The existence of a path-factor without small odd paths

Yoshimi Egawa

Department of Applied Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

disc_student_seminar@rs.tus.ac.jp

Michitaka Furuya*

College of Liberal Arts and Science
Kitasato University
1-15-1 Kitasato, Minami-ku, Sagamihara, Kanagawa 252-0373, Japan

michitaka.furuya@gmail.com

Submitted: Dec 12, 2015; Accepted: Dec 5, 2017; Published: Mar 2, 2018
Mathematics Subject Classifications: 05C70

Abstract

A $\{P_2, P_5\}$-factor of a graph is a spanning subgraph of the graph each of whose components is isomorphic to either P_2 or P_5, where P_n denote the path of order n. In this paper, we show that if a graph G satisfies $c_1(G-X)+\frac{2}{3}c_3(G-X) \leqslant \frac{4}{3}|X|+\frac{1}{3}$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_5\}$-factor, where $c_i(G-X)$ is the number of components C of $G-X$ with $|V(C)| = i$. Moreover, it is shown that above condition is sharp.

Keywords: path-factor; component-factor; matching.

1 Introduction

In this paper, all graphs are finite and simple. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The *order* of G is the cardinality $|V(G)|$ of $V(G)$. For $u \in V(G)$, the *neighborhood* of u, denoted by $N_G(u)$, is the set of vertices adjacent to u, and the *degree* of u, denoted by $d_G(u)$, is the number of vertices adjacent to u; thus $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. For

*Partly supported by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (B) 26800086

$U \subseteq V(G)$, let $N_G(U) = (\bigcup_{u \in U} N_G(u)) - U$. For disjoint sets $X, Y \subseteq V(G)$, let $E_G(X, Y)$ denote the set of edges of G joining a vertex in X and a vertex in Y. For $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G induced by X. If G is isomorphic to a graph H, we write $G \simeq H$. For two vertex-disjoint graphs H_1 and H_2, the *join* of H_1 and H_2, denoted by $H_1 + H_2$, is the graph obtained from H_1 and H_2 by joining each vertex of H_1 to all vertices of H_2. Let P_n denote the *path* of order n. For terms and symbols not defined here, we refer the reader to [2].

A subgraph of a graph G is *spanning* if the subgraph contains all vertices of G. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph F of a graph is called an *$\mathcal{H}$-factor* if each component of F is isomorphic to a graph in $\mathcal{H}$. A *path-factor* of a graph is a spanning subgraph whose components are paths of order at least 2. Since every path of order at least 4 can be partitioned into paths of orders 2 and 3, a graph has a path-factor if and only if it has a $\{P_2, P_3\}$-factor. Akiyama, Avis and Era [1] gave a necessary and sufficient condition for the existence of a path-factor (here $i(G)$ denotes the number of isolated vertices of a graph G).

Theorem A (Akiyama, Avis and Era [1]). *A graph G has a $\{P_2, P_3\}$-factor if and only if $i(G - X) \leqslant 2|X|$ for all $X \subseteq V(G)$.*

Now we consider a path-factor with additional conditions. For example, one may require a path-factor to consist of components of large order. Concerning such a problem, Kaneko [5] gave a necessary and sufficient condition for the existence of a path-factor whose components have order at least 3. On the other hand, for $k \geqslant 4$, it is not known that whether the existence problem of a path-factor whose components have order at least k is polynomially solvable or not, though some results about such a factor have been obtained (see, for example, Kano, Lee and Suzuki [6] and Kawarabayashi, Matsuda, Oda and Ota [7]).

In this paper, we study a different type of path-factor problem. Specifically, we focus on the existence of a $\{P_2, P_{2k+1}\}$-factor ($k \geqslant 2$).

The motivation to study such factors is related the notion of a hypomatchable graph. A graph H is *hypomatchable* if $H - x$ has a perfect matching for every $x \in V(H)$. A graph is a *propeller* if it is obtained from a hypomatchable graph H by adding new vertices a, b together with edge ab, and joining a to some vertices of H. Loebal and Poljak [8] proved the following theorem.

Theorem B (Loebal and Poljak [8]). *Let H be a connected graph. If either H has a perfect matching, or H is hypomatchable, or H is a propeller, then the existence problem of a $\{P_2, H\}$-factor is polynomially solvable. The problem is* **NP***-complete for all other graphs H.*

In particular, for $k \geqslant 2$, the existence problem of a $\{P_2, P_{2k+1}\}$-factor is **NP**-complete. Because of this fact, existence problems concerning $\{P_2, P_{2k+1}\}$-factors seem to have unjustly been ignored. However, in general, the fact that a problem is **NP**-complete in terms of algorithm may mean that one cannot obtain a necessary and sufficient condition, but it might be possible to obtain sufficient conditions. From this viewpoint, in this paper, we prove a theorem on the existence of a $\{P_2, P_5\}$-factor which, we hope, will serve as an initial attempt to develop the theory of $\{P_2, P_{2k+1}\}$-factors.

In order to state our theorem, we need some more definitions. For a graph H, let $\mathcal{C}(H)$ be the set of components of H, and for $i \geqslant 1$, let $\mathcal{C}_i(H) = \{C \in \mathcal{C}(H) \mid |V(C)| = i\}$ and $c_i(H) = |\mathcal{C}_i(H)|$. Note that $c_1(H)$ is the number of isolated vertices of H (i.e., $c_1(H) = i(H)$). If a graph G has a $\{P_2, P_5\}$-factor, then $c_1(G-X) + \frac{1}{2}c_3(G-X) \leqslant \frac{3}{2}|X|$ for all $X \subseteq V(G)$ (see Section 2). Thus if a condition concerning $c_1(G-X)$ and $c_3(G-X)$ for $X \subseteq V(G)$ assures us the existence of a $\{P_2, P_5\}$-factor, then it will make a useful sufficient condition.

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let G be a graph. If $c_1(G-X)+\frac{2}{3}c_3(G-X) \leqslant \frac{4}{3}|X|+\frac{1}{3}$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_5\}$-factor.*

We prove Theorem 1.1 in Sections 3 and 4. In Subsection 5.1, we show that the bound $\frac{4}{3}|X| + \frac{1}{3}$ in Theorem 1.1 is best possible.

In our proof of Theorem 1.1, we make use of the following fact.

Fact 1.1. *A graph G has a $\{P_2, P_5\}$-factor if and only if G has a path-factor F with $\mathcal{C}_3(F) = \emptyset$.*

We conclude this section with a conjecture concerning $\{P_2, P_{2k+1}\}$-factors with $k \geqslant 3$. If $k \geqslant 3$ and $k \equiv 0 \pmod 3$, then there exist infinitely many graphs G having no $\{P_2, P_{2k+1}\}$-factor such that $\sum_{0 \leqslant i \leqslant k-1} c_{2i+1}(G-X) \leqslant \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3}$ for all $X \subseteq V(G)$ (see Subsection 5.2). Thus we pose the following conjecture.[1]

Conjecture 1. *Let $k \geqslant 3$, and let G be a graph. If $\sum_{0 \leqslant i \leqslant k-1} c_{2i+1}(G-X) \leqslant \frac{4k+6}{8k+3}|X|$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_{2k+1}\}$-factor.*

2 A necessary condition for a $\{P_2, P_5\}$-factor

In this section, we give a necessary condition for the existence of a $\{P_2, P_5\}$-factor in terms of invariants c_1 and c_3. We show the following proposition.

Proposition 2.1. *If a graph G has a $\{P_2, P_5\}$-factor, then $c_1(G-X)+\frac{1}{2}c_3(G-X) \leqslant \frac{3}{2}|X|$ for all $X \subseteq V(G)$.*

Proof. Let F be a $\{P_2, P_5\}$-factor of G, and let $X \subseteq V(G)$. It can be verified that

$$c_1(P-X) + \frac{1}{2}c_3(P-X) \leqslant \frac{3}{2}|V(P) \cap X| \text{ for every } P \in \mathcal{C}(F). \tag{2.1}$$

Since every component C of $G-X$ with $|V(C)| = 1$ belongs to $\mathcal{C}_1(F-X)$, we have

$$|\mathcal{C}_1(G-X)| = |\mathcal{C}_1(F-X)| - |\mathcal{C}_1(F-X) - \mathcal{C}_1(G-X)|. \tag{2.2}$$

[1]Recently the conjecture for the case where $k \in \{3, 4\}$ was affirmatively settled in [3]. On the other hand, counterexamples to the conjecture for the case where $k \geqslant 36$ were constructed in [4] (see Section 6).

Furthermore,

$$|\mathcal{C}_3(G-X)| \leqslant |\mathcal{C}_3(F-X)| + |\mathcal{C}_3(G-X) - \mathcal{C}_3(F-X)|. \tag{2.3}$$

Let C be a component of $G-X$ with $|V(C)|=3$ which does not belong to $\mathcal{C}_3(F-X)$. Then C intersects with at least two components of $F-X$. Since $|V(C)|=3$, C contains a component of $F-X$ of order 1. Since C is arbitrary, this implies that

$$|\mathcal{C}_3(G-X) - \mathcal{C}_3(F-X)| \leqslant |\mathcal{C}_1(F-X) - \mathcal{C}_1(G-X)|. \tag{2.4}$$

By (2.1)–(2.4),

$$\begin{aligned}
&c_1(G-X) + \frac{1}{2}c_3(G-X) \\
&\quad \leqslant (|\mathcal{C}_1(F-X)| - |\mathcal{C}_1(F-X) - \mathcal{C}_1(G-X)|) \\
&\qquad + \frac{1}{2}(|\mathcal{C}_3(F-X)| + |\mathcal{C}_3(G-X) - \mathcal{C}_3(F-X)|) \\
&\quad \leqslant (|\mathcal{C}_1(F-X)| - |\mathcal{C}_1(F-X) - \mathcal{C}_1(G-X)|) \\
&\qquad + \frac{1}{2}(|\mathcal{C}_3(F-X)| + |\mathcal{C}_1(F-X) - \mathcal{C}_1(G-X)|) \\
&\quad \leqslant |\mathcal{C}_1(F-X)| + \frac{1}{2}|\mathcal{C}_3(F-X)| \\
&\quad = \sum_{P \in \mathcal{C}(F)} \left(c_1(P-X) + \frac{1}{2}c_3(P-X) \right) \\
&\quad \leqslant \frac{3}{2} \sum_{P \in \mathcal{C}(F)} |V(P) \cap X| \\
&\quad = \frac{3}{2}|X|.
\end{aligned}$$

Thus we get the desired conclusion. □

3 A path-factor in bipartite graph

Let G be a bipartite graph with bipartition (S,T). A subgraph F of G is *S-central* if $S \subseteq V(F)$ and $|V(A) \cap T| \geqslant |V(A) \cap S|$ for every $A \in \mathcal{C}(F)$.

In this section, we focus on the existence of a special path-factor in bipartite graphs, and show the following theorem, which will be used in our proof of Theorem 1.1.

Theorem 3.1. *Let S, T_1 and T_2 be disjoint sets with $1 \leqslant |S| \leqslant |T_1|+|T_2|$ and $|T_1|+\frac{2}{3}|T_2| \leqslant \frac{4}{3}|S|+\frac{1}{3}$, and set $T = T_1 \cup T_2$. Let G be a bipartite graph with bipartition (S,T) satisfying the property that for every $X \subseteq S$, we have either $|N_G(X) \cap T_1| + \frac{2}{3}|N_G(X) \cap T_2| \geqslant \frac{4}{3}|X|$ or $N_G(X) = T$. Then G has an S-central path-factor F such that $V(A) \cap T_2 \neq \emptyset$ for every $A \in \mathcal{C}_3(F)$.*

Our proof of Theorem 3.1 is rather technical, and thus readers not interested in technical details are advised to skip the rest of Section 3 and proceed to Section 4. Before proving the theorem, we prove a lemma.

Lemma 3.2. *Let S, T_1, T_2, T and G be as in Theorem 3.1. Then G has an S-central path-factor.*

Proof. Let $X \subseteq S$. If $|N_G(X) \cap T_1| + \frac{2}{3}|N_G(X) \cap T_2| \geqslant \frac{4}{3}|X|$, then $|N_G(X)| \geqslant |N_G(X) \cap T_1| + \frac{2}{3}|N_G(X) \cap T_2| \geqslant \frac{4}{3}|X| \geqslant |X|$; if $N_G(X) = T_1 \cup T_2$, then $|N_G(X)| = |T_1| + |T_2| \geqslant |S| \geqslant |X|$. In either case, we have $|N_G(X)| \geqslant |X|$. Since X is arbitrary, G has a matching covering S by Hall's marriage theorem. In particular, G has an S-central subgraph F such that every component of F is a path of order at least 2. Choose F so that $|V(F)|$ is as large as possible.

Suppose that $V(G) - V(F) \neq \emptyset$. Note that $V(G) - V(F) \subseteq T$. Now we define the set $\mathcal{A}$ of components of F as follows: Let $\mathcal{A}_1$ be the set of components A of F with $E_G(V(A) \cap S, V(G) - V(F)) \neq \emptyset$. For each $i \geqslant 2$, let $\mathcal{A}_i$ be the set of components A of F with $A \notin \bigcup_{1 \leqslant j \leqslant i-1} \mathcal{A}_j$ and $E_G(V(A) \cap S, \bigcup_{A' \in \mathcal{A}_{i-1}} (V(A') \cap T)) \neq \emptyset$. Let $\mathcal{A} = \bigcup_{i \geqslant 1} \mathcal{A}_i$.

Claim 3.1. *Every path belonging to $\mathcal{A}$ is isomorphic to P_3.*

Proof. Suppose that $\mathcal{A}$ contains a path which is not isomorphic to P_3. Let i be the minimum integer such that $\mathcal{A}_i$ contains a path $A_i = v_1^{(i)} \cdots v_l^{(i)}$ with $A_i \not\simeq P_3$. By the minimality of i, every path belonging to $\bigcup_{1 \leqslant j \leqslant i-1} \mathcal{A}_j$ is isomorphic to P_3. Hence by the definition of $\mathcal{A}_j$, there exists a vertex $v_1^{(0)} \in V(G) - V(F)$ and there exist paths $A_j = v_1^{(j)} v_2^{(j)} v_3^{(j)} \in \mathcal{A}_j$ $(1 \leqslant j \leqslant i-1)$ such that $E_G(V(A_1) \cap S, \{v_1^{(0)}\}) \neq \emptyset$ and $E_G(V(A_{j+1}) \cap S, V(A_j) \cap T) \neq \emptyset$ for every j $(1 \leqslant j \leqslant i-1)$. For each j $(1 \leqslant j \leqslant i-1)$, by renumbering the veritices $v_1^{(j)}, v_2^{(j)}, v_3^{(j)}$ of A_j backward (i.e., by tracing the path $v_1^{(j)} v_2^{(j)} v_3^{(j)}$ backward and numbering the vertices accordingly) if necessary, we may assume that $E_G(V(A_{j+1}) \cap S, \{v_1^{(j)}\}) \neq \emptyset$. Let m be an index such that $v_m^{(i)} v_1^{(i-1)} \in E(G)$. Note that $l \geqslant 2$ and $l \neq 3$. Thus by renumbering the vertices $v_1^{(i)}, \ldots, v_l^{(i)}$ of A_i backward if necessary, we may assume that $m \neq 2$ if l is odd, and m is odd if l is even. Let $B_j = v_1^{(j-1)} v_2^{(j)} v_3^{(j)}$ $(1 \leqslant j \leqslant i-1)$, $B_i = v_1^{(i-1)} v_m^{(i)} v_{m+1}^{(i)} \cdots v_l^{(i)}$ and $B_{i+1} = v_1^{(i)} \cdots v_{m-1}^{(i)}$ (see Figure 1). Note that $B_{i+1} = \emptyset$ if and only if l is even and $m = 1$. Then $|V(B_j) \cap T| \geqslant |V(B_j) \cap S|$ for every j $(1 \leqslant j \leqslant i+1)$. Therefore $F' = (F - (\bigcup_{1 \leqslant j \leqslant i} V(A_j))) \cup (\bigcup_{1 \leqslant j \leqslant i+1} B_j)$ is an S-central subgraph of G such that $V(F') = V(F) \cup \{v_1^{(0)}\}$ and every component of F' is a path of order at least 2, which contradicts the maximality of F. □

We continue with the proof of the lemma. Let $X_0 = (\bigcup_{A \in \mathcal{A}} V(A)) \cap S$ and $Y_0 = ((\bigcup_{A \in \mathcal{A}} V(A)) \cap T) \cup (V(G) - V(F))$. Since $V(G) - V(F) \neq \emptyset$ and $\mathcal{A} \subseteq \mathcal{C}_3(F)$ by Claim 3.1, we have

$$|Y_0 \cap T_1| + \frac{2}{3}|Y_0 \cap T_2| \geqslant \frac{2}{3}|Y_0| \geqslant \frac{2}{3}(2|X_0| + 1). \tag{3.1}$$

By the definition of $\mathcal{A}$, $N_G(S - X_0) \cap Y_0 = \emptyset$. In particular, $N_G(S - X_0) \neq T$, and hence $|N_G(S - X_0) \cap T_1| + \frac{2}{3}|N_G(S - X_0) \cap T_2| \geqslant \frac{4}{3}|S - X_0|$. This together with (3.1) implies

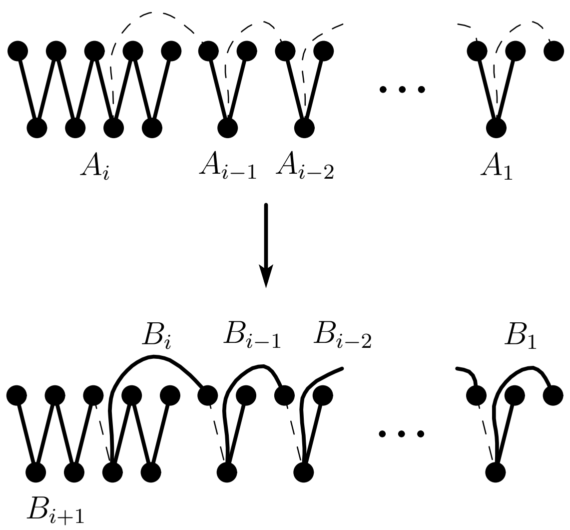

Figure 1: Construction of B_j

that

$$\begin{aligned}|T_1| + \frac{2}{3}|T_2| &\geqslant (|Y_0 \cap T_1| + |N_G(S - X_0) \cap T_1|) + \frac{2}{3}(|Y_0 \cap T_2| + |N_G(S - X_0) \cap T_2|)\\ &\geqslant \frac{2}{3}(2|X_0| + 1) + \frac{4}{3}|S - X_0|\\ &= \frac{4}{3}|S| + \frac{2}{3},\end{aligned}$$

which contradicts the assumption that $|T_1| + \frac{2}{3}|T_2| \leqslant \frac{4}{3}|S| + \frac{1}{3}$, completing the proof of the lemma. □

We here outline the proof of Theorem 3.1. We choose an S-central path-factor F_0 so that F_0 will satisfy certain minimality conditions (see the paragraph following the proof of Claim 3.3). We then introduce operations which turn F_0 into a new path-factor (see the paragraphs following Claim 3.5 and Claim 3.6), and show that the new path-factor contradicts our choice of F_0.

Proof of Theorem 3.1. We start with some definitions. Let F be an S-central path-factor of G. For each integer $i \geqslant 2$, let $\mathcal{C}_i^{(1)}(F) = \{A \in \mathcal{C}_i(F) \mid V(A) \cap T_2 = \emptyset\}$ and $\mathcal{C}_i^{(2)}(F) = \mathcal{C}_i(F) - \mathcal{C}_i^{(1)}(F)$. If there is no fear of confusion, we simply write $\mathcal{C}_i$ and $\mathcal{C}_i^{(h)}$ $(h \in \{1, 2\})$ instead of $\mathcal{C}_i(F)$ and $\mathcal{C}_i^{(h)}(F)$, respectively.

Let $\mathcal{D}_F$ be the digraph defined by $V(\mathcal{D}_F) = \mathcal{C}(F)$ and $E(\mathcal{D}) = \{AB \mid E_G(V(A) \cap S, V(B) \cap T) \neq \emptyset\}$. For each edge $AB \in E(\mathcal{D}_F)$, we fix an edge $\varphi_F(AB)$ in $E_G(V(A) \cap S, V(B) \cap T)$, and let $\sigma_F(AB) \in V(G)$ be the vertex of A incident with $\varphi_F(AB)$ and $\tau_F(AB) \in V(G)$ be the vertex of B incident with $\varphi_F(AB)$ (see Figure 2).

For a path $A = x_1x_2\cdots x_7 \in \mathcal{C}_7$, the vertex x_4 is called the *center* of A. A directed path $\mathcal{P} = A_1A_2\cdots A_l$ $(l \geqslant 2)$ of $\mathcal{D}_F$ is *admissible* if $A_1 \in \mathcal{C}(F) - (\mathcal{C}_3 \cup \mathcal{C}_5^{(1)})$ and $A_i \in \mathcal{C}_3^{(2)} \cup \mathcal{C}_5^{(1)}$ for every i $(2 \leqslant i \leqslant l-1)$. An admissible path $\mathcal{P} = A_1A_2\cdots A_l$ of $\mathcal{D}_F$ is *weakly admissible* if either

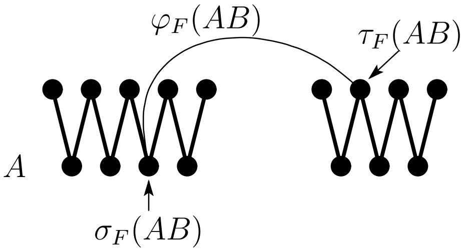

Figure 2: Edge $\varphi_F(AB)$ and vertices $\sigma_F(AB)$ and $\tau_F(AB)$

(W1) $A_1 \in \mathcal{C}_5^{(2)}$ and $|V(A_1) \cap T_2| = 1$, or

(W2) $A_1 \in \mathcal{C}_7^{(1)}$ and $\sigma_F(A_1A_2)$ is the center of A_1.

An admissible path $\mathcal{P}$ of $\mathcal{D}_F$ is *strongly admissible* if $\mathcal{P}$ is not weakly admissible.

A *path system* with respect to F is a sequence $(\mathcal{P}_1, \dots, \mathcal{P}_m)$ $(m \geqslant 0)$ of admissible paths such that

(P1) for each i $(1 \leqslant i \leqslant m)$, when we write $\mathcal{P}_i = A_1A_2 \cdots A_l$, $\{A_j \mid 1 \leqslant j \leqslant l-1\} \cap (\bigcup_{1 \leqslant j \leqslant i-1} V(\mathcal{P}_j)) = \emptyset$ and $A_l \in \mathcal{C}_3^{(1)} \cup (\bigcup_{1 \leqslant j \leqslant i-1} V(\mathcal{P}_j))$, and

(P2) for each i $(1 \leqslant i \leqslant m-1)$, $\mathcal{P}_i$ is weakly admissible.

A path system $(\mathcal{P}_1, \dots, \mathcal{P}_m)$ with respect to F is *complete* if $m \geqslant 1$ and $\mathcal{P}_m$ is strongly admissible.

By straightforward calculations, we get the following claim (and we omit its proof).

Claim 3.2. *Let F be an S-central path-factor of G. Then the following hold.*

(i) *For $A \in \mathcal{C}_3^{(1)}(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| = 2 = \frac{4}{3}|V(A) \cap S| + \frac{2}{3}$.*

(ii) *For $A \in \mathcal{C}_3^{(2)}(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| \geqslant \frac{4}{3}|V(A) \cap S|$.*

(iii) *For $A \in \mathcal{C}_5^{(1)}(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| > \frac{4}{3}|V(A) \cap S|$.*

(iv) *For $A \in \mathcal{C}_5^{(2)}(F)$ with $|V(A) \cap T_2| = 1$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| = \frac{4}{3}|V(A) \cap S|$.*

(v) *For $A \in \mathcal{C}_7^{(1)}(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| = \frac{4}{3}|V(A) \cap S|$.* □

The following claim plays a key role in the proof of the theorem.

Claim 3.3. *Let F be an S-central path-factor of G with $\mathcal{C}_3^{(1)}(F) \neq \emptyset$, and let $(\mathcal{P}_1, \dots, \mathcal{P}_m)$ be a path system with respect to F $(m \geqslant 0)$. Then the system can be extend to a complete path system $(\mathcal{P}_1, \dots, \mathcal{P}_m, \mathcal{P}_{m+1}, \dots, \mathcal{P}_{m'})$ with respect to F.*

Proof. We take a maximal path system $(\mathcal{P}_1, \ldots, \mathcal{P}_m, \mathcal{P}_{m+1}, \ldots, \mathcal{P}_{m'})$ with respect to F. We show that $(\mathcal{P}_1, \ldots, \mathcal{P}_{m'})$ is a complete path system. Suppose that $(\mathcal{P}_1, \ldots, \mathcal{P}_{m'})$ is not a complete path system. Then $\mathcal{P}_i$ is weakly admissible for each i with $1 \leqslant i \leqslant m'$ (this includes the case where $m' = 0$).

Set $\mathcal{A}_1 = \bigcup_{1 \leqslant i \leqslant m'} V(\mathcal{P}_i)$ (note that $\mathcal{A}_1 = \emptyset$ if and only if $m' = 0$). Let $X = (\bigcup_{A \in \mathcal{A}_1} V(A)) \cap S$ and $Y_h = (\bigcup_{A \in \mathcal{A}_1} V(A)) \cap T_h$ ($h \in \{1, 2\}$). Then by the definition of a weakly admissible path (and the definition of a path system), $\mathcal{A}_1 \subseteq \mathcal{C}_3 \cup \mathcal{C}_5 \cup \mathcal{C}_7^{(1)}$, and if $A \in \mathcal{A}_1 \cap \mathcal{C}_5^{(2)}$, then $|V(A) \cap T_1| = 1$. Furthermore, by condition (P1) in the definition of a path system, $\mathcal{A}_1 \neq \emptyset$ if and only if $\mathcal{A}_1 \cap \mathcal{C}_3^{(1)} \neq \emptyset$. Hence by Claim 3.2,

$$|Y_1| + \frac{2}{3}|Y_2| \geqslant \frac{4}{3}|X| \tag{3.2}$$

and

$$|Y_1| + \frac{2}{3}|Y_2| \geqslant \frac{4}{3}|X| + \frac{2}{3} \text{ if } \mathcal{A}_1 \neq \emptyset. \tag{3.3}$$

Let $\mathcal{A}_2 = \mathcal{C}_3^{(1)} - \mathcal{A}_1$, $X^* = (\bigcup_{A \in \mathcal{A}_2} V(A)) \cap S$ and $Y_h^* = (\bigcup_{A \in \mathcal{A}_2} V(A)) \cap T_h$ ($h \in \{1, 2\}$). By Claim 3.2(i),

$$|Y_1^*| + \frac{2}{3}|Y_2^*| \geqslant \frac{4}{3}|X^*| \tag{3.4}$$

and

$$|Y_1^*| + \frac{2}{3}|Y_2^*| = \frac{4}{3}|X^*| + \frac{2}{3} \text{ if } \mathcal{A}_2 \neq \emptyset, \tag{3.5}$$

Let $(B_1, \ldots, B_l)$ ($l \geqslant 0$) be a sequence such that for each i ($1 \leqslant i \leqslant l$), $B_i \in (\mathcal{C}_3^{(2)} \cup \mathcal{C}_5^{(1)}) - (\mathcal{A}_1 \cup \{B_j \mid 1 \leqslant j \leqslant i-1\})$ and there exists an edge of $\mathcal{D}_F$ from B_i to an element in $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{B_j \mid 1 \leqslant j \leqslant i-1\}$. We choose $(B_1, \ldots, B_l)$ so that l is as large as possible. Let $\mathcal{A}_3 = \{B_i \mid 1 \leqslant i \leqslant l\}$, $X^{**} = (\bigcup_{A \in \mathcal{A}_3} V(A)) \cap S$ and $Y_h^{**} = (\bigcup_{A \in \mathcal{A}_3} V(A)) \cap T_h$ ($h \in \{1, 2\}$). By Claim 3.2(ii)(iii),

$$|Y_1^{**}| + \frac{2}{3}|Y_2^{**}| \geqslant \frac{4}{3}|X^{**}|. \tag{3.6}$$

Let $X^0 = X \cup X^* \cup X^{**}$ and $Y_h^0 = Y_h \cup Y_h^* \cup Y_h^{**}$ ($h \in \{1, 2\}$). If $m' \geqslant 1$, then $\mathcal{A}_1 \neq \emptyset$; if $m' = 0$ (i.e., $\mathcal{A}_1 = \emptyset$), then $\mathcal{A}_2 \neq \emptyset$ because $\mathcal{C}_3^{(1)} \neq \emptyset$. Thus by (3.3) and (3.5), either $|Y_1| + \frac{2}{3}|Y_2| \geqslant \frac{4}{3}|X| + \frac{2}{3}$ or $|Y_1^*| + \frac{2}{3}|Y_2^*| = \frac{4}{3}|X^*| + \frac{2}{3}$. This together with (3.2), (3.4) and (3.6) leads to

$$|Y_1^0| + \frac{2}{3}|Y_2^0| \geqslant \frac{4}{3}|X^0| + \frac{2}{3}. \tag{3.7}$$

Since $|T_1| + \frac{2}{3}|T_2| \leqslant \frac{4}{3}|S| + \frac{1}{3}$, this implies $X^0 \neq S$ and hence $\mathcal{C}(F) - (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3) \neq \emptyset$.

Let $\tilde{\mathcal{A}} = \mathcal{C}(F) - (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$, $\tilde{X} = (\bigcup_{A \in \tilde{\mathcal{A}}} V(A)) \cap S$ and $\tilde{Y}_h = (\bigcup_{A \in \tilde{\mathcal{A}}} V(A)) \cap T_h$ ($h \in \{1, 2\}$). Note that S is the disjoint union of X^0 and $\tilde{X}$ and, for $h \in \{1, 2\}$, T_h is the disjoint union of Y_h^0 and $\tilde{Y}_h$. If $|\tilde{Y}_1| + \frac{2}{3}|\tilde{Y}_2| \geqslant \frac{4}{3}|\tilde{X}|$, then by (3.7), $|T_1| + \frac{2}{3}|T_2| = (|Y_1^0| + |\tilde{Y}_1|) + \frac{2}{3}(|Y_2^0| + |\tilde{Y}_2|) \geqslant \frac{4}{3}|X^0| + \frac{2}{3} + \frac{4}{3}|\tilde{X}| = \frac{4}{3}|S| + \frac{2}{3}$, which is a contradiction. Thus $|\tilde{Y}_1| + \frac{2}{3}|\tilde{Y}_2| < \frac{4}{3}|\tilde{X}|$. On the other hand, since $\mathcal{A}_1 \cup \mathcal{A}_2 \neq \emptyset$, we have $Y_1^0 \cup Y_2^0 \neq \emptyset$, and hence $\tilde{Y}_1 \cup \tilde{Y}_2 \neq T$. Consequently $N_G(\tilde{X}) \not\subseteq \tilde{Y}_1 \cup \tilde{Y}_2$ by the assumption of the theorem, which implies that there exists a vertex $x \in \tilde{X}$ with $N_G(x) \cap (Y_1^0 \cup Y_2^0) \neq \emptyset$. Let $\tilde{A} \in \tilde{\mathcal{A}}$ be the path containing x. By the definition of $\mathcal{A}_2$ and $\tilde{\mathcal{A}}$, $\tilde{A} \notin \mathcal{C}_3^{(1)}$. By the maximality of $(B_1, \dots, B_l)$, $\tilde{A} \notin \mathcal{C}_3^{(2)} \cup \mathcal{C}_5^{(1)}$. Thus $\tilde{A} \in \mathcal{C}(F) - (\mathcal{C}_3 \cup \mathcal{C}_5^{(1)})$. By the definition of $(B_1, \dots, B_l)$ and x, there exists a directed path $\mathcal{P}' = \tilde{A}_1 \cdots \tilde{A}_p$ of $\mathcal{D}_F$ such that $\tilde{A}_1 = \tilde{A}$, $\tilde{A}_i \in \mathcal{A}_3$ ($2 \leqslant i \leqslant p-1$) and $\tilde{A}_p \in \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\mathcal{P}'$ is an admissible path of $\mathcal{D}_F$. Now the sequence $(\mathcal{P}_1, \dots, \mathcal{P}_{m'}, \mathcal{P}')$ is a path system with respect to F, which contradicts the maximality of $(\mathcal{P}_1, \dots, \mathcal{P}_{m'})$. This contradiction completes the proof of the claim. □

We turn to the proof of Theorem 3.1. By way of contradiction, suppose that $\mathcal{C}_3^{(1)}(F) \neq \emptyset$ for every S-central path-factor F of G. By Lemma 3.2, G has an S-central path-factor F_0. Note that an empty sequence is a path system with respect to F_0. Hence by Claim 3.3, there exists a complete path system $(\mathcal{P}_1, \dots, \mathcal{P}_m)$ with respect to F_0. Choose F_0 and $(\mathcal{P}_1, \dots, \mathcal{P}_m)$ so that

(F1) $|\mathcal{C}_3^{(1)}(F_0)|$ is as small as possible, and

(F2) subject to (F1), $(|V(\mathcal{P}_1)|, \dots, |V(\mathcal{P}_m)|)$ is lexicographically as small as possible.

For each i ($1 \leqslant i \leqslant m$), write $\mathcal{P}_i = A_1^{(i)} \cdots A_{l_i}^{(i)}$. Then $\bigcup_{1 \leqslant i \leqslant m} \mathcal{P}_i$ contains a directed path $B_1 B_2 \cdots B_p$ of $\mathcal{D}_{F_0}$ with $B_1 = A_1^{(m)}$ and $B_p \in \mathcal{C}_3^{(1)}(F_0)$. For each i ($1 \leqslant i \leqslant p$), write $B_i = v_{i,1} v_{i,2} \cdots v_{i,q_i}$. For i ($1 \leqslant i \leqslant p-1$), let s_i be the integer with $v_{i,s_i} = \sigma_F(B_i B_{i+1})$, and for i ($2 \leqslant i \leqslant p$), let t_i be the integer with $v_{i,t_i} = \tau_{F_0}(B_{i-1} B_i)$. As in the proof of Claim 3.1, by renumbering the vertices of some of the B_i backward if necessary, we may assume that

(B1) $s_1 \geqslant \frac{q_1+1}{2}$ if q_1 is odd,

(B2) $\{v_{1,1}, v_{1,3}\} \cap T_2 \neq \emptyset$ if $B_1 \in \mathcal{C}_7^{(2)}(F_0)$ and $s_1 = 4$,

(B3) s_1 is odd if q_1 is even,

(B4) $t_i < s_i$ for each i ($2 \leqslant i \leqslant p-1$), and

(B5) $t_p = q_p$ ($= 3$).

Note that (B3) means that when q_1 is even, the vertices of B_1 are numbered so that $v_{1,q_1} \in T$. Thus $v_{i,q_i} \in T$ for each i ($1 \leqslant i \leqslant p$). We can divide the type of B_1 into three possibilities as follows:

Claim 3.4. *One of the following holds:*

(1) $|V(B_1)|$ *is even and* s_1 *is odd;*

(2) $B_1 \in \mathcal{C}_5^{(2)}(F_0) \cup \mathcal{C}_7^{(2)}(F_0)$, $s_1 = 4$ *and* $\{v_{1,1}, v_{1,3}\} \cap T_2 \neq \emptyset$*; or*

(3) $|V(B_1)| \geqslant 7$ *and* $s_1 \geqslant 6$.

Proof. If $|V(B_1)|$ is even, then (1) holds by (B3). Thus we may assume $|V(B_1)|$ is odd. Then by the definition of a strongly admissible path, $B_1 \in \mathcal{C}_5^{(2)}(F_0)$ and $|V(B_1) \cap T_2| \geqslant 2$, or $B_1 \in \mathcal{C}_7^{(1)}(F_0)$ and $s_1 \neq 4$, or $B_1 \in \mathcal{C}_7^{(2)}(F_0)$, or $|V(B_1)| \geqslant 9$. If $B_1 \in \mathcal{C}_5^{(2)}(F_0)$ and $|V(B_1) \cap T_2| \geqslant 2$, then (2) holds by (B1). If $B_1 \in \mathcal{C}_7^{(1)}(F_0) \cup \mathcal{C}_7^{(2)}(F_0)$ and $s_1 \neq 4$, then (3) holds by (B1). If $B_1 \in \mathcal{C}_7^{(2)}(F_0)$ and $s_1 = 4$, then (2) holds by (B2). If $|V(B_1)| \geqslant 9$, then (3) holds by (B1). □

As for B_i with $2 \leqslant i \leqslant p-1$, the following claim follows immediately from the definition of a weakly admissible path.

Claim 3.5. *Let* $2 \leqslant i \leqslant p-1$*. Then one of the following holds:*

(1) $B_i \in \mathcal{C}_3^{(2)}(F_0)$ *and* $s_i = 2$*;*

(2) $B_i \in \mathcal{C}_5(F_0)$ *and* $s_i = 2$ *or* 4*; or*

(3) $B_i \in \mathcal{C}_7^{(1)}(F_0)$ *and* $s_i = 4$. □

Let i_0 be the minimum integer i ($\geqslant 2$) satisfying one of the following two conditions:

(I1) $i = p$; or

(I2) $2 \leqslant i \leqslant p-1$ and $t_i = 1$.

Set $B'_1 = v_{1,1}v_{1,2}\cdots v_{1,s_1-1}$ and, for each i ($2 \leqslant i \leqslant i_0$), set

$$B'_i = v_{i-1,q_{i-1}}v_{i-1,q_{i-1}-1}\cdots v_{i-1,s_{i-1}}v_{i,t_i}v_{i,t_i-1}\cdots v_{i,1}$$

(see Figure 3). Let $2 \leqslant i \leqslant i_0 - 1$. By the definition of i_0, $t_i \geqslant 3$. On the other hand, $s_i \leqslant 4$ by Claim 3.5. Hence it follows from (B4) that $t_i = s_i - 1$. Since i ($2 \leqslant i \leqslant i_0 - 1$) is arbitrary, it follows that

$$B'_1, \ldots, B'_{i_0} \text{ are vertex-disjoint paths of } G \tag{3.8}$$

and

$$\bigcup_{1\leqslant i\leqslant i_0} V(B'_i) = \bigcup_{1\leqslant i\leqslant i_0} V(B_i) - \{v_{i_0,j} \mid t_{i_0}+1 \leqslant j \leqslant q_{i_0}\}. \tag{3.9}$$

Furthermore,

$$|V(B'_i) \cap T| \geqslant |V(B'_i) \cap S| \text{ for each } i \ (2 \leqslant i \leqslant i_0) \tag{3.10}$$

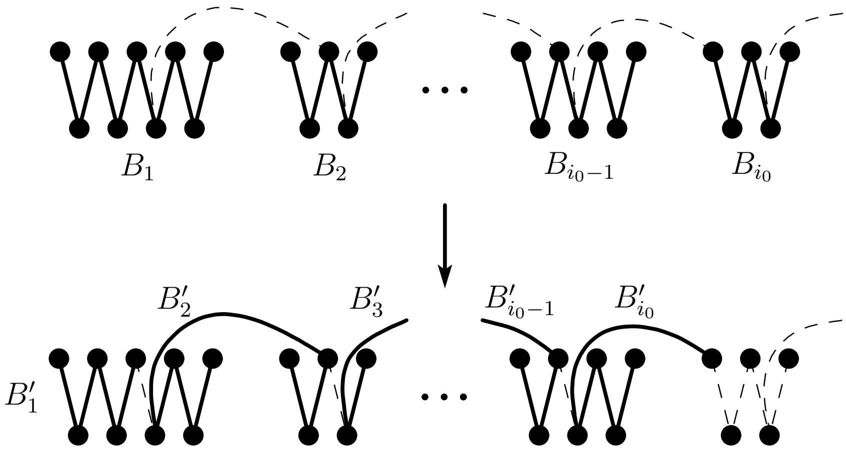

Figure 3: Construction of B'_i

because $v_{i-1,q_{i-1}} \in T$. If $B'_1 \neq \emptyset$, then $v_{1,s_1-1} \in T$, and hence

$$|V(B'_1) \cap T| \geqslant |V(B'_1) \cap S| \tag{3.11}$$

(if $B'_1 = \emptyset$, then (3.11) trivially holds). Also

$$|V(B'_i) \cap V(B_{i-1})| \text{ is even and } |V(B'_i) \cap V(B_{i-1})| \geqslant 2 \text{ for each } i\ (2 \leqslant i \leqslant i_0) \tag{3.12}$$

because $v_{i-1,s_{i-1}} \in S$ and $v_{i-1,q_{i-1}} \in T$. It follows from (3.12) that

$$|V(B'_i)| \geqslant 5 \text{ for each } i\ (2 \leqslant i \leqslant i_0 - 1) \tag{3.13}$$

because $|V(B'_i) \cap V(B_i)| = t_i \geqslant 3$. Since $|V(B'_1)| = s_1 - 1$, we see from Claim 3.4 that

$$|V(B'_1)| \text{ is even or } |V(B'_1)| \geqslant 3, \tag{3.14}$$

and

$$V(B'_1) \cap T_2 \neq \emptyset \text{ if } |V(B'_1)| = 3. \tag{3.15}$$

Combining (3.10) through (3.15), we get the following claim.

Claim 3.6. (i) *For each i with $1 \leqslant i \leqslant i_0$, we have $|V(B'_i) \cap T| \geqslant |V(B'_i) \cap S|$.*

(ii) *For each i with $1 \leqslant i \leqslant i_0 - 1$,*

(a) *$|V(B'_i)|$ is even or $|V(B'_i)| \geqslant 3$, and*

(b) *$V(B'_i) \cap T_2 \neq \emptyset$ if $B'_i \simeq P_3$.* □

Suppose that $i_0 = p$. Then

$$|V(B'_p)| \geqslant 5 \tag{3.16}$$

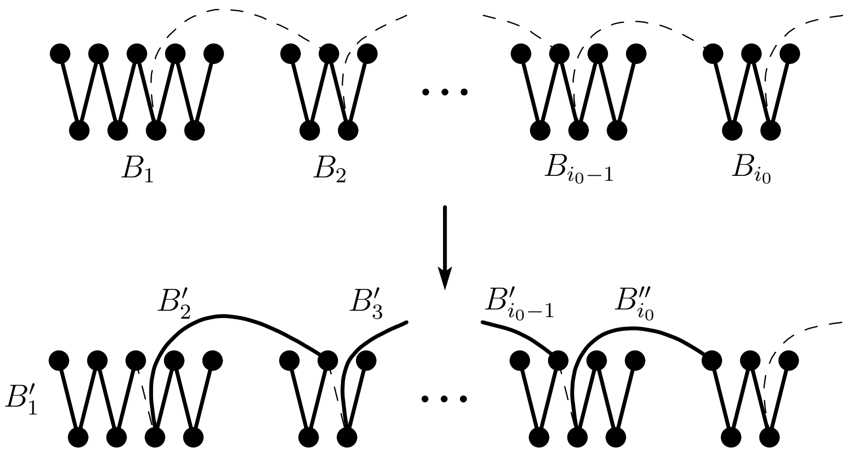

Figure 4: Construction of B''_{i_0}

by (3.12) and (B5). Let $F_1 = (F_0 - (\bigcup_{1 \leqslant i \leqslant p} V(B_i))) \cup (\bigcup_{1 \leqslant i \leqslant p} B'_i)$. Then by Claim 3.6, (3.16), (3.8), (3.9) and (B5), F_1 is an S-central path-factor of G, and $B'_i \notin \mathcal{C}_3^{(1)}(F_1)$ for each i $(1 \leqslant i \leqslant p)$. Since $B_1 \in \mathcal{C}_3^{(1)}(F_0)$, we have $|\mathcal{C}_3^{(1)}(F_1)| < |\mathcal{C}_3^{(1)}(F_0)|$, which contradicts the minimality of $|\mathcal{C}_3^{(1)}(F)|$. Thus $2 \leqslant i_0 \leqslant p-1$. Then by the definition of i_0, $t_{i_0} = 1$. Hence $B''_{i_0} = B_{i_0} \cup B'_{i_0}$ is a path of G with $|V(B''_{i_0}) \cap T| \geqslant |V(B''_{i_0}) \cap S|$ (see Figure 4). Set $F_2 = (F_0 - (\bigcup_{1 \leqslant i \leqslant i_0} V(B_i))) \cup (\bigcup_{1 \leqslant i \leqslant i_0 - 1} B'_i) \cup B''_{i_0}$. Then by Claim 3.6, (3.8) and (3.9), F_2 is an S-central path-factor of G, and $B'_i \notin \mathcal{C}_3^{(1)}(F_1)$ for each i $(1 \leqslant i \leqslant i_0 - 1)$. Furthermore,

$$|V(B''_{i_0})| = |V(B_{i_0})| + |V(B'_{i_0}) \cap V(B_{i_0-1})|. \tag{3.17}$$

Since $|V(B'_{i_0}) \cap V(B_{i_0-1})| \geqslant 2$ by (3.12), this implies $|V(B''_{i_0})| \geqslant 5$, and hence we also have $B''_{i_0} \notin \mathcal{C}_3^{(1)}(F_1)$. Thus $|\mathcal{C}_3^{(1)}(F_2)| = |\mathcal{C}_3^{(1)}(F_0)|$.

Set $k_0 = \min\{k \mid B_{i_0} \in V(\mathcal{P}_k)\}$, and write $B_{i_0} = A_{j_0}^{(k_0)}$. If $B_p \in V(\mathcal{P}_{k_0})$, then the fact that $B_{i_0} \neq B_p$ implies that $j_0 \leqslant l_{k_0} - 1$; if $B_p \notin V(\mathcal{P}_{k_0})$, then the minimality of k_0 implies that $j_0 \leqslant l_{k_0} - 1$. In either case, we have $j_0 \leqslant l_{k_0} - 1$.

Case 1: $j_0 = 1$.

Since $B_{i_0} = A_1^{(k_0)}$ and $i_0 \geqslant 2$, $B_1 \in \bigcup_{k_0+1 \leqslant i \leqslant m} V(\mathcal{P}_i)$. In particular, $k_0 \leqslant m-1$ and $\mathcal{P}_{k_0}$ is weakly admissible. Hence $B_{i_0} \in \mathcal{C}_5^{(2)}(F_0) \cup \mathcal{C}_7^{(1)}(F_0)$. This together with (3.17) and (3.12) implies that $B''_{i_0} \in \mathcal{C}_7^{(2)}(F_2)$ or $|V(B''_{i_0})| \geqslant 9$. Thus the directed path $\mathcal{P}'_{k_0} = B''_{i_0} A_2^{(k_0)} \cdots A_{l_{k_0}}^{(k_0)}$ of $\mathcal{D}_{F_2}$ is strongly admissible. Consequently $(\mathcal{P}_1, \ldots, \mathcal{P}_{k_0-1}, \mathcal{P}'_{k_0})$ is a complete path system with respect to F_2. Since $k_0 \leqslant m-1$ and $|V(\mathcal{P}_{k_0})| = |V(\mathcal{P}'_{k_0})|$, we see that $(|V(\mathcal{P}_1)|, \ldots, |V(\mathcal{P}_{k_0-1})|, |V(\mathcal{P}'_{k_0})|)$ is lexicographically less than $(|V(\mathcal{P}_1)|, \ldots, |V(\mathcal{P}_{k_0-1})|, |V(\mathcal{P}_{k_0})|, \ldots, |V(\mathcal{P}_m)|)$, which contradicts the minimality of $(|V(\mathcal{P}_1)|, \ldots, |V(\mathcal{P}_m)|)$.

Case 2: $2 \leqslant j_0 \leqslant l_{k_0} - 1$.

Since $B_{i_0} = A_{j_0}^{(k_0)}$, $B_{i_0} \in \mathcal{C}_3^{(2)}(F_0) \cup \mathcal{C}_5^{(1)}(F_0)$. This together with (3.17) and (3.12) implies that $B''_{i_0} \in \mathcal{C}_5^{(2)}(F_2)$ or $|V(B''_{i_0})| \geqslant 7$. Thus the directed path

$$\mathcal{P}'_{k_0} = B''_{i_0} A_{j_0+1}^{(k_0)} A_{j_0+2}^{(k_0)} \cdots A_{l_{k_0}}^{(k_0)}$$

of $\mathcal{D}_{F_2}$ is admissible. Consequently $(\mathcal{P}_1, \dots, \mathcal{P}_{k_0-1}, \mathcal{P}'_{k_0})$ is a path system with respect to F_2. By Claim 3.3, the system can be extend to a complete path system

$$(\mathcal{P}_1, \dots, \mathcal{P}_{k_0-1}, \mathcal{P}'_{k_0}, \mathcal{Q}_1, \dots, \mathcal{Q}_\alpha)$$

with respect to F_2 (it is possible that $\alpha = 0$). Since $j_0 \geqslant 2$, $|V(\mathcal{P}'_{k_0})| = l_{k_0} - j_0 + 1 < l_{k_0} = |V(\mathcal{P}_{k_0})|$, and hence $(|V(\mathcal{P}_1)|, \dots, |V(\mathcal{P}_{k_0-1})|, |V(\mathcal{P}'_{k_0})|, |V(\mathcal{Q}_1)|, \dots, |V(\mathcal{Q}_\alpha)|)$ is lexicographically less than $(|V(\mathcal{P}_1)|, \dots, |V(\mathcal{P}_{k_0-1})|, |V(\mathcal{P}_{k_0})|, \dots, |V(\mathcal{P}_m)|)$, which contradicts the minimality of $(|V(\mathcal{P}_1)|, \dots, |V(\mathcal{P}_m)|)$.

This completes the proof of Theorem 3.1. □

4 Proof of Theorem 1.1

Let G be as in Theorem 1.1. By assumption, we have $c_1(G) + \frac{2}{3}c_3(G) \leqslant \frac{4}{3}|\emptyset| + \frac{1}{3} = \frac{1}{3}$. Hence $c_1(G) = c_3(G) = 0$.

We now proceed by induction on $|V(G)| + |E(G)|$. We may assume $V(G) \neq \emptyset$. Note that if $E(G) = \emptyset$, then $c_1(G) = |V(G)| \geqslant 1$, which is a contradiction. This means that the theorem holds for graphs G with $E(G) = \emptyset$ in the sense that the assumption is not satisfied. We henceforth assume that $E(G) \neq \emptyset$ and the theorem holds for graphs G' with $|V(G')| + |E(G')| < |V(G)| + |E(G)|$.

Let $\mathcal{S} = \{X \subseteq V(G) \mid c_1(G - X) + c_3(G - X) \geqslant 1\}$. Since $c_1(G - N_G(x)) \geqslant 1$ for $x \in V(G)$, $\mathcal{S} \neq \emptyset$. Set

$$\beta = \min_{X \in \mathcal{S}} \left\{ \frac{4}{3}|X| + \frac{1}{3} - c_1(G - X) - \frac{2}{3}c_3(G - X) \right\}.$$

Claim 4.1. *If $\beta \geqslant 2$, then G has a $\{P_2, P_5\}$-factor.*

Proof. Let $e \in E(G)$, and suppose that $\mathcal{C}_1(G - e) \cup \mathcal{C}_3(G - e) \neq \emptyset$. Take $C \in \mathcal{C}_1(G - e) \cup \mathcal{C}_3(G - e)$. Since $c_1(G) = c_3(G) = 0$, e joins a vertex in $V(C)$ and a vertex y in $V(G) - V(C)$. This implies $C \in \mathcal{C}_1(G - y) \cup \mathcal{C}_3(G - y)$, and hence $\frac{4}{3}|\{y\}| + \frac{1}{3} - (c_1(G - y) + \frac{2}{3}c_3(G - y)) \leqslant \frac{4}{3} + \frac{1}{3} - \frac{2}{3} = 1$, which contradicts the assumption that $\beta \geqslant 2$. Thus $c_1(G - e) = c_3(G - e) = 0$ for all $e \in E(G)$. From the fact that $c_1(G - e) = \emptyset$ for all $e \in E(G)$, it follows that $d_G(x) \geqslant 2$ for all $x \in V(G)$. Assume for the moment that $d_G(x) = 2$ for all $x \in V(G)$. Then each component of G is a cycle. Since $c_3(G) = 0$, this implies that G has a path-factor F with $\mathcal{C}_3(F) = \emptyset$. Hence by Fact 1.1, G has a $\{P_2, P_5\}$-factor. Thus we may assume that there exists $x_0 \in V(G)$ such that $d_G(x_0) \geqslant 3$.

Fix an edge $e^* = x_0 y_0 \in E(G)$ incident with x_0, and let $G' = G - e^*$. By an assertion in the first paragraph of the proof the claim, $c_1(G') = c_3(G') = 0$. Let $X \subseteq V(G')$. We show that $\frac{4}{3}|X|+\frac{1}{3}-c_1(G'-X)-\frac{2}{3}c_3(G'-X)\geqslant 0$. We have $\frac{4}{3}|\emptyset|+\frac{1}{3}-c_1(G')-c_3(G')=\frac{1}{3}>0$. Thus we may assume $X\neq\emptyset$. Note that $c_1(G'-X)+c_3(G'-X)\leqslant c_1(G-X)+c_3(G-X)+2$, and hence

$$c_1(G'-X)+\frac{2}{3}c_3(G'-X)\leqslant c_1(G-X)+\frac{2}{3}c_3(G-X)+2. \tag{4.1}$$

Furthermore, if equality holds in (4.1), then $x_0, y_0\notin X$ and $\{x_0\},\{y_0\}\in\mathcal{C}_1(G'-X)$. If $c_1(G-X)+c_3(G-X)\geqslant 1$, then by the definition of β, $\frac{4}{3}|X|+\frac{1}{3}-c_1(G-X)-\frac{2}{3}c_3(G-X)\geqslant\beta\geqslant 2$ which, together with (4.1), leads to $\frac{4}{3}|X|+\frac{1}{3}-(c_1(G'-X)+\frac{2}{3}c_3(G'-X))\geqslant\frac{4}{3}|X|+\frac{1}{3}-(c_1(G-X)+\frac{2}{3}c_3(G-X)+2)\geqslant\beta-2\geqslant 0$. Thus we may assume that $c_1(G-X)+c_3(G-X)=0$. By (4.1), $c_1(G'-X)+c_3(G'-X)\leqslant c_1(G-X)+c_3(G-X)+2=2$. By way of contradiction, suppose that $\frac{4}{3}|X|+\frac{1}{3}-(c_1(G'-X)+\frac{2}{3}c_3(G'-X))<0$. Then $\frac{4}{3}|X|+\frac{1}{3}-2<0$. Since $X\neq\emptyset$, this forces $|X|=1$ and $c_1(G'-X)+\frac{2}{3}c_3(G'-X)=2$. Hence equality in (4.1), which implies $\{x_0\}\in\mathcal{C}_1(G'-X)$. Consequently $d_G(x_0)\leqslant|X\cup\{y_0\}|=2$, which contradicts the fact that $d_G(x_0)\geqslant 3$. Thus we have $\frac{4}{3}|X|+\frac{1}{3}-c_1(G'-X)-\frac{2}{3}c_3(G'-X)\geqslant 0$ for all $X\subseteq V(G')$. By the induction assumption, G' has a $\{P_2,P_5\}$-factor. Therefore G also has a $\{P_2,P_5\}$-factor. □

By Claim 4.1, we may assume that $\beta\leqslant\frac{5}{3}$.

Let $S\in\mathcal{S}$ be a maximum set with $\frac{4}{3}|S|-c_1(G-S)-\frac{2}{3}c_3(G-S)+\frac{1}{3}=\beta$.

Claim 4.2. *Let C be a component of $G-S$.*

(i) *If $|V(C)|\notin\{1,3\}$, then C has a $\{P_2,P_5\}$-factor.*

(ii) *If $|V(C)|=3$, then C is complete.*

Proof. (i) Suppose that C has no $\{P_2,P_5\}$-factor. Then by the induction assumption, there exists a set $S'\subseteq V(C)$ with $\frac{4}{3}|S'|+\frac{1}{3}-c_1(C-S')-\frac{2}{3}c_3(C-S')<0$. Set $S_0=S\cup S'$. Since $\mathcal{C}_1(G-S_0)=\mathcal{C}_1(G-S)\cup\mathcal{C}_1(C-S')$, $\mathcal{C}_3(G-S_0)=\mathcal{C}_3(G-S)\cup\mathcal{C}_3(C-S')$ and $\mathcal{C}_1(C-S')\cup\mathcal{C}_3(C-S')\neq\emptyset$, we have $S_0\in\mathcal{S}$. We also get $\frac{4}{3}|S_0|+\frac{1}{3}-c_1(G-S_0)-\frac{2}{3}c_3(G-S_0)=(\frac{4}{3}|S|+\frac{1}{3}-c_1(G-S)-\frac{2}{3}c_3(G-S))+(\frac{4}{3}|S'|-c_1(C-S')-\frac{2}{3}c_3(C-S'))<\beta$. This contradicts the definition of β.

(ii) Suppose that $|V(C)|=3$ and C is not complete (i.e., C is a path of order three). Let $x\in C$ be the vertex with $d_C(x)=2$. Then $c_1(C-x)=2$ and $c_3(C-x)=0$. Set $S_1=S\cup\{x\}$. Since $\mathcal{C}_1(G-S_1)=\mathcal{C}_1(G-S)\cup\mathcal{C}_1(C-x)$, $\mathcal{C}_3(G-S_1)=\mathcal{C}_3(G-S)-\{C\}$ and $\mathcal{C}_1(C-x)\neq\emptyset$, we have $S_1\in\mathcal{S}$. We also get $\frac{4}{3}|S_1|+\frac{1}{3}-c_1(G-S_1)-\frac{2}{3}c_3(G-S_1)=(\frac{4}{3}|S|+\frac{4}{3})+\frac{1}{3}-(c_1(G-S)+2)-\frac{2}{3}(c_3(G-S)-1)=\beta$. This contradicts the maximality of S. □

Set $T_1=\mathcal{C}_1(G-S)$, $T_2=\mathcal{C}_3(G-S)$ and $T=T_1\cup T_2$. Now we construct a bipartite graph H with bipartition (S,T) by letting $uC\in E(H)$ $(u\in S, C\in T)$ if and only if $N_G(u)\cap V(C)\neq\emptyset$.

Claim 4.3. *The following hold.*

(i) $|T_1|+\frac{2}{3}|T_2|\leqslant\frac{4}{3}|S|+\frac{1}{3}$.

(ii) $1 \leqslant |S| \leqslant |T_1| + |T_2|$.

(iii) *For every* $X \subseteq S_0$, *either* $|N_H(X) \cap T_1| + \frac{2}{3}|N_H(X) \cap T_2| \geqslant \frac{4}{3}|X|$ *or* $N_H(X) = T$.

Proof. (i) By the assumption of the theorem, $|T_1| + \frac{2}{3}|T_2| = c_1(G-S) + \frac{2}{3}c_3(G-S) \leqslant \frac{4}{3}|S| + \frac{1}{3}$.

(ii) Since $c_1(G) + c_3(G) = 0$ and $c_1(G-S) + c_3(G-S) \geqslant 1$, $S \neq \emptyset$ (i.e., $|S| \geqslant 1$). Since $\frac{4}{3}|S| + \frac{1}{3} - |T_1| - \frac{2}{3}|T_2| = \frac{4}{3}|S| + \frac{1}{3} - c_1(G-S) - \frac{2}{3}c_3(G-S) = \beta \leqslant \frac{5}{3}$, we get $|S| \leqslant \frac{3}{4}|T_1| + \frac{2}{4}|T_2| + 1 \leqslant \frac{3}{4}|T| + 1 < |T| + 1$, and hence $|S| \leqslant |T| = |T_1| + |T_2|$.

(iii) Suppose that there exists a set $X \subseteq S$ such that $|N_H(X)\cap T_1|+\frac{2}{3}|N_H(X)\cap T_2| < \frac{4}{3}|X|$ and $N_H(X) \neq T$. Since $T - N_H(X) \subseteq \mathcal{C}_1(G-(S-X)) \cup \mathcal{C}_3(G-(S-X))$ by the definition of H, we have $S-X \in \mathcal{S}$. We also get $c_1(G-(S-X))+\frac{2}{3}c_3(G-(S-X)) \geqslant (|T_1|-|N_H(X)\cap T_1|)+\frac{2}{3}(|T_2|-|N_H(X)\cap T_2|) = (c_1(G-S)+\frac{2}{3}c_3(G-S))-(|N_H(X)\cap T_1|+\frac{2}{3}|N_H(X)\cap T_2|)$. Consequently $\frac{4}{3}|S-X|+\frac{1}{3}-c_1(G-(S-X))-\frac{2}{3}c_3(G-(S-X)) \leqslant (\frac{4}{3}|S|+\frac{1}{3}-c_1(G-S)-\frac{2}{3}c_3(G-S))-(\frac{4}{3}|X|-|N_H(X)\cap T_1|-\frac{2}{3}|N_H(X)\cap T_2|) < \beta$, which contradicts the definition of β. □

By Claim 4.3 and Theorem 3.1, H has an S-central path-factor F such that $V(A)\cap T_2 \neq \emptyset$ for every $A \in \mathcal{C}_3(F)$. For $A \in \mathcal{C}(F)$, let $U_A = V(A) \cap S$, $\mathcal{L}_{A,h} = V(A) \cap T_h$ $(h \in \{1,2\})$, and $\mathcal{L}_A = \mathcal{L}_{A,1} \cup \mathcal{L}_{A,2}$. Let G_A be the graph obtained from $G[U_A \cup (\bigcup_{C\in\mathcal{L}_A} V(C))]$ by deleting all edges of $G[U_A]$.

Claim 4.4. *For each* $A \in \mathcal{C}(F)$, G_A *has a* $\{P_2, P_5\}$*-factor.*

Proof. Since A is a path of H, there exists a path Q_A of G_A such that $U_A \subseteq V(Q_A)$ and $V(Q_A) \cap V(C) \neq \emptyset$ for every $C \in \mathcal{L}_A$. Choose Q_A so that $|V(Q_A)|$ is as large as possible. Then for each $C \in \mathcal{L}_{A,2}$ (i.e., $C \in \mathcal{L}_A$ with $|V(C)| = 3$), since C is complete by Claim 4.2(ii), it follows that

$$\text{either } V(C) \subseteq V(Q_A) \text{ or } |V(C) \cap V(Q_A)| = 1,$$

and

$$\text{if } C \in \mathcal{L}_A \text{ is an endvertex of the path } A \text{ of } H, \text{ then } V(C) \subseteq V(Q_A).$$

Recall that $\mathcal{L}_{A,2} \neq \emptyset$ if $|V(A)| = 3$. Consequently $|V(Q_A)| \geqslant |V(A)|$ and, in the case where $|V(A)| \geqslant 3$, we have $|V(Q_A)| = 5$ or 7. Since $|V(A)| \geqslant 2$, this means that $|V(Q_A)| \geqslant 2$ and $|V(Q_A)| \neq 3$. Furthermore, for each $C \in \mathcal{L}_A$, $C - V(Q_A)$ is either empty or a path of order two. Therefore if we set $F_A = Q_A \cup (\bigcup_{C\in\mathcal{L}_A}(C - V(Q_A)))$, then F_A is a path-factor of G_A with $\mathcal{C}_3(F_A) = \emptyset$. By Fact 1.1, G_A has a $\{P_2, P_5\}$-factor. □

By Claims 4.2(i) and 4.4, G has a $\{P_2, P_5\}$-factor.

This completes the proof of Theorem 1.1.

5 Examples

In this section, we construct graphs having no $\{P_2, P_{2k+1}\}$-factor.

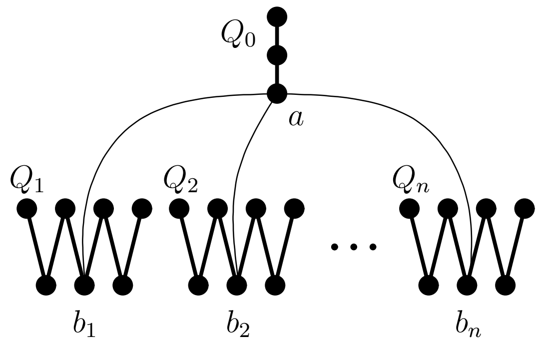

Figure 5: Graph H_n

5.1 Graphs without $\{P_2, P_5\}$-factor

Let $n \geqslant 1$ be an integer. Let Q_0 be a path of order 3, and let a be an endvertex of Q_0. Let $Q_1, \dots, Q_n$ be disjoint paths of order 7, and for each i ($1 \leqslant i \leqslant n$), let b_i be the center of Q_i. Let H_n denote the graph obtained from $\bigcup_{0 \leqslant i \leqslant n} Q_i$ by joining a to b_i for every i ($1 \leqslant i \leqslant n$) (see Figure 5).

Suppose that H_n has a $\{P_2, P_5\}$-factor F. Since Q_0 does not have a $\{P_2, P_5\}$-factor, F contains ab_i for some i ($1 \leqslant i \leqslant n$). Since $d_F(b_i) \leqslant 2$, this requires that at least one of the components of $Q_i - b_i$ should have a $\{P_2, P_5\}$-factor, which is impossible because each component of $Q_i - b_i$ is a path of order 3. Thus H_n has no $\{P_2, P_5\}$-factor.

Lemma 5.1. *For all $X \subseteq V(H_n)$, $c_1(H_n - X) + \frac{2}{3}c_3(H_n - X) \leqslant \frac{4}{3}|X| + \frac{2}{3}$.*

Proof. Let $X \subseteq V(H_n)$. Then we can verify that

$$c_1(Q_0 - X) + \frac{2}{3}c_3(Q_0 - X) \leqslant \frac{4}{3}|V(Q_0) \cap X| + \frac{2}{3} \tag{5.1}$$

and

$$c_1(Q_i - X) + \frac{2}{3}c_3(Q_i - X) \leqslant \frac{4}{3}|V(Q_i) \cap X| \text{ for every } i \ (1 \leqslant i \leqslant n) \tag{5.2}$$

Since every component C of $H_n - X$ with $|V(C)| = 1$ belongs to $\bigcup_{0 \leqslant i \leqslant n} \mathcal{C}_1(Q_i - X)$, we have

$$|\mathcal{C}_1(H_n - X)| = \sum_{0 \leqslant i \leqslant n} |\mathcal{C}_1(Q_i - X)| - \left| \left(\bigcup_{0 \leqslant i \leqslant n} \mathcal{C}_1(Q_i - X) \right) - \mathcal{C}_1(H_n - X) \right|. \tag{5.3}$$

Furthermore,

$$|\mathcal{C}_3(H_n - X)| \leqslant \sum_{0 \leqslant i \leqslant n} |\mathcal{C}_3(Q_i - X)| + \left| \mathcal{C}_3(H_n - X) - \left(\bigcup_{0 \leqslant i \leqslant n} \mathcal{C}_3(Q_i - X) \right) \right|. \tag{5.4}$$

Let C be a component of $H_n - X$ with $|V(C)| = 3$ which does not belong to $\bigcup_{0 \leqslant i \leqslant n} \mathcal{C}_3(Q_i - X)$. Then C intersects with at least two of the Q_i ($0 \leqslant i \leqslant n$). Since

$|V(C)| = 3$, C contains a component of $Q_i - X$ of order 1 for some i $(0 \leqslant i \leqslant n)$. Since C is arbitrary, this implies that

$$\left|\mathcal{C}_3(H_n - X) - \left(\bigcup_{0\leqslant i\leqslant n} \mathcal{C}_3(Q_i - X)\right)\right| \leqslant \left|\left(\bigcup_{0\leqslant i\leqslant n} \mathcal{C}_1(Q_i - X)\right) - \mathcal{C}_1(H_n - X)\right|. \quad (5.5)$$

By (5.1)–(5.5),

$$\begin{aligned}
c_1(H_n - X) &+ \frac{2}{3}c_3(H_n - X) \\
&\leqslant \left(\sum_{0\leqslant i\leqslant n} |\mathcal{C}_1(Q_i - X)| - \left|\left(\bigcup_{0\leqslant i\leqslant n} \mathcal{C}_1(Q_i - X)\right) - \mathcal{C}_1(H_n - X)\right|\right) \\
&\quad + \frac{2}{3}\left(\sum_{0\leqslant i\leqslant n} |\mathcal{C}_3(Q_i - X)| + \left|\mathcal{C}_3(H_n - X) - \left(\bigcup_{0\leqslant i\leqslant n} \mathcal{C}_3(Q_i - X)\right)\right|\right) \\
&\leqslant \left(\sum_{0\leqslant i\leqslant n} |\mathcal{C}_1(Q_i - X)| - \left|\left(\bigcup_{0\leqslant i\leqslant n} \mathcal{C}_1(Q_i - X)\right) - \mathcal{C}_1(H_n - X)\right|\right) \\
&\quad + \frac{2}{3}\left(\sum_{0\leqslant i\leqslant n} |\mathcal{C}_3(Q_i - X)| + \left|\left(\bigcup_{0\leqslant i\leqslant n} \mathcal{C}_1(Q_i - X)\right) - \mathcal{C}_1(H_n - X)\right|\right) \\
&\leqslant \sum_{0\leqslant i\leqslant n} |\mathcal{C}_1(Q_i - X)| + \frac{2}{3}\sum_{0\leqslant i\leqslant n} |\mathcal{C}_3(Q_i - X)| \\
&= \sum_{0\leqslant i\leqslant n} \left(c_1(Q_i - X) + \frac{2}{3}c_3(Q_i - X)\right) \\
&\leqslant \frac{4}{3}\sum_{0\leqslant i\leqslant n} |V(Q_i) \cap X| + \frac{2}{3} \\
&= \frac{4}{3}|X| + \frac{2}{3}.
\end{aligned}$$

Thus we get the desired conclusion. □

From Lemma 5.1, we get the following proposition, which implies that Theorem 1.1 is best possible.

Proposition 5.2. *There exist infinitely many graphs G having no $\{P_2, P_5\}$-factor such that $c_1(G - X) + \frac{2}{3}c_3(G - X) \leqslant \frac{4}{3}|X| + \frac{2}{3}$ for all $X \subseteq V(G)$.*

5.2 Graphs without $\{P_2, P_{2k+1}\}$-factor for $k \geqslant 3$

Let $k \geqslant 3$ be an integer with $k \equiv 0 \pmod 3$, and write $k = 3m$. Let $n \geqslant 1$ be an integer. Let R_0 be a complete graph of order n. For each i $(1 \leqslant i \leqslant 2n + 1)$, let K_i be a complete graph of order $2m - 1$, and let R_i denote the graph obtained from K_i by joining each vertex of the union of $2m + 1$ disjoint paths of order 2 to all vertices of K_i. Let $H'_n = R_0 + (\bigcup_{1\leqslant i\leqslant 2n+1} R_i)$ (see Figure 6).

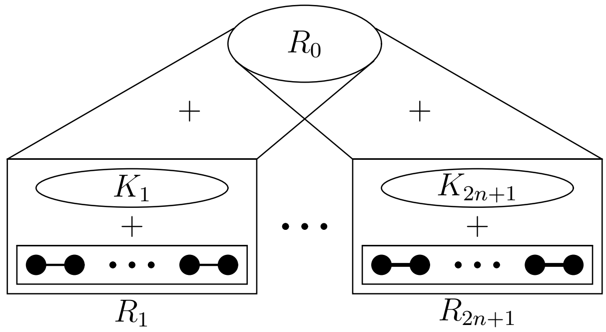

Figure 6: Graph H'_n

Since $|V(R_i)| = 2k+1$ and R_i does not contain a path of order $2k+1$, R_i has no $\{P_2, P_{2k+1}\}$-factor. Suppose that H'_n has a $\{P_2, P_{2k+1}\}$-factor F. Then for each i ($1 \leqslant i \leqslant 2n+1$), F contains an edge joining $V(R_i)$ and $V(R_0)$. Since $2n+1 > 2|V(R_0)|$, this implies that there exists $x \in V(R_0)$ such that $d_F(x) \geqslant 3$, which is a contradiction. Thus H'_n has no $\{P_2, P_{2k+1}\}$-factor.

Lemma 5.3. *For all* $X \subseteq V(H'_n)$, $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(H'_n - X) \leqslant \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3}$.

Proof. Let $X \subseteq V(H'_n)$.

Claim 5.1. *For each* i ($1 \leqslant i \leqslant 2n+1$), $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(R_i - X) \leqslant \frac{4k+6}{8k+3}|V(R_i) \cap X| + \frac{2k+3}{8k+3}$.

Proof. We first assume that $V(K_i) \not\subseteq X$. Then $R_i - X$ is connected. Clearly we may assume that $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(R_i - X) = 1$. Then $|V(R_i) \cap X| \geqslant 2$ because $|V(R_i)| = 2k+1$. Hence $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(R_i - X) = 1 < \frac{4k+6}{8k+3} \cdot 2 < \frac{4k+6}{8k+3}|V(R_i) \cap X| + \frac{2k+3}{8k+3}$. Thus we may assume that $V(K_i) \subseteq X$.

Let α be the number of components of $R_i - V(K_i)$ intersecting with X. Since $\alpha \leqslant 2m+1$, we have $(8m+1)\alpha \leqslant (4m+2)(2m-1+\alpha) + 2m+1$, and hence

$$\alpha \leqslant \frac{4m+2}{8m+1}(2m-1+\alpha) + \frac{2m+1}{8m+1} = \frac{4k+6}{8k+3}(2m-1+\alpha) + \frac{2k+3}{8k+3}.$$

Furthermore, $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(R_i - X) = c_1(R_i - X) \leqslant \alpha$ and $|V(R_i) \cap X| = |V(K_i)| + |(V(R_i) - V(K_i)) \cap X| \geqslant 2m-1+\alpha$. Consequently we get $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(R_i - X) \leqslant \frac{4k+6}{8k+3}|V(R_i) \cap X| + \frac{2k+3}{8k+3}$. ☐

Assume for the moment that $V(R_0) \not\subseteq X$. Then $H'_n - X$ is connected. Clearly we may assume that $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(H'_n - X) = 1$. Then $|X| \geqslant 2$ because $|V(H'_n)| \geqslant 2k+1$. Hence $\sum_{0 \leqslant j \leqslant k-1} c_{2j+1}(H'_n - X) = 1 < \frac{4k+6}{8k+3} \cdot 2 < \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3}$. Thus we may assume that $V(R_0) \subseteq X$. Then clearly

$$|\mathcal{C}_{2j+1}(H'_n - X)| = \sum_{1 \leqslant i \leqslant 2n+1} |\mathcal{C}_{2j+1}(R_i - X)|. \tag{5.6}$$

By Claim 5.1 and (5.6),

$$\begin{aligned}\sum_{0\leqslant j\leqslant k-1} c_{2j+1}(H'_n - X) &= \sum_{0\leqslant j\leqslant k-1}\left(\sum_{1\leqslant i\leqslant 2n+1} c_{2j+1}(R_i - X)\right)\\ &\leqslant \sum_{1\leqslant i\leqslant 2n+1}\left(\frac{4k+6}{8k+3}|V(R_i)\cap X| + \frac{2k+3}{8k+3}\right)\\ &= \frac{4k+6}{8k+3}(|X| - |V(R_0)|) + \frac{2k+3}{8k+3}(2n+1)\\ &= \frac{4k+6}{8k+3}(|X| - n) + \frac{2k+3}{8k+3}(2n+1)\\ &= \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3}.\end{aligned}$$

Thus we get the desired conclusion. □

From Lemma 5.3, we get the following proposition, which implies that if Conjecture 1 is true, then the coefficient of $|X|$ in the conjecture is best possible.

Proposition 5.4. *For an integer $k \geqslant 3$ with $k \equiv 0 \pmod 3$, there exist infinitely many graphs G having no $\{P_2, P_{2k+1}\}$-factor such that $\sum_{0\leqslant i\leqslant k-1} c_{2i+1}(G-X) \leqslant \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3}$ for all $X \subseteq V(G)$.*

6 Concluding remarks

In this paper, we prove that if a graph G satisfies $c_1(G-X) + \frac{2}{3}c_3(G-X) \leqslant \frac{4}{3}|X| + \frac{1}{3}$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_5\}$-factor. The result naturally suggests the following problem: For an integer $k \geqslant 3$, is there a number $\varepsilon_k > 0$ such that if a graph G satisfies $\sum_{0\leqslant j\leqslant k-1} c_{2j+1}(G-X) \leqslant \varepsilon_k|X|$ for all $X \subseteq V(G)$, then G has a $\{P_2, P_{2k+1}\}$-factor? Recently, Egawa, Furuya and Ozeki [4] gave an affirmative solution to the problem with $\varepsilon_k = \frac{5}{6k^2}$.

References

[1] J. Akiyama, D. Avis and H. Era, On a $\{1, 2\}$-factor of a graph, TRU Math. **16** (1980) 97–102.

[2] R. Diestel, "Graph Theory" (4th edition), Graduate Texts in Mathematics **173**, Springer (2010).

[3] Y. Egawa and M. Furuya, Path-factors involving paths of order seven and nine, Theory Appl. Graphs **3(1)** (2016) #5.

[4] Y. Egawa, M. Furuya and K. Ozeki, Sufficient conditions for the existence of a path-factor which are related to odd components, preprint.

[5] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, J. Combin. Theory Ser. B **88** (2003) 195–218.

[6] M. Kano, C. Lee and K. Suzuki, Path and cycle factors of cubic bipartite graphs, Discuss. Math. Graph Theory **28** (2008) 551–556.

[7] K. Kawarabayashi, H. Matsuda, Y. Oda and K. Ota, Path factors in cubic graphs, J. Graph Theory **39** (2002) 188–193.

[8] M. Loebl and S. Poljak, Efficient subgraph packing, J. Combin. Theory Ser. B **59** (1993) 106–121.