Eulerian Numbers Associated with Arithmetical Progressions

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Abstract

In this paper, we give a combinatorial interpretation of the r-Whitney-Eulerian numbers by means of coloured signed permutations. This sequence is a generalization of the well-known Eulerian numbers and it is connected to r-Whitney numbers of the second kind. Using generating functions, we provide some combinatorial identities and the log-concavity property. Finally, we show some basic congruences involving the r-Whitney-Eulerian numbers.

Keywords: Eulerian number, r-Whitney number, r-Whitney-Eulerian number, combinatorial identities, unimodality

1 Introduction

The Eulerian numbers were introduced by Euler in a noncombinatorial way. Euler was trying to obtain a formula for the alternating sum $\sum_{i=1}^{m} i^n(-1)^i$ (cf. [10]). Explicitly, *Eulerian numbers* $A(n,k)$ can be defined by the recurrence relation [6]

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k), \quad n \geqslant 1,\ k \geqslant 2, \tag{1}$$

with the initial values $A(n,1) = 1$ for $n \geqslant 0$ and $A(0,k) = 0$ if $k \geqslant 2$. Eulerian numbers can also be computed by the following expression

$$A(n,k) = \sum_{i=0}^{k} S(n,i)i!\binom{n-i}{k-i}(-1)^{k-i}, \tag{2}$$

where $S(n,m)$ are the Stirling numbers of the second kind.

Another interesting identity involving Eulerian numbers is called *Worpitzky's identity*

$$x^n = \sum_{k=1}^{n} \binom{x+k-1}{n} A(n,k), \quad n \geqslant 1.$$

It is well-known that Eulerian numbers have a combinatorial interpretation in term of permutations. In particular, the Eulerian number $A(n,k)$ counts the number of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ with $k-1$ descents, that is $k-1 = |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|$.

The *Eulerian polynomials* are defined by

$$A_n(x) := \sum_{k=1}^{n} A(n,k)x^k,$$

with $A_0(x) = 1$. These polynomials satisfy the following relation for any non-negative integer n [6, p. 245].

$$\frac{A_n(x)}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} k^n x^k.$$

The Eulerian numbers and their generalizations have been studied extensively (cf. [21]). In the present article, we are interested in a recent generalization called *r-Whitney-Eulerian numbers* and denoted by $A_{m,r}(n,k)$ in [19]. This new sequence is defined by the expression

$$A_{m,r}(n,k) = \sum_{j=0}^{n} W_{m,r}(n,j)m^j j!\binom{n-j}{k-j}(-1)^{k-j}, \tag{3}$$

where $W_{m,r}(n,k)$ are the r-Whitney numbers of the second kind.

The *r-Whitney numbers of the second kind* $W_{m,r}(n,k)$ were defined by Mező [16] as the connecting coefficients between some special polynomials. Specifically, for non-negative integers n, k and r with $n \geqslant k \geqslant 0$ and for any integer $m > 0$

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)x^{\underline{k}}, \tag{4}$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ if $n \geqslant 1$ and $x^{\underline{0}} = 1$.

The r-Whitney numbers of the second kind satisfy the recurrence [16]

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k). \tag{5}$$

Note that if $(m, r) = (1, 0)$ we obtain the Stirling numbers of the second kind, if $(m, r) = (1, r)$ we have the r-Stirling (or noncentral Stirling) numbers [4], and if $(m, r) = (m, 1)$ we have the Whitney numbers [1]. For more details on r-Whitney numbers see for example [5, 7, 14, 15, 18, 20, 23, 24].

From (3) and recurrence (5) we obtain that the r-Whitney-Eulerian numbers satisfy the recurrence relation

$$A_{m,r}(n, k) = (km + r)A_{m,r}(n - 1, k) + (m(n - (k - 1)) - r)A_{m,r}(n - 1, k - 1), \qquad (6)$$

with the initial values $A_{m,r}(0, 0) = 1$, $A_{m,r}(n, k) = 0$ if $k \geqslant n + 1$ or $k \leqslant -1$.

If $(m, r) = (1, 0)$ we recover the Eulerian numbers $A(n, k)$. If $(m, r) = (1, r)$ we obtain the cumulative numbers studied by Dwyer [8, 9], see also the Euler-Frobenius numbers studied by Gawronski and Neuschel [11]. If $(m, r) = (q + 1, 1)$ we obtain the q-Eulerian numbers studied by Brenti [3].

The *r-Whitney-Eulerian polynomials* are defined by

$$A_{n,m,r}(x) := \sum_{k=0}^{n} A_{m,r}(n, k)x^k.$$

For non-negative integers r, n and positive m, it is known [19] that they satisfy the following identity

$$\sum_{i=0}^{\infty}(mi + r)^n x^i = \frac{A_{n,m,r}(x)}{(1 - x)^{n+1}},$$

and their exponential generating function is

$$\sum_{n=0}^{\infty} A_{n,m,r}(x)\frac{y^n}{n!} = \frac{(1 - x)\exp(ry(1 - x))}{1 - x\exp(my(1 - x))}. \qquad (7)$$

For a similar class of Eulerian numbers connected to the Whitney numbers see the papers of Rahmani [22] and Mező [17].

In the present article, we give a combinatorial interpretation of the r-Whitney-Eulerian numbers by means of coloured signed permutations. Afterwards, we find several combinatorial identities in terms of this new sequence. Moreover, we prove that the r-Whitney-Eulerian numbers are log-concave and therefore unimodal. Finally, we establish some interesting congruences involving this sequence.

2 Combinatorial Interpretation

A *signed permutation* on $[n]$ is a map

$$\sigma : [n] \mapsto \{\pm 1, \pm 2, \dots, \pm n\}$$

which is bijective and $|\sigma|$ is a permutation ($|\sigma|$ is defined by $|\sigma|(i) = |\sigma(i)|$ for all $i \in [n]$). We denote by B_n the set of all signed permutations.

For any signed permutation $\sigma \in B_n$, we define a *descent* to be a position i such that $\sigma(i+1) < \sigma(i)$ with $i \in [n-1] \cup \{0\}$. We define $\sigma(0) = 0$. For example, if $\sigma = 3(-2)54(-1)$ then 1, 3 and 4 are the descents. If $\sigma = (-3)(-2)5(-4)1$ then 0 and 3 are the descents. The number of descents of a signed permutation σ is denoted by $\mathrm{des}_B(\sigma)$.

An *inversion* of a signed permutation σ in B_n is a pair (i, j) such that $i < j$, but $|\sigma(i)| > |\sigma(j)|$. The set of all inversion of σ is denoted by $\mathrm{Inv}_B(\sigma)$. For example, if $\sigma = (-3)(-2)5(-4)1$ then

$$\mathrm{Inv}_B(\sigma) = \{(1, 2), (1, 5), (2, 5), (3, 4), (3, 5), (4, 5)\}.$$

A signed permutation $\sigma \in B_n$ is (m, r)-*coloured* if it satisfies the following conditions:

- If $(i, \ell) \notin \mathrm{Inv}_B(\sigma)$ for all $\ell \geqslant i$, and $\sigma(i) > 0$ then $\sigma(i)$ is coloured with one of r colors. But, if $\sigma(i) < 0$ then it is coloured with one of $m - r$ colours.
- If the above inversion property does not hold, then we colour $\sigma(i)$ with one of $m - 1$ colours providing that $\sigma(i) < 0$, but if $\sigma(i)$ is positive we coloured it with one colour.

Let $n, k, m, r \geqslant 0$ be integers with $m \geqslant r$. Let $\mathbb{B}_{n,k}^{(m,r)}$ denote the set of (m, r)-coloured signed permutations of B_n with k descents.

Theorem 1. *For any integers $n, k, m, r \geqslant 0$, with $m \geqslant r$ we have*

$$|\mathbb{B}_{n,k}^{(m,r)}| = A_{m,r}(n, k).$$

Proof. Let $b_{n,k}^{(m,r)} = |\mathbb{B}_{n,k}^{(m,r)}|$. We are going to prove that the numbers $b_{n,k}^{(m,r)}$ satisfy the same recurrence that $A_{m,r}(n, k)$ with the same initial values. Indeed, note that any (m, r)-coloured signed permutation of $[n]$ with k descents can be obtained from a (m, r)-coloured signed permutation π' of $[n-1]$ with k or $k-1$ descents by inserting the entries n or $-n$ into π'.

In the first case, we have to put the entry n at the end of π', or we have to put the entries n or $-n$ between two entries that form one of the k descents of π'. Then we have the following possibilities:

$$(r + k + k(m-1))b_{n-1,k}^{(m,r)} = (km + r)b_{n-1,k}^{(m,r)}.$$

In the second case, we have to put the entries n or $-n$ at the beginning of π', or we have to put the entry $-n$ at the end of π' or we have to insert n or $-n$ between one of the $(n-2) - (k-1) = n - k - 1$ ascents of π'. Hence we have the following possibilities

$$(1+(m-1)+(m-r)+(n-k-1)+(n-k-1)(m-1))b_{n-1,k-1}^{(m,r)} = (m(n-(k-1))-r)b_{n-1,k-1}^{(m,r)}.$$

Therefore

$$b_{n,k}^{(m,r)} = (km + r)b_{n-1,k}^{(m,r)} + (m(n-(k-1)) - r)b_{n-1,k-1}^{(m,r)},$$

and the theorem is proved. □

Descents	Coloured signed permutations
0	12, 12, 12, 12.
1	21, 21, 2(-1), 1(-2), 1(-2), (-1)2, (-1)2, (-2)1, (-2)1, (-2)1, (-2)1, (-2)(-1), (-2)(-1).
2	(-1)(-2).

Table 1: (3,2)-Coloured signed permutations of size 2.

Example 2. Let $n = 2, m = 3$ and $r = 2$. The $m-1 = 2$ different colours of the elements will be fixed as red and green; the $r = 2$ different colours of the elements will be fixed as cyan and blue; while the $m - r = 1$ colours of the elements will be fixed as magenta. Therefore, $A_{3,2}(2, 0) = 4, A_{3,2}(2, 1) = 13$ and $A_{3,2}(2, 2) = 1$, where the coloured signed permutations are in Table 1.

Theorem 3. *The following identity holds*

$$m^n n! = \sum_{k=0}^{n} A_{m,r}(n, k).$$

Proof. Let $\sigma \in \mathbb{B}_{n,k}^{(m,r)}$. Consider the permutation $|\sigma|$ defined by $|\sigma|(i) = |\sigma(i)|$ for all $i \in [n]$. Let $P_\sigma = \{i \in [n] : (i, j) \notin \mathrm{Inv}(|\sigma|)$ for any $j > i\}$. We suppose that $\ell = |P_\sigma|$, and suppose there are t negative positions of these ℓ $(0 \leqslant t \leqslant \ell)$, then these negative positions can be coloured with one of $m - r$ colours, while the $\ell - t$ positive positions can be coloured with one of r-colours. Therefore by the product rule we have

$$\sum_{t=0}^{\ell} \binom{\ell}{t} (m-r)^t r^{\ell-t} \sum_{t=0}^{n-\ell} \binom{n-\ell}{t} (m-1)^t 1^{n-\ell-t} = m^\ell m^{n-\ell} = m^n$$

ways to colour each fixed permutation. So, summing over all possible non-signed permutations we get the desired identity. □

3 Some Combinatorial Identities

The goal of the current section is to extend some well-known identities for the classical Eulerian numbers to the r-Whitney-Eulerian numbers.

Theorem 4. *For $n, k \geqslant 0$, we have the following identity*

$$W_{m,r}(n, k) = \frac{1}{m^k k!} \sum_{i=0}^{k} A_{m,r}(n, i) \binom{n-i}{k-i}.$$

Proof. The proof follows by showing that the right side of the identity have the same recurrence relations as the r-Whitney numbers of the second kind. □

The r-Whitney-Eulerian numbers are not symmetric as the classical Eulerian numbers ($A(n,k) = A(n,n-k+1)$). However, we note that $A_{m,r}(n,n-k-1) = \widehat{A}_{m,r}(n,k)$, where $\widehat{A}_{m,r}(n,k)$ are the generalized Eulerian numbers defined by Xiong et al. [25]. From above relation and Lemmas 7 and 8 of [25], we obtain a generalization of the Worpitzky's identity.

Theorem 5. *For $n \geqslant 0$, we have the identities*

$$(mx+r)^n = \sum_{k=0}^{n} A_{m,r}(n,k)\binom{x+n-k}{n} = \sum_{k=1}^{n+1} A_{m,r}(n,n-k+1)\binom{x+k-1}{n}.$$

Theorem 6 gives a generalization of the well-known identity for the Eulerian numbers (cf. [6, p. 243])

$$A(n,k) = \sum_{i=0}^{k}(-1)^i(k-i)^n\binom{n+1}{i}.$$

Theorem 6. *For $n, k \geqslant 0$, we have the identity*

$$A_{m,r}(n,k) = \sum_{i=0}^{k}(-1)^i[(k-i)m+r]^n\binom{n+1}{i}.$$

Proof. By using the generating function (7) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{m,r}(n,k)x^k\frac{y^n}{n!} &= \frac{(1-x)\exp(ry(1-x))}{1-x\exp(my(1-x))} = (1-x)\exp(ry(1-x))\sum_{i=0}^{\infty}x^i e^{imy(1-x)}\\ &= (1-x)\sum_{i=0}^{\infty}x^i e^{y(1-x)(im+r)} = \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}(1-x)^{n+1}(im+r)^n x^i\frac{y^n}{n!}\\ &= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n+1}\binom{n+1}{\ell}(-1)^{\ell}(im+r)^n x^{i+\ell}\frac{y^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. □

Above identity gives us special values when k is small:

$$\begin{aligned}&A_{m,r}(n,0) = r^n, \quad A_{m,r}(n,1) = (m+r)^n - r^n(n+1),\\ &A_{m,r}(n,2) = (2m+r)^n - (m+r)^n(n+1) + r^n\binom{n+1}{2}.\end{aligned}$$

Finally, by using the generating function (7) we find a relation between the r-Whitney-Eulerian polynomials and the classical Eulerian polynomials.

Theorem 7. *For $n \geqslant 0$, we have the following identity*

$$A_{n,m,r}(x) = \sum_{j=0}^{n}\binom{n}{j}m^j r^{n-j}A_j(x)(1-x)^{n-j}.$$

4 Unimodality and Log-Concavity Properties

In this section we prove the log-concavity and therefore the unimodality of the r-Whitney-Eulerian numbers. Recall that a finite sequence of non negative real numbers $\{a_k\}_{0\leqslant k\leqslant n}$ is said to be *unimodal* if there is an index i such that $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{i-1} \leqslant a_i \geqslant a_{i+1} \geqslant \cdots \geqslant a_{n-1} \geqslant a_n$. A sequence of real numbers is log-concave if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for $0 < i < n$. It is well know that a sequence which is log-concave is also unimodal. We first prove the following equality.

Theorem 8. *For $n \geqslant 1$, the r-Whitney-Eulerian polynomials satisfy the recurrence*

$$A_{n,m,r}(x) = (mx - mx^2)A'_{n-1,m,r}(x) + (r + (mn - r)x)A_{n-1,m,r}(x). \tag{8}$$

Proof. From recurrence (6) we get

$$\begin{aligned}
A_{n,m,r}(x) &= \sum_{k=0}^{n} A_{m,r}(n,k)x^k \\
&= \sum_{k=0}^{n} \left[(km + r)A_{m,r}(n-1,k)x^k + (m(n-(k-1)) - r)A_{m,r}(n-1,k-1)x^k\right] \\
&= mx\sum_{k=0}^{n-1} kA_{m,r}(n-1,k)x^{k-1} + r\sum_{k=0}^{n-1} A_{m,r}(n-1,k)x^k + mnx\sum_{k=0}^{n-1} A_{m,r}(n-1,k)x^k \\
&\quad - mx^2\sum_{k=0}^{n-1} kA_{m,r}(n-1,k)x^{k-1} - rx\sum_{k=0}^{n-1} A_{m,r}(n-1,k)x^k \\
&= (mx - mx^2)A'_{n-1,m,r}(x) + (r + (mn - r)x)A_{n-1,m,r}(x). \qquad \square
\end{aligned}$$

The log-concavity property of the Eulerian numbers can be proved by means of the real zero property of the Eulerian polynomials $A_n(x)$ (cf. [2]). A sequence $\{a_0, a_1, \dots, a_n\}$ of the coefficients of a polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ of degree n with only real zeros is called the *Pólya frequency sequence (PF)*. It is well know that if a sequence is PF then it is log-concave (cf. [2]). We are going to prove that the sequence $A_{m,r}(n,k)$ is a PF-sequence. To reach this aim, we first prove the following general lemma.

Lemma 9. *Let $(T_n(x))_n$ be a sequence of functions for $n \geqslant 0$ defined by*

$$\begin{aligned}
T_{n+1}(x) &= p_n(x)T_n(x) + q_n(x)T'_n(x) \\
T_0(x) &= T(x),
\end{aligned}$$

for some sequence of functions $(p_n(x))_n, (q_n(x))_n$, then

$$T_{n+1}(x) = r_n(x)\frac{d}{dx}(u_n(x)T_n(x)),$$

where we define for some suitable real number α

$$r_n(x) = \frac{q_n(x)}{u_n(x)} \quad \textit{and} \quad u_n(x) = e^{\int_\alpha^x \frac{p_n(t)}{q_n(t)}dt}.$$

Proof. Observe that

$$\frac{d}{dx}u_n(x) = \frac{p_n(x)}{q_n(x)}u_n(x).$$

Then

$$\begin{aligned} r_n(x)\frac{d}{dx}(u_n(x)T_n(x)) &= \frac{q_n(x)}{u_n(x)}\frac{d}{dx}(u_n(x)T_n(x)) \\ &= \frac{q_n(x)}{u_n(x)}T_n(x)\frac{d}{dx}u_n(x) + \frac{q_n(x)}{u_n(x)}T_n'(x)u_n(x) \\ &= \frac{q_n(x)}{u_n(x)}T_n(x)\left(\frac{p_n(x)}{q_n(x)}u_n(x)\right) + q_n(x)T_n'(x) \\ &= p_n(x)T_n(x) + q_n(x)T_n'(x) \\ &= T_{n+1}(x). \end{aligned}$$

□

Theorem 10. *For $n \geqslant 1$, the r-Whitney-Eulerian polynomials $A_{n,m,r}(x)$ have only non-positive real roots if $m \geqslant r \geqslant 0$. Therefore $(A_{m,r}(n,k))_k$ is a PF-sequence.*

Proof. The case in which $m = r$ is clear because $A_{m,m}(n,k) = m^n A(n, n-k-1)$. Let us assume that $m > r$, this implies $0 < 1 - \frac{r}{m}$. Using our previous lemma and identity (8) we have that

$$A_{n,m,r}(x) = mx^{1-\frac{r}{m}}(1-x)^{n+1}\frac{d}{dx}(x^{\frac{r}{m}}(1-x)^{-n}A_{n-1,m,r}(x)). \tag{9}$$

We now proceed by using induction over n. For $n = 1$ we get

$$A_{1,m,r}(x) = r + (m-r)x$$

which have only one real root being

$$x = -\frac{r}{m-r} < 0.$$

By the inductive hypothesis for $n-1$ the term

$$x^{\frac{r}{m}}(1-x)^{-n}A_{n-1,m,r}(x)$$

has $n-1$ non-positive real roots plus the root in $x = 0$. So by Rolle's Theorem the derivative of this term must have exactly $n-1$ non-positive real roots and by Equation (9) the polynomial $A_{n,m,r}(x)$ must have $n-1$ non-positive real roots. Since complex roots appear in conjugate pairs the only choice for the last root of $A_{n,m,r}(x)$ is to be real and non positive since the polynomial $A_{n,m,r}(x)$ has positive coefficients. □

Therefore we have the following theorem.

Theorem 11. *If $0 \leqslant r \leqslant m$, the r-Whitney Eulerian sequence $(A_{m,r}(n,k))_k$ is log-concave and therefore unimodal.*

Note that the proof of the Theorem 10 actually provide more information than what is stated. It also shows that the polynomials $A_{n,m,r}(x)$ and $A_{n-1,m,r}(x)$ are interlacing if $m \geqslant r$.

Let $(r_i)_{i\in\mathbb{N}}$ and $(s_j)_{j\in\mathbb{N}}$ be the sequences of the real zeros of polynomials f of degree n and g of degree $n-1$ in nonincreasing order, respectively. We say that g *interlaces* f [13], denoted by $g \preccurlyeq f$, if

$$r_n \leqslant s_{n-1} \leqslant \cdots \leqslant s_2 \leqslant r_2 \leqslant s_1 \leqslant r_1.$$

So, by using the argument of the proof, we can state that

$$A_{n-1,m,r}(x) \preccurlyeq A_{n,m,r}(x).$$

5 Some Congruences

In this section, we will show some properties regarding prime congruences over generalized Eulerian numbers. These results generalize those of Knopfmacher and Robbins [12]. We make use of the following lemmas [12].

Lemma 12. *If p is a prime number and $\ell \geqslant 1$, $1 \leqslant k \leqslant p^\ell - 1$, then*

$$\binom{p^\ell}{k} \equiv 0 \pmod p.$$

Lemma 13. *If p is a prime number and $\ell \geqslant 1$, $1 \leqslant k \leqslant p^\ell - 1$, then*

$$\binom{p^\ell+1}{k} \equiv \begin{cases} 1 \pmod p, & \text{if } k = 0,\, 1,\, p^l,\, p^l+1; \\ 0 \pmod p, & \text{if } 2 \leqslant k \leqslant p^\ell - 1. \end{cases}$$

Remember that $A_{m,r}(n, n-k-1) = \widehat{A}_{m,r}(n,k)$. Now we can prove the main results of this section.

Theorem 14. *If p is a prime number and $\ell \geqslant 1$, $1 \leqslant k+1 \leqslant p^\ell - 1$, then*

$$\widehat{A}_{m,r}(p^\ell - 1, k) \equiv \begin{cases} 1 \pmod p, & \text{if } p \nmid m(k+2) - r; \\ 0 \pmod p, & \text{otherwise.} \end{cases}$$

Proof. From Theorem 6 we can establish the identity

$$\widehat{A}_{m,r}(n,k) = \sum_{i=0}^{k+1} (-1)^i [(k+2-i)m - r]^n \binom{n+1}{i}. \tag{10}$$

Therefore

$$\begin{aligned} \widehat{A}_{m,r}(p^\ell - 1, k) &= \sum_{i=0}^{k+1} (-1)^i [(k+2-i)m - r]^{p^\ell - 1} \binom{p^\ell}{i} \\ &\equiv [m(k+2) - r]^{p^\ell - 1} = ([m(k+2) - r]^{p-1})^{\frac{p^\ell - 1}{p-1}} \pmod p. \end{aligned}$$

From Fermat little's theorem we get the desired result. □

In particular, if $m = r = 1$ we have the following congruence for the Eulerian numbers.

Corollary 15 ([12], Theorem 1). *If p is a prime number and $\ell \geqslant 1$, $1 \leqslant k \leqslant p^\ell - 1$, then*

$$A(p^\ell - 1, k) \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \nmid k; \\ 0 \pmod{p}, & \text{otherwise.} \end{cases}$$

Theorem 16. *Let n be an integer such that n does not divide the integers $(k+2)m - r$, $(k+1)m - r$ and m. Then n is prime if and only if $\widehat{A}_{m,r}(n-1, k) \equiv 1 \pmod{n}$, for $1 \leqslant k+1 \leqslant n-1$ and $m^{n-1} \equiv [(k+2)m - r]^{n-1} \equiv [(k+1)m - r]^{n-1} \equiv 1 \pmod{n}$.*

Proof. If we assume that n is prime, then the implication follows from Theorem 14. For the converse observe that

$$\begin{aligned}(n-1)! \equiv m^{n-1}(n-1)! = \sum_{k=0}^{n-1} \widehat{A}_{m,r}(n-1, k) \equiv \widehat{A}_{m,r}(n-1, 0) + \sum_{k=1}^{n-1} 1 \\ = [(k+2)m - r]^{n-1} - [(k+1)m - r]^{n-1} + (n-1) \equiv -1 \pmod{n}.\end{aligned}$$

From Wilson's Theorem we deduce that n is a prime number. □

Theorem 17. *Suppose that p does not divide $(k+2)m - r$ and $(k+1)m - r$. If p is a prime number, $\ell \geqslant 1$, and $1 \leqslant k+1 \leqslant p^\ell$, then*

$$\widehat{A}_{m,r}(p^\ell, k) \equiv m \pmod{p}.$$

Proof. From identity (10) we have

$$\begin{aligned}\widehat{A}_{m,r}(p^\ell, k) &= \sum_{i=0}^{k+1} (-1)^i [(k+2-i)m - r]^{p^\ell} \binom{p^\ell + 1}{i} \\ &\equiv [(k+2)m - r]^{p^\ell} - [(k+1)m - r]^{p^\ell} \\ &\equiv [(k+2)m - r] - [(k+1)m - r] \\ &\equiv m \pmod{p}.\end{aligned}$$

□

Theorem 18. *Suppose that p does not divide $(k+2)m - r$ and $(k+1)m - r$. If p is a prime number, $\ell \geqslant 1$ and $2 \leqslant k+1 \leqslant p^\ell$, then*

$$\widehat{A}_{m,r}(p^\ell + 1, k) \equiv 2m^2 \pmod{p}.$$

Proof. By recurrence (6) we have

$$\widehat{A}_{m,r}(p^\ell + 1, k) = ((k+2)m - r)\widehat{A}_{m,r}(p^\ell, k) + (r + (p^m - k)m)\widehat{A}_{m,r}(p^\ell + 1, k).$$

From the previous theorem we have

$$\widehat{A}_{m,r}(p^\ell + 1, k) \equiv ((k+2)m - r)m + (r + (p^m - k)m)m \equiv 2m^2 \pmod{p}.$$

□

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References

[1] M. Benoumhani. On Whitney numbers of Dowling lattices. *Discrete Math.*, 159: 13–33, 1996.

[2] M. Bóna. *Combinatorics of Permutations*. Chapmann & Hall/CRC, 2004.

[3] F. Brenti. q-Eulerian polynomials arising from Coxeter groups. *Europ. J. Combin.*, 15: 417–441, 1994.

[4] A. Z. Broder. The r-Stirling numbers. *Discrete Math.*, 49: 241–259, 1984.

[5] G.-S. Cheon and J.-H. Jung. r-Whitney numbers of Dowling lattices. *Discrete Math.*, 312(15): 2337–2348, 2012.

[6] L. Comtet. *Advanced Combinatorics*. D. Reidel Publishing Company, 1974.

[7] C. Corcino, R. Corcino, I. Mező, and J. L. Ramírez. Some polynomials associated with the r-Whitney numbers. *Proc. Indian Acad. Sci. Math. Sci.*, To appear.

[8] P. S. Dwyer. The computation of moments with the use of cumulative totals. *Ann. Math Stat.*, 9: 288–304, 1938.

[9] P. S. Dwyer. The cumulative numbers and their polynomials. *Ann. Math. Stat.*, 11: 66–71, 1940.

[10] D. Foata. Eulerian polynomials: from Euler's time to the present. In: The Legacy of Alladi Ramakrishnan in the Mathematical Sciences, 253–273. Springer, New York, 2010.

[11] W. Gawronski and T. Neuschel. Euler-Frobenius numbers. *Integral Transforms Spec. Funct.*, 24(10): 817–830, 2013.

[12] A. Knopfmacher and N. Robbins. Some arithmetical properties of Eulerian numbers. *J. Combin. Math. Combin. Comput.*, 36: 31–42, 2001.

[13] L. L. Liu and Y. Wang. A unified approach to polynomial sequences with only real zeros. *Adv. in Appl. Math.*, 38(4): 542–560, 2007.

[14] T. Mansour, J. L. Ramírez, and M. Shattuck. A generalization of the r-Whitney numbers of the second kind. *J. Comb.*, 8(1): 29–55, 2017.

[15] M. Merca. A note on the r-Whitney numbers of Dowling lattices. *C. R. Math. Acad. Sci. Paris*, 351: 649–655, 2013.

[16] I. Mező. A new formula for the Bernoulli polynomials. *Result. Math.*, 58(3): 329–335, 2010.

[17] I. Mező. A kind of Eulerian numbers connected to Whitney numbers of Dowling lattices. *Discrete Math.*, 328: 88–95, 2014.

[18] I. Mező and J. L. Ramírez. The linear algebra of the r-Whitney matrices. *Integral Transforms Spec. Funct.*, 26(3): 213–225, 2015.

[19] I. Mező and J. L. Ramírez. Some identities of the r-Whitney numbers. *Aequationes Math.*, 90(2): 393–406, 2016.

[20] M. Mihoubi and M. Tiachachat. Some applications of the r-Whitney numbers. *C. R. Math. Acad. Sci. Paris*, 352(12): 965–969, 2014.

[21] T. K. Petersen. *Eulerian Numbers.* Birkhaüser Advanced Texts Basler Lehrbücher, Springer, New York, 2015.

[22] M. Rahmani. Some results on Whitney numbers of Dowling lattices. *Arab. J. of Math. Sci.*, 20(1): 11–27, 2014.

[23] J. L. Ramírez and M. Shattuck. Generalized r-Whitney numbers of the first kind. *Ann. Math. Inform.*, 46: 175–193, 2016.

[24] J. L. Ramírez and M. Shattuck. (p, q)-Analogue of the r-Whitney-Lah numbers. *J. Integer Seq.*, 19, Article 16.5.6, 2016.

[25] T. Xiong, H. Tsao, and J. I. Hall. General Eulerian numbers and Eulerian polynomials. *J. Math.*, Article ID 629132: 1–9, 2013.