The Cayley isomorphism property for Cayley maps

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Submitted: Feb 21, 2016; Accepted: Feb 3, 2018; Published: Mar 2, 2018
Mathematics Subject Classifications: 05E18, 05C10, 20B25

Abstract

The *Cayley Isomorphism* property for combinatorial objects was introduced by L. Babai in 1977. Since then it has been intensively studied for binary relational structures: graphs, digraphs, colored graphs etc. In this paper we study this property for oriented Cayley maps. A Cayley map is a Cayley graph provided by a cyclic rotation of its connection set. If the underlying graph is connected, then the map is an embedding of a Cayley graph into an oriented surface with the same cyclic rotation around every vertex.

Two Cayley maps are called *Cayley isomorphic* if there exists a map isomorphism between them which is a group isomorphism too. We say that a finite group H is a *CIM-group*[1] if any two Cayley maps over H are isomorphic if and only if they are Cayley isomorphic.

The paper contains two main results regarding CIM-groups. The first one provides necessary conditons for being a CIM-group. It shows that a CIM-group should be one of the following

$$\mathbb{Z}_m \times \mathbb{Z}_2^r,\ \mathbb{Z}_m \times \mathbb{Z}_4,\ \mathbb{Z}_m \times \mathbb{Z}_8,\ \mathbb{Z}_m \times Q_8,\ \mathbb{Z}_m \rtimes \mathbb{Z}_{2^e}, e = 1, 2, 3,$$

where m is an odd square-free number and r a non-negative integer[2]. Our second main result shows that the groups $\mathbb{Z}_m \times \mathbb{Z}_2^r,\ \mathbb{Z}_m \times \mathbb{Z}_4,\ \mathbb{Z}_m \times Q_8$ contained in the above list are indeed CIM-groups.

1 Introduction

The history of the Isomorphism problem for Cayley graphs started in 1967 when the famous Ádám's conjecture was posed [1]. In 1977, L. Babai generalized this problem to

*Supported by the Israeli Ministry of Absorption
[1]CIM stands for Cayley Isomorphism property for Maps
[2]The cases of $m = 1$ and $r = 0$ are allowed.

any class of combinatorial structures via concrete categories [3]. He introduced Cayley Isomorphism property (*CI-property*, for short) for arbitrary combinatorial structures and developed group-theoretic approach to the isomorphism problem of Cayley structures (see [18, 20] for the recent developments in this area). Although the basic tools and techniques developed by Babai are applicable to any class of combinatorial objects, the mainstream of the research in the area was focused on Cayley graphs and other binary relational Cayley structures. Only recently the study of isomorphism problem for non-graphical Cayley structures (ternary relational structures [7, 8, 9], linear codes [22], balanced configurations [17]) was started.

In this paper we start the investigation of the isomorphism problem for a quite natural class of Cayley structures which was not studied before, namely: the class of Cayley maps. Although this class of structures is being actively studied during recent two decades, the isomorphism problem for Cayley maps was not considered at all. This paper aims to fill this lacuna. In order to formulate our main results we need to introduce basic definitions.

Let H be a finite group and S a subset of $H \setminus \{1\}$. A *Cayley (di)graph* $\mathsf{Cay}(H,S)$ is defined by having the vertex set H and g is adjacent to h if and only if $g^{-1}h \in S$. The set S is called the *connection set* of the Cayley graph $\mathsf{Cay}(H,S)$. A Cayley graph $\mathsf{Cay}(H,S)$ is *undirected* if and only if $S = S^{-1}$, where $S^{-1} = \{\, s^{-1} \in H \mid s \in S \,\}$. Every left multiplication by an element of H is an automorphism of $\mathsf{Cay}(H,S)$, so the automorphism group of every Cayley graph over H contains a regular subgroup isomorphic to H. Moreover, this property characterises the Cayley graphs of H. The permutation group consisting of the left multiplications will be denoted by $\widehat{H}$ and the left multiplication by $h \in H$ by $\widehat{h}$ (that is $\widehat{h}(x) = hx$).

A group H is called a *CI-group* with respect to graphs if two Cayley graphs of H are isomorphic if and only if they are isomorphic by a group automorphism as well. For an old but excellent survey about CI-groups, see [18] and further results can be found in [20].

An *(oriented) Cayley map* $\mathsf{CM}(H,S,\rho)$ is built on an undirected Cayley graph $\mathsf{Cay}(H,S)$, which is endowed with a cyclic ordering $\rho \in \mathsf{Sym}(S)$ of the connection set. We say that a map $\mathsf{CM}(H,S,\rho)$ is *connected* if the underlying Cayley graph is connected, that is $\langle S\rangle = H$. Every connected Cayley map determines a 2-cell embedding of a Cayley graph into oriented surface with the same cyclic rotation around each vertex. For precise definiton of embedding of graphs into orientable surfaces, see [16].

Note that our definition of a Cayley map is a bit different from the standard one where the underlying Cayley graph is required to be connected. The main reason for this change is that traditionally the analisys of the isomorphism problem for Cayley combinatorial objects was not split into “connected” and “non-connected” parts. It is pretty clear that if a group has CI-proprety for all Cayley maps then it also has the same property for connected ones. The converse is not true. Theorems 1.1 and 1.2 below show that connectedness does matter for Cayley maps. This is quite different from the situation with Cayley graphs where we have no difference between the connected and disconnected CI-properties. The reason for that is that a complement of disconnected Cayley graph is always a connected one, while for a Cayley map there is no natural choice for the cyclic rotation of the connection set on the complement graph. In general, the analysis (of the

CI property) of connected Cayley maps seems to be more complicated than the one of ordinary Cayley maps.

Several different subclasses of Cayley maps have been investigated. The notion of a Cayley map first appeared in the paper of Biggs [5] who investigated balanced Cayley maps. A Cayley map $\mathsf{CM}(H, S, \rho)$ is called *balanced* if $\rho(s^{-1}) = \rho(s)^{-1}$ and it is called *antibalanced* if $\rho(s^{-1}) = \rho^{-1}(s)^{-1}$.

Given two Cayley maps $M_1 = \mathsf{CM}(H_1, S_1, \rho_1)$ and $M_2 = \mathsf{CM}(H_2, S_2, \rho_2)$, a bijection $\phi : H_1 \to H_2$ is a *map isomorphism* from M_1 to M_2 if ϕ is an isomorphism of the underlying Cayley graphs and for all $h \in H_1, s \in S_1$ it holds that $\phi(h)^{-1}\phi(h\rho_1(s)) = \rho_2(\phi(h)^{-1}\phi(hs))$. Although the standard definition of map isomorphism is based on a bijection between the dart sets (see [12]), our definition is equivalent to the standard one in the case of Cayley maps. We refer the reader to Section 2 where all necessary definitions and proofs related to map isomorphism are given.

In what follows we say that M_1 and M_2 are *Cayley isomorphic* if there exists a group isomorphism $\phi : H_1 \to H_2$ which is simulteneously a map isomorphism, that is $\phi(S_1) = S_2$ and $\phi(\rho_1(s)) = \rho_2(\phi(s))$ hold for each $s \in S_1$.

The *automorphism group* of a Cayley map $M = \mathsf{CM}(H, S, \rho)$ is the set of all map isomorphisms from M to M and it will be denoted by $\mathsf{Aut}(M)$. Thus $\mathsf{Aut}(\mathsf{CM}(H, S, \rho))$ contains the regular subgroup $\widehat{H}$. Every group automorphism $\sigma \in \mathsf{Aut}(H)$ induces Cayley isomorphism between the maps $\mathsf{CM}(H, S, \rho)$ and $\mathsf{CM}(H, \sigma(S), \sigma'^{-1}\rho\sigma')$ where $\sigma' = \sigma|_S$ is the restriction of σ on S. Thus a group automorphism σ is an automorphism of the map $\mathsf{CM}(H, S, \rho)$ if and only if $\sigma(S) = S$ and $\sigma|_S\rho = \rho\sigma|_S$. Since ρ is a full cycle, the latter condition is equivalent to $\sigma|_S = \rho^k$ for some integer k. A Cayley map M is called *regular* if its automorphism group is transitive on the arcs as well. It follows from the definition of the automorphism group of a Cayley map that this is the largest possible automorphism group of a connected Cayley map. Following Jajcay and Širáň [14], we say that for a group H a permutation $\phi \in \mathsf{Sym}(H)$ is a *skew-morphism* if it fixes the identity and there exists a mapping $\pi : H \mapsto \mathbb{N}$ such that $\phi(gh) = \phi(g)\phi^{\pi(g)}(h)$ for every $g, h \in G$.

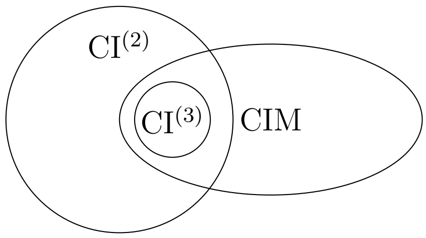
Similarly to the original definition of the CI property, we say that a Cayley map $M = \mathsf{CM}(H, S, \rho)$ is a *CI-map* of H if every Cayley map M' over H isomorphic to M is also Cayley isomorphic to M. We call a group H a *CIM-group* if every Cayley map $\mathsf{CM}(H, S, \rho)$ is a CI-map.

A Cayley map $M = \mathsf{CM}(H, S, \rho)$ can also be considered as a ternary relational structure on the vertices of the underlying graph. Three vertices (x, y, z) are in the relation $\mathcal{R}$ if and only if $x^{-1}y, x^{-1}z \in S$ and $\rho(x^{-1}y) = x^{-1}z$. The automorphism group $\mathsf{Aut}(M)$ consists of all those permutations of the vertices which preserve the relation $\mathcal{R}$. In particular, it is a 3-closed permutation group. This observation allows us to apply the group-theoretic technique developed by Babai [3] for arbitrary Cayley combinatorial structures to a particular case of Cayley maps.

Moreover, a theorem of Pálfy [24] shows that the groups which are CI-groups for every m-ary relational structure are the cyclic groups of order n, where $(n, \phi(n)) = 1$ and the Klein group. Pálfy also proved that if a group is not a CI-group with respect to some m-ary relation, then it is not a CI-group with respect to 4-ary relational structures.

CI-groups with respect to ternary relations (CI$^{(3)}$-groups, for short) were investigated by Dobson [7],[8] and later by Dobson and Spiga [9]. Although the class of CI$^{(3)}$-groups is rather narrow, its full classification is not finished yet. The latest results can be found in [8] and [9]. Since every map automorphism group is 3-closed, each CI$^{(3)}$-group is a CIM-group. The converse is not true. For example, every elementary abelian 2-group of rank at least 6 is a CIM-group (see Theorem 1.2) but not a CI$^{(3)}$-group [23].

As it was also pointed out by Dobson and Spiga [9] every CI$^{(3)}$-group is also a CI$^{(2)}$-group, that is a group which has a the CI-property with respect to binary relational structures. However, we will prove that there are CIM-groups which are not CI$^{(2)}$-groups. The Venn diagram below reflects the relationships between the three classes of CI-groups.



Our first result formulates necessary conditions for being a CIM-group.

Theorem 1.1. *Let H be a CIM-group. Then H is isomorphic to one of the following groups*

(a) $\mathbb{Z}_m \times \mathbb{Z}_2^r,\ \mathbb{Z}_m \times \mathbb{Z}_4,\ \mathbb{Z}_m \times \mathbb{Z}_8,\ \mathbb{Z}_m \times Q_8$;

(b) $\mathbb{Z}_m \rtimes \mathbb{Z}_{2^e}, e = 1, 2, 3,$

where m is an odd square-free number.

The second main result provides several infinite series of CIM-groups.

Theorem 1.2. *The following groups are CI-groups with respect to Cayley maps.*

$$\mathbb{Z}_m \times \mathbb{Z}_2^r,\ \mathbb{Z}_m \times \mathbb{Z}_4,\ \mathbb{Z}_m \times Q_8$$

where m is an odd square-free number.

As an immediate corollary of the above theorems we obtain the following criterion. Further we obtain similar result for abelian groups.

Theorem 1.3. *Let H be an abelian group of odd order. Then H is a connected CIM-group if and only if $|H|$ is square-free.*

Notice that obtained results do not provide a complete classification of cyclic CIM-group. This is because we do not know which of the groups $\mathbb{Z}_m \times \mathbb{Z}_8$, m is odd and square-free, are CIM-groups. Proposition 6.7 shows that $\mathbb{Z}_8$ is a CIM-group. We believe that all groups of the above structure have the CIM-property.

The last main Theorem of the paper provides a useful source of connected CIM-groups which are not CIM-group in general.

Theorem 1.4. *Let H be a simple group. Then H is a connected CIM-group provided that $H \not\cong A_5$.*

Notice that the group A_5 is not a connected CIM-group because of the following reason. There exists a regular Cayley map over A_5 the automorphism group of which is $PSL_2(11)$ ([19], Example 3.9). According to [2] $PSL_2(11)$ has two conjugacy classes of maximal subgroups isomorphic to A_5.

Our paper is organised as follows. In Section 3 we collect a few general results about the CIM-property which will be used later. In Section 4 we characterize Sylow subgroups of CIM-groups. Section 5 is devoted to the proof of Theorem 1.1. The last section provides proofs of Theorems 1.2 and 1.3.

Most of the group-theoretical notation used in the paper are standard and can be found in [26].

2 Cayley maps and ternary relational structures

First we repeat some basic definitions related to Cayley maps. The set of *darts* $D(\mathcal{M})$ of a Cayley map $\mathcal{M} = \mathsf{CM}(H, S, \rho)$ consists of all directed arcs of the underlying graph, i.e. $D(\mathcal{M}) := \{(x, y) \in H \times H \mid x^{-1}y \in S\} \subseteq H \times H$. There are two permutations of the dart set $D(\mathcal{M})$, namely: the *dart reversing involution* $\mathcal{S}$ and the *dart rotation* $\mathcal{R}$. Their action is defined by the following formulae:

$$\mathcal{S}((x, y)) = (y, x), \;\; \mathcal{R}(x, y) = (x, x\rho(x^{-1}y)).$$

Notice that $\mathcal{S}$ is an involution while $\mathcal{R}$ is a semi-regular permutation of order $|S|$. The orbits of $\langle\mathcal{R}\rangle$ on D are in one-to-one correspondence with the elements of H. More precisely, two darts $(x, y), (u, v) \in D$ belong to the same $\langle\mathcal{R}\rangle$-orbit if and only if $x = u$.

In addition, we define a ternary relation $T(\mathcal{M})$ on H associated to the map $\mathcal{M}$:

$$T(\mathcal{M}) := \{(x, y, z) \in H \times H \times H \mid x^{-1}y, x^{-1}z \in S \text{ and } \rho(x^{-1}y) = x^{-1}z\}.$$

Notice that $|T(\mathcal{M})| = |H||S|$.

Given two maps $\mathcal{M} = \mathsf{CM}(H, S, \rho)$ and $\mathcal{M}' = \mathsf{CM}(H', S', \rho')$, a *(map) isomorphism* is a bijection $\Phi : D \to D'$ which satisfies $\Phi\mathcal{S} = \mathcal{S}'\Phi$ and $\Phi\mathcal{R} = \mathcal{R}'\Phi$ [12]. The automorphism group $\mathsf{Aut}(\mathcal{M})$ of a map $\mathcal{M}$ consists of all permutations Φ of the dart set which commute with $\mathcal{R}$ and $\mathcal{S}$. In other words $\mathsf{Aut}(\mathcal{M})$ is the centralizer of the elements $\mathcal{S}, \mathcal{R}$ in the symmetric group $\mathsf{Sym}(D(\mathcal{M}))$.

Theorem 2.1. *Let $\mathcal{M} = \mathsf{CM}(H, S, \rho)$ and $\mathcal{M}' = \mathsf{CM}(H', S', \rho')$ be two Cayley maps. A bijection $\Phi : D(\mathcal{M}) \to D(\mathcal{M}')$ is a map isomorphism if and only if there exists a bijection $\phi : H \to H'$ such that $\Phi((x, y)) = (\phi(x), \phi(y))$ and $\phi(T(\mathcal{M})) = T(\mathcal{M}')$.*

Proof. Let $\mathcal{R}, \mathcal{S}$ and $\mathcal{R}', \mathcal{S}'$ denote the dart rotation and dart reversing involution of the maps $\mathcal{M}$ and $\mathcal{M}'$ respectively. We also abbreviate $T := T(\mathcal{M})$, $T' := T(\mathcal{M}')$, $D := D(\mathcal{M})$, $D' := D(\mathcal{M}')$.

Let $\Phi : D \to D'$ be a map isomorphism. Then $\Phi\mathcal{R} = \mathcal{R}'\Phi$ implies $\Phi\mathcal{R}\Phi^{-1} = \mathcal{R}'$. Therefore Φ induces a bijection between the orbits of $\langle\mathcal{R}\rangle$ and $\langle\mathcal{R}'\rangle$. Since the orbits of $\langle\mathcal{R}\rangle$ (resp. $\langle\mathcal{R}'\rangle$) are in one-to-one correspondence with the elements of H (resp. H'), there exists a bijection $\phi : H \to H'$ such that $\Phi((x, y)) = (\phi(x), \Psi(x, y))$ for some $\Psi : D \to H'$.

Now it follows from $\mathcal{S}'\Phi = \Phi\mathcal{S}$ that $\Psi(x, y) = \phi(y)$. Thus $\Phi((x, y)) = (\phi(x), \phi(y))$ implying $\phi(D) = D'$.

Pick an arbitrary triple $(x, y, z) \in T$. Then $(x, y), (x, z) \in D$ and $z = \rho(x^{-1}y)$. It follows from $\phi(D) = D'$, that $(\phi(x), \phi(y)), (\phi(x), \phi(z)) \in D'$.

Now $\Phi(\mathcal{R}((x, y))) = \mathcal{R}'(\Phi((x, y)))$ implies

$$(\phi(x), \phi(z)) = (\phi(x), \phi(x)\rho'(\phi(x)^{-1}\phi(y)))$$

so we have $\phi(z) = \phi(x)\rho'(\phi(x)^{-1}\phi(y))$. We have already seen $(\phi(x), \phi(z)) \in D'$ thus $(\phi(x), \phi(y), \phi(z)) \in T'$. Hence $\phi(T) \subseteq T'$. Comparing the cardinalities we conclude $\phi(T) = T'$.

Assume now that $\phi : H \to H'$ is a bijection which satisfy $\phi(T) = T'$. Since the projection of T (resp. T') onto the first two coordinates is D (resp. D'), we conclude that $\phi(D) = D'$. Therefore the mapping $\Phi((x, y)) := (\phi(x), \phi(y))$ is a bijection between the dart sets of $\mathcal{M}$ and $\mathcal{M}'$.

It remains to show that Φ is a map isomorphism, i.e. $\Phi\mathcal{S} = \mathcal{S}'\Phi$ and $\Phi\mathcal{R} = \mathcal{R}'\Phi$. The first equality is immediate. To prove the second one we notice that if $(x, y, z) \in T$, then $\mathcal{R}((x, y)) = (x, z)$. Therefore $\Phi(\mathcal{R}((x, y)) = (\phi(x), \phi(z))$. It follows from $\phi(T) = T'$ that $(\phi(x), \phi(y), \phi(z)) \in T'$. Therefore $\mathcal{R}'((\phi(x), \phi(y)) = (\phi(x), \phi(z))$. Finally we get $\mathcal{R}'(\Phi((x, y))) = \Phi(\mathcal{R}((x, y)))$, finishing the proof of Theorem 2.1. ■

It follows from the above Theorem that regarding map isomorphism of Cayley maps we can replace dart bijection by a bijection between the point sets. Thus in what follows a map isomorphism or automorphism will always mean a bijection or a permutation between the point sets. In particular, the automorphism group $\mathsf{Aut}(\mathsf{CM}(H, S, \rho)) = \mathsf{Aut}(T(\mathsf{CM}(H, S, \rho))$ will be considered as a subgroup of the symmetric group $\mathsf{Sym}(H)$.

Proposition 2.2. *A bijection $\phi : H \to H'$ is an isomorphism between the maps $\mathcal{M} = \mathsf{CM}(H, S, \rho)$ and $\mathcal{M}' = \mathsf{CM}(H', S', \rho')$ if and only if for any $h \in H$ the "differential" map $\Delta_h\phi : H \to H'$ defined via $\Delta_h\phi(x) := \phi(h)^{-1}\phi(hx)$ satisfies the conditions*

(a) $\Delta_h\phi(S) = \{\Delta_h\phi(S) \mid s \in S\} = S'$;

(b) $\Delta_h\phi(\rho(s)) = \rho'(\Delta_h\phi(s))$ holds for each $s \in S$.

Proof. By Theorem 2.1 ϕ is a map isomorphism if and only if $\phi(T(\mathcal{M})) = T(\mathcal{M}')$.

Assume first that ϕ satisfies the assumptions (a) and (b). Pick and arbitrary triple $(x, y, z) \in T(\mathcal{M})$. Then $y = xs$ and $z = x\rho(s)$ for some $s \in S$. It follows from (a) that $\Delta_x\phi(s) = s' \in S'$. Therefore $\phi(x)^{-1}\phi(xs) = s'$ implying $\phi(y) = \phi(x)s'$. Now (b) implies $\phi(x)^{-1}\phi(z) = \phi(x)^{-1}\phi(x\rho(s)) = \rho'(\phi(x)^{-1}\phi(xs)) = \rho'(s')$. Therefore $\phi(y) = \phi(x)s' = \phi(x)\rho'(s')$ implying $(\phi(x), \phi(y), \phi(z)) \in T(\mathcal{M}')$. Thus $\phi(T(\mathcal{M})) \subseteq \mathcal{M}'$. Combining this with $|T(\mathcal{M})| = |H||S| = |H'||S'| = |T(\mathcal{M}')|$ we obtain that $\phi(T(\mathcal{M})) = T(\mathcal{M}')$.

Let now $\phi : H \to H'$ be a bijection which maps $T(\mathcal{M})$ onto $T(\mathcal{M}')$. Then $\phi(D(\mathcal{M})) = D(\mathcal{M}')$ because the dart set is a projection of the set of triples onto the first two coordinates. Therefore $x^{-1}y \in S \iff \phi(x)^{-1}\phi(y) \in S'$ holds for all $x, y \in H$. This yields us (a).

Pick an arbtitrary pair $x \in H, s \in S$. Then $(x, xs, x\rho(s)) \in T(\mathcal{M})$ implying $(\phi(x), \phi(xs), \phi(x\rho(s))) \in T(\mathcal{M}')$. By definition of $T(\mathcal{M}')$ we obtain that $\phi(xs) = \phi(x)s', \phi(\rho(xs)) = \phi(x)\rho'(s')$ for some $s' \in S$. Now the claim follows. ■

As a consequence we obtain the following result the second part of which was proved in [12]. To make the text self-contained we provide here a complete proof of the statement.

Theorem 2.3. *Let G be the full automorphism group of a Cayley map $\mathcal{M} = \mathsf{CM}(H, S, \rho)$. Then the following two statements hold.*

(a) If $\mathcal{M}$ is disconnected (that is $\langle S \rangle \neq H$), then G is permutation equivalent to the wreath product $G_0 \wr S_m$ in an imprimitive action where $G_0 := \mathsf{Aut}(\mathsf{CM}(\langle S \rangle, S, \rho))$ and $m := [H : \langle S \rangle]$;

(b) If $\mathcal{M}$ is connected, then the point stabilizer G_e is cyclic and it acts faithfully on S with $(G_e)|_S \in \langle \rho \rangle$. In particular G_e acts semiregularly on S.

Proof. We abbreviate $T := T(\mathcal{M})$.

Proof of part (a). Let $H = h_1F \cup \ldots \cup h_mF$, where $h_1 = e$ be a decomposition of H into a disjoint union of left cosets of the subgroup $F := \langle S \rangle$. Then $T(\mathcal{M})$ is a point disjoint union of the relations

$$T_i := T \cap ((h_iF) \times (h_iF) \times (h_iF)), i = 1, \ldots, m.$$

Since the relations T_i are pairwise isomorphic, we obtain that $G = \mathsf{Aut}(T) = \mathsf{Aut}(T_1) \wr S_m$, as required.

Proof of part (b). It follows from the definition of T that $\widehat{H} \leqslant G$. Since $\widehat{H}$ acts regularly on the point set H, the group G admits a decomposition $G = G_e\widehat{H}$. Pick an arbitrary $\phi \in G_e$. Then $\phi(e) = e$ so $\Delta_e\phi = \phi$ and, by part (b) of Proposition 2.2, the restriction $\phi|_S$ commutes with ρ. Therefore $(G_e)|_S$ centralizes ρ, and, consequently, $(G_e)|_S \leqslant \langle \rho \rangle$. It remains to show that G_e acts faithfully on S.

The group G_e acts on S semiregularly. Therefore for each $s \in S$ the subgroup $G_{e,s}$ acts trivially on S. Since S is a neighbourhood of e in the Cayley graph $\Gamma := \mathsf{Cay}(H, S)$ and $G \leqslant \mathsf{Aut}(\Gamma)$, the group G has the following property: for any arc x, y of Γ the subgroup $G_{x,y}$ acts trivially on the neighbourhood $\Gamma(x) = xS$ of x. Now one can easily show that the subgroup $G_{e,s}$ is trivial. Indeed, if not, then there exists a non-identical permutation $g \in G_{e,s}$. Among all the points moved by g we choose one, say x, which has minimal distance, denoted by d, from e in Γ. Since $G_{e,s}$ acts trivially on S and Γ is connected, we obtain $2 \leqslant d < \infty$. Let $x_0 = e, x_1 = s, \ldots, x_d = x$ be the shortest path in Γ connecting e and x. By the choice of g and x, the permutation g fixes the points $x_0, \ldots, x_{d-1}$. Therefore $g \in G_{x_{d-2}x_{d-1}}$ implying that g fixes all points of $\Gamma(x_{d-1})$. But the point $x = x_d \in \Gamma(x_{d-1})$ is moved by g, a contradiction. ■

3 General observations

The original CI property for graphs is inherited by subgroups which gives us a strong tool to determine the list of possible CI-groups. Similar, but weaker, property holds for CIM-groups as well. Let us call a group H a *connected* CIM-group if it is a CI-group with respect to connected maps.

Lemma 3.1. *Every subgroup of a CIM-group is a connected CIM-group.*

Proof. Let G be a CIM-group and $H \leqslant G$. Let us assume that $\mathsf{CM}(H, S, \rho)$ and $\mathsf{CM}(H, S', \rho')$ are isomorphic connected Cayley maps of H. Let ϕ be a map isomorphism from $\mathsf{CM}(H, S, \rho)$ to $\mathsf{CM}(H, S', \rho')$. Then $\widehat{g_2}\phi\widehat{g_1}^{-1}$ is an isomorphism between the connected component of $\mathsf{CM}(G, S, \rho)$ on $g_1 H$ and the one of $\mathsf{CM}(G, S', \rho')$ on $g_2 H$. This shows that the connected components of $\mathsf{CM}(G, S, \rho)$ and $\mathsf{CM}(G, S', \rho')$ are isomorphic. Therefore $\mathsf{CM}(G, S, \rho)$ and $\mathsf{CM}(G, S', \rho')$ are isomorphic Cayley maps. Since G is a CIM-group there exists $\alpha \in \mathsf{Aut}(G)$, which induces an isomorphism from $\mathsf{CM}(G, S, \rho)$ to $\mathsf{CM}(G, S', \rho')$. Since the Cayley map $\mathsf{CM}(H, S, \rho)$ is a connected component of $\mathsf{CM}(G, S, \rho)$, its image $\mathsf{CM}(\alpha(H), \alpha(S), \alpha|_S \rho (\alpha|_S)^{-1})$ is a connected component of $\mathsf{CM}(G, S', \rho')$. Therefore $\alpha(H)$ is a left coset of H implying $\alpha(H) = H$. Hence $\alpha|_H$ is a Cayley isomorphism between the above maps. ■

This result suggests that it is worth investigating p-groups which arise as the Sylow p-subgroups of finite groups.

Another important observation is that if $\mathsf{CM}(H, S, \rho)$ is a Cayley map with $|S| \leqslant 2$, then the Cayley graph $\mathsf{Cay}(H, S,)$ has to be a CI-graph since there exists only one cyclic ordering on one or two elements. This shows that the automorphism group of a CIM-group H has only one orbit on the elements of order 2 and for every $g, h \in H$ with the same order there exists $\alpha \in \mathsf{Aut}(H)$ with $\alpha(g) = h$ or $\alpha(g) = h^{-1}$. Groups having this property were investigated by Li and Praeger [21].

The following lemma is due to Babai [3] and applies to every Cayley relational structure[3].

Lemma 3.2 (Babai)**.** *Let* $\mathsf{Cay}(H, \mathcal{R})$ *be a Cayley relational structure. Then* $\mathsf{Cay}(H, \mathcal{R})$ *has the CI-property if and only if for every regular subgroup* $\mathring{H} \cong H$ *of* $\mathsf{Aut}(\mathsf{Cay}(G, \mathcal{R}))$ *there exists* $\alpha \in \mathsf{Aut}(\mathsf{Cay}(G, \mathcal{R}))$ *with* $\alpha(\mathring{H}) = \widehat{H}$.

In what follows we refer to a regular permutation subgroup isomorphic to H as H-*regular* subgroup.

The statement below describes the structure of the Cayley map automorphism group. Although it was proved by Jajcay [12] we prefer to provide its proof here to make the paper self-contained.

[3] Recall that a Cayley relational structure over a group H is any set of relations over H which is $\widehat{H}$-invariant.

Lemma 3.3. *Let $M := \mathsf{CM}(H,S,\rho)$ be a connected Cayley map and $G := \mathsf{Aut}(M)$ its automorphism group. Then G_e acts faithfully on S and its restriction $(G_e)|_S$ is contained in $\langle\rho\rangle$. In particular, G_e is cyclic.*

Proof. Pick an arbitrary $\phi \in G_e$. Then $\Delta_e\phi = \phi|_S$ implying $\rho\phi|_S = \phi|_S\rho$. Since ρ is a full cycle on S, any permutation commuting with it belongs to $\langle\rho\rangle$. Therefore $(G_e)|_S \leqslant \langle\rho\rangle$. This inclusion also implies that for each $s \in S$ the two-point stabilizer $G_{e,s}$ acts trivially on S. Therefore $G_{h,hs}$ acts trivially on hS for any $h \in H$ and $s \in S$. Thus if ϕ fixes e and $s \in S$, then it fixes pointwise the sets S, S^2, S^3 etc. Since $\mathsf{Cay}(H,S)$ is connected, we conclude that $G_{e,s}$ is trivial, i.e. G_e acts faithfully on S. ∎

4 Sylow subgroups of CIM-groups

Similarly to the classical case of CI-groups, it follows from Lemma 3.1 that it is important to investigate p-groups. Babai and Frankl [4] proved that if a group H is a $CI^{(2)}$-group of prime power order, then H is either elementary abelian p-group, the quaternion group of order 8 or a cyclic group of small order. In this section we investigate the possible Sylow p-subgroups of of a CIM-group and we give necessary condition for the structure of abelian groups of odd order which are connected CIM-groups.

4.1 Groups of odd order

The statement below describes odd order Sylow subgroups of a CIM-group.

Lemma 4.1. *A Sylow p-subgroup of a CIM-group H corresponding to an odd prime divisor p of $|H|$ has order p.*

Proof. It follows from Lemma 3.1 that it is sufficient to show that any subgroup of order p^2 is not a connected CIM-subgroup.

Let K be a group of order p^2. Then either $K \cong \mathbb{Z}_p^2$ or $K \cong \mathbb{Z}_{p^2}$. In both cases there exists an automorphism $\beta \in \mathsf{Aut}(K)$ of order p (the concrete examples of β are given below). A direct check shows that the bijection $\alpha \in \mathsf{Sym}(K)$ defined via $\alpha(x) = -\beta(x)$ is an automorphism of K of order $2p$. It follows from $\alpha^p = -1$ that each non-zero α-orbits is symmetric, and, therefore, has even cardinality. This implies that at least one orbit of α contains $2p$ elements. Let us denote this orbit by S. Clearly $\langle S\rangle = K$. Consider a Cayley map $M = \mathsf{Cay}(K,S,\alpha|_S)$. The group $G := \mathsf{Aut}(M)$ contains the semidirect product $\widehat{K} \rtimes \langle\alpha\rangle \leqslant \mathsf{Sym}(K)$. Combining this with $|\mathsf{Aut}(M)| \leqslant |K||S| = |K||\langle\alpha\rangle|$ we conclude that $G = \widehat{K} \rtimes \langle\alpha\rangle$. Thus M is a balanced map, which is regular by $\alpha^p = -1$. We claim that M is not a CI-map.

According to Lemma 3.2 it is enough to find two K-regular subgroups of G which are not conjugate in G. It can be seen from the description of the automorphism group of M that $\widehat{K}$ is a normal subgroup of G but it is also clear from the fact that M is a connected balanced Cayley map [25]. Since $\widehat{K}$ is normal in G, it is sufficient to find a K-regular subgroup of G distinct from $\widehat{K}$. To point out such a subgroup we consider the cases of $K \cong \mathbb{Z}_{p^2}$ and $K \cong \mathbb{Z}_p^2$ separately.

Case of $K \cong \mathbb{Z}_{p^2}$.
In this case we choose $\beta \in \mathsf{Aut}(K)$ defined via $\beta(x) = (1+p)x$. The permutation $\gamma(x) := \beta(x) + 1 = (1+p)x + 1$ belongs to the group G because $\gamma = \widehat{1}\beta$. A direct check shows that $\gamma^p(x) = x + p$ implying $o(\gamma^p) = p$, and, consequently, $o(\gamma) = p^2$. Therefore $\langle\gamma\rangle$ is a regular cyclic subgroup of G different from $\widehat{K}$.

Case of $K \cong \mathbb{Z}_p^2$.
In this case we choose $\beta \in \mathsf{Aut}(\mathbb{Z}_p^2)$ defined via $\beta((x,y)) = (x+y,y)$. Then the group G contains the subgroup $\widehat{\mathbb{Z}}_p^2 \rtimes \langle\beta\rangle$ which consists of all permutations of the form $(x,y) \mapsto (x+ay+u, y+v)$ where $a,u,v \in \mathbb{Z}_p$. A direct check shows that the permutations $\tau_{a,b} : (x,y) \mapsto (x+ay+b, y+a), a,b \in \mathbb{Z}_p$ form a subgroup, say T, of G isomorphic to $\mathbb{Z}_p^2$. It is easy to check that T acts regularly on $\mathbb{Z}_p^2$. ∎

In order to classify connected abelian CIM-groups we generalize the previous lemma.

Lemma 4.2. *Let p be an odd prime. Then $\mathbb{Z}_{p^k}$ is not a connected CIM-group if $k \geqslant 2$.*

Proof. As in the previous case we prove that $\gamma(x) = (1+p)x+1$ is a permutation of order p^k. First we verify that $\beta(x) = (1+p)x$ is of order p^{k-1}. The group of units of the ring $\mathbb{Z}_{p^k}$ is of order p^{k-1} so we only have to verify that $\beta^{p^{k-2}} \neq id$.

$$(1+p)^{p^{k-2}} = 1 + \binom{p^{k-2}}{1}p + \binom{p^{k-2}}{2}p^2 + \ldots + p^{p^{k-2}}.$$

One can verify that if $1 \leqslant l = p^s i \leqslant p^j$, where i is prime to p, then $\binom{p^j}{l} = p^{j-s}m$, where m is prime to p. Therefore all but the first two terms are not divisible by p^k so β is of order p^{k-1}. Now

$$\gamma^l(x) = (1+p)^l x + \sum_{i=0}^{l-1}(1+p)^i = (1+p)^l x + \frac{(1+p)^l - 1}{p}.$$

If $\gamma^l = id$, then l is divisible by p^{k-1} since the order of β is p^{k-1}. Further $\frac{(1+p)^l-1}{p}$ is divisible by p^k, which means that p^{k+1} divides $(1+p)^l - 1$. As we have already seen for β this means that l is multiple of p^k.

Repeating the same construction as in the previous lemma we obtain a connected balanced Cayley map the automorphism group of which contains two H-regular subgroups. One of them is a normal subgroup, so they are not conjugate. ∎

Proposition 4.3. *Let A be an abelian group of non square-free odd order. Then A is not a connected CIM-group.*

Proof. Let p be a prime the square of which divides $|A|$. Then A admits a decomposition $A = K \times H$ where $K \cong \mathbb{Z}_{p^k}, k \geqslant 2$ or $K \cong \mathbb{Z}_p^2$. In both cases K admits a connected balanced regular Cayley map $M = \mathsf{CM}(K,S,\rho)$ over K which is not CI (see the proofs of Lemmas 4.1 and 4.2). In particular S is symmetric and hence $|S|$ is even. Since the map is balanced and connected, there exists a unique $\alpha \in \mathsf{Aut}(K)$ with $\alpha|_S = \rho$. We extend α to an automorphism of A by $\bar{\alpha} = (\alpha, -1)$. It follows from the construction of α (see the proofs of Lemmas 4.1 and 4.2) that $\alpha^{|S|/2}$ inverts the elements of K. Since $|S|/2$ is odd, the automorphism $\bar{\alpha}^{|S|/2}$ coincides with -1_A.

Now we pick a symmetric generating set of H of the form $T = \{1, g_1, g_1^{-1}, \dots, g_i, g_i^{-1}\}$. The set $S\times T$ is a symmetric $\bar{\alpha}$-invariant generating set of G. Therefore $S\times T$ is the disjoint union of $\langle\bar{\alpha}\rangle$-orbits, all having length $|S|$. Therefore, there exists a cyclic permutation π of the set $S\times T$ such that $\pi^{|T|} = \bar{\alpha}|_{S\times T}$. It follows from $\bar{\alpha}^{|S|/2} = -1_A$ that $\pi^{\frac{|T||S|}{2}} = -1_{S\times T}$. Therefore $\pi(x)^{-1} = \pi(x^{-1})$ for every $x \in S \times T$. Thus we get a balanced Cayley map $M_1 = \mathsf{CM}(G, S \times T, \pi)$.

It was proved in Lemma 4.1 and Lemma 4.2 that the map M is not a CI-map over K. Therefore $Aut(M)$ contains two K-regular subgroups K_1 and K_2. Then $K_1 \times \hat{H}$ and $K_2 \times \hat{H}$ are A-regular subgroups of $Aut(M_1)$. One of them is $\hat{K} \times \hat{H}$ which is a normal subgroup in $Aut(M_1)$ since M_1 is balanced again, hence these subgroups are not conjugate in $Aut(M_1)$. ■

Combining this result with Theorem 1.2 (which will be proved in Section 6) we get Theorem 1.3.

4.2 Sylow 2-subgroups of CIM-groups

Proposition 4.4. *For every $n \geqslant 4$ the cyclic group $\mathbb{Z}_{2^n}$ is not a connected CIM-group.*

Proof. The element $a = 1 + 2^{n-1} \in \mathbb{Z}_{2^n}$ has multiplicative order 2. Therefore the automorphism $\alpha \in \mathsf{Aut}(\mathbb{Z}_{2^n})$ defined via $\alpha(x) = ax$ has order two as well. We construct an antibalanced Cayley map the automorphism group of which contains the subgroup $\widehat{\mathbb{Z}}_{2^n} \rtimes \langle\alpha\rangle$. Let $S = \{1, -1, 3, -3a, a, -a, 3a, -3\}$ be a set of 8 different elements, and let $\rho = (1, -1, 3, -3a, a, -a, 3a, -3)$ be an 8-cycle. The group automorphism α is an automorphism of the map $M := \mathsf{CM}(\mathbb{Z}_{2^n}, S, \rho)$, because $\alpha(S) = S$ and $\alpha|_S = \rho^4$. Thus the full automorphism group $G := \mathsf{Aut}(M)$ contains the subgroup $A := \widehat{\mathbb{Z}}_{2^n} \rtimes \langle\alpha\rangle$.

Straightforward calculation shows that $(\widehat{1}\alpha)^2(x) = x + a + 1 = x + 2^{n-1} + 2$ implying that $(\widehat{1}\alpha)^2$ has order 2^{n-1}. Hence the order of $\widehat{1}\alpha$ is 2^n. Therefore the subgroup A of G contains at least two regular subgroups isomorphic to $\mathbb{Z}_{2^n}$, both of index two. These subgroups are not conjugate in A, since they are normal in A. Thus it is enough to prove that $A = G$. The latter is equivalent to showing that the point stabilizer G_0 has order two. Assume, towards a contradiction, that $|G_0| > 2$. The group G_0 is cyclic and acts on S faithfully and semi-regularly. Therefore there exists an element $\sigma \in G_0$ such that $\sigma^2 = \alpha$. In particular, σ has order 4. Since $\sigma|_S$ commutes with ρ, we conclude that $\sigma|_S = \rho^2 = (1, 3, a, 3a)(-1, -3a, -a, -3)$ or $\sigma|_S = (1, 3a, a, 3)(-1, -3, -a, -3a)$.

Assume first that $\sigma|_S = (1, 3, a, 3a)(-1, -3a, -a, -3)$. Consider the subset $T := \{x \in \mathbb{Z}_{2^n} \,|\, |S \cap (S + x)| = 6\}$[4]. Since σ is an automorphism of $\mathsf{Cay}(\mathbb{Z}_{2^n}, S)$ stabilizing 0, it

[4] These are elements at distance two from 0 in $\mathsf{Cay}(\mathbb{Z}_{2^n}, S)$, each of them is connected to 0 by six paths of length two.

satisfies the equation $\sigma(x+S)=\sigma(x)+S$ for every $x\in\mathbb{Z}_{2^n}$. Thus T is σ-invariant. A direct calculation yields us $T=\{2,-2,2+2^{n-1},-2+2^{n-1}\}$.

Consider the set $\sigma(S\setminus(S+2))=\sigma(\{-3,-3a\})=\sigma(\{-3,-3+2^{n-1}\})$. Since σ is an automorphism of the graph $\mathsf{Cay}(\mathbb{Z}_{2^n},S)$, we can write $\sigma(S+2)=S+\sigma(2)$. Therefore $\sigma(S\setminus(S+2))=\sigma(\{-3,-3+2^{n-1}\})$, thus $S\setminus(S+\sigma(2)))=\{-1,-1+2^{n-1}\}$. Since T is σ-invariant $\sigma(2)\in T$, none of the elements $t\in T$ satisfies $S\setminus(S+t))=\{-1,-1+2^{n-1}\}$, a contradiction.

Similar calculation gives the result if $\sigma|_S=(1,3a,a,3)(-1,-3,-a,-3a)$. ■

Proposition 4.5. *Let $P\leqslant H$ be a Sylow 2-subgroup of a CIM-group H. Then P is either elementary abelian or cyclic C_{2^n}, where $n\leqslant 3$ or Q_8.*

Proof. Assume that $\exp(P)>2$. Then P contains a cyclic subgroup $C_4=\langle c\rangle$ of order 4. We claim that P doesn't contain the Klein subgroup $K_4\cong\mathbb{Z}_2^2$, Indeed, if $K_4=\{1,u,v,w\}\leqslant P$ is the Klein subgroup, then the Cayley map $\mathsf{CM}(K_4,\{u,v\},(u,v))$ is isomorphic, as a map, to the Cayley map $\mathsf{CM}(C_4,\{c,c^{-1}\},(c,c^{-1}))$. Hence there should exists an automorphism $\alpha\in\mathsf{Aut}(H)$ which maps the first map onto the second one. Since both maps are connected, this would imply $\alpha(C_4)=K_4$, a contradiction.

Thus P does not contain K_4. By Burnside's Theorem [6], P is either cyclic or generalized quaternion. If P is cyclic, then by Proposition 4.4 its order is bounded by 8.

Assume now that P is a generalized quaternion group distinct from Q_8. Then P contains a characteristic cyclic subgroup $C=\langle c\rangle$ of index 2. Then it follows from Lemma 3.1 and Proposition 4.4 that $|C|\leqslant 8$. Together with $P\not\cong Q_8$ we obtain that $|C|=8$, and, consequently $|P|=16$.

Let $a\in P$ denote an element of order 4 outside of C. Then $\langle a,c^2\rangle\cong Q_8$. Let α be an automorphism of $\langle a,c^2\rangle$ whose action on the generating set is described by the formulas $\alpha(a)=c^2$ and $\alpha(c^2)=a^{-1}$. Its orbit $\{a,c^2,a^{-1},c^{-2}\}$ is symmetric and generates $\langle a,c^2\rangle$. Therefore $M=\mathsf{CM}(\langle a,c^2\rangle,\{a,c^2,a^{-1},c^{-2}\},\alpha)$ is a regular balanced Cayley map with $\mathsf{Aut}(M)=\langle a,c^2\rangle\rtimes\alpha$. The element $\widehat{a}\alpha\in\langle a,c^2\rangle\rtimes\alpha$ has order 8 and acts regularly on the point set $\langle a,c^2\rangle$ of the map M. Therefore there exists a regular Cayley map M' over the cyclic group of order 8 isomorphic to M. Thus $M\cong M'=\mathsf{CM}(C,S,\rho)$ for some $S\subseteq C$ and an appropriate rotation ρ.

Therefore if $H=Q_{16}$ is a CIM-group, there exists $\beta\in\mathsf{Aut}(H)$ which maps M on M'. But in this case $\langle a,c^2\rangle\cong C$, a contradiction. ■

5 Proof of Theorem 1.1

We start with the following lemma which deals with Cayley maps over semi-direct product of special type. Note, that maps appearing in the lemma are examples of half-regular Cayley maps introduced in [13].

Lemma 5.1. *Let H be a group which admits a decomposition $H=CK$ such that $K\cap C=\{1\}$ and $K\lhd H$ and $C=\langle c\rangle$ is cyclic of odd order m. Assume that a there exists a faithful C-orbit $O=\{k,k^c,\dots,k^{c^{m-1}}\}$ such that $\langle O^{(-1)}O\rangle=K$. Then H is not a connected CIM-group.*

Proof. It is sufficient to provide an example of a connected non-CI map over H. Take $S := cO = \{ck_0, ck_1, \ldots, ck_{m-1}\}$ where $k_i := k^{c^i}, i = 0, \ldots, m-1$. Then $S^{(-1)} \cap S = \emptyset$ because the images of S and $S^{(-1)}$ in $H/K \cong C$ are c and c^{-1}, respectively.

Take a Cayley map $M = \mathsf{CM}(H, S \cup S^{(-1)}, \rho)$ where

$$\rho = (ck_0, (ck_\ell)^{-1}, ck_1, (ck_{\ell+1})^{-1}, \ldots, ck_{m-1}, (ck_{\ell+m-1})^{-1})$$

and $\ell = \frac{m+1}{2}$ and the indices are taken modulo m. Notice that the condition $\langle OO^{(-1)} \rangle = K$ implies that the map is connected.

It follows from the construction that $\rho^2 = \sigma|_{S \cup S^{(-1)}}$, where σ is the inner automorphism of H mapping x to x^c. Therefore $\sigma \in \mathsf{Aut}(M)$ and $G := \widehat{H} \rtimes \langle \sigma \rangle \leqslant \mathsf{Aut}(M)$.

In order to build a regular subgroup of $\mathsf{Aut}(M)$ different from $\widehat{H}$ we notice first, that G is isomorphic to a direct product $H \times C$, where the isomorphism is defined via $\psi : \widehat{h}\sigma^i \mapsto (c^i h, c^{-i})$. Under this isomorphism the point stabilizer $G_1 = \langle \sigma \rangle$ is mapped onto the subgroup $\psi(G_1) = \{(d^{-1}, d) \mid d \in C\}$.

Let $\pi : H \to C$ be a projection homomorphism defined via $\pi(xk) := x$ for $x \in C$ and $k \in K$. Then $F := \{(h, \pi(h)) \mid h \in H\}$ is a subgroup of $H \times C$ which intersects $\psi(G_1)$ trivially. Indeed,

$$(h, \pi(h)) \in \psi(G_1) \iff \pi(h) = h^{-1} \implies h \in C \implies \pi(h) = h \implies h = h^{-1}.$$

By assumption C is of odd order. Therefore $h = 1$.

It follows from $F \cap \psi(G_1) = 1$ and from the fact that F is a normal subgroup of $H \times C$ that $\psi^{-1}(F)$ has trivial intersection with all stabilizers. Therefore F is a regular subgroup of G. Thus G contains two H-regular subgroups, which are $\widehat{H}$ and $\psi^{-1}(F)$. Since $\widehat{H} \triangleleft G$, it is not conjugate to $\psi^{-1}(F)$ inside G.

Since G_1 has two orbits on the connection set $S \cup S^{-1}$, either $\mathsf{Aut}(M) = G$ or $[\mathsf{Aut}(M) : G] = 2$. In the first case we already have two H-regular subgroups of G which are non-conjugate in G. In the second case it follows from $\rho(x^{-1}) = \rho(x)^{-1}$ that M is a regular balanced map over H. It was proved in [25] that $\widehat{H} \trianglelefteq \mathsf{Aut}(M)$. Since G contains a H-regular subgroup distinct from $\widehat{H}$, it is not conjugate to $\widehat{H}$ inside $\mathsf{Aut}(M)$. ■

Remark. The condition $\langle OO^{(-1)} \rangle = K$ is always fulfilled if K does not contain a proper non-trivial C-normalized subgroups. For example, if K is of prime order, then $\langle OO^{(-1)} \rangle = K$ holds for any orbit O with $|O| > 1$.

Now we are ready to prove Theorem 1.1.

Proof. Let T denote a Sylow 2-subgroup of H. Our proof is divided into few steps.

Step 1. Any normal subgroup N of H of odd order is cyclic.
Since all Sylow subgroups of N have prime order[5], it is sufficient to prove that any Sylow subgroup of N is normal in H. This would follow if we prove that each Sylow subgroup of N has a normal complement. To show that let us fix a Sylow p-subgroup P of order p,

[5]Note that the assumptions of Theorem 1.1 imply that the order od N is square-free

where p is prime. By Burnside's Theorem the existence of a normal complement follows from $\mathbf{N}_N(P) = \mathbf{C}_N(P)$. Assume towards a contradiction that there exists $g \in \mathbf{N}_N(P)$ which does not centralize P. Since every element of a group can be written as the product of elements of prime power order we may assume that $o(g)$ is a prime power. By Lemma 4.1 any Sylow subgroup of N has prime order. Therefore $o(g)$ is prime distinct from p since elements of order p centralize g. In this case by the previous Remark, the group $\langle g\rangle P$ satisfies the assumptions of Lemma 5.1 and therefore, is not a connected CIM group. A contradiction.

Step 2. T has a normal complement.
By Proposition 4.5, T is isomorphic to one of the groups $\mathbb{Z}_2^r, \mathbb{Z}_4, \mathbb{Z}_8$ or Q_8. If T is cyclic, then the result follows from the Cayley normal 2-complement theorem.

Assume now that T is not cyclic, i.e. $T \cong \mathbb{Z}_2^r$ or $T \cong Q_8$. By Frobenius normal p-complement Theorem it is sufficient to show that $\mathbf{N}_H(U)/\mathbf{C}_H(U)$ is a 2-group for every non-trivial subgroup of T. Notice that $\mathbf{N}_H(U)/\mathbf{C}_H(U)$ embeds into $\mathsf{Aut}(T)$.

If $U \ncong Q_8$, then $U \cong \mathbb{Z}_2^e$ for some $e \geqslant 1$. Assume, towards a contradiction, that $\mathbf{N}_H(U)/\mathbf{C}_H(U)$ is not a 2-group. Then there exists an element $g \in \mathbf{N}_H(U)$ of odd order which acts on U non-trivially. Without loss of generality, we may assume that $o(g)$ is a p-power for some odd prime divisor p of $|H|$. Again, we conclude $o(g) = p$. Since U is an elementary abelian 2-group, it contains a minimal g-invariant subgroup U_1 on which g acts non-trivially. The group $\langle g\rangle U_1$ satisfies the assumptions of Proposition 5.1. Therefore $\langle g\rangle U_1$ is not a connected CIM-group. A contradiction.

If $U \cong Q_8$ and $\mathbf{N}_H(U)/\mathbf{C}_H(U)$ is not a 2-group, then this group contains an element of order 3. Hence $\mathbf{N}_H(U)$ contains an element g of order 3 which acts on U non-trivially. Applying Lemma 5.1 once more we get a contradiction.

If $U \cong \mathbb{Z}_4$, then its automorphism group is a 2-group. Therefore the condition on $\mathbf{N}_H(U)/\mathbf{C}_H(U)$ is trivially satisfied.

Step 3. If T is non-cyclic, then $H \cong N \times T$.
As it was mentioned before, a CIM-group H has the property that any two elements of the same order are either conjugate or inverse conjugate by an automorphism of H. In particular, this implies that all involutions of H are $\mathsf{Aut}(H)$-conjugate.

If T is non-cyclic group, then either it is elementary abelian or Q_8. Let us assume first that T is an elementary abelian 2-group of order at least 4. Then all non-trivial elements of T are $\mathsf{Aut}(H)$-conjugate. Since N is characteristic in H, the subgroups $\mathbf{C}_N(s)$ and $\mathbf{C}_N(t)$ are $\mathsf{Aut}(H)$-conjugate for any $s \neq t \in T \setminus \{1\}$. Since any subgroup of N is characteristic in N, we have that any subgroup of N is characteristic in H. We conclude that $K := \mathbf{C}_N(s) = \mathbf{C}_N(t) = \mathbf{C}_N(ts)$. Let $L \leqslant N$ be the unique subgroup complementary to K in N. Such a subgroup exists since N is a cyclic group of square-free odd order.

L is a subgroup of N so it is the direct sum of cyclic groups of different prime order, therefore an automorphism of order 2 fixing only the identity element must invert all elements of L. Then both s and t invert the elements of L. Therefore st acts trivially on L implying $L \leqslant K$, and consequently $L = 1$. Thus any element of T centralizes N. Therefore $H \cong N \times T$.

It remains to settle the case when $T \cong Q_8$. In this case all cyclic subgroups of order 4 are $\mathsf{Aut}(H)$-conjugate. Since $\mathsf{Aut}(N)$ is abelian the commutator subgroup Z of T acts trivially on N. Hence $Z \trianglelefteq H$. The quotient group $\bar{H} = H/Z$ is isomorphic to $N \rtimes \mathbb{Z}_2^2$. Moreover all involutions of $\mathbb{Z}_2^2$ are $\mathsf{Aut}(\bar{H})$-conjugate. From the previous paragraph we obtain that $\mathbb{Z}_2^2$ acts trivially on N. So, the semi-direct product $N \rtimes \mathbb{Z}_2^2$ is, in fact, a direct one. Z is central, therefore $H \cong N \times T$. ∎

6 Proof of Theorem 1.2

In what follows we denote by $\mathscr{M}$ the set of groups of the form $\mathbb{Z}_m \times \mathbb{Z}_2^r$, $\mathbb{Z}_m \times \mathbb{Z}_4$, $\mathbb{Z}_m \times Q_8$, where m is a square-free odd number. Thus, Theorem 1.2 states that every $H \in \mathcal{M}$ is a CIM-group. The proof of this property is divided into several steps. First, we show that is sufficient to check that groups in $\mathcal{M}$ are connected CIM-groups. Then we prove Theorem 6.3 which resolves the connected case. Note, that the proof of the latter theorem is divided into several propositions following it.

The statement below collects the properties of groups in class $\mathcal{M}$. We omit the proof because it is straightforward.

Proposition 6.1. *Let $H \in \mathcal{M}$. The following properties hold:*

(a) Every subgroup and factor group of $H \in \mathscr{M}$ belong to $\mathscr{M}$;

(b) Any two subgroups $A, B \leqslant H \in \mathscr{M}$ of the same order are conjugate by an automorphism of H;

(c) Any subgroup automorphism $\beta \in \mathsf{Aut}(A)$, where $A \leqslant H$ may be extended to an automorphism of H;

(d) The groups in $\mathscr{M}$ are Dedekind groups.

Our first step provides a reduction of Theorem 1.2 to the connected case.

Proposition 6.2. *If the groups of $\mathscr{M}$ are connected CIM-groups, then they are CIM-groups.*

Proof. Let $M = \mathsf{CM}(H, S, \rho)$ and $M' = \mathsf{CM}(H, S', \rho')$ be two isomorphic maps over a group $H \in \mathscr{M}$. Then $|\langle S\rangle| = |\langle S'\rangle|$, and by Proposition 6.1 (b) there exists an automorphism $\alpha \in \mathsf{Aut}(H)$ such that $\alpha(\langle S'\rangle) = \langle S\rangle$. Thus replacing M' by $\alpha(M')$ we may assume that $\langle S\rangle = \langle S'\rangle$. Since M and M' are isomorphic, their connected components $M_1 := \mathsf{CM}(\langle S\rangle, S, \rho)$ and $M_1' := \mathsf{CM}(\langle S\rangle, S', \rho')$ are isomorphic too. Both M_1 and M_1' are connected maps over the group $\langle S\rangle \in \mathscr{M}$. Therefore there exists $\beta \in \mathsf{Aut}(\langle S\rangle)$ such that $\beta(M_1) = M_1'$. By Proposition 6.1 (c) β can be extended up to an automorphism of H, α say. Then $\alpha(M) = M'$, hereby proving the claim. ∎

To prove Theorem 1.2 for connected maps we provide a little bit more general result.

Theorem 6.3. *Let $G \leqslant \mathsf{Sym}(\Omega)$ be a transitive permutation group with cyclic point stabilizer which contains a regular subgroup $H \in \mathscr{M}$. Then any H-regular subgroup of G is conjugate to H in G.*

Kegel-Wielandt theorem says that if a finite group is the product of two nilpotent groups, then it is solvable. In our case this gives that $G = HG_e$ is solvable.

We will prove Theorem 6.3 by induction on $|G|$ and assume that G is a counterexample of a minimal order. In particular, this implies that the theorem is correct for any proper subgroup X where $H \leqslant X < G$. Since G is a counterexample, there exists an H-regular subgroup F of G which is not conjugate to $\widehat{H}$ inside G. We fix F till the end of the proof. By the minimality of G, we may assume that $\langle \widehat{H}, F^g \rangle = G$ for each $g \in G$. We write the order of H as $2^r m$. Recall that m is an odd square-free number.

Below the following notiation is used. If G is a group acting on a set X, then G_X denotes the kernel of this action and G^X denotes the image of G in $\mathsf{Sym}(X)$. Thus $G^X \cong G/G_X$.

We call a system of imprimitivity system minimal if it is not the trivial imprimitivity system consisting of one element sets and there is no nontrivial block contained in any of the elements of the system. Note that the trivial imprimitivity system consisting of only one set might be minimal using our definition. Hence there is at least one minimal imprimitivity system.

Proposition 6.4. *Let G be a minimal counterexample to Theorem 6.3 and $\mathcal{D}$ be a proper non-trivial imprimitivity system of G. Then $G^{\mathcal{D}} = H^{\mathcal{D}}$, or equivalently, $G = HG_{\mathcal{D}}$, which is further equivalent to $G_\omega \leqslant G_{\mathcal{D}}$ (here G_ω is the stabilizer of an arbitrary $\omega \in \Omega$).*

Proof. Note that $\mathcal{D}$ is an imprimitivity system of H too. Since H is regular, the setwise stabilizer $H_{\{D\}}$ of a block $D \in \mathcal{D}$ acts regularly on D. Since the block stabilizers are conjugate in H and H is a Hamiltonian group, the subgroup $H_{\{D\}}$ does not depend on a choice of $D \in \mathcal{D}$. Therefore the subgroup $H_{\{D\}}, D \in \mathcal{D}$ coincides with $H_{\mathcal{D}}$ implying that D is an orbit of $H_{\mathcal{D}}$. It follows from $H_{\mathcal{D}} \leqslant G_{\mathcal{D}}$ that $G_{\mathcal{D}}$ acts transitively on each block of $\mathcal{D}$. The group $H^{\mathcal{D}}$ is a regular subgroup of $G^{\mathcal{D}}$. Also $H^{\mathcal{D}} \cong H/H_{\mathcal{D}} \in \mathscr{M}$. The point stabilizer of $G^{\mathcal{D}}$ is isomorphic to $G_\omega G_{\mathcal{D}}/G_{\mathcal{D}} \cong G_\omega/(G_\omega \cap G_{\mathcal{D}})$, and, therefore, is cyclic. Thus $G^{\mathcal{D}} \leqslant \mathsf{Sym}(\mathcal{D})$ satisfies the assumptions of Theorem 6.3. Since $|G^{\mathcal{D}}| = |G|/|G_{\mathcal{D}}| < |G|$, we may apply the induction hypothesis to $G^{\mathcal{D}}$. It yields us that $F^{\mathcal{D}}$ and $H^{\mathcal{D}}$ are conjugate in $G^{\mathcal{D}}$. Therefore there exists $g \in G$ such that $(F^g)^{\mathcal{D}} = H^{\mathcal{D}}$ implying $G^{\mathcal{D}} = \langle F^g, H \rangle^{\mathcal{D}} = \langle (F^g)^{\mathcal{D}}, H^{\mathcal{D}} \rangle = H^{\mathcal{D}}$. ■

Proposition 6.5. *Let G be a minimal counterexample to Theorem 6.3. Then G admits a unique minimal imprimitivity system.*

Proof. Assume, towards a contradiction, that G admits two minimal imprimitivity systems, say $\mathcal{D}$ and $\mathcal{E}$. By Proposition 6.4 $G_\omega \leqslant G_{\mathcal{D}}$ and $G_\omega \leqslant G_{\mathcal{E}}$. It follows from the minimality of $\mathcal{E}$ and $\mathcal{D}$ that $G_{\mathcal{D}} \cap G_{\mathcal{E}} = \{1\}$. Therefore $G_\omega = \{1\}$ implying $G = H$ contrary to G being a counterexample. ■

For a set of elements S of a group acting on a set X, we denote by $\mathsf{Fix}(S)$, the elements of X fixed by every $s \in S$.

Proposition 6.6. *Let $G \leqslant \mathsf{Sym}(\Omega)$ be a transitive permutation group with cyclic point stabilizer. Then for each $S \leqslant G_\omega$ the set $\mathsf{Fix}(T)$ is a block of G.*

Proof. Assume that $\mathsf{Fix}(T) \cap \mathsf{Fix}(T)^g = \mathsf{Fix}(T) \cap \mathsf{Fix}(T^g)$ is non-empty, and pick an arbitrary $\delta \in \mathsf{Fix}(T) \cap \mathsf{Fix}(T^g)$. Then $T, T^g \leqslant G_\delta$. Since G_δ is cyclic, any two subgroups of G_δ of the same order coincide. Therefore $T = T^g$ implying that $\mathsf{Fix}(T^g) = \mathsf{Fix}(T)$. ∎

Proof of Theorem 6.3

Let $\mathcal{P}$ be a minimal imprimitivity system of G. Pick an arbitrary block $\Pi \in \mathcal{P}$. Then $G_{\{\Pi\}}^{\Pi}$ is a solvable primitive permutation subgroup of $\mathsf{Sym}(\Pi)$. Therefore $|\Pi|$ is a power of a prime divisor p of $|H|$.

We split the proof into few steps.

Step 1. We claim that $|\mathcal{P}| > 1$, or equivalently G is imprimitive.
Assume the contrary, that is $|\mathcal{P}| = 1$, or, equivalently, $\Pi = \Omega$. In this case H is a p-group. By Proposition 6.6 the set $\mathsf{Fix}(G_{\alpha,\beta})$ is a block of G for any pair of points $\alpha, \beta \in \Omega$. Together with the primitivity of G this implies that $G_{\alpha,\beta} = 1$ whenever $\alpha \neq \beta$. Therefore G is a Frobenius group (the stabilizer G_e is clearly non-trivial) the kernel of which, K say, has order $|H|$. Since K is a characteristic subgroup of G, it is a unique Sylow p-subgroup of G. Therefore $H = K = F$, a contradiction.

Step 2. We claim $G_{\mathcal{P}}$ is a p-group.
Assume that there exists a prime divisor $q \neq p$ of $|G_{\mathcal{P}}|$. Since $G_{\mathcal{P}}$ acts transitively on each block $\Pi \in \mathcal{P}$ and $G_\omega \leqslant G_{\mathcal{P}}$ by Proposition 6.4, we conclude that $|G_{\mathcal{P}}| = |G_\omega| \cdot |\Pi|$. This implies that q divides $|G_\omega|$. Thus G_ω contains a subgroup Q of order q. By Proposition 6.6 the set $\mathsf{Fix}(Q)$ is a block of G. Since $Q \leqslant G_\omega \leqslant G_P$ we have that Q fixes each block of $\mathcal{P}$ setwise. Since blocks of $\mathcal{P}$ have a p-power size, the set $\mathsf{Fix}(Q)$ intersects each block of $\mathcal{P}$ non-trivially. By Proposition 6.5 $\mathcal{P}$ is a unique minimal imprimitivity system of G. Therefore each block of G is a union of some blocks of $\mathcal{P}$. Thus $\mathsf{Fix}(Q) = \Omega$ implying that $Q = \{1\}$. A contradiction.

Step 3. We claim that $G_{p'} = H_{p'} = \mathbf{O}_{p'}(G) \neq \{1\}$[6].
By Step 2 $G_{\mathcal{P}}$ is a p-group. Therefore $|H|_{p'} = |H^{\mathcal{P}}|_{p'}$ and $|G|_{p'} = |G^{\mathcal{P}}|_{p'}$. By Proposition 6.4 $H^{\mathcal{P}} = G^{\mathcal{P}} \cong G/G_{\mathcal{P}}$. Therefore $|G_{p'}| = |H_{p'}|$ implying $G_{p'} = H_{p'}$. By Hall's Theorem there exists $g \in G$ such that $(F^g)_{p'} = (F_{p'})^g = H_{p'}$. Hence F^g normalizes $H_{p'}$. Combining this with $G = \langle H, F^g \rangle$ we conclude that $H_{p'} \trianglelefteq G$. Together with $H_{p'} = G_{p'}$ we obtain that $H_{p'} = G_{p'} = \mathbf{O}_{p'}(G)$.

To finish the proof we need to eleminate the case of $G_{p'} = \{1\}$. Assume $G_{p'}$ is trivial. Then G and H are p-groups of order greater than p, so they are not primitive permutation groups. Thus $\mathcal{P}$ is non-trivial and $|\mathcal{P}| > 1$, we conclude that $|H| = |\Omega| \geqslant p^2$. Together with $H \in \mathscr{M}$ this implies that $p = 2$ and H is one of the groups: $\mathbb{Z}_2^r,\ \mathbb{Z}_4,\ Q_8$.

[6]Recall that $\mathbf{O}_{p'}$ is a maximal normal sugroup of G of order coprime to p.

Since $G^{\Pi}_{\{\Pi\}}$ is a primitive 2-group, we conclude that $|\Pi| = 2$. Therefore $G_{\mathcal{P}}$ is an elementary abelian 2-group. By Proposition 6.4 $G_\omega \leqslant G_{\mathcal{P}}$. Therefore $|G_\omega| = 2$ since $G_\omega \neq \{1\}$ is cyclic. Both H and F are index two subgroups of G. So, both of them are normal in G and, in particular, $G \leqslant \mathbf{N}_{\mathsf{Sym}(\Omega)}(H)$. If H is isomorphic to one of $\mathbb{Z}_4, Q_8$, then H is a unique H-regular subgroup of $\mathbf{N}_{\mathsf{Sym}(\Omega)}(H)$, contrary to $F \leqslant G \leqslant \mathbf{N}_{\mathsf{Sym}(\Omega)}(H)$. Therefore $H \cong \mathbb{Z}_2^r$, where $r \geqslant 2$.

It follows from $H \neq F$ that $G = HF$ and $H \cap F \leqslant \mathbf{Z}(G)$. Further the unique involution $s \in G_\omega$ has a presentation $s = h_0 f_0$ with $h_0 \in H$ and $f_0 \in F$. Notice that $h_0 \notin H \cap F$ and $f_0 \notin H \cap F$ (otherwise we would have $s \in (H \cup F) \setminus \{1\}$ which cannot happen because $(H \cup F) \setminus \{1\}$ contains only fixed-point-free permutations). Thus $G = HF = \langle f_0 \rangle \langle h_0 \rangle (H \cap F)$. It follows from $s^2 = 1$ that $[f_0, h_0] = 1$. Together with $H \cap F \leqslant \mathbf{Z}(G)$ we conclude that G is an abelian group. Thus G should be regular, contrary to $|G_\omega| = 2$.

Step 4. Getting the final contradiction. It follows from Step 3 that $\mathbf{O}_{p'}(G)$ is non-trivial. Therefore the orbits of $\mathbf{O}_{p'}(G)$ form a non-trivial imprimitivity system of G with block size coprime to p. Since $\mathcal{P}$ is a unique minimal imprimitivity system (Proposition 6.5), the orbits of $\mathbf{O}_{p'}(G)$ are unions of blocks of $\mathcal{P}$. But this is impossible, since the cardinality of blocks of $\mathcal{P}$ is a p-power. ∎

We finish this section by resolving the status of the cyclic group of order 8.

Proposition 6.7. *The cyclic group $\mathbb{Z}_8$ is a CIM-group.*

Proof. Assume towards a contradiction that $M := \mathsf{CM}(\mathbb{Z}_8, S, \rho)$ is a non-CI map over $\mathbb{Z}_8$. Let P be a Sylow 2-subgroup of $G := \mathsf{Aut}(M)$ which contains $\widehat{\mathbb{Z}}_8$. Then P contains a regular cyclic subgroup which is not conjugate to $\widehat{\mathbb{Z}}_8$ inside P. In particular, $|P| \geqslant 16$. Therefore $|\mathbf{N}_P(\widehat{\mathbb{Z}}_8)| \geqslant 16$. The point stabilizer $\mathbf{N}_P(\widehat{\mathbb{Z}}_8)_0$ is cyclic and is contained in $\mathsf{Aut}(\mathbb{Z}_8)$. Therefore $|\mathbf{N}_P(\widehat{\mathbb{Z}_8})_0| = 2$, or, equivalently, $\mathbf{N}_P(\widehat{\mathbb{Z}_8})_0 = \langle \alpha \rangle$ for some $\alpha \in \mathsf{Aut}(\mathbb{Z}_8)$.

If $\widehat{\mathbb{Z}}_8$ is a unique regular cyclic subgroup of $\mathbf{N}_P(\widehat{\mathbb{Z}}_8)$, then $\mathbf{N}_P(\mathbf{N}_P(\widehat{\mathbb{Z}}_8))$ normalizes $\widehat{Z}_8$. So, in this case $\mathbf{N}_P(\mathbf{N}_P(\widehat{\mathbb{Z}}_8)) = \mathbf{N}_P(\widehat{\mathbb{Z}}_8)$ implying $P = \mathbf{N}_P(\widehat{\mathbb{Z}}_8)$, since a p-group is nilpotent and hence the normalizer of a proper subgroup is strictly bigger than the subgroup itself. The latter equality contradicts our assumption that P contains non-conjugate regular cyclic subgroups. Thus $\mathbf{N}_P(\widehat{\mathbb{Z}}_8) = \widehat{\mathbb{Z}}_8 \rtimes \langle \alpha \rangle$ contains non-conjugate regular cyclic subgroups. This yields a unique choice for $\alpha \in \mathsf{Aut}(H)$, namely: $\alpha(x) = 5x, x \in \mathbb{Z}_8$. Notice that $\widehat{\mathbb{Z}}_8 \rtimes \langle \alpha \rangle$ contains exactly two regular cyclic subgroups $\widehat{\mathbb{Z}}_8$ and $\langle \widehat{1}\alpha \rangle$. Each of these subgroups is normal in $\widehat{\mathbb{Z}}_8 \rtimes \langle \alpha \rangle$.

Since $\alpha \in G_0$, it acts semiregularly on S. Combining this with $\langle S \rangle = \mathbb{Z}_8$ and $S = -S$ we obtain that the only possibility for S is $\{1, 5, 3, 7\}$. It follows from $\rho^2 = \alpha|_S$ that either $\rho = (1, 3, 5, 7)$ or $\rho = (1, 7, 5, 3)$. In both cases M is an antibalanced map the full automorphism group of which has order 32 and has a decomposition $G = \widehat{\mathbb{Z}}_8 \langle \rho \rangle$ where ρ acts trivially on the subgroup $2\mathbb{Z}_8$. In both cases all regular cyclic subgroups are conjugate in G. ∎

7 Connected CIM-groups

The main goal of this section is to prove Theorem 1.4. We start with the following.

Lemma 7.1. *Let G be a group which possesses a factorization $G = CS$ where C is cyclic and S is a non-abelian simple group. If S is not normal in G and $\mathsf{core}_G(C) = \{1\}$, then one of the following holds*[7]

(a) $G \cong A_n$, *where n is odd* $S \cong A_{n-1}$, $C \cong \mathbb{Z}_n$;

(b) $G = PSL_2(11)$, $S \cong A_5$, $C \cong Z_{11}$;

(c) $G = M_{23}$, $S \cong M_{22}, C \cong \mathbb{Z}_{23}$.

Proof. Induction on $|G|$.

Consider the action of G on the set G/S of left cosets of S. This action is faithful, because its point stabilizer is a non-normal simple subgroup of G, therefore its core is trivial. Since the action of C on G/S is transitive, it should be regular. Thus G is a transitive subgroup of $\mathsf{Sym}(G/S)$ which contains a regular cyclic subgroup C point stabilizer of which is simple.

Assume first that the action of G is primitive. Since G is non-solvable, a more detailed version of Feit's Theorem [10] proved by Jones [15] implies that G is one of the following groups (here $m := |C|$):

(i) $G = \mathsf{Sym}(m)$ for some $m \geqslant 2$ or $G = \mathsf{Alt}(m)$ for somme odd $m \geqslant 3$;

(ii) $PSL_d(q) \leqslant G \leqslant P\Gamma L_d(q)$ with $m = (q^d - 1)/(q - 1)$ for some $d \geqslant 2$;

(iii) $G = PSL_2(11), M_{11}$ or M_{23} with $m = 11, 11$ or 23, respectively.

Since G_x is a non-abelian simple group, the only possibillities for G are (i) or (iii). This proves our statement in the case of primitive action.

Assume now that G acts imprimitively on the coset space G/S. Then there exists an intermediate subgroup $S < H < G$. It follows from $G = SC$ that $H = S(H \cap C)$ with $|H \cap C| = [H : S] > 1$. The core N of H in G is a normal subgroup in G. It is non-trivial because

$$N = \bigcap_{g \in G} H^g = \bigcap_{g \in C} H^g \geqslant H \cap C$$

since for $x \in H \cap C$ we have $x^g = x$ for every $g \in C$. Note, that $H \cap C \leqslant N \leqslant H = S(H \cap C)$ implies that $N = (N \cap S)(H \cap C)$. Since S is simple, either $S \leqslant N$ or $S \cap N = \{1\}$.

Case A: $S \leqslant N$.

In this case $N = H$, i.e. H is normal in G. Since $H < G$, we can apply the induction hypothesis to H. Thus either $S \trianglelefteq H$ or $\mathsf{core}_H(H \cap C) \neq \{1\}$ or H is one of the groups in the list.

[7]Recall that $\mathsf{core}_G(C)$ is the maximal normal sugroup of G containing in C.

In the first case, S is characteristic in H, and, therefore, normal in G. A contradiction.

In the second case, $core_H(H \cap C) \neq \{1\}$ is normalized by H and hence by $S \leqslant H$ and it is clearly normalized or even centralized by C. Since $G = CS$ we have $core_H(H \cap C) \trianglelefteq G$. Therefore $H \cap C$ is a non-trivial normal sugroup of G contained in C. A contradiction.

In the third case, H is one of the groups in the list with a decomposition $H = S(H \cap C)$. Let c be a generator of C. Then c induces an automorphism of H: $x \mapsto x^c$ which centralizes the cyclic factor $H \cap C$. A direct check shows that the only automorphisms of the groups in (a)-(c) which centralizes the cyclic factor are the internal ones induced by elements of this factor. Therefore $x^c = x^d, x \in H$ for some $d \in H \cap C < C$. This implies that cd^{-1} centralizes H. But this element is centralized by C too. Hence $cd^{-1} \in \mathbf{Z}(G) \cap C$, contrary to $\mathsf{core}_G(C) = \{1\}$.

Note that this last case could also be handled using the fact that in 7 and 7 the group G are simple. Thus the action of G on G/S is faithful and clearly primitive, giving a contradiction in these cases.

Case B: $S \cap N = \{1\}$.

It follows from the decomposition $N = (N \cap S)(H \cap C)$ that $N = H \cap C$.

The automorphism group of the cyclic group $H \cap C$ is abelian while S is non-abelian and simple. Therefore S acts trivially on $H \cap C$ (by conjugation). Thus S centralizes $H \cap C$ implying $H \cap C \leqslant \mathbf{Z}(G)$. But in this case $H \cap C \leqslant \mathsf{core}_G(C) = \{1\}$. A contradiction. ■

Proof of Theorem 1.4. Let G be the automorphism group of a connected map $\mathsf{CM}(H, S, \rho)$. We have to show that any H-regular subgroup of G is conjugate to H inside G.

By Theorem 2.3 the point stabilizer G_e is cyclic. Therefore G has a factorization $G = G_e H$ which satisfies the assumptions of Lemma 7.1. Thus either H is normal in G or G is one of the groups mentioned in the list of Lemma 7.1. If H is normal in G, then $[G, G] = H$ implying that any non-abelian simple subsgroup of G is contained in H. Therefore any H-regular subgroup of G coincides with H.

Assume for the rest of the proof that H is not normal in G. Then one of the cases (a)-(c) happens.

Since $H \ncong A_5$, the case (b) is impossible.

In the case of (a) $G \cong A_n, n$ odd and $H \cong A_{n-1}$. Since for $n \geqslant 7$ all subgroups of A_n isomorphic to A_{n-1} are conjugate in A_n, our statement is true.

In the case of (c) $G \cong M_{23}$ the group H is isomorphic to M_{22}. According to [2] all subgroups of M_{23} isomorphic to M_{22} are conjugate in M_{23}. ■

Acknowledgements

The second author was supported by a postdoctoral fellowship funded by the Skirball Foundation via the Center of Advanced Studies at the Ben-Gurion University of the Negev, and, in addition, by the research grant with PI Dmitry Kerner.

Both authors are grateful to Mikhail Klin who created the opportunity to work together. Without his efforts this cooperation would not have led to writing this paper.

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