Coloring cross-intersecting families

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Abstract

Intersecting and cross-intersecting families usually appear in extremal combinatorics in the vein of the Erdős–Ko–Rado theorem [4]. On the other hand, P. Erdős and L. Lovász in the noted paper [6] posed problems on coloring intersecting families as a restriction of classical hypergraph coloring problems to a special class of hypergraphs. This note deals with the mentioned coloring problems stated for cross-intersecting families.

1 Introduction

A hypergraph is a pair (V, E), where V is a finite set whose elements are called vertices and E is a family of subsets of V, called edges. A hypergraph is n-uniform if every edge has size n.

Definition 1.1. *Intersecting family is a hypergraph $H = (V, E)$ such that $e \cap f \neq \emptyset$ for every $e, f \in E$.*

Intersecting families in extremal combinatorics appeared in the paper by P. Erdős, C. Ko and R. Rado [4], where they determine the maximal number of edges in an n-uniform intersecting family on a given vertex set. A large branch of extremal combinatorics starts from the mentioned paper.

Then P. Erdős and L. Lovász in [6] introduced several problems on coloring intersecting families (*cliques* in the original notation), i. e. hypergraphs without a pair of disjoint edges. Obviously, an intersecting family could have chromatic number 2 or 3 only; the main interest refers to chromatic number 3.

Definition 1.2. *Cross-intersecting family is a hypergraph $H = (V, E)$, equipped by a (not necessarily disjoint) covering $E = A \cup B$ by nonempty sets A and B of edges, such that every $a \in A$ intersects every $b \in B$. Slightly abusing proper notation we allow the use of both $H = (V, E)$ and $H = (V, A, B)$.*

Cross-intersecting families were introduced to study maximal and almost-maximal intersecting families (the notation appears in [13]). The Hilton–Milner theorem [10] uses this notion to determine the maximal number of edges in an n-uniform intersecting family with empty common intersection on a given vertex set. The Frankl theorem [7] is a sharpening of the Hilton–Milner theorem in the case of the bounded maximal vertex degree of an intersecting family. Recently a general approach to mentioned problems was introduced by A. Kupavskii and D. Zakharov [12] (the reader can also see it for a survey).

1.1 The chromatic number

A vertex r-coloring of a hypergraph (V, E) is a map $c : V \to \{1, \dots, r\}$. We are interested in vertex colorings of cross-intersecting families. Coloring is *proper* if there are no monochromatic edges. *Chromatic number* $\chi(H)$ is the minimal number of colors such that H admits a proper coloring. First, note that a cross-intersecting family could have an arbitrarily large chromatic number.

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Example 1.3. *Consider an arbitrary integer $r > 1$. Consider a hypergraph $H_0 = (V_0, E_0)$ with chromatic number r. Put $A := E_0$, $B := \{V_0\}$. Obviously, $H := (V_0, A, B)$ is a cross-intersecting family with chromatic number r.*

However, under a natural assumption (note that it holds for any n-uniform hypergraph) a chromatic number of a cross-intersecting family is bounded.

Proposition 1.4. *Let $H = (V, A, B)$ be a cross-intersecting family. Suppose that A and B both have minimal elements of E, i. e. there are such $a \in A$, $b \in B$ that a, b both have no subedge in H. Then $\chi(H) \leq 4$.*

Proof. Let us color $a \cap b$ in color 1, $a \setminus b$ in color 2, $b \setminus a$ in color 3 and all other vertices in color 4. One can see that the coloring is proper because both a and b have no subedge. □

It turns out, that if there is no pair e_1, $e_2 \in E$ such that $e_1 \subset e_2$ and every edge has a size of at least 3, then the cross-intersecting family can have chromatic number 2 or 3 only. Moreover, the following theorem holds.

Theorem 1.5. *Let $H = (V, A, B)$ be a cross-intersecting family such that there is no pair $e_1, e_2 \in A \cup B$ such that $e_1 \subset e_2$ (i. e. (V, E) is a Sperner system). Then $\chi(H) \leq 3$ or $V := \{v_1, \ldots, v_m, u_1, \ldots u_l\}$; $B := \{\{v_1, \ldots, v_m\}, \{u_1, \ldots u_l\}\}$; $A := \{\{v_i, u_j\}$ for all $i, j\}$ (modulo A-B symmetry), where m, $l \geq 2$.*

Corollary 1.6. *Let $H = (V, A, B)$ be an n-uniform cross-intersecting family. Then $\chi(H) \leq 3$ or $n = 2$ and H is a complete graph on 4 vertices.*

Corollary 1.7. *Let $H = (V, A, B)$ be an n-uniform cross-intersecting family and $\min(|A|, |B|) \geq 3$. Then $\chi(H) \leq 3$.*

1.2 Maximal number of edges

It turns out that the maximal number of edges in a "nontrivial" n-uniform intersecting family is bounded. There are two ways to formalize the notion "nontrivial". The first one is to say that $\chi(H) \geq 3$ (denote the corresponding maximum by $M(n)$). The second one says that H is nontrivial if and only if $\tau(H) = n$ (denote the corresponding maximum by $r(n)$), where $\tau(H)$ is defined below.

Definition 1.8. *Let $H = (V, E)$ be a hypergraph. The covering number $\tau(H)$ (also known as transversal number or blocking number) of the hypergraph H is the smallest size of a set $A \subset V$ such that every $e \in E$ intersects A.*

1.2.1 Upper bounds.

Note that $M(n) \leq r(n)$, because if $\tau(H) < n$ one can color an arbitrary minimal covering set in the first color, and the rest vertices in the second, producing a proper 2-coloring. P. Erdős and L. Lovász proved in [6] that $r(n) \leq n^n$ (one can find slightly better bound in [2]). The best current upper bound is $r(n) \leq cn^{n-1}$ (see [1]). Surprisingly, we can prove a very similar statement for cross-intersecting families. Let us introduce a "nontriviality" notion for cross-intersecting families.

Definition 1.9. *Let us call a cross-intersecting family $H = (V, A, B)$ critical if*

- *for any edge $a \in A$ and any $v \in a$ there is $b \in B$ such that $a \cap b = \{v\}$;*
- *for any edge $b \in B$ and any $v \in b$ there is $a \in A$ such that $a \cap b = \{v\}$.*

Note that if an n-uniform intersecting family $H = (V, E)$ has $\tau(H) = n$ then (V, E, E) is a critical cross-intersecting family.

Theorem 1.10. *Let $H = (V, A, B)$ be a critical cross-intersecting family. Denote*

$$n := \max_{e \in A \cup B} |e|.$$

Then

$$\max(|A|, |B|) \leq n^n.$$

1.2.2 Lower bounds.

L. Lovász conjectured that $M(n) = [(e-1)n!]$ (an example was constructed in [6]). This was disproved by P. Frankl, K. Ota and N. Tokushige [8]. They have provided an explicit example of an n-uniform hypergraph H with $\tau(H) = n$ and at least

$$c\left(\frac{n}{2}\right)^{n-1} \tag{1}$$

edges. For cross-intersecting families Example 1.16 shows that Theorem 1.10 is tight.

1.3 The set of the pairwise edge intersection sizes

Definition 1.11. *For a hypergraph $H = (V, E)$ let us consider the set of the sizes of pairwise edge intersections:*

$$Q(H) := \{|e_1 \cap e_2|, e_1, e_2 \in E\}.$$

Again, P. Erdős and L. Lovász showed that for an n-uniform intersecting family H with $\chi(H) = 3$ one has $3 \leq |Q(H)|$ for sufficiently large n, but there is no example with $|Q(H)| < \frac{n-1}{2}$. For cross-intersecting families there is a simple example with $|Q(H)| = 4$.

Theorem 1.12. *There is an n-uniform cross-intersecting family H with $Q(H) = \{0, 1, 2, n-1\}$ and $\chi(H) = 3$.*

See Example 1.17 for the proof.

1.4 Examples

Unlike the case of intersecting families there is a method of constructing a large set of (critical) cross-intersecting families with chromatic number 3, based on percolation. This method makes it possible to construct a cross-intersecting family from a planar triangulation, which in turn may be generated by well-known random processes.

Example 1.13. *Consider an arbitrary planar triangulation with external face F that has a size of at least 4. Split F into 4 disjoint connected parts F_1, F_2, F_3, F_4. Let A_0 be the set of collections of vertices that form a simple path from F_1 to F_3; B_0 be the set of collections of vertices that form a simple path from F_2 to F_4. Finally, let $A \subset A_0$, $B \subset B_0$ be the sets of all minimal (by the inclusion relation) subsets; $H := (V, A, B)$.*

Obviously, $\chi(H) = 3$ (one may see that no example with chromatic number 4 could be obtained from a planar triangulation).

Remark 1.14. *The same procedure can be generalized on the intersecting families as follows. Consider a planar triangulation with a marked point in the interior. Then every set containing a loop around the marked point intersects all other such sets. Passing to minimal (by the inclusion relation) sets we got an intersecting hypergraph with chromatic number 3.*

For a given $n > 2$ there exists an n-uniform cross-intersecting family (not critical) with chromatic number 3 and an arbitrarily large number of edges.

Example 1.15. *Let m be an arbitrary integer number. Put $V(H) := \{v_1, \dots, v_{2n-1}\} \cup \{u_1, \dots, u_m\}$; $E(H) := A_1 \cup A_2 \cup B_1 \cup B_2$, where $A_1 \cup B_1$ is the set of all n-subsets of $\{v_1, \dots, v_{2n-1}\}$, A_1 contains edges intersecting $\{v_1, \dots v_{n-1}\}$, B_1 contains edges intersecting $\{v_1, v_n, \dots, v_{2n-3}\}$ (so $A_1 \cap B_1 \neq \emptyset$),*

$$A_2 := \{\{v_1, \dots v_{n-1}, u_i\} \text{ for every } i\},$$

$$B_2 := \{\{v_1, v_n, \dots, v_{2n-3}, u_i\} \text{ for every } i\}.$$

Note that $H_1 := (V_1, A_1 \cup B_1)$ has chromatic number 3, so $\chi(H) \geq 3$, hence by Corollary 1.6 we have $\chi(H) = 3$.

Let us show that H is a cross-intersecting family. Clearly, since A_1, $B_1 \subset V_1$, every edge from A_1 intersects with every edge from B_1. By the definition every edge of A_2 contains $\{v_1, \dots v_{n-1}\}$, so it intersects with every edge from B_1; by symmetry the same holds for B_2 and A_1. Also every edge from A_2 intersects with every edge from B_2 at the point v_1.

Example 1.16. *Consider an arbitrary $n > 1$. Let $V := \{v_{ij} \mid 1 \le i, j \le n\}$, $A := \{\{v_{i1}, \dots v_{in}\} \mid 1 \le i \le n\}$, $B := \{\{v_{1i_1}, v_{2i_2}, \dots, \ v_{ni_n}\} \mid 1 \le i_1, i_2, \dots, i_n \le n\}$. Note that $|A| = n$, $|B| = n^n$. Obviously, $H := (V, A, B)$ is a cross-intersecting family and $\chi(H) = 3$.*

Example 1.17 (Proof of Theorem 1.12)**.** *Our construction is based on the following object.*

Definition 1.18. *A hypergraph is called simple if every two edges share at most one vertex.*

Let us take an $(n-1)$-uniform simple hypergraph $H_0 = (V_0, E_0)$ such that $\chi(H) = 3$ (see [6, 11] for constructions). Denote $V := V_0 \sqcup \{u_1, \dots, u_n\}$, $B := \{\{u_1, \dots, u_n\}\}$, $A := \{e \cup \{u_i\} | e \in E_0, 1 \le i \le n\}$. By the construction, H is an n-uniform cross-intersecting family.

Let us show that $\chi(H) = 3$. Suppose the contrary, i.e. there is a 2-coloring of V without monochromatic edges of $A \cup B$. By the definition of H_0, every 2-coloring of V_0 gives a monochromatic (say, blue) edge $e \in E_0$. Then every u_i is red, otherwise $e \cup \{u_i\}$ is monochromatic. So $\{u_1, \dots, u_n\}$ is red, a contradiction.

Note that $Q(H_0) = \{0, 1\}$, so $Q(H) = \{0, 1, 2, n-1\}$.

2 Proofs

Proof of Theorem 1.5. First, suppose that there is no edge of size 2. We show that in this case $\chi(H) \le 3$. Consider such a pair $a \in A$, $b \in B$ that $|a \cup b|$ is the smallest. Pick arbitrary vertices $v_a \in a \setminus b$ and $v_b \in b \setminus a$. Let us color v_a and v_b in color 1, $a \cup b \setminus \{v_a, v_b\}$ in color 2 and the remaining vertices in color 3.

Let us show that this coloring is proper. Since there is no edge of size 2, there is no edge of color 1. Every edge intersects a or b, so there is no edge of color 3. Suppose that there is an edge e of color 2. Without loss of generality $e \in A$. Then $e \subset |a \cup b \setminus \{v_a\}|$, so $|e \cup b| < |a \cup b|$, a contradiction.

Now let us consider the case $\{u, v\} \in E(H)$. We suppose that $\chi(H) > 3$ and show that H has the claimed structure.

Lemma 2.1. *Let $a = \{u, v\} \in A$, $u \in b \in B$. Then for every $w \in B$ there is the edge $\{v, w\} \in E(H)$ or $\chi(H) \le 3$.*

Proof. Suppose that $\chi(H) > 3$. Then for every $w \in b$ there is edge $\{w, v\} \in E(H)$, otherwise one can color v, w in color 1, $b \setminus w$ in color 2 and all other vertices in color 3, producing a proper 3-coloring. □

Without loss of generality $\{u, v\} \in A$. Consider any edge $b \in B$ (without loss of generality $u \in b$). By Lemma 2.1 for $w \in B$ edge $\{v, w\}$ is contained in $E(H)$. Suppose that for some $w \in b$ there is edge $\{v, w\} \in B$. Then, by Lemma 2.1 (for $a = \{u, v\}$ and $b = \{v, w\}$) we have $\{u, w\} \in E(H)$, so $b = \{u, w\}$. Thus H contains a triangle on $\{u, v, w\}$ with edges both in A and B ($\star$). If H coincides with the triangle on $\{u, v, w\}$, then $\chi(H) = 3$. Otherwise, H contains an edge e which does not intersect one of the edges $\{u, v\}$, $\{u, w\}$, $\{v, w\}$. So, we can change denotation as follows: $\{u, v, w\} = \{q, r, s\}$, such that e, $\{q, r\} \in B$ and $e \cap \{q, r\} = \emptyset$. Note that one of the edges $\{q, s\}$, $\{r, s\}$ lies in A (without loss of generality it is $\{q, s\}$).

By Lemma 2.1 (for $a = \{q, s\}$ and $b = e$) for every $t \in e$ there is edge $\{q, t\}$. If $\{r, s\} \in B$, then for every $t \in e \setminus s$ one has $\{q, t\} \in A$. So by Lemma 2.1 (for $a = \{q, t\}$ and $b = \{q, r\}$) there is an edge $\{r, t\}$ for every $t \in e$. If $\{r, s\} \in A$, then by Lemma 2.1 again (for $a = \{r, s\}$ and $b = e$) there is an edge $\{r, t\}$ for every $t \in e$. Summing up, we have edges $\{q, r\}$, $e \in B$, $\{q, s\} \in A$ and $\{x, t\} \in E(H)$ for every choice $x \in \{q, r\}$ and $t \in e$.

If $|e| = 2$, then H is a complete graph on 4 vertices, and $\chi(H) = 4$.

Let us deal with the case $|e| > 2$. It means that there are different $s, t_1, t_2 \in e$. Note that $\{r, t_1\} \in A$ since $\{q, s\} \in A$, so $\{q, t_2\} \in A$. Thus for every choice $x \in \{q, r\}$ and $t \in e$, A contains an edge $\{x, t\}$. Obviously, every edge of hypergraph either intersects both $\{q, r\}$ and e (then it coincides with an edge $\{x, t\}$, where

$x \in \{q, r\}$ and $t \in e$) or contains one of them (then it coincides with $\{q, r\}$ or e). Thus we have listed all the edges of the hypergraph, so we proved the claim in this case. Note also that the set of colors in $\{q, r\}$ does not intersect the set of colors in e, so again $\chi(H) = 4$.

In the remaining case we have all $\{v, w\}$ in A. If $|B| = 1$, then $\chi(H) \leq 3$, so there is an edge $b' \in B$, such that it does not contain u. Suppose that $b \cap b' = \emptyset$. Then for every $w \in b$ and $t' \in b'$ by Lemma 2.1 (for $a = \{v, w\}$ and b') we have edges $\{w, t'\}$ in E. Obviously, all these edges lie in A, otherwise we are done by the first case (if some $\{w, t'\} \in B$, then we have $\{w, v\} \in A$, $\{w, t'\}$, $b' \in B$). Thus, H has the claimed structure.

Finally, if $b \cap b' \neq \emptyset$, then by Lemma 2.1 for $a = \{u, v\}$ and b we have edge $\{v, t\}$ for some $t \in b \cap b'$. Then $b' = \{v, t\}$. Analogously, $b = \{u, t\}$. So the condition $(\star)$ holds, and we are done.

□

Proof of Theorem 1.10. First, we need the following definition.

Definition 2.2. *Let $H = (V, E)$ be a hypergraph and W be a subset of V. Define*

$$H_W := (V \setminus W, \{e \setminus W \mid e \in E\}).$$

Then H is a flower with k petals with core W if $\tau(F_W) \geq k$.

The following Lemma was proved by J. Håstad, S. Jukna and P. Pudlák [9]. We provide its proof for the completeness of presentation.

Lemma 2.3. *Let $H = (V, E)$ be a hypergraph; $n := \max_{e \in E} |e|$. If $|E| > (k-1)^n$ then F contains a flower with k petals.*

Proof. Induction on n. The basis $n = 1$ is trivial.

Now suppose that the statement is true for $n-1$ and prove it for n. If $\tau(H) \geq k$, then H itself is a flower with at least k petals (and an empty core). Otherwise, some set of size $k-1$ intersects all the edges of H, and hence, at least $|E|/(k-1)$ of the edges must contain some vertex x. The hypergraph $H_{\{x\}} = (V_{\{x\}}, E_{\{x\}})$ has

$$|E_{\{x\}}| \geq \frac{|E|}{k-1} > (k-1)^{n-1}$$

edges, each of cardinality at most $n-1$. By the induction hypothesis, $H_{\{x\}}$ contains a flower with k petals and some core Y. Adding the element x back to the sets in this flower, we obtain a flower in H with the same number of petals and the core $Y \cup \{x\}$. □

Now let us prove Theorem 1.10. Suppose the contrary, that is, without loss of generality, $|A| \geq n^n + 1$. Then by Lemma 2.3 the hypergraph (V, A) contains a flower with $n+1$ petals. It means that every $b \in B$ intersects the core of the flower, and H is not critical. A contradiction. □

3 Open questions

The most famous problem in hypergraph coloring is to determine the minimal number of edges in an n-uniform hypergraph with $\chi(H) = 3$ (it is usually denoted by $m(n)$). The best known bounds ([5, 14, 3]) are

$$c\sqrt{\frac{n}{\ln n}}2^n \leq m(n) \leq \frac{e \cdot \ln 2}{4} n^2 2^n (1 + o(1)). \tag{2}$$

P. Erdős and L. Lovász in [6] posed the same question for the class of intersecting families. Even though the intersecting condition is very strong, it does not provide a better lower bound. On the other hand, the upper bound in (2) is probabilistic, so it does not work for intersecting families. The current asymptotically best upper bound [6] is $7^{\frac{n-1}{2}}$ for $n = 3^k$, which is given by the iterated Fano plane.

Another question is to determine the minimal size $a(n)$ of the largest intersection in an n-uniform intersecting family. The best bounds at this time are

$$\frac{n}{\log_2 n} \leq a(n) \leq n-2.$$

Studying the mentioned problems for cross-intersecting families is also of interest.

Recall that Example 1.16 shows that Theorem 1.10 is tight. On the other hand, $\max \min(|A|,|B|)$ over all cross-intersecting families with chromatic number 3 is unknown. Obviously, one may take the example (V,E) by P. Frankl, K. Ota and N. Tokushige and put $A = B = E$ to get lower bound (1).

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