Improved Bounds for the Graham-Pollak Problem for Hypergraphs

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Abstract

For a fixed r, let $f_r(n)$ denote the minimum number of complete r-partite r-graphs needed to partition the complete r-graph on n vertices. The Graham-Pollak theorem asserts that $f_2(n) = n - 1$. An easy construction shows that $f_r(n) \leqslant (1 + o(1))\binom{n}{\lfloor r/2 \rfloor}$, and we write c_r for the least number such that $f_r(n) \leqslant c_r(1 + o(1))\binom{n}{\lfloor r/2 \rfloor}$.

It was known that $c_r < 1$ for each even $r \geqslant 4$, but this was not known for any odd value of r. In this short note, we prove that $c_{295} < 1$. Our method also shows that $c_r \to 0$, answering another open problem.

Keywords: Hypergraph, Decomposition, Graham-Pollak

1 Introduction

The edge set of K_n, the complete graph on n vertices, can be partitioned into $n-1$ complete bipartite subgraphs: this may be done in many ways, for example by taking $n - 1$ stars centred at different vertices. Graham and Pollak [4, 5] proved that the number $n - 1$ cannot be decreased. Several other proofs of this result have been found, by Tverberg [8], Peck [7], and Vishwanathan [9, 10], among others.

Generalising this to hypergraphs, for $n \geqslant r \geqslant 1$, let $f_r(n)$ be the minimum number of complete r-partite r-graphs needed to partition the edge set of $K_n^{(r)}$, the complete r-uniform hypergraph on n vertices (i.e., the collection of all r-sets from an n-set). Thus the Graham-Pollak theorem asserts that $f_2(n) = n - 1$. For $r \geqslant 3$, an easy upper bound of $\binom{n-\lceil r/2 \rceil}{\lfloor r/2 \rfloor}$ may be obtained by generalising the star example above. Indeed, for r even, having ordered the vertices, consider the collection of r-sets whose $2nd, 4th, \ldots, rth$

vertices are fixed. This forms a complete r-partite r-graph, and the collection of all $\binom{n-r/2}{r/2}$ such is a partition of $K_n^{(r)}$. For r odd, we instead fix the $2nd, 4th, \ldots, (r-1)th$ vertices, yielding a partition into $\binom{n-(r+1)/2}{(r-1)/2}$ parts.

Alon [1] showed that $f_3(n) = n-2$. More generally, for each fixed $r \geqslant 1$, he showed that

$$\frac{2}{\binom{2\lfloor r/2\rfloor}{\lfloor r/2\rfloor}}(1+o(1))\binom{n}{\lfloor r/2\rfloor} \leqslant f_r(n) \leqslant (1-o(1))\binom{n}{\lfloor r/2\rfloor},$$

where the upper bound follows from the construction above. Writing c_r for the least c such that $f_r(n) \leqslant c(1+o(1))\binom{n}{\lfloor r/2\rfloor}$, the above results assert that $c_2 = 1$, $c_3 = 1$, and $\frac{2}{\binom{2\lfloor r/2\rfloor}{\lfloor r/2\rfloor}} \leqslant c_r \leqslant 1$ for all r. How do the c_r behave?

Cioabă, Kündgen and Verstraëte [2] gave an improvement (in a lower-order term) to Alon's lower bound, and Cioabă and Tait [3] showed that the construction above is not sharp in general, but Alon's asymptotic bounds (i.e., the above bounds on c_r) remained unchanged. Recently, Leader, Milićević and Tan [6] showed that $c_r \leqslant \frac{14}{15}$ for each even $r \geqslant 4$. However, they could not improve the bound of $c_r \leqslant 1$ for any odd r – the point being that the construction above is better for r odd than for r even (the exponent of n is $(r-1)/2$ for r odd versus $r/2$ for r even), and so is harder to improve.

In this note, we give a simple argument to show that $c_{295} < 1$. Our method also shows that $c_r \to 0$, answering another question from [6].

It would be interesting to know what happens for smaller odd values of r: for example, is $c_5 < 1$? Determining the precise value of c_4 (i.e., the asymptotic behaviour of $f_4(n)$) would also be of great interest, as would determining the decay rate of the c_r. See [6] for several related questions and conjectures.

2 Main Result

The motivation for our proof is as follows. The key to the approach used in [6] in proving $c_r < 1$ for each even $r \geqslant 4$ was to investigate the minimum number of products of complete bipartite graphs, that is, sets of the form $E(K_{a,b}) \times E(K_{c,d})$, needed to partition the set $E(K_n) \times E(K_n)$. Writing $g(n)$ for this minimum value, it is trivial that $g(n) \leqslant (n-1)^2$, by taking the products of the complete bipartite graphs appearing in a decomposition of K_n into $n-1$ complete bipartite graphs. It was shown in [6] that

$$g(n) \leqslant \left(\frac{14}{15} + o(1)\right) n^2. \tag{1}$$

It turned out that this upper bound on $g(n)$ was enough (via an iterative construction) to bound c_r below 1 for each even $r \geqslant 4$.

Now, as remarked above, for r odd the construction in the Introduction is much better than for r even. In fact, while there are many iterative ways to redo the construction when r is even, passing from $n/2$ to n, these fail when r is odd: it turns out that an extra factor is introduced at each stage. However, rather unexpectedly, we will see that (at least if r is large) if we partition into *many* pieces, instead of just two pieces, then the gain we obtain from the 14/15 improvement in $g(n)$ outweighs the loss arising from this extra factor – even though this extra factor grows as the number of pieces grows.

A *minimal decomposition* of a complete r-partite r-graph $K_n^{(r)}$ is a partition of the edge set into $f_r(n)$ complete r-partite r-graphs. A *block* is a product of the edge sets of two complete bipartite graphs. Similarly, a *minimal decomposition* of $E(K_n) \times E(K_n)$ is a partition of $E(K_n) \times E(K_n)$ into $g(n)$ blocks. Finally, for a set V, we may write $E(V)$ to denote the edge set of the complete graph on V, that is, the set of all 2-subsets of V.

Theorem 1. *Let $r = 2d+1$ be fixed. Then for each k there exists ϵ_k, with $\epsilon_k \to 0$ as $k \to \infty$, such that for all n we have*

$$f_r(kn) \leqslant \left(\left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} + d \left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor} + \epsilon_k \right) (1+o(1)) \binom{kn}{d}.$$

(Here the $o(1)$ term is as $n \to \infty$, with k and d fixed.)

Proof. In order to decompose the edge set of $K_{kn}^{(r)}$, we start by splitting the kn vertices into k equal parts, say $V\left(K_{kn}^{(r)}\right) = V_1 \cup V_2 \cup \cdots \cup V_k$, where $|V_i| = n$ for each i. We consider the r-edges based on their intersection sizes with the k vertex classes. For each partition of r into positive integers $r_1 + r_2 + \cdots + r_l$ with $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_l$ and for each collection of l vertex classes $V_{i_1}, V_{i_2}, \ldots, V_{i_l}$, the set of r-edges e with $|e \cap V_{i_j}| = r_j$ for all j can be decomposed into $f_{r_1}(n) f_{r_2}(n) \cdots f_{r_l}(n)$ complete r-partite r-graphs: take a complete r_j-partite r_j-graph from a minimal decomposition of $K_n^{(r_j)}$ for each j, and form a complete r-partite r-graph by taking the product of them.

Note that if at least three values of the r_j are odd, then $f_{r_1}(n) f_{r_2}(n) \cdots f_{r_l}(n) = O(n^{d-1})$, as $f_s(n) \leqslant \binom{n}{\lfloor s/2 \rfloor}$ for any s. So the set of r-edges e with $|e \cap V_i|$ is odd for at least three distinct V_i can be decomposed into Cn^{d-1} complete r-partite r-graphs, for some constant C depending on d and k.

Let C' be the number of partitions of r into at most $d-1$ positive integers where exactly one of them is odd. Then we observe that the set of r-edges e such that e intersects with at most $d-1$ vertex classes and $|e \cap V_i|$ is odd for exactly one V_i can be decomposed into at most $C' k^{d-1} n^d$ complete r-partite r-graphs.

We are now only left with two partitions of r: $r = 1+2+2+\cdots+2$ and $r = 2+2+\cdots+2+3$. The first case corresponds to the set of r-edges with $r_1 = 1, r_2 = \cdots = r_{d+1} = 2$. For each of the $\binom{k}{d}$ collections of d vertex classes $V_{i_1}, V_{i_2}, \ldots, V_{i_d}$, we claim that the set of r-edges $\{e : |e \cap V_{i_j}| = 2, j = 1, 2, \ldots, d\}$ can be decomposed into $g(n)^{d/2}$ or $ng(n)^{(d-1)/2}$ complete r-partite r-graphs, depending on whether d is even or odd. This is done by pairing up the V_{i_j}s (or all but one of the V_{i_j}s if d is odd), and forming complete r-partite r-graphs using products of blocks in a minimal decomposition of $E(K_n) \times E(K_n)$. [For example, for $d = 4$, we would take a decomposition of $E(V_{i_1}) \times E(V_{i_2})$ into blocks $E_x \times F_x, 1 \leqslant x \leqslant g(n)$, and similarly a decomposition of $E(V_{i_3}) \times E(V_{i_4})$ into blocks $G_x \times H_x, 1 \leqslant x \leqslant g(n)$, and now the set of all 9-edges e with $|e \cap V_{i_j}| = 2$ for all

$1 \leqslant j \leqslant 4$ may be decomposed into $g(n)^2$ complete 9-partite 9-graphs by taking the $E_x \times F_x \times G_y \times H_y \times (V_{i_1} \cup V_{i_2} \cup V_{i_3} \cup V_{i_4})^c$ for $1 \leqslant x, y \leqslant g(n)$.]

Finally, the second case corresponds to the set of r-edges with $r_1 = r_2 = \cdots = r_{d-1} = 2, r_d = 3$. These can be decomposed in a similar fashion. Indeed, for each collection of d vertex classes $V_{i_1}, V_{i_2}, \ldots, V_{i_d}$, the set of r-edges $\{e : |e \cap V_{i_d}| = 3 \text{ and } |e \cap V_{i_j}| = 2, j = 1, 2, \ldots, d-1\}$ can be decomposed into $n^2 g(n)^{(d-2)/2}$ or $ng(n)^{(d-1)/2}$ complete r-partite r-graphs, depending on whether d is even or odd. There are $d\binom{k}{d}$ such sets of r-edges.

Combining the above and the bound on $g(n)$ given in inequality (1), we have

$$\begin{aligned}
f_r(kn) &\leqslant \begin{cases} \binom{k}{d} g(n)^{\frac{d}{2}} + d\binom{k}{d} n^2 g(n)^{\frac{d-2}{2}} + C'k^{d-1}n^d + Cn^{d-1} & \text{(if } d \text{ even)} \\ \binom{k}{d} ng(n)^{\frac{d-1}{2}} + d\binom{k}{d} ng(n)^{\frac{d-1}{2}} + C'k^{d-1}n^d + Cn^{d-1} & \text{(if } d \text{ odd)} \end{cases} \\
&\leqslant \binom{k}{d}\left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} n^d + d\binom{k}{d}\left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor} n^d + C'k^{d-1}n^d + o(n^d) \\
&\leqslant \left(\left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} + d\left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor} + \frac{d!C'}{k}\right)\binom{k}{d} n^d + o(n^d) \\
&\leqslant \left(\left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} + d\left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor} + \epsilon_k\right)(1 + o(1))\binom{kn}{d}. \qquad \square
\end{aligned}$$

Corollary 2. *Let $r \geqslant 295$ be a fixed odd number. Then there exists $c < 1$ such that*

$$f_r(n) \leqslant c(1 + o(1))\binom{n}{\lfloor r/2 \rfloor}.$$

Proof. As above, write $r = 2d+1$. It is straightforward to check that for $d \geqslant 147$ we have $\left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} + d\left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor} < 1$. Choosing k such that

$$c = \left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} + d\left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor} + \epsilon_k < 1,$$

we have $f_r(kn) \leqslant c(1 + o(1))\binom{kn}{d}$ for all n. However since the function $f_r(n)$ is monotone in n, and k is constant as n varies, it follows that $f_r(n) \leqslant c(1 + o(1))\binom{n}{d}$ for all n. $\square$

From Theorem 1, we have

$$c_{2d+1} \leqslant \left(\frac{14}{15}\right)^{\lfloor \frac{d}{2} \rfloor} + d\left(\frac{14}{15}\right)^{\lfloor \frac{d-1}{2} \rfloor}$$

for every d. Also, it is easy to see that $c_{2d} \leqslant c_{2d+1}$. Indeed, by excluding a vertex in the complete $(2d+1)$-graph on $n+1$ vertices, the complete $(2d)$-partite $(2d)$-graphs induced from the complete $(2d+1)$-partite $(2d+1)$-graphs in a minimal decomposition of $K_{n+1}^{(2d+1)}$ form a decomposition of $K_n^{(2d)}$, implying that $f_{2d}(n) \leqslant f_{2d+1}(n+1)$. Hence we have the following.

Corollary 3. *The numbers c_r satisfy*

$$c_r \leqslant \frac{r}{2}\left(\frac{14}{15}\right)^{r/4} + o(1).$$

(Here the $o(1)$ term is as $r \to \infty$.)

Corollary 3 implies that $c_r \to 0$ as $r \to \infty$, proving Conjecture 16 in [6].

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