Obstructions to combinatorial formulas for plethysm

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Abstract

Motivated by questions of Mulmuley and Stanley we investigate quasi-polynomials arising in formulas for plethysm. We demonstrate, on the examples of $S^3(S^k)$ and $S^k(S^3)$, that these need not be counting functions of inhomogeneous polytopes of dimension equal to the degree of the quasi-polynomial. It follows that these functions are not, in general, counting functions of lattice points in any scaled convex bodies, even when restricted to single rays. Our results also apply to special rectangular Kronecker coefficients.

Many problems in representation theory have combinatorial solutions. A well-known, important example is the Littlewood–Richardson rule which gives the multiplicities of isotypic components in the tensor product of two irreducible $GL(n)$ representations [FH91]. The solution is combinatorial in the sense that the answer is given by the number of lattice points in explicit rational convex polyhedra, i.e. the multiplicities are equal to values of an Ehrhart quasi-polynomial. Berenstein and Zelevinsky provided another interpretation of the Littlewood–Richardson rule [BZ92], isomorphic, however, on the level of polytopes [PV05]. The study of different polyhedral structures turned out to be very useful. Knutson and Tao [KT99] showed that the honeycomb polytopes (in spirit similar to [BZ92]) are nonempty if and only if they contain a lattice point. This was the crucial last step in the solution to the Horn problem [Hor62] which goes back to Weyl's work on eigenvalues of partial differential equations [Wey12].

In this paper we study the more complicated plethysm coefficients $m_\lambda^{d,k}$ defined by

$$S^d(S^k W) = \bigoplus_\lambda m_\lambda^{d,k} S^\lambda W,$$

where the sum is over partitions λ of dk.

The computation of the multiplicity functions in plethysms can be seen as an operation on Schur polynomials [Mac98]. Viewed like this, it is surprising that when the plethysm is written in terms of other Schur polynomials, the coefficients are always nonnegative. Of course this must be true, because the coefficients are multiplicities of irreducible representations, but it would be desirable to have a combinatorial explanation for nonnegativity. In [Sta00, Problem 9] Richard Stanley asked for a *positive* combinatorial method to compute plethysm coefficients. A connection between plethysm and lattice point counting was shown at least in [KM16, Col17, CDW12]. These connections are not direct in the sense that plethysm coefficients are not seen to equal counts, but always involve some opaque arithmetics.

The functions $f(s) = m_{s\lambda}^{d,sk}$ and $g(s) = m_{s\lambda}^{sd,k}$ share many properties with Ehrhart functions of rational polytopes:

- Both f and g are nonnegative and $f(0) = g(0) = 1$.
- Both f and g are quasi-polynomials. This deep fact follows from [MS99, Corollary 2.12]. See [KM16, Remark 3.12].
- For regular λ, the leading term of f is proportional to the Ehrhart function of a rational polytope corresponding to the Littlewood–Richardson rule [KM16, Section 4].

Remark 1. The scaling factor between the leading coefficient of f and the Ehrhart function in the third bullet is the Plancherel measure on Young diagrams. It is conjectured that the same is true for any λ, however we are not aware of a proof.

Example 2. The multiplicity of $\lambda = s \cdot (2k-1, 1)$ inside $S^2(S^{sk})$ equals 0 when s is odd and 1 when s even. In particular, it is equal to the number of integral points in the (zero-dimensional) polytope $\{s/2\}$. The multiplicity of $s \cdot (2k, k)$ inside $S^3(S^{sk})$ is equal to the number of integral points in the (one-dimensional) polytope $s \cdot [1/3, 1/2]$ or $s \cdot [1/2, 2/3]$.

Ketan Mulmuley posed several conjectures concerning the behavior of plethysm and Kronecker coefficients in relation to geometric complexity theory [MS01, Mul09].

Question 3. *Fix positive integers k, d, and λ a partition of kd. Are the quasi-polynomials $s \mapsto m_{s\lambda}^{d,sk}$ and $s \mapsto m_{s\lambda}^{sd,k}$ Ehrhart functions of rational polytopes?*

The question about $m_{s\lambda}^{sd,k}$ comes directly from [Mul09, Hypothesis 1.6.4]. We give a negative answer for both functions in Remark 6. Our main goal, however, is a generalized version of these questions for which we need some additional terminology. Following

[Mul11, Section 5.1] we define a *shifted* or *inhomogeneous rational polytope* as a system of inequalities

$$P(A, b, c) = \{x \in \mathbb{Q}^m : Ax \leqslant b + c\},$$

where b, c are arbitrary rational vectors and A is a rational matrix. Splitting the right hand side as $b + c$ is motivated by the definition of the dilations of P as

$$P(A, sb, c) = \{x \in \mathbb{Q}^m : Ax \leqslant sb + c\}.$$

An *asymptotic Ehrhart quasi-polynomial* is a counting function of the form

$$s \mapsto \#(P(A, sb, c) \cap \mathbb{Z}^m).$$

The *dimension* of a shifted rational polytope P is by definition the dimension of $P(A, sb, c)$ for large s (for small s the polytope can be empty). Contrary to the case of Ehrhart quasi-polynomials, an asymptotic Ehrhart quasi-polynomial does not need to be a quasi-polynomial, although it is for large arguments. Moreover, a quasi-polynomial may be an asymptotic Ehrhart quasi-polynomial but not an Ehrhart quasi-polynomial. Asymptotic Ehrhart quasi-polynomials do not have to satisfy Ehrhart reciprocity. Further, the dimension of a shifted rational polytope can be strictly greater than the degree of the associated asymptotic Ehrhart quasi-polynomial. See [Sta82] and [Sta96, Chapter I] for structure theory of asymptotic Ehrhart quasi-polynomials and the relation to linear diophantine equations.

In [Mul11, Hypothesis 5.3] it is conjectured that the multiplicity of $s \cdot \pi$ in $S^{s \cdot \lambda}(S^\mu)$ is an asymptotic Ehrhart quasi-polynomial with additional complexity-theoretic properties. The asymptotic part of the conjecture is motivated by the failure of Ehrhart type explanations already for Kronecker coefficients [BOR09, KW09] which are often considered less complicated than plethysm coefficients. In the present paper we make progress towards a negative answer of the following simplified version.

Question 4. *Is $m_{s\lambda}^{sd,k}$ an asymptotic Ehrhart quasi-polynomial?*

Our main result, Theorem 5, implies a negative answer to both cases in Question 3 and strong restrictions on a positive solution to Question 4. We show that $m_{s\lambda}^{sd,k}$ need not be an asymptotic Ehrhart quasi-polynomial of a polytope of dimension equal to the degree of its growth. At the moment we are not able either to exclude or confirm that $m_{s\lambda}^{sd,k}$ is an asymptotic Ehrhart quasi-polynomial for an inhomogeneous polytope of dimension strictly larger than its degree.

In previous work the authors gave a formula for plethysm coefficients, which is a sum of Ehrhart functions of various polytopes with (positive and negative) coefficients [KM16]. This allows to gather experimental data on the questions for many rays. A-posteriori, the specific plethysm in Theorem 5 can also be confirmed using well-known formulas for $S^3(S^k)$. General formulas for $S^k(S^3)$ are unknown, though. Our methods are inspired by [KW09, BOR09].

Theorem 5. *The multiplicity functions of $S^{s\cdot(7,5,0)}$ in $S^3(S^{4s})$ and $S^{4s}(S^3)$ equal*

$$\phi : s \mapsto \frac{s + r(s)}{3}$$

where $r(s)$ has period 6 and takes the values $3, -1, 1, 0, 2, -2$ on respectively the integers $0, \ldots, 5$. There do not exist rational vectors $a, b, c \in \mathbb{Q}^n$ (of arbitrary length n) such that ϕ equals the counting function of a one-dimensional inhomogeneous polytope $P(a, b, c)$. In particular, ϕ is not an Ehrhart function of any rational polyhedron.

Remark 6. Before giving a general proof, it is easy to see that ϕ cannot be an Ehrhart function: it violates Ehrhart–Macdonald reciprocity [Ehr67, Mac71]. The value $f(-s)$ of an Ehrhart quasi-polynomial f of a rational polytope P at a negative integer argument counts (up to a global sign) the number of interior lattice points in $s \cdot P$. In particular, $|f(-s)| \leqslant f(s)$. However,

$$-\phi(-1) = 1 > 0 = \phi(1).$$

Interestingly, the jumps in values that make Ehrhart reciprocity fail are also the crucial ingredient for our proof of non-representability by a one-dimensional inhomogeneous polytope.

Proof. The equality of the multiplicities of the given ray in both plethysms is a consequence of Hermite reciprocity [Her54], [FH91, Exercise 6.18]. There are now various ways to determine the formula from the interpretation as a multiplicity in $S^3(S^{4s})$. One is to simply evaluate the explicit formula from [KM16] along a ray. Another way is to observe that the function must exhibit linear growth and that its period is at most six [CGR84, MM15]. With this information and some values the function can be interpolated.

Let P be an inhomogeneous line segment as in the statement. Then the s-th dilation of P can be written as

$$P(s) = \{x \in \mathbb{Q} : \max_i(sb_i + c_i) \leqslant x \leqslant \min_i(sb'_i + c'_i)\}.$$

Asymptotically, $P(s)$ becomes an interval of length $\frac{s}{3}$. This means that there exists an s_0 such that for all $s > s_0$

$$P(s) = \{x \in \mathbb{Q} : sb + c \leqslant x \leqslant s(b + \tfrac{1}{3}) + c')\} =: Q(s), \tag{1}$$

for some $b, c, c' \in \mathbb{Q}$. Making s_0 even larger it can be assumed that $s_0 b$ are integers and that s_0 is divisible by 6 (the period of ϕ). Since $\#Q(s)$ and $\#P(s)$ have the same linear term and constants that are s_0-periodic (by the divisibility assumptions) they agree for all s.

The proof of nonexistence of the family $Q(s)$ such that $\phi(s) = \#Q(s)$ is by examination of constraints on b, coming from the known values of the counting function ϕ. Without loss of generality we have

$$0 \leqslant b < 1. \tag{2}$$

Indeed, changing b by an integer also only shifts Q by an integral value.

As $\phi(5) + 2 = \phi(6)$ there must be at least two integers x such that $5(b + \frac{1}{3}) + c' < x \leqslant 6(b + \frac{1}{3}) + c'$. Hence, $b + \frac{1}{3} > 1$. In particular, for any s there is always at least one integer x satisfying $s(b + \frac{1}{3}) + c' < x \leqslant (s+1)(b + \frac{1}{3}) + c'$ (an interval of length > 1 must contain an integer). On the other hand, by (2) there is at most one integer x satisfying $sb + c' < x \leqslant (s+1)b + c'$ (an interval of length < 1 may contain at most one integer). The above two intervals are the difference between $P(s)$ and $P(s+1)$. It follows that $\phi(s)$ is nondecreasing. This contradicts $\phi(0) = 1 > 0 = \phi(1)$. $\square$

Remark 7. We have confirmed that all examples with two rows and fewer boxes are in fact Ehrhart functions. In this sense, our counter-example is minimal.

Remark 8. A different way to describe an inhomogeneous polytope is by linear integral equations and nonnegativity. In this setting Stanley gave a general reciprocity theorem which relates the Hilbert series (expanded at ∞) of the module of positive integral solutions of linear Diophantine equations to the Hilbert series of negative solutions to the same equations. In [Sta82, Theorem 4.2] there are, however, combinatorially defined correction terms.

Remark 9. While this is not visible from the representation in [KM16], a general theorem of Meinrenken and Sjamaar [MS99] implies that the chambers of plethysm quasi-polynomials are polyhedral cones (see [KM16, Remark 3.12], [PV15]). One may thus ask the stronger question if they are Ehrhart quasi-polynomials in general. Of course, Theorem 5 gives a negative answer to this much stronger property too.

Lattice point counting in polytopes and Ehrhart functions are arguably the most natural combinatorial explanations one may hope for here, but they are not the only ones. Our theorem also excludes other possibilities.

Corollary 10. *The plethysm coefficients are not positive combinations of counting functions of lattice points in integral scalings of any convex regions.*

Proof. One-dimensional convex sets are polyhedra and the counting function in Theorem 5 would need to specialize to a positive combination of Ehrhart functions. $\square$

The plethysm counting function in Theorem 5 cannot be written as a positive combination of Ehrhart quasi-polynomials as these also have to satisfy Ehrhart reciprocity. However, it can be written as a sum of an asymptotic Ehrhart quasi-polynomial and an honest Ehrhart quasi-polynomial as in the following proposition (which is very easy to confirm).

Proposition 11. *Fix a small rational $\varepsilon > 0$. Let $P(s) = [\varepsilon, s \cdot \frac{1}{3} + \varepsilon]$ and $Q(s) = s \cdot \{\frac{1}{2}\}$. Then $\phi(s) = \#(P(s) \cap \mathbb{Z}) + \#(Q(s) \cap \mathbb{Z})$.*

Remark 12. An interesting open question is the following. Suppose we are given a piecewise quasi-polynomial whose chambers are cones. Is there a finite computational test if the quasi-polynomial is an (asymptotic) Ehrhart function? Assume it is not an Ehrhart function globally, but is an Ehrhart function on each individual ray. Is there a finite number of rays such that finding polytopes for each of the finitely many rays proves the Ehrhart nature for all rays? Is it some multi-dimensional generalization of Apéry sets known from numerical semigroups?

Remark 13. According to [KM16, Remark 3.8] the function ϕ in Theorem 5 also counts, for example, the multiplicity of $\lambda = s \cdot (7+2t, 5+2t, 2t)$ in $S^3(S^{s\cdot(5+2t)})$ for any positive t. Therefore counterexamples also exist for strictly interior rays.

Recently, there has been a lot of interest in *Kronecker coefficients* [Man16, BOR09, SS16, BV15]. These are different from plethysm, but also hard to compute and important in geometric complexity theory. It is known that on rays they are given by quasi-polynomials [Man15, Theorem 1] that do not have to be Ehrhart functions [KW09]. However, the problem is open for specific rays that are important in GCT. Emmanuel Briand pointed us towards following observation. Two row partitions are useful because Hermite reciprocity applies to them. Further, by the work of Vallejo [Val14], cf. [Mac98], the plethysm coefficients in Theorem 5 are equal to special Kronecker coefficients:

$$\phi(s) = K(3^{4s}, 3^{3s}, (7s, 5s)) = K((4s, 4s, 4s), (4s, 4s, 4s), (7s, 5s)),$$

where K depends on three partitions and is the Kronecker coefficient. Indeed, let $p_n(a, b)$ denote the number of Young diagrams with n boxes contained inside an $a \times b$ rectangle. Both the multiplicity of λ with two rows in the plethysm $S^a(S^b)$ and the Kronecker coefficient $K(a^b, a^b, \lambda)$ can be expressed as the difference

$$p_{\lambda_2}(a, b) - p_{\lambda_2 - 1}(a, b).$$

For example by transposition, this immediately implies Hermite reciprocity.

Example 14. The multiplicity of $(7, 5)$ inside $S^4(S^3)$ equals zero. Here we have 4 diagrams with 5 boxes inside a 4×3 rectangle: $(4, 1), (3, 2), (3, 1, 1), (2, 2, 1)$. We also have 4 such diagrams with 4 boxes: $(4, 0), (3, 1), (2, 2), (2, 1, 1)$, hence the difference is indeed zero.

The rectangular Kronecker coefficients $K(a^b, a^b, \lambda)$ are of particular interest in GCT (see e.g. [IP17]). Moreover, they exactly govern unimodality of q-binomial coefficients, observed already in 1878 by Sylvester (see [PP14, p. 2]).

In a different direction, the following interesting question was recently brought to our attention by Michèle Vergne.

Question 15. *Are the plethysm coefficients Ehrhart polynomials after rescaling? Specifically, are $\tilde{f}(s) := f(ks)$ or $\tilde{g}(s) := g(ks)$ Ehrhart polynomials for some k?*

One can prove that the answer is positive when f or g are of degree one, thus to find a counterexample one needs to consider partitions with more than one row.

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References

[BOR09] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas, *Reduced Kronecker coefficients and counter–examples to Mulmuley's strong saturation conjecture SH*, Computational Complexity **18** (2009), no. 4, 577–600.

[BV15] Velleda Baldoni and Michèle Vergne, *Multiplicity of compact group representations and applications to Kronecker coefficients*, preprint, arXiv:1506.02472 (2015).

[BZ92] Arkady D Berenstein and Andrei V Zelevinsky, *Triple multiplicities for* $sl(r+1)$ *and the spectrum of the exterior algebra of the adjoint representation*, Journal of Algebraic Combinatorics **1** (1992), no. 1, 7–22.

[CDW12] Matthias Christandl, Brent Doran, and Michael Walter, *Computing multiplicities of Lie group representations*, IEEE 53rd Annual Symposium on Foundations of Computer Science (FOCS), IEEE, 2012, pp. 639–648.

[CGR84] Y. M. Chen, A. M. Garsia, and J. Remmel, *Algorithms for plethysm*, Combinatorics and algebra (Boulder, Colo., 1983), Contemporary Mathematics, vol. 34, AMS, Providence, RI, 1984, pp. 109–153.

[Col17] Laura Colmenarejo, *Stability properties of the plethysm: A combinatorial approach*, Discrete Mathematics **340** (2017), no. 8, 2020–2032.

[Ehr67] Eugène Ehrhart, *Sur un problème de géométrie diophantienne linéaire I*, Journal für die reine und angewandte Mathematik **226** (1967), 1–29.

[FH91] William Fulton and Joe Harris, *Representation theory. A first course*, Springer, Berlin, 1991.

[Her54] Charles Hermite, *Sur la theorie des fonctions homogènes à deux indéterminées*, Cambridge and Dublin Mathematical Journal **9** (1854), 172–217.

[Hor62] Alfred Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific Journal of Mathematics **12** (1962), no. 1, 225–241.

[IP17] Christian Ikenmeyer and Greta Panova, *Rectangular Kronecker coefficients and plethysms in geometric complexity theory*, Advances in Mathematics **319** (2017), 40–66.

[KM16] Thomas Kahle and Mateusz Michałek, *Plethysm and lattice point counting*, Foundations of Computational Mathematics **16** (2016), no. 5, 1241–1261.

[KT99] Allen Knutson and Terence Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture*, Journal of the American Mathematical Society **12** (1999), no. 4, 1055–1090.

[KW09] Ronald King and Trevor Welsh, *Some remarks on characters of symmetric groups, Schur functions, Littlewood–Richardson and Kronecker coefficients*, work in progress http://congreso.us.es/enredo2009/Workshop_files/Sevilla_King.pdf (2009).

[Mac71] Ian G Macdonald, *Polynomials associated with finite cell-complexes*, Journal of the London Mathematical Society **2** (1971), no. 1, 181–192.

[Mac98] Ian Grant Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1998.

[Man15] Laurent Manivel, *On the asymptotics of Kronecker coefficients*, Journal of Algebraic Combinatorics **42** (2015), no. 4, 999–1025.

[Man16] ———, *On the asymptotics of Kronecker coefficients, 2*, Séminaire Lotharingien de Combinatoire **75** (2016), B75d, 13p.

[MM15] Laurent Manivel and Mateusz Michałek, *Secants of minuscule and cominuscule minimal orbits*, Linear Algebra and its Applications **481** (2015), 288–312.

[MS99] Eckhard Meinrenken and Reyer Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), no. 4, 699–762.

[MS01] Ketan D Mulmuley and Milind Sohoni, *Geometric complexity theory I: An approach to the P vs. NP and related problems*, SIAM Journal on Computing **31** (2001), no. 2, 496–526.

[Mul09] Ketan D Mulmuley, *Geometric complexity theory VI: the flip via saturated and positive integer programming in representation theory and algebraic geometry*, preprint, arXiv:0704.0229 (2009).

[Mul11] Ketan Mulmuley, *Geometric complexity theory VI: The flip via positivity*, http://gct.cs.uchicago.edu/gct6.pdf, 2011, Online; accessed 14 December 2015.

[PP14] Igor Pak and Greta Panova, *Bounds on Kronecker and q-binomial coefficients*, preprint, arXiv:1410.7087 (2014).

[PV05] Igor Pak and Ernesto Vallejo, *Combinatorics and geometry of Littlewood–Richardson cones*, European Journal of Combinatorics **26** (2005), no. 6, 995–1008.

[PV15] Paul-Emile Paradan and Michèle Vergne, *Witten non abelian localization for equivariant K-theory, and the $[Q, R] = 0$ theorem*, preprint, arXiv:1504.07502 (2015).

[SS16] Steven V Sam and Andrew Snowden, *Proof of Stembridge's conjecture on stability of Kronecker coefficients*, Journal of Algebraic Combinatorics **43** (2016), no. 1, 1–10.

[Sta82] Richard P Stanley, *Linear Diophantine equations and local cohomology*, Inventiones mathematicae **68** (1982), no. 2, 175–193.

[Sta96] Richard P. Stanley, *Combinatorics and commutative algebra*, 2nd ed., Birkhäuser, 1996.

[Sta00] Richard P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, Mathematics: Frontiers and Perspectives, AMS, Providence, RI, 2000, pp. 295–319.

[Val14] Ernesto Vallejo, *A diagrammatic approach to Kronecker squares*, Journal of Combinatorial Theory, Series A **127** (2014), 243–285.

[Wey12] Hermann Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Mathematische Annalen **71** (1912), no. 4, 441–479.