A Brooks type theorem for the maximum local edge connectivity

Michael Stiebitz*
Technische Universität Ilmenau
Inst. of Math.
PF 100565
D-98684 Ilmenau, Germany
michael.stiebitz@tu-ilmenau.de

Bjarne Toft*
University of Southern Denmark
IMADA
Campusvej 55
DK-5320 Odense M, Denmark
btoft@imada.sdu.dk

Submitted: Mar 31, 2016; Accepted: Feb 18, 2018; Published: Mar 2, 2018
Mathematics Subject Classifications: 05C15

Abstract

For a graph G, let $\chi(G)$ and $\lambda(G)$ denote the chromatic number of G and the maximum local edge connectivity of G, respectively. A result of Dirac implies that every graph G satisfies $\chi(G) \leqslant \lambda(G) + 1$. In this paper we characterize the graphs G for which $\chi(G) = \lambda(G) + 1$. The case $\lambda(G) = 3$ was already solved by Aboulker, Brettell, Havet, Marx, and Trotignon. We show that a graph G with $\lambda(G) = k \geqslant 4$ satisfies $\chi(G) = k + 1$ if and only if G contains a block which can be obtained from copies of K_{k+1} by repeated applications of the Hajós join.

Keywords: graph coloring; connectivity; critical graphs; Brooks' theorem

1 Introduction and main result

The paper deals with the classical vertex coloring problem for graphs. The term graph refers to a finite undirected graph without loops and without multiple edges. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of G such that each vertex receives a color and adjacent vertices receive different colors. There are several degree bounds for the chromatic number. For a graph G, let $\delta(G) = \min_{v \in V(G)} d_G(v)$ and $\Delta(G) = \max_{v \in V(G)} d_G(v)$ denote the *minimum degree* and the *maximum degree* of G, respectively. Furthermore, let

$$\mathrm{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$$

*Booth authors thank the Danish Research Council for support through the program Algodisc.

denote the *coloring number* of G, and let

$$\operatorname{mad}(G) = \max_{\varnothing \neq H \subseteq G} \frac{2|E(H)|}{|V(H)|}$$

denote the *maximum average degree* of G. By $H \subseteq G$ we mean that H is a subgraph of G. If G is the *empty graph*, that is, $V(G) = \varnothing$, we briefly write $G = \varnothing$ and define $\delta(G) = \Delta(G) = \operatorname{mad}(G) = 0$ and $\operatorname{col}(G) = 1$. A simple sequential coloring argument shows that $\chi(G) \leqslant \operatorname{col}(G)$, which implies that every graph G satisfies

$$\chi(G) \leqslant \operatorname{col}(G) \leqslant \lfloor \operatorname{mad}(G) \rfloor + 1 \leqslant \Delta(G) + 1.$$

These inequalities were discussed in a paper by Jensen and Toft [10]. Brooks' famous theorem provides a characterization for the class of graphs G satisfying $\chi(G) = \Delta(G) + 1$. Let $k \geqslant 0$ be an integer. For $k \neq 2$, let $\mathcal{B}_k$ denote the class of complete graphs having order $k+1$; and let $\mathcal{B}_2$ denote the class of odd cycles. A graph in $\mathcal{B}_k$ has maximum degree k and chromatic number $k+1$. Brooks' theorem [2] is as follows.

Theorem 1 (Brooks 1941)**.** *Let G be a non-empty graph. Then $\chi(G) \leqslant \Delta(G) + 1$ and equality holds if and only if G has a connected component belonging to the class $\mathcal{B}_{\Delta(G)}$.*

In this paper we are interested in connectivity parameters of graphs. Let G be a graph with at least two vertices. The *local connectivity* $\kappa_G(v, w)$ of distinct vertices v and w is the maximum number of internally vertex disjoint v-w paths of G. The *local edge connectivity* $\lambda_G(v, w)$ of distinct vertices v and w is the maximum number of edge-disjoint v-w paths of G. The *maximum local connectivity* of G is

$$\kappa(G) = \max\{\kappa_G(v, w) \mid v, w \in V(G), v \neq w\},$$

and the *maximum local edge connectivity* of G is

$$\lambda(G) = \max\{\lambda_G(v, w) \mid v, w \in V(G), v \neq w\}.$$

For a graph G with $|G| \leqslant 1$, we define $\kappa(G) = \lambda(G) = 0$. Clearly, the definition implies that $\kappa(G) \leqslant \lambda(G)$ for every graph G. By a result of Mader [11] it follows that $\delta(G) \leqslant \kappa(G)$. Since κ is a monotone graph parameter in the sense that $H \subseteq G$ implies $\kappa(H) \leqslant \kappa(G)$, it follows that every graph G satisfies $\operatorname{col}(G) \leqslant \kappa(G) + 1$. Consequently, every graph G satisfies

$$\chi(G) \leqslant \operatorname{col}(G) \leqslant \kappa(G) + 1 \leqslant \lambda(G) + 1 \leqslant \Delta(G) + 1. \tag{1}$$

Our aim is to characterize the class of graphs G for which $\chi(G) = \lambda(G) + 1$. For such a characterization we use the fact that if we have an optimal coloring of each block of a graph G, then we can combine these colorings to an optimal coloring of G by permuting colors in the blocks if necessary. For every non-empty graph G, we thus have

$$\chi(G) = \max\{\chi(H) \mid H \text{ is a block of } G\}. \tag{2}$$

We also need a famous construction, first used by Hajós [9]. Let G_1 and G_2 be two vertex-disjoint graphs and, for $i = 1, 2$, let $e_i = v_i w_i$ be an edge of G_i. Let G be the graph obtained from G_1 and G_2 by deleting the edges e_1 and e_2 from G_1 and G_2, respectively, identifying the vertices v_1 and v_2, and adding the new edge $w_1 w_2$. We then say that G is the *Hajós join* of G_1 and G_2 and write $G = (G_1, v_1, w_1) \bigtriangleup (G_2, v_2, w_2)$ or briefly $G = G_1 \bigtriangleup G_2$.

For an integer $k \geqslant 0$ we define a class $\mathcal{H}_k$ of graphs as follows. If $k \leqslant 2$, then $\mathcal{H}_k = \mathcal{B}_k$. The class $\mathcal{H}_3$ is the smallest class of graphs that contains all odd wheels and is closed under taking Hajós joins. Recall that an *odd wheel* is a graph obtained from on odd cycle by adding a new vertex and joining this vertex to all vertices of the cycle. If $k \geqslant 4$, then $\mathcal{H}_k$ is the smallest class of graphs that contains all complete graphs of order $k+1$ and is closed under taking Hajós joins. Our main result is the following counterpart of Brooks' theorem. In fact, Brooks' theorem may easily be deduced from it.

Theorem 2. *Let G be a non-empty graph. Then $\chi(G) \leqslant \lambda(G) + 1$ and equality holds if and only if G has a block belonging to the class $\mathcal{H}_{\lambda(G)}$.*

For the proof of this result, let G be a non-empty graph with $\lambda(G) = k$. By (1), we obtain $\chi(G) \leqslant k + 1$. By an observation of Hajós [9] it follows that every graph in $\mathcal{H}_k$ has chromatic number $k+1$. Hence if some block of G belongs to $\mathcal{H}_k$, then (2) implies that $\chi(G) = k + 1$. So it only remains to show that if $\chi(G) = k + 1$, then some block of G belongs to $\mathcal{H}_k$. For proving this, we shall use the critical graph method, see [12].

A graph G is *critical* if every proper subgraph H of G satisfies $\chi(H) < \chi(G)$. We shall use the following two properties of critical graphs. As an immediate consequence of (2) we obtain that if G is a critical graph, then $G = \varnothing$ or G contains no separating vertex, implying that G is its only block. Furthermore, every graph contains a critical subgraph with the same chromatic number.

Let G be a non-empty graph with $\lambda(G) = k$ and $\chi(G) = k + 1$. Then G contains a critical subgraph H with chromatic number $k + 1$, and we obtain that $\lambda(H) \leqslant \lambda(G) = k$. So the proof of Theorem 2 is complete if we can show that H is a block of G which belongs to $\mathcal{H}_k$. For an integer $k \geqslant 0$, let $\mathcal{C}_k$ denote the class of graphs H such that H is a critical graph with chromatic number $k + 1$ and with $\lambda(H) \leqslant k$. We shall prove that the two classes $\mathcal{C}_k$ and $\mathcal{H}_k$ are the same.

2 Connectivity of critical graphs

In this section we shall review known results about the structure of critical graphs. First we need some notation. Let G be an arbitrary graph. For an integer $k \geqslant 0$, let $\mathcal{CO}_k(G)$ denote the set of all colorings of G with color set $\{1, 2, \ldots, k\}$. Then a function $f : V(G) \to \{1, 2, \ldots, k\}$ belongs to $\mathcal{CO}_k(G)$ if and only if $f^{-1}(c)$ is an independent vertex set of G (possibly empty) for every color $c \in \{1, 2, \ldots, k\}$. A set $S \subseteq V(G) \cup E(G)$ is called a *separating set* of G if $G - S$ has more components than G. A vertex v of G is called a *separating vertex* of G if $\{v\}$ is a separating set of G. An edge e of G is called a *bridge* of G if $\{e\}$ is a separating set of G. For a vertex set $X \subseteq V(G)$, let $\partial_G(X)$

denote the set of all edges of G having exactly one end in X. Clearly, if G is connected and $\varnothing \neq X \subsetneq V(G)$, then $F = \partial_G(X)$ is a separating set of edges of G. The converse is not true. However if F is a minimal separating edge set of a connected graph G, then $F = \partial_G(X)$ for some vertex set X. As a consequence of Menger's theorem about edge connectivity, we obtain that if v and w are distinct vertices of G, then

$$\lambda_G(v, w) = \min\{|\partial_G(X)| \mid X \subseteq V(G), v \in X, w \notin X\}.$$

Color critical graphs were first introduced and investigated by Dirac in the 1950s. He established the basic properties of critical graphs in a series of papers [3], [4] and [5]. Some of these basic properties are listed in the next theorem.

Theorem 3 (Dirac 1952)**.** *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geqslant 0$. Then the following statements hold:*

(a) $\delta(G) \geqslant k$.

(b) *If $k \in \{0, 1\}$, then G is a complete graph of order $k+1$; and if $k = 2$, then G is an odd cycle.*

(c) *No separating vertex set of G is a clique of G. As a consequence, G is connected and has no separating vertex, i.e., G is a block.*

(d) *If v and w are two distinct vertices of G, then $\lambda_G(v, w) \geqslant k$. As a consequence G is k-edge-connected.*

Theorem 3(a) leads to a very natural way of classifying the vertices of a critical graph into two classes. Let G be a critical graph with chromatic number $k+1$. The vertices of G having degree k in G are called *low vertices* of G, and the remaining vertices are called *high vertices* of G. So any high vertex of G has degree at least $k+1$ in G. Furthermore, let G_L be the subgraph of G induced by the low vertices of G, and let G_H be the subgraph of G induced by the high vertices of G. We call G_L the *low vertex subgraph* of G and G_H the *high vertex subgraph* of G. This classification is due to Gallai [8] who proved the following theorem. Note that statements (b) and (c) of Gallai's theorem are simple consequences of statement (a), which is an extension of Brooks' theorem.

Theorem 4 (Gallai 1963)**.** *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geqslant 1$. Then the following statements hold:*

(a) *Every block of G_L is a complete graph or an odd cycle*

(b) *If $G_H = \varnothing$, then G is a complete graph of order $k+1$ if $k \neq 2$, and G is an odd cycle if $k = 2$.*

(c) *If $|G_H| = 1$, then either G has a separating vertex set of two vertices or $k = 3$ and G is an odd wheel.*

As observed by Dirac, a critical graph is connected and contains no separating vertex. Dirac [3] and Gallai [8] characterized critical graphs having a separating vertex set of size two. In particular, they proved the following theorem, which shows how to decompose a critical graph having a separating vertex set of size two into smaller critical graphs.

Theorem 5 (Dirac 1952 and Gallai 1963)**.** *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geqslant 3$, and let $S \subseteq V(G)$ be a separating vertex set of G with $|S| \leqslant 2$. Then S is an independent vertex set of G consisting of two vertices, say v and w, and $G - S$ has exactly two components H_1 and H_2. Moreover, if $G_i = G[V(H_i) \cup S]$ for $i \in \{1, 2\}$, we can adjust the notation so that for some coloring $f_1 \in \mathcal{CO}_k(G_1)$ we have $f_1(v) = f_1(w)$. Then the following statements hold:*

(a) *Every coloring $f \in \mathcal{CO}_k(G_1)$ satisfies $f(v) = f(w)$ and every coloring $f \in \mathcal{CO}_k(G_2)$ satisfies $f(v) \neq f(w)$.*

(b) *The subgraph $G_1' = G_1 + vw$ obtained from G_1 by adding the edge vw is critical and has chromatic number $k+1$.*

(c) *The vertices v and w have no common neighbor in G_2 and the subgraph $G_2' = G_2/S$ obtained from G_2 by identifying v and w is critical and has chromatic number $k+1$.*

Dirac [6] and Gallai [8] also proved the converse theorem, that G is critical and has chromatic number $k+1$ provided that G_1' is critical and has chromatic number $k+1$ and G_2 obtained from the critical graph G_2' with chromatic number $k+1$ by splitting a vertex into v and w has chromatic number k.

Hajós [9] invented his construction to characterize the class of graphs with chromatic number at least $k+1$. Another advantage of the Hajós join is the well known fact that it not only preserve the chromatic number, but also criticality. It may be viewed as a special case of the Dirac–Gallai construction, described above.

Theorem 6 (Hajós 1961)**.** *Let $G = G_1 \triangle G_2$ be the Hajós join of two graphs G_1 and G_2, and let $k \geqslant 3$ be an integer. Then G is critical and has chromatic number $k+1$ if and only if both G_1 and G_2 are critical and have chromatic number $k+1$.*

If G is the Hajós join of two graphs that are critical and have chromatic number $k+1$, where $k \geqslant 3$, then G is critical and has chromatic number $k+1$. Moreover, G has a separating set consisting of one edge and one vertex. Theorem 5 implies that the converse statement also holds.

Theorem 7. *Let G be a critical graph graph with chromatic number $k+1$ for an integer $k \geqslant 3$. If G has a separating set consisting of one edge and one vertex, then G is the Hajós join of two graphs.*

Next we will discuss a decomposition result for critical graphs having chromatic number $k+1$ an having an separating edge set of size k. Let G be an arbitrary graph. By an *edge cut* of G we mean a triple (X, Y, F) such that X is a non-empty proper subset of

$V(G)$, $Y = V(G) \setminus X$, and $F = \partial_G(X) = \partial_G(Y)$. If (X, Y, F) is an edge cut of G, then we denote by X_F (respectively Y_F) the set of vertices of X (respectively, Y) which are incident to some edge of F. An edge cut (X, Y, F) of G is non-trivial if $|X_F| \geqslant 2$ and $|Y_F| \geqslant 2$. The following decomposition result was proved independently by T. Gallai and Toft [13].

Theorem 8 (Toft 1970)**.** *Let G be a critical graph with chromatic number $k+1$ for an integer $k \geqslant 3$, and let $F \subseteq E(G)$ be a separating edge set of G with $|F| \leqslant k$. Then $|F| = k$ and there is an edge cut (X, Y, F) of G satisfying the following properties:*

(a) *Every coloring $f \in \mathcal{CO}_k(G[X])$ satisfies $|f(X_F)| = 1$ and every coloring $f \in \mathcal{CO}_k(G[Y])$ satisfies $|f(Y_F)| = k$.*

(b) *The subgraph G_1 obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of Y_F, so that Y_F becomes a clique of G_1, is critical and has chromatic number $k+1$.*

(c) *The subgraph G_2 obtained from $G[Y]$ by adding a new vertex v and joining v to all vertices of Y_F is critical and has chromatic number $k+1$.*

A particular nice proof of this result is due to T. Gallai (oral communication to the second author). Recall that the *clique number* of a graph G, denoted by $\omega(G)$, is the largest cardinality of a clique in G. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. For the proof of the next lemma, due to Gallai, we use the fact that complements of bipartite graphs are perfect.

Lemma 9. *Let H be a graph and let $k \geqslant 3$ be an integer. Suppose that (A, B, F') is an edge cut of H such that $|F'| \leqslant k$ and A as well as B are cliques of H with $|A| = |B| = k$. If $\chi(H) \geqslant k+1$, then $|F'| = k$ and $F' = \partial_H(\{v\})$ for some vertex v of H.*

Proof. The graph H is perfect and so $\omega(H) = \chi(H) \geqslant k+1$. Consequently, H contains a clique X with $|X| = k+1$. Let $s = |A \cap X|$ and hence $k+1-s = |B \cap X|$. Since $|A| = |B| = k$, this implies that $s \geqslant 1$ and $k+1-s \geqslant 1$. Since X is a clique of H, the set E' of edges of H joining a vertex of $A \cap X$ with a vertex of $B \cap X$ satisfies $E' \subseteq F'$ and $|E'| = s(k+1-s)$. The function $g(s) = s(k+1-s)$ is strictly concave on the real interval $[1, k]$ as $g''(s) = -2$. Since $g(1) = g(k) = k$, we conclude that $g(s) > k$ for all $s \in (1, k)$. Since $g(s) = |E'| \leqslant |F'| \leqslant k$, this implies that $s = 1$ or $s = k$. In both cases we obtain that $|E'| = |F'| = k$, and hence $E' = F' = \partial_H(\{v\})$ for some vertex v of H. □

Based on Lemma 9 it is easy to give a proof of Theorem 8, see also the paper by Dirac, Sørensen, and Toft [7]. Theorem 8 is a reformulation of a result by Toft [13, Chapter 4] in his Ph.D thesis. Toft gave a complete characterization of the class of critical graphs, having chromatic number $k+1$ and containing a separating edge set of size k. The characterization involves critical hypergraphs.

3 Proof of the main result

Theorem 10. *Let $k \geqslant 0$ be an integer. Then the two graph classes $\mathcal{C}_k$ and $\mathcal{H}_k$ coincide.*

Proof. That the two classes $\mathcal{C}_k$ and $\mathcal{H}_k$ coincide if $0 \leqslant k \leqslant 2$ follows from Theorem 3(b). In this case both classes consists of all critical graphs with chromatic number $k+1$. In what follows we therefore assume that $k \geqslant 3$. The proof of the following claim is straightforward and left to the reader.

Claim 1. *The odd wheels belong to the class $\mathcal{C}_3$ and the complete graphs of order $k+1$ belong to the class $\mathcal{C}_k$.*

Claim 2. *Let $k \geqslant 3$ be an integer, and let $G = G_1 \triangle G_2$ the Hajós join of two graphs G_1 and G_2. Then G belongs to the class $\mathcal{C}_k$ if and only if both G_1 and G_2 belong to the class $\mathcal{C}_k$.*

Proof: We may assume that $G = (G_1, v_1, w_1) \triangle (G_2, v_2, w_2)$ and v is the vertex of G obtained by identifying v_1 and v_2. First suppose that $G_1, G_2 \in \mathcal{C}_k$. From Theorem 6 it follows that G is critical and has chromatic number $k+1$. So it suffices to prove that $\lambda(G) \leqslant k$. To this end let u and u' be distinct vertices of G and let $p = \lambda_G(u, u')$. Then there is a system $\mathcal{P}$ of p edge disjoint u-u' paths in G. If u and u' belong both to G_1, then only one path P of $\mathcal{P}$ may contain vertices not in G_1. In this case P contains the vertex v and the edge w_1w_2. If we replace in P the subpath vPw_1 by the edge v_1w_1, we obtain a system of p edge disjoint u-u' paths in G_1, and hence $p \leqslant \lambda_{G_1}(u, u') \leqslant k$. If u and u' belong to G_2, a similar argument shows that $p \leqslant k$. It remains to consider the case that one vertex, say u, belongs to G_1 and the other vertex u' belongs to G_2. By symmetry we may assume that $u \neq v$. Again at most one path P of $\mathcal{P}$ uses the edge w_1w_2 and the remaining paths of $\mathcal{P}$ all uses the vertex $v(= v_1 = v_2)$. If we replace P by the path $uPw_1 + w_1v_1$, then we obtain p edge disjoint u-v_1 path in G_1, and hence $p \leqslant \lambda_{G_1}(u, v_1) \leqslant k$. This shows that $\lambda(G) \leqslant k$ and so $G \in \mathcal{C}_k$.

Suppose conversely that $G \in \mathcal{C}_k$. From Theorem 6 it follows that G_1 and G_2 are critical graphs, both with chromatic number $k+1$. So it suffices to show that $\lambda(G_i) \leqslant k$ for $i = 1, 2$. By symmetry it suffices to show that $\lambda(G_1) \leqslant k$. To this end let u and u' be distinct vertices of G_1 and let $p = \lambda_{G_1}(u, u')$. Then there is a system $\mathcal{P}$ of p edge disjoint u-u' paths in G_1. At most one path P of $\mathcal{P}$ can contain the edge v_1w_1. Since $k \geqslant 3$, there is a v_2-w_2 path P' in G_2 not containing the edge v_2w_2. So if we replace the edge v_1w_1 of P by the path $P' + w_2w_1$, we get p edge disjoint u-u' paths of G, and hence $p \leqslant \lambda_G(u, u') \leqslant k$. This shows that $\lambda(G_1) \leqslant k$ and by symmetry $\lambda(G_2) \leqslant k$. Hence $G_1, G_2 \in \mathcal{C}_k$. $\triangle$

As a consequence of Claim 1 and Claim 2 and the definition of the class $\mathcal{H}_k$ we obtain the following claim.

Claim 3. *Let $k \geqslant 3$ be an integer. Then the class $\mathcal{H}_k$ is a subclass of $\mathcal{C}_k$.*

Claim 4. *Let $k \geqslant 3$ be an integer, and let G be a graph belonging to the class $\mathcal{C}_k$. If G is 3-connected, then either $k = 3$ and G is an odd wheel, or $k \geqslant 4$ and G is a complete graph of order $k+1$.*

Proof: The proof is by contradiction, where we consider a counterexample G whose order $|G|$ is minimum. Then $G \in \mathcal{C}_k$ is a 3-connected graph, and either $k = 3$ and G is not an odd wheel, or $k \geqslant 4$ and G is not a complete graph of order $k+1$. First we claim that $|G_H| \geqslant 2$. If $G_H = \varnothing$, then Theorem 4(b) implies that G is a complete graph of order $k+1$, a contradiction. If $|G_H| = 1$, then Theorem 4(c) implies that $k = 3$ and G is an odd wheel, a contradiction. This proves the claim that $|G_H| \geqslant 2$. Then let u and v be distinct high vertices of G. Since $G \in \mathcal{C}_k$, Theorem 3(d) implies that $\lambda_G(u,v) = k$ and, therefore, G contains a separating edge set F of size k which separates u and v. From Theorem 8 it then follows that there is an edge cut (X, Y, F) satisfying the three properties of that theorem. Since F separates u and v, we may assume that $u \in X$ and $v \in Y$. By Theorem 8(a), $|Y_F| = k$ and hence each vertex of Y_F is incident to exactly one edge of F. Since Y contains the high vertex v, we conclude that $|Y_F| < |Y|$. Now we consider the graph G' obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of Y_F, so that Y_F becomes a clique of G'. By Theorem 8(b), G' is a critical graph with chromatic number $k+1$. Clearly, every vertex of Y_F is a low vertex of G' and every vertex of X has in G' the same degree as in G. Since X contains the high vertex u of G, this implies that $|X_F| < |X|$. Since G is 3-connected, we conclude that $|X_F| \geqslant 3$ and that G' is 3-connected.

Now we claim that $\lambda(G') \leqslant k$. To prove this, let x and y be distinct vertices of G'. If x or y is a low vertex of G', then $\lambda_{G'}(x,y) \leqslant k$ and there is nothing to prove. So assume that both x and y are high vertices of G'. Then both vertices x and y belong to X. Let $p = \lambda_{G'}(x,y)$ and let $\mathcal{P}$ be a system of p edge disjoint x-y paths in G'. We may choose $\mathcal{P}$ such that the number of edges in $\mathcal{P}$ is minimum. Let $\mathcal{P}_1$ be the paths in $\mathcal{P}$ which uses edges of F. Since $|Y_F| = k$ and each vertex of Y_F is incident with exactly one edge of F, this implies that each path P in $\mathcal{P}_1$ contains exactly two edges of F. Since $|X_F| < |X|$ and $|Y_F| < |Y|$, there are vertices $u' \in X \setminus X_F$ and $v' \in Y \setminus Y_F$. By Theorem 3(d) it follows that $\lambda_G(u',v') = k$ and, therefore, there are k edge disjoint u'-v' paths in G. Since $|Y_F| = k$, for each vertex $z \in Y_F$, there is a v'-z path P_z in $G[Y]$ such that these paths are edge disjoint. Now let P be an arbitrary path in $\mathcal{P}_1$. Then P contains exactly two vertices of Y_F, say z and z', and we can replace the edge zz' of the path P by a z-z' path contained in $P_z \cup P_{z'}$. In this way we obtain a system of p edge disjoint x-y paths in G, which implies that $p \leqslant \lambda_G(x,y) \leqslant k$. This proves the claim that $\lambda(G') \leqslant k$. Consequently $G' \in \mathcal{C}_k$. Clearly, $|G'| < |G|$ and either $k = 3$ and G' is not an odd wheel, or $k \geqslant 4$ and G is not a complete graph of order $k+1$. This, however, is a contradiction to the choice of G. Thus the claim is proved. △

Claim 5. *Let $k \geqslant 3$ be an integer, and let G be a graph belonging to the class $\mathcal{C}_k$. If G has a separating vertex set of size 2, then $G = G_1 \triangle G_2$ is the Hajós sum of two graphs G_1 and G_2, which both belong to $\mathcal{C}_k$.*

Proof: If G has a separating set consisting of one edge and one vertex, then Theorem 7 implies that G is the Hajoś join of two graphs G_1 and G_2. By Claim 2 it then follows that both G_1 and G_2 belong to $\mathcal{C}_k$ and we are done. It remains to consider the case that G does not contain a separating set consisting of one edge and one vertex. By assumption, there is a separating vertex set of size 2, say $S = \{u, v\}$. Then Theorem 5 implies that $G - S$ has exactly two components H_1 and H_2 such that the graphs $G_i = G[V(H_i) \cup S]$ with $i \in \{1, 2\}$ satisfies the three properties of that theorem. In particular, we have that $G_1' = G_1 + uv$ is critical and has chromatic number $k+1$. By Theorem 3(d), it then follows that $\lambda_{G_1'}(u, v) \geqslant k$ implying that $\lambda_{G_1}(u, v) \geqslant k - 1$. Since $G \in \mathcal{C}_k$, we then conclude that $\lambda_{G_2}(u, v) \leqslant 1$. Since G_2 is connected, this implies that G_2 has a bridge e. Since $k \geqslant 3$, we conclude that $\{u, e\}$ or $\{v, e\}$ is a separating set of G, a contradiction. △

As a consequence of Claim 4 and Claim 5, we conclude that the class $\mathcal{C}_k$ is a subclass of the class $\mathcal{H}_k$. Together with Claim 3 this yields $\mathcal{H}_k = \mathcal{C}_k$ as wanted. □

Proof of of Theorem 2: For the proof of this theorem let G be a non-empty graph with $\lambda(G) = k$. By inequality (1) we obtain that $\chi(G) \leqslant k + 1$. If one block H of G belongs to $\mathcal{H}_k$, then $H \in \mathcal{C}_k$ (by Theorem 10) and hence $\chi(G) = k + 1$ (by (2)).

Assume conversely that $\chi(G) = k+1$. Then G contains a subgraph H which is critical and has chromatic number $k + 1$. Clearly, $\lambda(H) \leqslant \lambda(G) \leqslant k$, and, therefore, $H \in \mathcal{C}_k$. By Theorem 3(c), H contains no separating vertex. We claim that H is a block of G. For otherwise, H would be a proper subgraph of a block G' of G. This implies that there are distinct vertices u and v in H which are joined by a path P of G with $E(P) \cap E(H) = \varnothing$. Since $\lambda_H(u, v) \geqslant k$ (by Theorem 3(c)), this implies that $\lambda_G(u, v) \geqslant k + 1$, which is impossible. This proves the claim that H is a block of G. By Theorem 10, $\mathcal{C}_k = \mathcal{H}_k$ implying that $H \in \mathcal{H}_k$. This completes the proof of the theorem □

The case $\lambda = 3$ of Theorem 2 was obtained earlier by Aboulker, Brettell, Havet, Marx, and Trotignon [1]; their proof is similar to our proof. Let $\mathcal{L}_k$ denote the class of graphs G satisfying $\lambda(G) \leqslant k$. It is well known that membership in $\mathcal{L}_k$ can be tested in polynomial time. It is also easy to show that there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, decides whether G or one of its blocks belong to $\mathcal{H}_k$. So it can be tested in polynomial time whether a graph $G \in \mathcal{L}_k$ satisfies $\chi(G) \leqslant k$. Moreover, the proof of Theorem 2 yields a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring of $\mathcal{CO}_k(G)$ when such a coloring exists. This result provides a positive answer to a conjecture made by Aboulker *et al.* [1, Conjecture 1.8]. The case $k = 3$ was solved by Aboulker *et al.* [1].

Theorem 11. *For fixed $k \geqslant 1$, there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring in $\mathcal{CO}_k(G)$ or a block of G belonging to $\mathcal{H}_k$.*

Sketch of Proof: The Theorem is evident if $k \in \{1, 2\}$; and the case $k = 3$ was solved by Alboulker *et al.* [1]. Hence we assume that $k \geqslant 4$ and $G \in \mathcal{L}_k$. If we find for each block H of G a coloring in $\mathcal{CO}_k(H)$, we can piece these colorings together by permuting colors to obtain a coloring in $\mathcal{CO}_k(G)$. Hence we may assume that G is a block. Since $\lambda(G) \leqslant k$ and $\lambda(H) = k$ for every graph $H \in \mathcal{H}_k$, it then follows that no proper subgraph of G belongs to $\mathcal{H}_k$.

First, we check whether G has a separating set S consisting of one vertex and one edge. If we find such a set, say $S = \{v, e\}$ with $v \in V(G)$ and $e \in E(G)$, then $G - e$ is the union of two connected graphs G_1 and G_2 having only vertex v in common where $e = w_1w_2$ and

$w_i \in V(G_i)$ for $i = 1, 2$. Both blocks $G'_1 = G_1 + vw_1$ and $G'_2 = G_2 + vw_2$ belong to $\mathcal{L}_k$. Now we check whether these blocks belong to $\mathcal{H}_k$. If both blocks G'_1 and G'_2 belong to $\mathcal{H}_k$, then $vw_i \notin E(G_i)$ for $i = 1, 2$, and hence G belongs to $\mathcal{H}_k$ and we are done. If one of the blocks, say G'_1 does not belong to $\mathcal{H}_k$, we can construct a coloring $f_1 \in \mathcal{CO}_k(G'_1)$. Since no block of G_2 belongs to $\mathcal{H}_k$, we can construct a coloring $f_2 \in \mathcal{CO}_k(G_2)$. Then $f_1 \in \mathcal{CO}_k(G_1)$ and $f_1(v) \neq f_1(w_1)$. Since $k \geqslant 4$, we can permute colors in f_2 such that $f_1(v) = f_2(v)$ and $f_1(w_1) \neq f_2(w_2)$. Consequently, $f = f_1 \cup f_2$ belongs to $\mathcal{CO}_k(G)$ and we are done.

It remains to consider the case that G contains no separating set consisting of one vertex and one edge. Then let p denote the number of vertices of G whose degree is greater that k. If $p \leqslant 1$, then let v be a vertex of maximum degree in G. Color v with color 1 and let L be a list assignment for $H = G - v$ satisfying $L(u) = \{2, 3, \ldots, k\}$ if $vu \in E(G)$ and $L(u) = \{1, 2, \ldots, k\}$ otherwise. Then H is connected and $|L(u)| \geqslant d_H(u)$ for all $u \in V(H)$. Now we can use the degree version of Brooks' theorem, see [12, Theorem 2.1]. Either we find a coloring f of H such that $f(u) \in L(u)$ for all $u \in V(H)$, yielding a coloring of $\mathcal{CO}_k(G)$, or $|L(u)| = d_H(u)$ for all $u \in V(H)$ and each block of H is a complete graph or an odd cycle. In this case, $d_H(u) \in \{k, k-1\}$ for all $u \in V(H)$ and, since $k \geqslant 4$, each block of H is a K_k or a K_2. Since G contains no separating set consisting of one vertex and one edge, this implies that $H = K_k$ and so $G = K_{k+1} \in \mathcal{H}_k$ and we are done.

If $p \geqslant 2$, then we choose two vertices u and u' whose degrees are greater that k. Then we construct an edge cut (X, Y, F) with $u \in X$, $u' \in Y$, and $|F| = \lambda_G(u, u')$. We may assume that $a = |X_F|$ and $b = |Y_F|$ satisfies $a \leqslant b \leqslant k$.

If $b \leqslant k - 1$, then both graphs $G[X]$ and $G[Y]$ belong to $\mathcal{L}_k$ and there are colorings $f_X \in \mathcal{CO}_k(G[X])$ and $f_Y \in \mathcal{CO}_k(G[Y])$. Note that no block of these two graphs can belong to $\mathcal{H}_k$. By permuting colors in f_Y, we can combine the two colorings f_X and f_Y to obtain a coloring $f \in \mathcal{CO}_k(G)$. To see this, we apply Lemma 9 to the auxiliary graph $H = H(f_X, f_Y)$ obtained from two disjoint complete graphs of order k, one with vertex set $A = \{a_1, a_2, \ldots, a_k\}$ and the other one with vertex set $B = \{b_1, b_2, \ldots, b_k\}$, by adding all edges of the form $a_i b_j$ for which there exists an edge $e = vv' \in F$ such that $f_X(v) = i$ and $f_Y(v') = j$. By the assumption on the edge cut (X, Y, F) it follows from Lemma 9 that $\chi(H) \leqslant k$, which leads to to the desired coloring f.

If $a < b = k$, then we consider the graph G_1 obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of Y_F, so that Y_F becomes a clique of G_1. Then G_1 belongs to $\mathcal{L}_k$ (see the proof of Claim 4) and, since G contains no separating set consisting of one vertex and one edge, the block G_1 does not belongs to $\mathcal{H}_k$. Hence there are colorings $f_1 \in \mathcal{CO}_k(G_1)$ and $f_Y \in \mathcal{CO}_k(G[Y])$. Then the restriction of f_1 to X yields a coloring $f_X \in \mathcal{CO}_k(G[X])$ such that $|f_X(X_F)| \geqslant 2$. By permuting colors in f_Y, we can combine the two colorings f_X and f_Y to obtain a coloring $f \in \mathcal{CO}_k(G)$ (by applying Lemma 9 to the auxiliary graph $H = H(f_X, f_Y)$ as in the former case).

It remains to consider the case $a = b = k$. Then let G_2 be the graph obtained from $G[Y \cup X_F]$ by adding all edges between the vertices of X_F, so that X_F becomes a clique of G_2. Then we find colorings $f_1 \in \mathcal{CO}_k(G_1)$ and $f_2 \in \mathcal{CO}_k(G_2)$ and, hence, colorings $f_X \in \mathcal{CO}_k(G[X])$ and $f_Y \in \mathcal{CO}_k(G[Y])$ such that $|f_X(X_F)| \geqslant 2$ and $|f_Y(Y_F)| \geqslant 2$. By permuting colors in f_Y, we can combine the two colorings f_X and f_Y to obtain a coloring $f \in \mathcal{CO}_k(G)$ (by using Lemma 9). $\square$

References

[1] P. Aboulker, N. Brettell, F. Havet, D. Marx, and N. Trotignon. Colouring graphs with constraints on connectivity. *J. Graph Theory*, 85(4):814–838, 2017.

[2] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[3] G. A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1951.

[4] G. A. Dirac. The structure of k-chromatic graphs. *Fund. Math.*, 40:42–55, 1953.

[5] G. A. Dirac. A theorem of R. L. Brooks and a conjecture of H. Hadwiger. *Proc. London Math. Soc.*, 7(3):161–195, 1957.

[6] G. A. Dirac. On the structure of 5- and 6-chromatic graphs. *J. Reine Angew. Math.*, 214/215:43–52, 1974.

[7] G. A. Dirac, B. A. Sørensen, and B. Toft. An extremal result for graphs with an application to their colorings. *J. Reine Angew. Math.*, 268/269:216–221, 1974.

[8] T. Gallai. Kritische Graphen I. *Publ. Math. Inst. Hungar. Acad. Sci.*, 8:165–192, 1963.

[9] G. Hajós. Über eine Konstruktion nicht n-färbbarer Graphen. *Wiss. Z. Martin Luther Univ. Halle-Wittenberg, Math.-Natur. Reihe*, 10:116–117, 1961.

[10] T. Jensen and B. Toft. Choosability versus chromaticity. *Geombinatorics*, 5:45–64, 1995.

[11] W. Mader. Grad und lokaler Zusammenhang in endlichen Graphen. *Math. Ann.*, 205:9–11, 1973.

[12] M. Stiebitz and B. Toft. Brooks's theorem. In L. W. Beineke and R. Wilson, eds., *Topics in Chromatic Graph Theory* volume 165 of *Encyyclopedia of Mathematics and Its Application*, pages 36–55, Cambridge Press, 2015.

[13] B. Toft. Some contribution to the theory of colour-critical graphs. Ph.D thesis, University of London 1970. Published as No. 14 in Various Publication Series, Matematisk Institut, Aarhus Universitet 1970.

[14] B. Toft. Colour-critical graphs and hypergraphs. *J. Combin. Theory (B)*, 16:145–161, 1974.

[15] B. Toft. Critical subgraphs of colour critical graphs. *Discrete Math.*, 7:377–392, 1974.