-to-right-maxima such that their positions belong to set P. The ls-weight of these letters will be the q-binomial coefficient $\begin{bmatrix} j+k-1 \\ j \end{bmatrix}_q$. Fourth, insert into the word z the letters of the left-to-right-maxima. There is a unique way to do this insertion. Finally, insert the 1's corresponding to positions not in the set P, which were already been chosen by the binomial coefficient. $\square$

Note that the above proof fails when $n = k = 0$ since we are using that a non-empty *RG*-word must begin with the letter 1, whereas the empty *RG*-word does not.

The next identity is due to Carlitz [3, equation (9)]. It was stated by Gould [15, equation (3.10)]. As a warning to the reader, Gould's notation $S_2(n, k)$ for the q-Stirling number of the second kind is related to ours by $S_2(n, k) = S_q[n + k, n]$. This identity also appears in the paper of de Médicis, Stanton and White [9, equation (3.3)] using modern notation.

Theorem 9.2 (Carlitz, 1933)**.** *For two non-negative integers n and k the following identity holds:*

$$(1-q)^{n-k} \cdot S_q[n,k] = \sum_{j=0}^{n-k} (-1)^j \cdot \binom{n}{k+j} \cdot \begin{bmatrix} j+k \\ j \end{bmatrix}_q . \tag{9.2}$$

Proof. When $n = k = 0$ the statement is direct. It is enough to show that the right-hand sides of equations (9.1) and (9.2) agree. We have

$$\begin{aligned}
&\sum_{j=0}^{n-k} (-1)^j \cdot \binom{n-1}{n-k-j} \cdot q^j \cdot \begin{bmatrix} j+k-1 \\ j \end{bmatrix}_q \\
&\quad= \sum_{j=0}^{n-k} (-1)^j \cdot \binom{n-1}{n-k-j} \cdot \left(\begin{bmatrix} j+k \\ j \end{bmatrix}_q - \begin{bmatrix} j+k-1 \\ j-1 \end{bmatrix}_q \right) \\
&\quad= \sum_{j=0}^{n-k} (-1)^j \cdot \binom{n-1}{n-k-j} \cdot \begin{bmatrix} j+k \\ j \end{bmatrix}_q - \sum_{j=1}^{n-k} (-1)^j \cdot \binom{n-1}{n-k-j} \cdot \begin{bmatrix} j+k-1 \\ j-1 \end{bmatrix}_q \\
&\quad= \sum_{j=0}^{n-k} (-1)^j \cdot \binom{n-1}{n-k-j} \cdot \begin{bmatrix} j+k \\ j \end{bmatrix}_q - \sum_{j=0}^{n-k-1} (-1)^{j+1} \cdot \binom{n-1}{n-k-j-1} \cdot \begin{bmatrix} j+k \\ j \end{bmatrix}_q \\
&\quad= \sum_{j=0}^{n-k} (-1)^j \cdot \binom{n}{n-k-j} \cdot \begin{bmatrix} j+k \\ j \end{bmatrix}_q .
\end{aligned}$$

Here we used the Pascal recursion for the q-binomial coefficients in the first step, shifted j to $j + 1$ in the second sum in the third step, and applied the Pascal recursion for the binomial coefficients in the last step. $\square$

The next identity is also due to Carlitz; see [3, equation (8)]. It is equivalent to the previous identity, but we provide a proof using *RG*-words.

Theorem 9.3 (Carlitz, 1933)**.** *For n and k two non-negative integers the following identity holds:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=k}^{n} (q-1)^{j-k} \cdot \binom{n}{j} \cdot S_q[j,k].$$

Proof. The right-hand side describes the following collection of triplets (A, u, P) where A is a subset of the set $[n]$, u is a word in $RG(|A|, k)$ and P is a subset of $\mathrm{NLRM}(u)$. Define the weight of the triple (A, u, P) to be the product $(-1)^{j-k-|P|} \cdot q^{|P|} \cdot \mathrm{wt}(u)$. To better visualize the pair (A, u), define the word $w = w_1 w_2 \cdots w_n$ of length n with the letters in the set $\{0\} \cup [k]$ as follows. Write A as the increasing set $\{a_1 < a_2 < \cdots < a_j\}$ and let $w_{a_r} = u_r$. The remaining letters of w are set to be 0, that is, if $i \notin A$ let $w_i = 0$. Note that the word w uniquely encodes the pair (A, u). Let Q be the set $Q = \{a_r : r \in P\}$, that is, the set Q encodes the subset P, where these non-left-to-right-maxima occur in the longer word w.

Similar to the proof of Theorem 9.1 we define a sign-reversing involution by selecting the smallest $r \in \mathrm{NLRM}(w)$ such that $r \notin Q$ and $w_r \geqslant 2$, or $r \in Q$ and $1 \leqslant w_r \leqslant b(r) - 1$. In the first case decrease w_r by 1 and join r to the subset Q. In the second case, increase w_r by 1 and remove r from Q. The remaining words w satisfy the following: for a non-left-to-right-maxima r such that $w_r \geqslant 1$ either $r \notin Q$ and $w_r = 1$ both hold, or $r \in Q$ and $w_r = b(r)$ hold.

On the remaining pairs (w, Q) define yet again a sign-reversing involution. Let $i > a_1$ be the smallest index i such that $w_i = 0$, or $w_i = 1$ and $i \notin Q$. This involution replaces w_i with $1 - w_i$. Note that this involution changes the sign.

The pairs (w, Q) which remain unmatched under this second involution are those where the word w is weakly increasing and the subset Q consists of all non-left-to-right-maxima r with $w_r \geqslant 1$. Note that the weight of the pair (w, Q) is the weight $q^{|Q|} \cdot q^{\sum_{r \in Q} w_r - 1} = q^{\sum_{r \in Q} w_r}$. Finally, the sum of the weights of these pairs is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. ◻

10 An extension of Mercier's identities

We simultaneously generalize two identities of Mercier by introducing a two parameter identity involving q-Stirling numbers. The proof of this identity depends on a different decomposition of RG-words.

Let $([n]_q)_k$ denote the q-analogue of the lower factorial, that is, $([n]_q)_k = [n]_q!/[n-k]_q!$. Alternatively, one can expand it as the product

$$([n]_q)_k = [n]_q \cdot [n-1]_q \cdots [n-k+1]_q.$$

Theorem 10.1. *For three non-negative integers n, r and s such that $s < r \leqslant n$ the following holds:*

$$\sum_{k=r}^{n} (-q^s)^{k-r} \cdot ([k-s-1]_q)_{k-r} \cdot S_q[n, k] = \sum_{i=r-1}^{n-1} S_q[i, r-1] \cdot [s]_q^{n-i-1}. \tag{10.1}$$

Proof. On the set of RG-words $S = \bigcup_{r \leqslant k \leqslant n} RG(n, k)$ define a weight function by

$$f(w) = (-q^s)^{k-r} \cdot ([k-s-1]_q)_{k-r} \cdot \mathrm{wt}(w).$$

Our objective is to evaluate the sum $\sum_{w \in S} f(w)$, which is the left-hand side of (10.1). We do this in two steps. First we will partition the set S into blocks and extend the weight f

to a block B by $f(B) = \sum_{w \in B} f(w)$. Secondly, on the set of blocks we will define a sign-reversing involution such that if two blocks B and C are matched, their f-weights cancel, that is, $f(B) + f(C) = 0$. Hence the right-hand side of (10.1) will be equal to the sum over all the blocks that have not been matched.

We now define an equivalence relation on the set S. The blocks of our partition will be the equivalence classes. For any integer k where $r \leqslant k \leqslant n$, we say two words $u, v \in RG(n,k)$ are equivalent if there exists an index i such that $s+1 \leqslant u_i, v_i \leqslant k$ and $u = x \cdot u_i \cdot y$, $v = x \cdot v_i \cdot y$ for some word $x \in RG(i-1,k)$ and y a word in $[1,s]^*$. Note that when $s = 0$ that y is the empty word ϵ. If a word $v \in RG(n,k)$ is of the form $v = x \cdot k \cdot y$ for some $x \in RG(i-1,k-1)$ and $y \in [1,s]^*$, then this word is not equivalent to any other words and hence it belongs to a singleton block. Since for any RG-word we can find a decomposition described as above, this is a partition of the set $RG(n,k)$ and hence the set S.

Match the singleton block $B = \{x \cdot k \cdot y\}$ where $x \in RG(i-1,k-1)$, $y \in [1,s]^*$ and $r < k$ with the block

$$C = \{x \cdot j \cdot y \ : \ s+1 \leqslant j \leqslant k-1\}.$$

It is straightforward to check that $C \subseteq RG(n,k-1) \subseteq S$. Note that the weight $\mathrm{wt}(x \cdot j \cdot y)$ factors as $\mathrm{wt}(x) \cdot q^{j-1} \cdot \mathrm{ls}(y)$. Moreover, the f-weight of the block C satisfies

$$\begin{aligned} f(C) &= \sum_{j=s+1}^{k-1} (-q^s)^{k-r-1} \cdot ([k-s-2]_q)_{k-r-1} \cdot \mathrm{wt}(x) \cdot q^{j-1} \cdot \mathrm{ls}(y) \\ &= (-q^s)^{k-r-1} \cdot ([k-s-2]_q)_{k-r-1} \cdot \mathrm{wt}(x) \cdot q^s \cdot [k-s-1]_q \cdot \mathrm{ls}(y) \\ &= -(-q^s)^{k-r} \cdot ([k-s-1]_q)_{k-r} \cdot \mathrm{wt}(x) \cdot \mathrm{ls}(y) \\ &= -f(x \cdot k \cdot y). \end{aligned} \tag{10.2}$$

Hence the weight of the two blocks B and C cancel each other.

It remains to determine the weight of the unmatched blocks. Observe that every block of the form $\{x \cdot j \cdot y : s+1 \leqslant j \leqslant k\}$ where $x \in RG(i,k)$ and $y \in [1,s]^*$ has been matched by the above construction. Hence the unmatched blocks are singleton blocks. Given a word $u \in RG(n,k) \subseteq S$, it has a unique factorization as $u = x \cdot j \cdot y$ where $y \in [1,s]^*$ and $s < j$. If x is a word in $RG(i,k)$ then the word u belongs to a block that has been matched. If $x \in RG(i,k-1)$ then $m = k$ and the word u belongs to a singleton block which has been matched if $k > r$. Hence the unmatched blocks are of the form $\{x \cdot r \cdot y\}$ where $x \in RG(i,r-1)$ and $y \in [1,s]^{n-i-1}$. The sum of their f-weights are

$$\begin{aligned} \sum_{i=r-1}^{n-1} \sum_{\substack{x \in RG(i,r-1) \\ y \in [1,s]^{n-i-1}}} f(x \cdot r \cdot y) &= \sum_{i=r-1}^{n-1} \sum_{\substack{x \in RG(i,r-1) \\ y \in [1,s]^{n-i-1}}} \mathrm{wt}(x) \cdot \mathrm{ls}(y) \\ &= \sum_{i=r-1}^{n-1} S_q[i,r-1] \cdot [s]_q^{n-i-1}, \end{aligned}$$

which is the right-hand side of the desired identity. □

Setting $(r,s)=(1,0)$ and $(r,s)=(2,1)$ in Theorem 10.1 we obtain two special cases, both of which are due to Mercier [23, Theorem 2]. The second identity is a q-analogue of a result due to Jordan [18, equation 5, page 186].

Corollary 10.2 (Mercier, 1990)**.** *For $n \geqslant 2$, the following two identities hold:*

$$\sum_{k=1}^{n}(-1)^k \cdot [k-1]_q! \cdot S_q[n,k] = 0, \tag{10.3}$$

$$\sum_{k=2}^{n}(-1)^k \cdot q^{k-2} \cdot [k-2]_q! \cdot S_q[n,k] = n-1. \tag{10.4}$$

Mercier's identity (10.4) reappears in work of Ernst [13, Corollary 3.30].

The next result is the case $r = s+1$ in Theorem 10.1. Here we provide a different expression.

Proposition 10.3. *The following identity holds:*

$$\sum_{k=r}^{n}(-q^{r-1})^{k-r} \cdot ([k-r]_q)_{k-r} \cdot S_q[n,k] = \sum_{c_1+c_2+\cdots+c_{r-1}=n-r+1} c_{r-1} \cdot [1]_q^{c_1} \cdot [2]_q^{c_2} \cdots [r-2]_q^{c_{r-2}} \cdot [r-1]_q^{c_{r-1}-1}. \tag{10.5}$$

Proof. The f-weights of the unmatched words $u = x \cdot r \cdot y$ in the proof of Theorem 10.1 can be determined in a different manner. Since $x \in RG(i, r-1)$ we can factor x according to equation (2.4). Hence the unmatched word u has the form

$$u = 1 \cdot x_1 \cdot 2 \cdot x_2 \cdot 3 \cdots (r-1) \cdot x_{r-1} \cdot r \cdot y,$$

where x_i belongs to $[1,i]^*$ and y belongs to $[1,r-1]^*$. All possible words x_i of length c_i give total weight of $[i]_q^{c_i}$ for $i \leqslant r-2$. For the word $x_{r-1} \cdot r \cdot y$, suppose its length is c_{r-1}. Then we have c_{r-1} choices to place the letter r and the total weight of such words of the form $x_{r-1} \cdot y$ will be $[r-1]_q^{c_{r-1}-1}$. Hence the f-weight for unmatched words is given by equation (10.5). □

Recall the Stirling numbers of the second kind are specializations of the homogeneous symmetric function; see equation (2.3). Thus one can view Theorem 10.1 from a symmetric function perspective.

Theorem 10.4. *The following polynomial identity holds:*

$$\sum_{i=0}^{n-r} h_i(x_1, x_2, \ldots, x_{r-1}) \cdot x_s^{n-r-i} = \sum_{k=r}^{n}(x_s - x_r) \cdot (x_s - x_{r+1}) \cdots (x_s - x_{k-1}) \cdot h_{n-k}(x_1, x_2, \ldots, x_k).$$

Proof. Let $[t^n]f(t)$ denote the coefficient of t^n in $f(t)$. We consider the function

$$G_r(t) = \frac{1}{1 - x_s \cdot t} \cdot \prod_{j=1}^{r-1} \frac{1}{1 - x_j \cdot t} = \prod_{j=1}^{r} \frac{1}{1 - x_j \cdot t} + \frac{(x_s - x_r) \cdot t}{1 - x_s \cdot t} \cdot \prod_{j=1}^{r} \frac{1}{1 - x_j \cdot t},$$

and compute the coefficient of t^{n-r} in two ways. Using the first expression of $G_r(t)$ and that

$$\frac{1}{1 - x_s \cdot t} = \sum_{i \geqslant 0} x_s^i \cdot t^i \quad \text{and} \quad \prod_{j=1}^{r-1} \frac{1}{1 - x_j \cdot t} = \sum_{i \geqslant 0} h_i(x_1, x_2, \ldots, x_{r-1}) \cdot t^i,$$

we obtain

$$[t^{n-r}]G_r(t) = \sum_{i=0}^{n-r} h_j(x_1, x_2, \ldots, x_{r-1}) \cdot x_s^{n-r-i}. \tag{10.6}$$

Using the second expression of $G_r(t)$ we have

$$\begin{aligned} [t^{n-r}]G_r(t) &= [t^{n-r}] \prod_{j=1}^{r} \frac{1}{1 - x_j \cdot t} + [t^{n-r}] \frac{(x_s - x_r) \cdot t}{1 - x_s \cdot t} \cdot \prod_{j=1}^{r} \frac{1}{1 - x_j \cdot t} \\ &= h_{n-r}(x_1, x_2, \ldots, x_r) + (x_s - x_r) \cdot [t^{n-r-1}]G_{r+1}(t). \end{aligned} \tag{10.7}$$

Iterate equation (10.7) $n - r$ times yields

$$[t^{n-r}]G_r(t) = \sum_{k=r}^{n} (x_s - x_r) \cdot (x_s - x_{r+1}) \cdots (x_s - x_{k-1}) \cdot h_{n-k}(x_1, x_2, \ldots, x_k). \tag{10.8}$$

Now combine equations (10.6) and (10.8) we obtain the desired identity. □

Second proof of Theorem 10.1. Substituting $x_i = [i]_q$ in Theorem 10.4 yields the result using that $[s]_q - [i]_q = -q^s \cdot [i - s]_q$ when $i > s \geqslant 0$. □

11 Concluding remarks

There are many extensions of q-Stirling numbers of the second kind. Remmel and Wachs [27] develop (p, q)-analogues of q-Stirling numbers of the first and second kinds, and give a colored restricted growth word interpretation of Hsu and Shiue's generalized Stirling numbers [16]. An earlier two-parameter (p, q)-Stirling number of the second kind by Wachs and White [34] include interpretations via rook placements and restricted growth functions. What identities arise in these settings? As an example, Briggs and Remmel [1] have a (p, q)-analogue of Garsia and Remmel's q-Frobenius formula (Theorem 7.1) via three statistics: number of descents, major index and comajor index.

One can also be interested in other applications of restricted growth words. See the recent paper of Ehrenborg, Hedmark and Hettle for an application to partitions with block sizes of the same parity [11], as well as Steingrímsson's new statistics on ordered set partitions [30] and the related papers by Ishikawa, Kasraoui and Zeng [17, 20].

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