Upper bound on the circular chromatic number of the plane

Konstanty Junosza-Szaniawski

Faculty of Mathematics and Information Science,
Warsaw University of Technology,
Poland

k.szaniawski@mini.pw.edu.pl

Submitted: Jul 17, 2015; Accepted: Feb 20, 2018; Published: Mar 2, 2018
Mathematics Subject Classifications: 05C15, 05C10, 05C62

Abstract

We consider a circular version of the famous Hadwiger-Nelson problem. It is known that 4 colors are necessary and 7 colors suffice to color the Euclidean plane so that points at distance one get different colors. In an r-circular coloring we assign to points arcs of length one of a circle whose circumference is equal to r so that points at distance one get disjoint arcs. No r-circular coloring of the plane with $r < 7$ was known so far. In this paper we show the existence of an r-circular coloring of the plane for $r = 4 + \frac{4\sqrt{3}}{3} \approx 6.30$. We also prove existence of a variant of the r-circular coloring of the plane with $r < 7$ in which we require that the arcs assigned to points at distance belonging to the interval $[0.9327, 1.0673]$ are disjoint.

Keywords: circular coloring, Hadwiger-Nelson problem, coloring of the plane.

1 Introduction

The Hadwiger-Nelson problem asks about the minimum number of colors required to color the Euclidean plane so that no two points at distance 1 receive the same color. This number is called the *chromatic number* of the plane and its exact value is not known. We know that at least 4 colors are needed [5] and that 7 colors suffice [3]. For a comprehensive history of the Hadwiger-Nelson problem see the monograph [7].

An r-circular coloring of a graph $G = (V, E)$ is a function $c : V \to [0, r)$ such that $1 \leqslant |c(u) - c(v)| \leqslant r - 1$ holds for every edge uv of G. Notice that an r-circular coloring can be seen as an assignment of arcs of length 1 of a circle with circumference r to vertices of G so that adjacent vertices get disjoint arcs. The circular chromatic number of G is the number $\chi_c(G) = \inf\{r \in \mathbb{R} :$ there exists an r-circular coloring of $G\}$. A circular coloring was first introduced by Vince [8]. For a survey on this subject see Zhu [9]. The circular coloring finds applications in scheduling theory, to minimize the average time of a process that is repeated many times. It is known (see [8]) that the circular chromatic number $\chi_c(G)$ of a graph G does not exceed its chromatic number but it is bigger than the chromatic number minus one, formally

$$\chi(G) - 1 < \chi_c(G) \leqslant \chi(G). \tag{1}$$

In this paper we consider the circular coloring of the Euclidean plane. More precisely, let G_0 be an infinite graph with the set of all points of the plane as its vertex set and the set of all pairs of points at distance one as its edge set. Clearly, the Hadwiger-Nelson problem concerns finding the chromatic number of G_0. If we combine known bounds for the chromatic number of G_0 with with the inequalities (1), we obtain $3 < \chi_c(G_0) \leqslant 7$. The lower bound can be improved by applying some results on the fractional chromatic number of the graph G_0.

Let us consider a j-fold coloring of a graph G which is an assignment of j-element sets of natural numbers to the vertices of G so that adjacent vertices get disjoint sets. The fractional chromatic number of G is defined by

$$\chi_f(G) = \inf \left\{ \frac{k}{j} : \text{ there exists a } j\text{-fold coloring of } G \text{ using } k \text{ colors} \right\}.$$

It is known [9] that $\chi_f(G) \leqslant \chi_c(G)$ for any graph G. The best known lower bound for the fractional chromatic number of G_0 is $\frac{32}{9} \approx 3.55$ (see Scheinerman and Ullman [6]). Hence $\chi_c(G) \geqslant \frac{32}{9}$. DeVos, Ebrahimi, Ghebleh, Goddyn, Mohar, Naserasr [1] improved this bound by showing that $\chi_c(G) \geqslant 4$. In this paper we give the first non-trivial upper bound on the circular chromatic number of the plane: $\chi_c(G_0) \leqslant 6.3095$.

Exoo [2] considered a more restricted coloring of the plane. He asked for the minimum number of colors to color the plane so that any points at distance belonging to a given interval $[1-\epsilon, 1+\epsilon]$ get different colors. For $\epsilon = 0$ the problem reduces to the Hadwiger-Nelson problem. A fractional and a j-fold Exoo type colorings were studied in [4]. The approach to circular coloring of the plane presented in this paper can be adapted to Exoo type colorings of the plane.

2 Main results

For $x \in \mathbb{R}$ and $\ell \in \mathbb{R}_+$ we define $\lfloor x \rfloor_\ell = \lfloor \frac{x}{\ell} \rfloor \cdot \ell$ and $(x)_\ell = x - \lfloor x \rfloor_\ell$. Notice that for $\ell = 1$ the function $\lfloor x \rfloor_\ell$ is the standard floor function $\lfloor x \rfloor$. For a pair of points $p_1, p_2 \in \mathbb{R}^2$, $d(p_1, p_2)$ denotes the ordinary Euclidean distance. Let G_ε be a graph with the vertex set $\mathbb{R}^2$ and the edge set $\{p_1p_2 : d(p_1, p_2) \in [1-\varepsilon, 1+\varepsilon]\}$. Here is our main result.

Theorem 1.

$$\chi_c(G_0) \leqslant 4 + \frac{4\sqrt{3}}{3} \approx 6.3095$$

Proof. We construct a $(4+\frac{4\sqrt{3}}{3})$-circular coloring of the plane. Let $\ell = 2+2\sqrt{3}$ and let $r = \frac{2\sqrt{3}}{3}\ell = 4+\frac{4\sqrt{3}}{3}$. We denote by R the rectangle $[0,\ell)\times[0,\frac{1}{2})$. We define a coloring c of the points of R by $c(x,y) = \frac{2\sqrt{3}}{3}x$ (where $(x,y)\in R$). Then, we extend this coloring in a "circular way" on the strip $S = \mathbb{R}\times[0,\frac{1}{2})$. More precisely, we join copies of the rectangle R along their vertical sides so that they form a strip S. Each copy of R is colored in the same way as the original rectangle R. Then we take copies of the strip S and join them along the horizontal sides. Each strip S is colored the same way, but when we move up, we shift each next copy of S to the right by $(1+\frac{\sqrt{3}}{2})$ (see Figures 1-3).

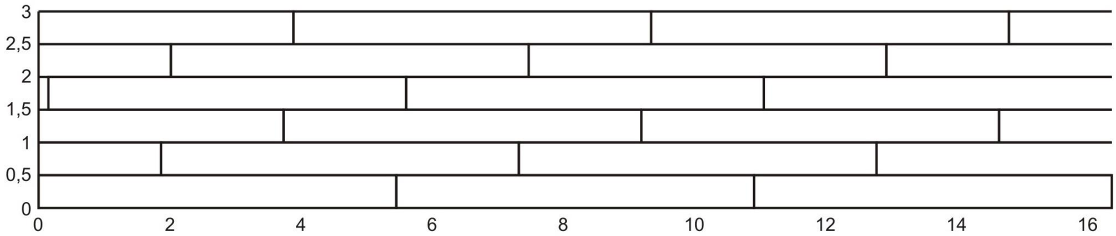

Figure 1: A partition of the plane into copies of the rectangle R

Figure 2 may give some intuition about the numbers r and ℓ.

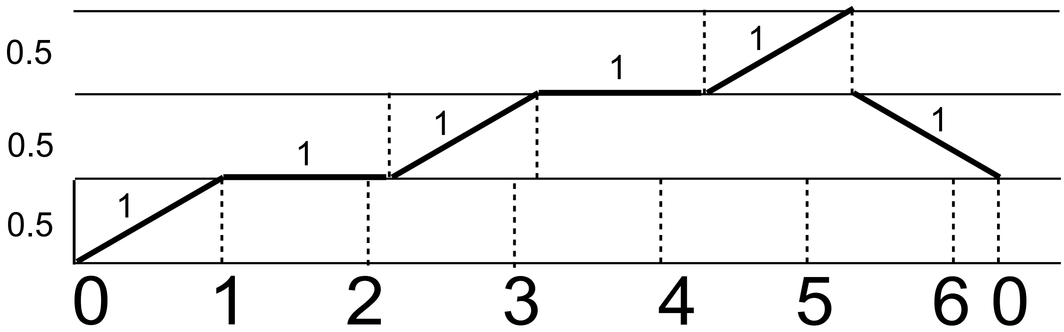

Figure 2: An ilustration of the coloring

Formally, the coloring of G_0 is defined by the formula:

$$c(x,y) = \frac{2\sqrt{3}}{3}\left(x-(2+\sqrt{3})\lfloor y\rfloor_{\frac{1}{2}}\right)_{\ell}$$

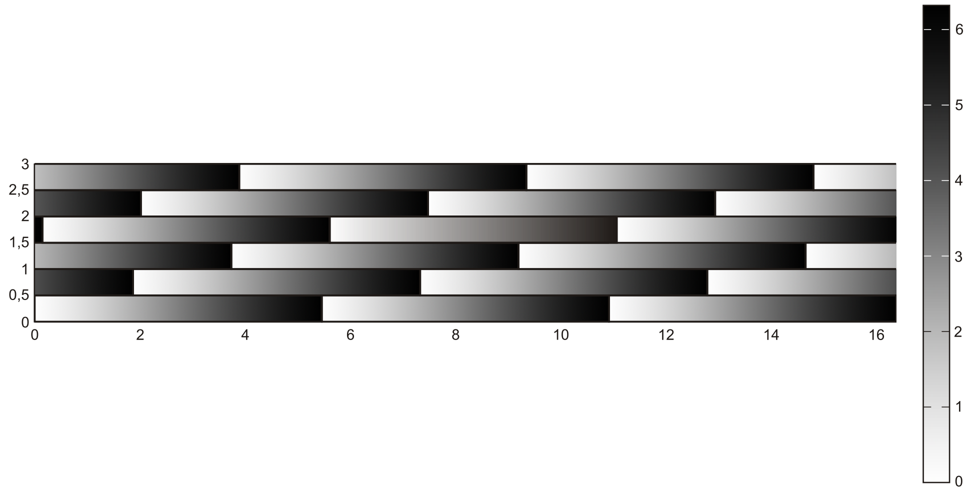

Figure 3: A $(4+\frac{4\sqrt{3}}{3})$-circular coloring of the plane, black and white

To prove that c is an r-circular coloring of G_0 we show that for any pair of points (x_1, y_1) and (x_2, y_2) at distance 1 the condition $1 \leqslant |c(x_1, y_1) - c(x_2, y_2)| \leqslant r-1$ is satisfied. Without loss of generality we assume $c(x_1, y_1) \geqslant c(x_2, y_2)$, i.e. $(x_1 - (2+\sqrt{3})\lfloor y_1 \rfloor_{\frac{1}{2}})_\ell \geqslant (x_2 - (2+\sqrt{3})\lfloor y_2 \rfloor_{\frac{1}{2}})_\ell$. Let $(x_1', y_1') = (x_1 - x_2, y_1 - \lfloor y_2 \rfloor_{\frac{1}{2}})$ and $(x_2', y_2') = (0, y_2 - \lfloor y_2 \rfloor_{\frac{1}{2}})$. Notice that $d((x_1', y_1'), (x_2', y_2')) = d((x_1, y_1), (x_2, y_2))$.

Moreover, using the equalities $\lfloor y_1 - \lfloor y_2 \rfloor_{\frac{1}{2}} \rfloor_{\frac{1}{2}} = \lfloor y_1 \rfloor_{\frac{1}{2}} - \lfloor y_2 \rfloor_{\frac{1}{2}}$ and $(a-b)_\ell = (a)_\ell - (b)_\ell$ for $(a)_\ell \geqslant (b)_\ell$ we obtain:

$$|c(x_1', y_1') - c(x_2', y_2')| = \frac{2\sqrt{3}}{3}\left|(x_1 - x_2 - (2+\sqrt{3})\lfloor y_1 - \lfloor y_2 \rfloor_{\frac{1}{2}} \rfloor_{\frac{1}{2}})_\ell - 0\right| =$$

$$= \frac{2\sqrt{3}}{3}\left|(x_1 - x_2 - (2+\sqrt{3})(\lfloor y_1 \rfloor_{\frac{1}{2}} - \lfloor y_2 \rfloor_{\frac{1}{2}}))_\ell\right| =$$

$$= \frac{2\sqrt{3}}{3}\left|(x_1 - (2+\sqrt{3})\lfloor y_1 \rfloor_{\frac{1}{2}} - (x_2 - (2+\sqrt{3}))\lfloor y_2 \rfloor_{\frac{1}{2}}))_\ell\right| =$$

$$= \frac{2\sqrt{3}}{3}\left|(x_1 - (2+\sqrt{3})\lfloor y_1 \rfloor_{\frac{1}{2}}))_\ell - (x_2 - (2+\sqrt{3})\lfloor y_2 \rfloor_{\frac{1}{2}})_\ell\right| = |c(x_1, y_1) - c(x_2, y_2)|\,.$$

Therefore, without loss of generality, we may assume that $x_2 = 0$ and $y_2 \in [0, \frac{1}{2})$. Recall that $d((x_1, y_1), (x_2, y_2)) = 1$.

It remains to prove that $|c(x_1, y_1) - c(x_2, y_2)| = |c(x_1, y_1)| \in [1, 3+\frac{4\sqrt{3}}{3}]$. Consider the following cases:

Case 1: $\lfloor y_1 \rfloor_{\frac{1}{2}} = 0, x_1 > 0$. In this case $x_1 \in (\frac{\sqrt{3}}{2}, 1]$ and $c(x_1, y_1) \in (1, \frac{2\sqrt{3}}{3}] \subseteq [1, 3+\frac{4\sqrt{3}}{3}]$.

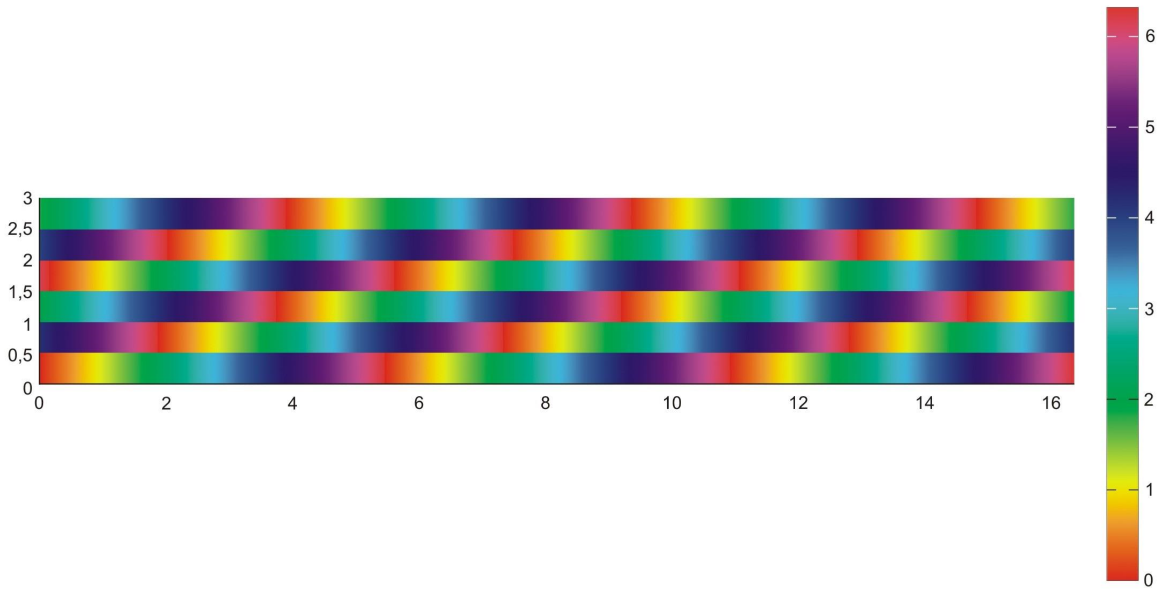

Figure 4: A $(4+\frac{4\sqrt{3}}{3})$-circular coloring of the plane, color

Case 2: $\lfloor y_1 \rfloor_{\frac{1}{2}} = 0, x_1 < 0$. In this case $x_1 \in [-1, -\frac{\sqrt{3}}{2})$ and $c(x_1, y_1) \in [4+\frac{2\sqrt{3}}{3}, 3+\frac{4\sqrt{3}}{3}) \subseteq [1, 3+\frac{4\sqrt{3}}{3}]$.

Case 3: $\lfloor y_1 \rfloor_{\frac{1}{2}} = \frac{1}{2}$. In this case $x_1 \in (-1, 1)$ and $c(x_1, y_1) \in (3, 3+\frac{4\sqrt{3}}{3}) \subseteq ([1, 3+\frac{4\sqrt{3}}{3}]$.

Case 4: $\lfloor y_1 \rfloor_{\frac{1}{2}} = 1$. In this case $x_1 \in (-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$ and $c(x_1, y_1) \in (1, 3) \subseteq [1, 3+\frac{4\sqrt{3}}{3}]$.

Case 5: $\lfloor y_1 \rfloor_{\frac{1}{2}} = -\frac{1}{2}$. In this case $x_1 \in (-1, 1)$ and $c(x_1, y_1) \in (1, 1+\frac{4\sqrt{3}}{3}) \subseteq [1, 3+\frac{4\sqrt{3}}{3}]$.

Case 6: $\lfloor y_1 \rfloor_{\frac{1}{2}} = -1$. In this case $x_1 \in (-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$ and $c(x_1, y_1) \in (1+\frac{4\sqrt{3}}{3}, 3+\frac{4\sqrt{3}}{3}) \subseteq [1, 3+\frac{4\sqrt{3}}{3}]$.

□

The approach described in the proof of Theorem 1 can also be used to color G_ε in the "circular way". In Theorem 2 we show that for small values of ε the number $\chi_c(G_\varepsilon)$ is strictly smaller than the known bound on the classic chromatic number of G_ε.

Theorem 2. *For* $\varepsilon \in [0, 0.0673)$,

$$\chi_c(G_\varepsilon) \leqslant 3 + \frac{(4+\sqrt{3})(1+\varepsilon)}{\sqrt{3\varepsilon^2 - 10\varepsilon + 3}} < 7.$$

Proof. The proof is analogous to the proof of Theorem 1. We partition the plane into copies of a rectangle, and color each copy in the same way. The difference is in the size of the rectangle and in the definition of the coloring function. Figure 5 may give some intuition about the number $3 + \frac{(4+\sqrt{3})(1+\varepsilon)}{\sqrt{3\varepsilon^2-10\varepsilon+3}} < 7$.

Let $\ell = \frac{3}{2}\sqrt{3\varepsilon^2 - 10\varepsilon + 3} + (2+\frac{\sqrt{3}}{2})(1+\varepsilon)$ and $r = 3 + \frac{(4+\sqrt{3})(1+\varepsilon)}{\sqrt{3\varepsilon^2-10\varepsilon+3}}$. To prove the theorem we define an r-circular coloring of G_ϵ by:

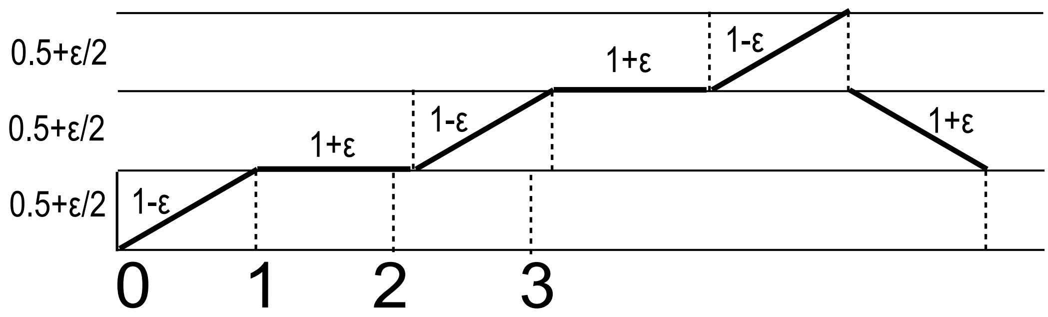

Figure 5: A coloring of G_ε

$$c(x,y) = \frac{2}{\sqrt{3\varepsilon^2 - 10\varepsilon + 3}} \left(x - \left(2 + \frac{\sqrt{3\varepsilon^2 - 10\varepsilon + 3}}{1+\varepsilon} \right) \lfloor y \rfloor_{\frac{1+\varepsilon}{2}} \right)_\ell .$$

To make formulas shorter, we define $a = \frac{1}{2}\sqrt{3\varepsilon^2 - 10\varepsilon + 3}$. Then $r = 3 + \frac{(2+\frac{\sqrt{3}}{2})(1+\varepsilon)}{a}$ and

$$c(x,y) = \frac{1}{a} \left(x - (1+\varepsilon+a) \frac{2}{1+\varepsilon} \lfloor y \rfloor_{\frac{1+\varepsilon}{2}} \right)_\ell .$$

To prove that c is an r-circular coloring of G_ε we show that for any pair of points (x_1, y_1) and (x_2, y_2) at distance in the interval $[1-\varepsilon, 1+\varepsilon]$ the condition $1 \leqslant |c(x_1,y_1) - c(x_2,y_2)| \leqslant r-1$ holds. Without loss of generality we assume that $c(x_1,y_1) \geqslant c(x_2,y_2)$, i.e.

$$\left(x_1 - (1+\varepsilon+a) \frac{2}{1+\varepsilon} \lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} \right)_\ell \geqslant \left(x_2 - (1+\varepsilon+a) \frac{2}{1+\varepsilon} \lfloor y_2 \rfloor_{\frac{1+\varepsilon}{2}} \right)_\ell$$

Let $(x_1', y_1') = (x_1 - x_2, y_1 - \lfloor y_2 \rfloor_{\frac{1+\varepsilon}{2}})$ and $(x_2', y_2') = (0, y_2 - \lfloor y_2 \rfloor_{\frac{1+\varepsilon}{2}})$. Notice that $d((x_1', y_1'), (x_2', y_2')) = d((x_1, y_1), (x_2, y_2))$.

Moreover, similarly as in the proof of Theorem 1, one can prove that $|c(x_1', y_1') - c(x_2', y_2')| = |c(x_1, y_1) - c(x_2, y_2)|$. Therefore, without loss of generality, we may assume that $x_2 = 0$ and $y_2 \in [0, \frac{1+\varepsilon}{2})$.

Recall that $d((x_1, y_1), (x_2, y_2)) \in [1-\varepsilon, 1+\varepsilon]$. It remains to prove that $|c(x_1, y_1) - c(x_2, y_2)| = |c(x_1, y_1)| \subseteq [1, r-1]$. Consider the following cases:

Case 1: $\lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} = 0, x_1 > 0$. In this case $x_1 \in (a, 1+\varepsilon]$ and $c(x_1, y_1) \in (1, \frac{1+\varepsilon}{a}] \subseteq [1, r-1]$.

Case 2: $\lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} = 0, x_1 < 0$. In this case $x_1 \in [-1-\varepsilon, -a)$ and $c(x_1, y_1) \in [r - \frac{1+\varepsilon}{a}, r-1) \subseteq [1, r-1]$.

Case 3: $\lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} = \frac{1+\varepsilon}{2}$. In this case $x_1 \in (-(1+\varepsilon), 1+\varepsilon)$ and $c(x_1, y_1) \in (2 + \sqrt{3}\frac{1+\varepsilon}{\sqrt{3\varepsilon^2 - 10\varepsilon + 3}}, r-1) \subseteq [1, r-1]$.

Case 4: $\lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} = 1+\varepsilon$. In this case $x_1 \in (-\frac{\sqrt{3}}{2}(1+\varepsilon), \frac{\sqrt{3}}{2}(1+\varepsilon))$ and $c(x_1, y_1) \in (1, 1 + 2\sqrt{3}\frac{1+\varepsilon}{\sqrt{3\varepsilon^2 - 10\varepsilon + 3}}) \subseteq [1, r-1]$.

Case 5: $\lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} = -\frac{1+\varepsilon}{2}$. In this case $x_1 \in (-(1+\varepsilon), 1+\varepsilon)$ and $c(x_1, y_1) \in (1, 1 + 4\frac{1+\varepsilon}{\sqrt{3\varepsilon^2 - 10\varepsilon + 3}}) \subseteq [1, r-1]$.

Case 6: $\lfloor y_1 \rfloor_{\frac{1+\varepsilon}{2}} = -(1+\varepsilon)$. In this case $x_1 \in (-\frac{\sqrt{3}}{2}(1+\varepsilon), \frac{\sqrt{3}}{2}(1+\varepsilon))$ and $c(x_1, y_1) \in (2 + (4-\sqrt{3})\frac{1+\varepsilon}{\sqrt{3\varepsilon^2-10\varepsilon+3}}, 2 + (4+\sqrt{3})\frac{1+\varepsilon}{\sqrt{3\varepsilon^2-10\varepsilon+3}}) \subseteq [1, r-1]$.

□

Acknowledgements

I am grateful to Zbigniew Lonc, Jarosław Grytczuk, Jacek Wojciechowski, Zbigniew Walczak and Krzysztof Węsek for the inspiration. Moreover, I would like to thank Zbigniew Lonc, Agnieszka Żychowska and Jerzy Junosza-Szaniawski for their help in preparing the paper. I am also very grateful to referees for their remarks.

References

[1] M. DeVos, J. Ebrahimi, M. Ghebleh, L. Goddyn, B. Mohar, R. Naserasr, Circular Coloring the Plane, SIAM J. Discrete Math., 21 (2):461–465, 2007.

[2] G. Exoo, ε-Unit Distance Graphs, Discrete Comput. Geom., (33): 117–123, 2005.

[3] H. Hadwiger, Ungelöste Probleme, Elem. Math., (16): 103–104, 1961.

[4] J. Grytczuk, K. Junosza-Szaniawski, J. Sokół, K. Węsek, Fractional and j-fold coloring of the plane, Discrete Comput. Geom. 55 (3), 594–609, 2016.

[5] L. Moser, W. Moser, Problems for Solution, Canadian Bulletin of Mathematics, (4): 187–189, 1961.

[6] E. R. Scheinerman, D.H. Ullman, Fractional Graph Theory, John Wiley and Sons, 2008.

[7] A. Soifer, The Mathematical Coloring Book, Springer, 2008.

[8] A. Vince, Star chromatic number. J. Graph Theory, (12): 551–559, 1988.

[9] X. Zhu, Circular chromatic number: a survey. Discrete Math., (229): 371–410, 2001.