On bipartite Q-polynomial distance-regular graphs with diameter 9, 10, or 11

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Abstract

Let Γ denote a bipartite distance-regular graph with diameter D. In [J. S. Caughman, Bipartite Q-polynomial distance-regular graphs, *Graphs Combin.* **20** (2004), 47–57], Caughman showed that if $D \geqslant 12$, then Γ is Q-polynomial if and only if one of the following (i)-(iv) holds: (i) Γ is the ordinary $2D$-cycle, (ii) Γ is the Hamming cube $H(D, 2)$, (iii) Γ is the antipodal quotient of the Hamming cube $H(2D, 2)$, (iv) the intersection numbers of Γ satisfy $c_i = (q^i - 1)/(q - 1)$, $b_i = (q^D - q^i)/(q - 1)$ $(0 \leqslant i \leqslant D)$, where q is an integer at least 2. In this paper we show that the above result is true also for bipartite distance-regular graphs with $D \in \{9, 10, 11\}$.

Keywords: bipartite distance-regular graph; Q-polynomial property

1 Introduction

As a classification of all distance-regular graphs is currently beyond our reach, classifications of some subclasses of distance-regular graphs are also very important projects. One such subclass is the class of Q-polynomial bipartite distance-regular graphs. This paper is part of an effort to understand and classify Q-polynomial bipartite distance-regular graphs (see [3, 4, 5, 6, 7] for relevant literature). A crucial step towards a classification of this class was made by Caughman, who proved the following result.

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Theorem 1.1 *([7, Theorem 1.1]) Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 12$. Then Γ is Q-polynomial if and only if one of the following (i)-(iv) holds:*

(i) Γ *is the ordinary $2D$-cycle.*

(ii) Γ *is the Hamming cube $H(D, 2)$.*

(iii) Γ *is the antipodal quotient of the Hamming cube $H(2D, 2)$.*

(iv) *The intersection numbers of* Γ *satisfy*

$$c_i = \frac{q^i - 1}{q - 1}, \qquad b_i = \frac{q^D - q^i}{q - 1} \qquad (0 \leqslant i \leqslant D),$$

where q is an integer at least 2.

In this paper we prove an analogue of Theorem 1.1 for bipartite distance-regular graphs with diameter $D \in \{9, 10, 11\}$. We follow the ideas of Caughman [7] and use the Terwilliger algebra of Γ to prove our result. Generalization of Theorem 1.1 to bipartite distance-regular graphs with diameter less than 12 is also mentioned as an open problem in the recent survey paper *Distance-regular graphs* by van Dam, Koolen and Tanaka, see [10, Section 18.3].

The paper is organized as follows. In Sections 2, 3, 4 we review some basic definitions and results about distance-regular graphs, the Q-polynomial property of distance-regular graphs, and the Terwilliger algebra of distance-regular graphs. In Section 5 we review and prove some results concerning multiplicities of irreducible modules of the Terwilliger algebra. In Sections 6 and 7 we prove our main result.

2 Preliminaries

In this section we review some definitions and basic concepts. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Throughout this paper, $\Gamma = (X, R)$ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path length distance function ∂, and diameter $D := \max\{\partial(x, y) | x, y \in X\}$. For a vertex $x \in X$ define $\Gamma_i(x)$ to be the set of vertices at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. Let k denote a nonnegative integer. Then Γ is said to be *regular* with *valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant D)$, and all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\{z \mid z \in X,\ \partial(x, z) = i,\ \partial(y, z) = j\}| \tag{1}$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1i-1}$ for $1 \leqslant i \leqslant D$, $a_i := p^i_{1i}$ for $0 \leqslant i \leqslant D$, $b_i := p^i_{1i+1}$ for $0 \leqslant i \leqslant D-1$, $k_i := p^0_{ii}$ for $0 \leqslant i \leqslant D$, and $c_0 = b_D = 0$. We observe $a_0 = 0$, $c_1 = 1$.

Moreover, Γ is regular with valency $k = b_0$, and $c_i + a_i + b_i = k$ for $0 \leqslant i \leqslant D$. It is well-known that

$$k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i} \qquad (0 \leqslant i \leqslant D). \tag{2}$$

Observe that Γ is bipartite if and only if $a_i = 0$ for $0 \leqslant i \leqslant D$. In this case $b_i + c_i = k$ for $0 \leqslant i \leqslant D$.

From now on we assume Γ is distance-regular with diameter $D \geqslant 3$ and valency $k \geqslant 3$. We recall the Bose-Mesner algebra of Γ. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices over $\mathbb{C}$ which have rows and columns indexed by X. For $0 \leqslant i \leqslant D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X). \tag{3}$$

We call A_i the i-th *distance matrix* of Γ. We abbreviate $A = A_1$ and call A the *adjacency matrix* of Γ. The matrices $A_0, A_1, \ldots, A_D$ form a basis for a commutative semi-simple $\mathbb{C}$-algebra M, known as the *Bose-Mesner algebra*, see for example [11, Lemma 11.2.2]. The algebra M has a second basis $E_0, E_1, \ldots, E_D$ such that $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant D)$, see [2, Theorem 2.6.1]. The $E_0, E_1, \ldots, E_D$ are known as the *primitive idempotents* of Γ, and E_0 is the *trivial* idempotent.

For $0 \leqslant i \leqslant D$ define a real number θ_i by $A = \sum_{i=0}^{D} \theta_i E_i$. Then $AE_i = E_i A = \theta_i E_i$ for $0 \leqslant i \leqslant D$. The scalars $\theta_0, \theta_1, \ldots, \theta_D$ are distinct, since A generates M [1, p. 197]. The $\theta_0, \theta_1, \ldots, \theta_D$ are known as the *eigenvalues* of Γ. We remark $k \geqslant \theta_i \geqslant -k$ for $0 \leqslant i \leqslant D$, and $\theta_0 = k$ [2, p. 45].

Let θ denote an eigenvalue of Γ, and let E denote the associated primitive idempotent. For $0 \leqslant i \leqslant D$ define a real number θ_i^* by $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$. We call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ the *dual eigenvalue sequence* associated with θ, E.

3 The Q-polynomial property

In this section we recall the Q-polynomial property of distance-regular graphs. Let Γ denote a distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$, and let $A_0, A_1, \ldots, A_D$ denote the distance matrices of Γ. Observe that $A_i \circ A_j = \delta_{ij} A_i$ $(0 \leqslant i, j \leqslant D)$, where $\circ$ denotes entrywise multiplication, and so algebra M is closed under $\circ$. Let $E_0, E_1, \ldots, E_D$ denote the primitive idempotents of Γ. The *Krein parameters* q_{ij}^h $(0 \leqslant h, i, j \leqslant D)$ of Γ are defined by

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leqslant i, j \leqslant D). \tag{4}$$

We say Γ is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents), whenever for all distinct integers i, j $(0 \leqslant i, j \leqslant D)$ the following holds: $q^1_{ij} \neq 0$ if and only if $|i-j| = 1$. Let E denote a nontrivial primitive idempotent of Γ. We say Γ is *Q-polynomial with respect to E* whenever there exists an ordering $E_0, E_1 = E, \ldots, E_D$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial.

We have the following important result about bipartite Q-polynomial distance-regular graphs, see [7, Lemma 3.2, Lemma 3.3].

Lemma 3.1 *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$, and intersection numbers b_i, c_i $(0 \leqslant i \leqslant D)$. We assume Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$. For $0 \leqslant i \leqslant D$ let θ_i denote the eigenvalue associated with E_i. Let $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ denote the dual eigenvalue sequence associated with E_1. Assume Γ is not the D-cube or the antipodal quotient of the $2D$-cube. Then there exist scalars $q, s^* \in \mathbb{R}$ such that* (i)–(iii) *hold below.*

(i) $|q| > 1, \;\; s^*q^i \neq 1 \qquad (2 \leqslant i \leqslant 2D+1)$;

(ii) $\theta_i = h(q^{D-i} - q^i), \;\; \theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$ *for* $0 \leqslant i \leqslant D$, *where*

$$h = \frac{1-s^*q^3}{(q-1)(1-s^*q^{D+2})}, \quad h^* = \frac{(q^D+q^2)(q^D+q)}{q(q^2-1)(1-s^*q^{2D})}, \quad \theta^*_0 = \frac{h^*(q^D-1)(1-s^*q^2)}{q(q^{D-1}+1)};$$

(iii) $k = c_D = h(q^D-1)$, *and*

$$c_i = \frac{h(q^i-1)(1-s^*q^{D+i+1})}{1-s^*q^{2i+1}}, \qquad b_i = \frac{h(q^D-q^i)(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \qquad (1 \leqslant i \leqslant D-1).$$

4 The Terwilliger algebra

In this section we recall the Terwilliger algebra of a distance-regular graph. Let Γ denote a distance-regular graph with diameter $D \geqslant 3$, valency $k \geqslant 3$, and distance matrices $A_0, A_1, \ldots, A_D$. Fix any vertex $x \in X$. For $0 \leqslant i \leqslant D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with y, y entry $(A_i)_{xy}$ $(y \in X)$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A and $E^*_0, \ldots, E^*_d$. We call T the *Terwilliger algebra* of Γ with respect to x. We remark that T is finite dimensional and semisimple.

Let V denote the $\mathbb{C}$-vector space consisting of the column vectors over $\mathbb{C}$ which have rows indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We refer to V as the *standard module of* T. By a *T-module* we mean a subspace W of the standard module V such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than zero and W.

Let W and W' denote T-modules. By a *T-isomorphism* from W to W', we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The modules W and W' are said to be *isomorphic* whenever there exists a T-isomorphism from W to W'.

Let W denote a T-module and let W' denote a T-module contained in W. Then the orthogonal complement of W' in W is a T-module. From this we find W is an orthogonal direct sum of irreducible T-modules. Taking $W = V$ we find V is an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity* with which W appears in V, we mean the number of irreducible T-modules in this sum which are isomorphic to W. It is known that the multiplicity of W is independent of the decomposition of V.

Let W denote an irreducible T-module. We define the *endpoint* r and the *diameter* d of W by $r = \min\{i \mid 0 \leqslant i \leqslant D,\ E_i^* W \neq 0\}$ and $d = |\{i \mid 0 \leqslant i \leqslant D,\ E_i^* W \neq 0\}| - 1$. Similarly, the *dual endpoint* t and *dual diameter* $d^*$of W are defined by $t = \min\{i \mid 0 \leqslant i \leqslant D,\ E_i W \neq 0\}$ and $d^* := |\{i \mid 0 \leqslant i \leqslant D,\ E_i W \neq 0\}| - 1$. We say W is *thin*, whenever $\dim(E_i^* W) \leqslant 1$ for every $0 \leqslant i \leqslant D$.

Assume now that our distance-regular graph Γ is Q-polynomial. Let W denote an irreducible T-module with endpoint r, dual endpoint t, diameter d and dual diameter d^*. Then W is thin by [4, Theorem 9.3]. We comment on r, t, d and d^*. By [4, Lemma 5.1(ii)] we have $2r + d^* \geqslant D$, and by [4, Lemma 9.2(ii)] we have that $d = d^*$. It follows that $(D-d)/2 \leqslant r$. It is also clear that $r + d \leqslant D$, and so we have that

$$\frac{D-d}{2} \leqslant r \leqslant D - d.$$

We have $2t + d = D$ by [4, Theorem 9.4(ii)], and so $D - d$ is even. By [4, Theorem 13.1], the isomorphism class of W is determined by r and d. For the rest of the paper we will consider the following situation.

Notation 4.1 *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geqslant 4$, valency $k \geqslant 3$, intersection numbers b_i, c_i, and distance matrices A_i $(0 \leqslant i \leqslant D)$. We fix $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$ and $T = T(x)$ denote the dual idempotents and the Terwilliger algebra of Γ with respect to x, respectively. Let V denote the standard module for Γ. Let $H(D,2)$ denote the D-dimensional hypercube, and let $\overline{H}(2D,2)$ denote the antipodal quotient of $H(2D,2)$.*

5 Multiplicities of the irreducible T-modules

With reference to Notation 4.1, assume that Γ is Q-polynomial. In this section we review and prove some results about the multiplicities of irreducible T-modules (see also [4, Section 14]).

Fix a decomposition of the standard module V into an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. Recall that the multiplicity of W equals the number of irreducible T-modules in this sum which are isomorphic to W. As the isomorphism class of W is determined by its endpoint and diameter, we introduce the following notation. For any integers r, d $(0 \leqslant r, d \leqslant D)$, we define $\mathrm{mult}(r, d)$ to be the number of irreducible T-modules in this decomposition which have endpoint r and diameter d. If no such modules exist, then we set $\mathrm{mult}(r, d) = 0$. Note that if W has endpoint r and diameter d, then the multiplicity of W equals $\mathrm{mult}(r, d)$.

Definition 5.1 *([4, Definition 14.2]) With reference to Notation 4.1, assume that* Γ *is* Q*-polynomial. Define a set* Υ *by*

$$\Upsilon := \{(r,d) \in \mathbb{Z}^2 \mid 0 \leqslant d \leqslant D, D-d \text{ even }, \frac{D-d}{2} \leqslant r \leqslant D-d\}.$$

Observe that $\operatorname{mult}(r,d) = 0$ for all integers r,d such that $(r,d) \notin \Upsilon$. We define a partial order $\preccurlyeq$ on Υ by

$$(r',d') \preccurlyeq (r,d) \quad \text{if and only if} \quad r' \leqslant r \text{ and } r+d \leqslant r'+d'.$$

To further describe $\operatorname{mult}(r,d)$, we need a definition. Let $(r,d) \in \Upsilon$. Following [4, pp. 87-88] we define scalars $c_i(r,d)$ $(1 \leqslant i \leqslant d)$ and $b_i(r,d)$ $(0 \leqslant i \leqslant d-1)$ by

$$c_i(r,d) = \frac{\theta_t(\theta^*_{r+i+1} - \theta^*_{r+1}) - \theta_{t+1}(\theta^*_{r+i} - \theta^*_r)}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \qquad (1 \leqslant i \leqslant d-1), \tag{5}$$

$$b_i(r,d) = \frac{\theta_t(\theta^*_{r+1} - \theta^*_{r+i-1}) + \theta_{t+1}(\theta^*_{r+i} - \theta^*_r)}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \qquad (1 \leqslant i \leqslant d-1), \tag{6}$$

where $t = (D-d)/2$. We also set $b_0(r,d) = c_d(r,d) = \theta_t$, $b_d(r,d) = c_0(r,d) = 0$. Assume now that Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. Then using Lemma 3.1(ii) we get that

$$c_i(r,d) = \frac{h(q^i-1)(1-s^*q^{2r+d+i+1})}{q^{d+t}(1-s^*q^{2r+2i+1})} \qquad (1 \leqslant i \leqslant d-1), \tag{7}$$

$$b_i(r,d) = \frac{h(q^d-q^i)(1-s^*q^{2r+i+1})}{q^{d+t}(1-s^*q^{2r+2i+1})} \qquad (1 \leqslant i \leqslant d-1), \tag{8}$$

and $b_0(r,d) = c_d(r,d) = h(q^{-t} - q^{t-D})$, $b_d(r,d) = c_0(r,d) = 0$.

Theorem 5.2 *([4, Theorem 14.7]) With reference to Definition 5.1, fix any* $(r,d) \in \Upsilon$. *Then*

$$k_r \prod_{i=r}^{r+d-1} b_i c_{r+d-i} = \sum_{\substack{(r',d') \in \Upsilon \\ (r',d') \preccurlyeq (r,d)}} \operatorname{mult}(r',d') \prod_{i=r-r'}^{r+d-r'-1} b_i(r',d') c_{i+1}(r',d').$$

Corollary 5.3 *With reference to Definition 5.1, fix any* $(r,d) \in \Upsilon$ *such that* $r+d = D$ $(1 \leqslant r \leqslant D-1)$. *Then* r *is even and*

$$k_r \prod_{i=r}^{D-1} b_i c_{D-i} = \sum_{\ell=0}^{r/2} \operatorname{mult}(2\ell, D-2\ell) \prod_{i=r-2\ell}^{D-2\ell-1} b_i(2\ell, D-2\ell) c_{i+1}(2\ell, D-2\ell).$$

Proof. Recall that $r = D-d$ is even by definition of the set Υ. As $r+d = D$, it follows from the definition of $\preccurlyeq$ that the only pairs $(r',d') \in \Upsilon$ such that $(r',d') \preccurlyeq (r,d)$ are pairs with $r' \leqslant r$ and $r'+d' = D$. Note that $r'+d' = D$ implies r' is even, and so the result follows from Theorem 5.2. ∎

Lemma 5.4 *With reference to Definition 5.1, fix any $(r, d) \in \Upsilon$ such that $r+d = D$ ($1 \leqslant r \leqslant D-1$). Assume that Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. If there exists an irreducible T-module with endpoint r and diameter d, then*

$$\prod_{i=0}^{D-r-1} b_i(r, D-r) c_{i+1}(r, D-r) \neq 0.$$

Proof. Let q, s^* be as in Lemma 3.1. Recall that $|q| > 1$ and $s^* q^i \neq 1$ for $2 \leqslant i \leqslant 2D+1$. Using this, equations (7), (8) and $b_0(r,d) = c_d(r,d) = h(q^{-t} - q^{t-D})$, we find that $b_i(r, D-r) \neq 0$ and $c_{i+1}(r, D-r) \neq 0$ for $0 \leqslant i \leqslant D-r-1$. The result follows. ∎

Corollary 5.5 *With reference to Definition 5.1, fix any $(r,d) \in \Upsilon$ such that $r+d = D$ ($1 \leqslant r \leqslant D-1$). Assume that Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. Then* $\mathrm{mult}(r,d)$ *is equal to the quantity*

$$k_r \prod_{i=r}^{D-1} b_i c_{D-i} - \sum_{\ell=0}^{r/2-1} \mathrm{mult}(2\ell, D-2\ell) \prod_{i=r-2\ell}^{D-2\ell-1} b_i(2\ell, D-2\ell) c_{i+1}(2\ell, D-2\ell),$$

divided by the quantity

$$\prod_{i=0}^{D-r-1} b_i(r, D-r) c_{i+1}(r, D-r).$$

Proof. Immediately from Corollary 5.3 and Lemma 5.4. ∎

We now give explicit formulae for *mult*(r,d) for some specific values of r, d. To do this we need the following definition. For $a, b \in \mathbb{R}$ and for a non-negative integer n we set

$$(a; b)_n = \prod_{i=1}^{n} (1 - ab^{i-1}).$$

Theorem 5.6 *With reference to Definition 5.1 assume Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters q, s^* be as in Lemma 3.1. Then the following* (i)–(iii) *hold.*

(i) $\mathrm{mult}(0, D) = 1$.

(ii) *If $r \in \{2, 4\}$, then*

$$\mathrm{mult}(r, D-r) = \frac{(-1)^t q^{t(t-1)} (1 - s^* q^{2r}) (q^{d+1}; q)_r (-s^* q^{D+1}; q)_t (s^* q^2; q^2)_{t-1}}{(q^2; q^2)_t (s^* q^{D+t+1}; q)_t (s^* q^{D+d+2}; q^2)_t};$$

where $d = D - r$ and $t = (D-d)/2 = r/2$.

(iii) *If $r \in \{6, 8\}$ and $D \in \{9, 10, 11\}$, then*

$$\mathrm{mult}(r, D-r) = \frac{(-1)^t q^{t(t-1)} (1 - s^* q^{2r}) (q^{d+1}; q)_r (-s^* q^{D+1}; q)_t (s^* q^2; q^2)_{t-1}}{(q^2; q^2)_t (s^* q^{D+t+1}; q)_t (s^* q^{D+d+2}; q^2)_t};$$

where $d = D - r$ and $t = (D-d)/2 = r/2$.

Proof. (i), (ii) This is [4, Theorem 15.6 (i),(iii),(vii)].
(iii) The proof (although a bit tedious and lengthy) follows straightforward from Corollary 5.5 using (7), (8). We omit the details. ∎

Remark 5.7 *With reference to Definition 5.1 assume* Γ *is not* $H(D,2)$ *or* $\overline{H}(2D,2)$*. We conjecture that the formula for* $\mathrm{mult}(r, D-r)$ *given in Theorem 5.6 holds for any diameter* D *and for any even number* $r \leqslant D$*. See also [4, Conjecture 15.8] for an extended conjecture about the multiplicities of irreducible* T*-modules of* Γ*. However, for the purpose of this paper, Theorem 5.6 suffices.*

6 Some results about parameter s^*

With reference to Notation 4.1 assume Γ is Q-polynomial, and let parameters q, s^* be as in Lemma 3.1. In this section we derive some restrictions on parameter s^*. We first recall some results of Caughman.

Theorem 6.1 *([7, Theorem 4.1, Lemma 5.1, Lemma 6.6]) With reference to Notation 4.1 assume* Γ *is* Q*-polynomial and assume that* Γ *is not* $H(D,2)$ *or* $\overline{H}(2D,2)$*. Let parameters* q, s^* *be as in Lemma 3.1. Then the following* (i)–(iii) *hold.*

(i) *If* $D \geqslant 6$*, then* $q > 1$.

(ii) *If* $q > 1$*, then* $-q^{-D-1} \leqslant s^* < q^{-2D-1}$.

(iii) *If* $D \geqslant 7$ *and* $-q^{-13} \leqslant s^* \leqslant q^{-13}$*, then* $s^* = 0$.

Lemma 6.2 *With reference to Notation 4.1 assume* Γ *is* Q*-polynomial and assume that* Γ *is not* $H(D,2)$ *or* $\overline{H}(2D,2)$*. Let parameters* q, s^* *be as in Lemma 3.1. Set* $\beta = q + 1/q$*. If* $D \geqslant 5$*, then* β *is a rational number.*

Proof. Assume that Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$ and let θ_1 denote the eigenvalue of Γ corresponding to E_1. By [6, Lemma 3.2] we have that $\theta_1 \neq -1$ and

$$\beta = \frac{\theta_1^2 + c_2\theta_1 + b_2(k-2)}{b_2(\theta_1 + 1)}.$$

If $D \geqslant 5$ then θ_1 is integer by [9, Theorem 8.1.3], see also [6, Lemma 3.3(i)]. The result follows. ∎

We note that it is easy to see that $q^2 + q^{-2} = \beta^2 - 2$, $q^3 + q^{-3} = \beta^3 - 3\beta$ and $q^4 + q^{-4} = \beta^4 - 4\beta^2 + 2$. Also, if $D \geqslant 6$, then $q > 1$ by Theorem 6.1(i), and so $\beta > 2$ in this case.

The following result was proved by Lang.

Proposition 6.3 *([12, Lemma 9.3]) With reference to Notation 4.1 assume* Γ *is* Q*-polynomial and assume that* Γ *is not* $H(D,2)$ *or* $\overline{H}(2D,2)$*. Let parameters* q, s^* *be as in Lemma 3.1. If* $D \geqslant 5$*, then* $s^* \neq -q^{-D-3}$.

Proposition 6.4 *With reference to Notation 4.1 assume Γ is Q-polynomial and assume that Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters q, s^* be as in Lemma 3.1. If $D \geqslant 9$, then $s^* \neq -q^{-D-2}$.*

Proof. Assume on contrary that $D \geqslant 9$ and $s^* = -q^{-D-2}$. Observe that this implies $s^* < 0$ as $q > 1$ by Theorem 6.1(i). If $D \geqslant 11$, then $s^* = -q^{-D-2}$ implies $-q^{-13} \leqslant s^* \leqslant q^{-13}$. But then $s^* = 0$ by Theorem 6.1(iii), a contradiction.

If $D \in \{9, 10\}$ then the proof is similar to the proof of Proposition 6.3 for the case $D = 9$. Let β be as in Lemma 6.2. Assume first that $D = 10$. In this case we have

$$2c_2 = \beta^2 + 2\beta - 1 - \frac{2\beta^2 - \beta - 3}{\beta^3 - \beta^2 - 2\beta + 1},$$

which shows that β is an algebraic integer. As β is rational by Lemma 6.2, this implies that β is an integer, and so $(2\beta^2 - \beta - 3)/(\beta^3 - \beta^2 - 2\beta + 1)$ is an integer. Observe that $2\beta^2 - \beta - 3$ and $\beta^3 - \beta^2 - 2\beta + 1$ are both positive for $\beta \geqslant 2$, so $2\beta^2 - \beta - 3 \geqslant \beta^3 - \beta^2 - 2\beta + 1$. But this implies $\beta = 2$, a contradiction (recall that $\beta > 2$).

Assume now $D = 9$. In this case we have

$$2c_2 = \beta^2 + 2\beta - 1 - \frac{2\beta^2 + \beta - 4}{\beta(\beta^2 - 3)},$$

which again shows that β is an algebraic integer. As β is rational by Lemma 6.2, this implies that β is an integer, and so $(2\beta^2 + \beta - 4)/(\beta(\beta^2 - 3))$ is an integer. Observe that $2\beta^2 + \beta - 4$ and $\beta(\beta^2 - 3)$ are both positive for $\beta \geqslant 2$, so $2\beta^2 + \beta - 4 \geqslant \beta(\beta^2 - 3)$. But this implies $\beta = 2$, a contradiction. This shows that $s^* \neq -q^{-D-2}$. ∎

Proposition 6.5 *With reference to Notation 4.1 assume Γ is Q-polynomial and assume that Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters q, s^* be as in Lemma 3.1. If $D \geqslant 6$, then $s^* \neq -q^{-D-1}$.*

Proof. We assume $s^* = -q^{-D-1}$ and derive a contradiction. By Theorem 5.6(ii) we find that $\text{mult}(2, D-2) = 0$. But now Γ has, up to isomorphism, a unique irreducible T-module with endpoint 2, and this module has diameter $D - 4$. By [8, Theorem 3.12], Γ is 2-homogeneous in the sense of Nomura [13]. However, as $D \geqslant 6$ we have that Γ is $H(D,2)$ by [14, Theorem 1.2] (see also [8, Theorem 4.1]), a contradiction. This shows that $s^* \neq -q^{-D-1}$. ∎

7 Proof of the main theorem

In this section we prove our main theorem. To do this we first need the following result.

Theorem 7.1 *With reference to Notation 4.1 assume Γ is Q-polynomial and assume that Γ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters q, s^* be as in Lemma 3.1. Assume further that $D \in \{9, 10, 11\}$. Then $s^* \geqslant -q^{-D-4}$.*

Proof. The result obviously holds if $s^* \geqslant 0$ (recall that $q > 1$ by Theorem 6.1(i)), so assume that $s^* < 0$. Consider $\mathrm{mult}(2, D-2)$ and recall that this number is non-negative. It follows from Theorem 5.6(ii) that

$$(1 + s^* q^{D+1}) \geqslant 0.$$

As $s^* \neq -1/q^{D+1}$ by Proposition 6.5, we have that $s^* > -q^{-D-1}$.

Consider now $\mathrm{mult}(4, D-4)$ and recall that this number is non-negative. It follows from Theorem 5.6(ii) that

$$(1 + s^* q^{D+1})(1 + s^* q^{D+2}) \geqslant 0.$$

We have just proved that $s^* > -q^{-D-1}$ and so $1+s^* q^{D+1} > 0$, implying that $1+s^* q^{D+2} \geqslant 0$. As $s^* \neq -q^{-D-2}$ by Proposition 6.4, this shows that $s^* > -q^{-D-2}$.

Consider now $\mathrm{mult}(6, D-6)$ and recall that this number is non-negative. It follows from Theorem 5.6(iii) that

$$(1 + s^* q^{D+1})(1 + s^* q^{D+2})(1 + s^* q^{D+3}) \geqslant 0.$$

We have just proved that $s^* > -q^{-D-2}$ and so $(1 + s^* q^{D+1})(1 + s^* q^{D+2}) > 0$, implying that $1 + s^* q^{D+3} \geqslant 0$. As $s^* \neq -q^{-D-3}$ by Proposition 6.3, this shows that $s^* > -q^{-D-3}$.

Finally, consider $\mathrm{mult}(8, D-8)$ and recall that this number is non-negative. It follows from Theorem 5.6(iii) that

$$(1 + s^* q^{D+1})(1 + s^* q^{D+2})(1 + s^* q^{D+3})(1 + s^* q^{D+4}) \geqslant 0.$$

We have just proved that $s^* > -q^{-D-3}$ and so $(1 + s^* q^{D+1})(1 + s^* q^{D+2})(1 + s^* q^{D+3}) > 0$, implying that $1 + s^* q^{D+4} \geqslant 0$. This shows that $s^* \geqslant -q^{-D-4}$. ∎

We are now ready to prove our main result.

Theorem 7.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 9$. Then Γ is Q-polynomial if and only if one of the following (i)–(iv) holds:*

(i) Γ is the ordinary $2D$-cycle.

(ii) Γ is the Hamming cube $H(D, 2)$.

(iii) Γ is the antipodal quotient of the Hamming cube $H(2D, 2)$.

(iv) The intersection numbers of Γ satisfy

$$c_i = \frac{q^i - 1}{q - 1}, \qquad b_i = \frac{q^D - q^i}{q - 1} \qquad (0 \leqslant i \leqslant D), \tag{9}$$

where q is an integer at least 2.

Proof. If $D \geqslant 12$ then this is [7, Theorem 1.1], therefore we assume $D \in \{9, 10, 11\}$. Assume first that Γ is Q-polynomial and that Γ is not a $2D$-cycle, $H(D, 2)$ or $\overline{H}(2D, 2)$. Let parameters q, s^* be as in Lemma 3.1. By Theorem 7.1 we have $s^* \geqslant -q^{-D-4}$. Together with Theorem 6.1(ii) this implies that $-q^{-13} \leqslant s^* \leqslant q^{-13}$, and so $s^* = 0$ by Theorem 6.1(iii). It follows from Lemma 3.1 that

$$c_i = \frac{q^i - 1}{q - 1}, \qquad b_i = \frac{q^D - q^i}{q - 1} \qquad (0 \leqslant i \leqslant D).$$

But now $c_2 = q + 1$, and so q is an integer. As $q > 1$ we have that $q \geqslant 2$.

Concerning the converse, assume that one of the cases (i)-(iv) holds. If (i) or (iii) holds, then Γ is Q-polynomial by [2, Corollary 8.5.3(i),(iii)]. If (ii) or (iv) holds, then Γ has classical parameters and it is Q-polynomial by [2, Corollary 8.4.2]. ∎

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