A classification of Motzkin numbers modulo 8

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Abstract

The well-known Motzkin numbers were conjectured by Deutsch and Sagan to be nonzero modulo 8. The conjecture was first proved by Sen-Peng Eu, Shu-chung Liu and Yeong-Nan Yeh by using the factorial representation of the Catalan numbers. We present a short proof by finding a recursive formula for Motzkin numbers modulo 8. Moreover, such a recursion leads to a full classification of Motzkin numbers modulo 8.

Keywords: Motzkin numbers, congruence classes

1 Introduction

Much work has been done in calculating the congruences of various combinatorial numbers modulo a prime power p^r. We begin by introduce some notations. We will use the p-adic notations $[n]_p = \langle n_d n_{d-1} \cdots n_0 \rangle_p$ to denote the sequence of digits representing n in base p [15]. The *p-adic order* or *p-adic valuation* $\omega_p(n)$ of n is defined by

$$\omega_p(n) = \max\{t \in \mathbb{N} : p^t | n\}.$$

In words, it is the highest power of p dividing n, or equivalently, the number of 0's to the right of the rightmost nonzero digit in $[n]_p$. The value $\omega_p(n)$ indicates the divisibility by powers of p, which can be found in many previous studies [5].

Many results have been established for the binomial coefficients. The most famous as well as age-old one is the Pascal's fractal which is formed by the parities of the binomial coefficients $\binom{n}{k}$ [20]. Pascal's triangle also has versions modulo 4 and 8 [3, 10]. The behavior of Pascal's triangle modulo higher powers of p is more complicated. Some rules

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for this behavior are discussed by Granville [9]. Kummer computed the p-adic order of $\binom{m+n}{m}$ [13], by counting the number of carries that occur when $[m]_p$ and $[n]_p$ are added. The elegant result of Lucas [15] states that $\binom{n}{k} \equiv_p \prod_i \binom{n_i}{k_i}$ where n_i and k_i come from $[n]_p$ and $[k]_p$, and $\equiv_p$ denotes the congruence class modulo p. A generalization of Lucas' theorem for a prime power was established by Davis and Webb [2].

The most useful combinatorial numbers other than the binomial coefficients are the well-known *Catalan numbers*

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}, \quad n \in \mathbb{N}.$$

They have more than 200 combinatorial interpretations, as collected by Stanley in [18]. The congruence class of C_n modulo 2^r was studied in [6, 11, 14, 21]. Several other combinatorial numbers have been studied for their congruences, for example, Apéry numbers [8, 16], Central Delannoy numbers [7] and weighted Catalan numbers [17].

In this paper we will focus on the well-known *Motzkin numbers*

$$M(n) = M_n = \sum_{k \geqq 0} \binom{n}{2k} C_k, \quad n \in \mathbb{N}. \tag{1}$$

Their congruences were only studied very recently. Klazar and Luca proved that the Motzkin numbers are never periodic modulo any prime number [12]. Deutsch and Sagan [4] studied the congruences of M_n modulo $2, 3$ and 5 and made the following two ex-conjectures.

Conjecture 1 ([4]). *We have $M_n \equiv_4 0$ if and only if $n = (4i+1)4^{j+1} - 1$ or $n = (4i+3)4^{j+1} - 2$, where i and j are nonnegative integers.*

Conjecture 2 ([4]). *The Motzkin numbers are never congruent to 0 modulo 8.*

The two conjectures were first proved by Eu-Liu-Yeh in [6]. They first derived the congruence class of the Catalan numbers C_n modulo 8 by using their factorial representations. Then they proved Conjecture 1 by careful analyzing formula (1) modulo 8. Finally they proved Conjecture 2 by confirming that $M(n) \equiv_8 4$ when n belongs to the two cases in Conjecture 1.

Our main result is the following explicit formula for M_n modulo 8, from which Conjectures 1 and 2 clearly follow.

Theorem 3. *The congruence class of $M(n)$ modulo 8 can be characterized as follows:*

$$M(4s) \equiv_8 \begin{cases} 1 - 4\mathcal{Z}(\alpha) - 2\chi(\|\alpha\| \equiv_2 1) + 4\alpha, & s = 2\alpha, \\ 1 - 4\mathcal{Z}(\alpha) - 2\chi(\|\alpha\| \equiv_2 1), & s = 2\alpha + 1. \end{cases}$$

$$M(4s+1) \equiv_8 \begin{cases} 1 - 4\mathcal{Z}(\alpha) - 2\chi(\|\alpha\| \equiv_2 1) + 4\alpha, & s = 2\alpha, \\ 1 - 4\mathcal{Z}(\alpha) - 2\chi(\|\alpha\| \equiv_2 1) + 4, & s = 2\alpha + 1. \end{cases}$$

$$M(4s+2) \equiv_8 \begin{cases} 4, & s = (4\alpha+3)2^{2j}-1, \\ 2-4\|\alpha\|, & s = (4\alpha+1)2^{2j}-1, \\ -1+4\mathcal{Z}(\alpha)+2\chi(\|\alpha\| \equiv_2 1), & s = (4\alpha+3)2^{2j+1}-1, \\ 3+4\mathcal{Z}(\alpha)+2\chi(\|\alpha\| \equiv_2 1)+4\alpha, & s = (4\alpha+1)2^{2j+1}-1. \end{cases}$$

$$M(4s+3) \equiv_8 \begin{cases} -2+4\|\alpha\|, & s = (4\alpha+3)2^{2j}-1, \\ 4, & s = (4\alpha+1)2^{2j}-1, \\ -1+4\mathcal{Z}(\alpha)+2\chi(\|\alpha\| \equiv_2 1), & s = (4\alpha+3)2^{2j+1}-1, \\ -1+4\mathcal{Z}(\alpha)+2\chi(\|\alpha\| \equiv_2 1)+4\alpha, & s = (4\alpha+1)2^{2j+1}-1. \end{cases}$$

Here $\chi(S)$ equals 1 if the statement S is true and equals 0 if otherwise, $\|\alpha\|$ is the sum of the digits of $[\alpha]_2$, and $\mathcal{Z}(\alpha)$ is the number of zero runs of α as described later in Proposition 11.

Our approach is along the line of [21], by using the following recursive formula:

$$C_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 2^{n-2i} C_i.$$

This formula can be easily proved by using Zeilberger's creative telescoping method [22], or by two different combinatorial interpretations of C_n (see [21]). By combining the above formula with (1), we derive the following recursive formulas for $M(n)$.

$$M(2k+2) - M(2k) \equiv_8 (-1)^k M(k) + f(k), \tag{2}$$
$$f(k) = 4\left(\binom{k+1}{2} - (-1)^k k\right) M(k-1) - 4\binom{k}{2} M(k-2);$$
$$M(2k+1) \equiv_8 (2k+1)M(2k) + g(k), \tag{3}$$
$$g(k) = -2\binom{2k+1}{2} M(2k-1) + 4\binom{2k+1}{3} M(2k-2).$$

By using these recursive formulas, we give a simple way to compute the congruences of $M(n)$ modulo $2, 4, 8$.

The paper is organized as follow. In Section 2, we derive the recurrence formulas of M_n, which are the staring point of our approach. We also introduce basic tools for further calculations. In Section 3, we compute the congruences classes of Motzkin numbers modulo 2 and 4. Finally, we compute the congruences classes of Motzkin numbers modulo 8 in Section 4.

2 Weighted Motzkin paths and the recursion

Let $F(x;u) = \sum_{n \geqslant 0} M_u(n) x^n$ be the unique power series defined by the functional equation

$$F(x;u) = \frac{1}{1 - ux - x^2 F(x;u)}.$$

Then $F(x;u)$ is the generating function of weighted Motzkin paths (see, e.g. [19]). That is, $M_u(n)$ counts weighted lattice paths from $(0,0)$ to $(n,0)$ that never go below the horizontal axis and use only steps $U=(1,1)$ $H=(1,0)$, or $D=(1,-1)$ and weights $1,u,1$ respectively.

The well-known Motzkin number $M(n)$ is our $M_1(n)$, and the Catalan number C_n is our $M_0(2n)$. We also have $C_{n+1}=M_2(n)$, which is written as

$$M_0(2n)=M_2(n-1), \qquad \text{for } n\geqslant 1. \tag{4}$$

Lemma 4. *For any constants u and v, we have*

$$M_{u+v}(n)=\sum_{i=0}^{n}\binom{n}{i}v^iM_u(n-i), \tag{5}$$

$$M_u(2k+1)=\sum_{i=1}^{n}\binom{2k+1}{i}(-2)^{i-1}u^iM_u(2k+1-i). \tag{6}$$

Proof. Equation (5) is routine. For (6), we need the easy fact $M_{-u}(n)=(-1)^nM_u(n)$. By setting $v=-2u$ in (5), we obtain

$$M_{-u}(n)=M_u(n)+\sum_{i=1}^{n}\binom{n}{i}(-2u)^iM_u(n-i).$$

Thus for $n=2k+1$, we obtain

$$M_u(2k+1)=\sum_{i=1}^{n}\binom{2k+1}{i}(-2)^{i-1}u^iM_u(2k+1-i).$$

This is equation (6). □

Theorem 5. *We have the recursion* (2) *and* (3) *with initial condition $M(0)=1$.*

Proof. Setting $u=1$ in (6) and simplifying gives

$$M(2k+1)\equiv_8 (2k+1)M(2k)-2\binom{2k+1}{2}M(2k-1)+4\binom{2k+1}{3}M(2k-2).$$

This is (3). Note that no recursion for $M(2k)$ can be obtained in this way.

For (2), we start with

$$\begin{aligned} M(2k) &= \sum_{i=0}^{k}\binom{2k}{2i}C_{k-i} \\ &= \sum_{i=0}^{k}\sum_{j=0}^{i}2^{2j}\binom{k}{2j}\binom{k-2j}{i-j}C_{k-i}, \end{aligned}$$

which can be easily proved using Zeilberger's creative telescoping method [22]. When reduced to modulo 8, this gives

$$\begin{aligned} M(2k) &\equiv_8 \sum_{i=0}^{k} \binom{k}{i} C_{k-i} + 4\binom{k}{2} \sum_{i=1}^{k-1} \binom{k-2}{i-1} C_{k-i} \\ &\equiv_8 1 + \sum_{i=0}^{k-1} \Big(\sum_{j=1}^{i} \binom{k-j}{i-j+1} \Big) M_2(k-i-1) + 4\binom{k}{2} \sum_{i'=0}^{k-2} \binom{k-2}{i'} M_2(k-2-i') \\ &\equiv_8 1 + \sum_{j=1}^{k-1} \sum_{i=j}^{k-1} \binom{k-j}{i-j+1} M_2((k-j)-(i-j+1)) + 4\binom{k}{2} M_3(k-2) \\ &\equiv_8 1 + \sum_{j=1}^{k-1} M_3(k-j) + 4\binom{k}{2} M_3(k-2). \end{aligned}$$

(We remark that the computation modulo 2^r when $r \geqslant 4$ becomes complicated.) Thus,

$$\begin{aligned} M(2k+2) - M(2k) &\equiv_8 M_3(k) + 4\binom{k+1}{2} M_3(k-1) - 4\binom{k}{2} M_3(k-2) \\ &\equiv_8 M_{-1}(k) + 4\binom{k}{1} M_{-1}(k-1) + 4\binom{k+1}{2} M(k-1) - 4\binom{k}{2} M(k-2) \\ &\equiv_8 (-1)^k M(k) + 4\left(\binom{k+1}{2} - (-1)^k k \right) M(k-1) - 4\binom{k}{2} M(k-2). \end{aligned}$$

This is just equation (2). □

We derive explicit formulas of $M(n) \mod 2^r$ successively for $r = 1, 2, 3$. The idea is based on the fact that

$$2^{r-r'} M(n) \mod 2^r = 2^{r-r'}(M(n) \mod 2^{r'}), \qquad \text{for } r' < r.$$

This fact will be frequently used without mentioning.

Lemma 6. *We keep the notations from Theorem 5. Assume that we have obtained explicit formulas for $M(n) \mod 2^{r-1}$. Then there are explicit formulas for $f(k)$ and $g(k)$. Moreover, the recursion is reduced as follows.*

$$M(2k+1) \equiv_8 (2k+1)M(2k) + g(k), \tag{7}$$

$$M(4s) \equiv_{2^r} M(0) + \sum_{j=2}^{2s-1} f(j) - \sum_{j=1}^{s-1} (2jM(2j) + g(j)), \tag{8}$$

$$M(4s+2) \equiv_{2^r} M(2\beta - 2) + \sum_{i=0}^{a} \left(M(\beta 2^{i+2} - 4) + f(\beta 2^{i+1} - 2) \right), \tag{9}$$

where in (9), $s + 1 = \beta 2^a$ *for some odd number* β *and* $a \geqslant 0$.

Proof. By (7), we can eliminate those $M(2k+1)$ so that our formulas only involve $f(k), g(k)$ and $M(2k)$. We have to split by cases $k = 2s$ and $k = 2s+1$ in (2):

$$\begin{aligned} M(4s+4) - M(4s+2) &\equiv_{2^r} -M(2s+1) + f(2s+1) \\ &\equiv_{2^r} -(2s+1)M(2s) - g(s) + f(2s+1), \\ M(4s+2) - M(4s) &\equiv_{2^r} M(2s) + f(2s) \equiv_{2^r} M(2s) + f(2s). \end{aligned}$$

Taking the sum of the above two equations gives the following recursion:

$$M(4s+4) - M(4s) \equiv_{2^r} -2sM(2s) - g(s) + f(2s) + f(2s+1). \tag{8'}$$

(Note that we have explicit formulas of $-2sM(2s) \mod 2^r$ by the induction hypothesis.) This is equivalent to (8).

Next for $M(4s+2)$ we rewrite as follows:

$$M(4(s+1)-2) - M(4s) \equiv_{2^r} M(2(s+1)-2) + f(2s).$$

If $s = \beta 2^a - 1$ with β odd and $a \geqslant 0$, then the above equation can be rewritten as

$$M(\beta 2^{a+2} - 2) - M(\beta 2^{a+1} - 2) \equiv_{2^r} M(\beta 2^{a+2} - 4) + f(\beta 2^{a+1} - 2). \tag{9'}$$

This is equivalent to (9). $\square$

We remark that (8') and (9') are easier to use than (8) and (9).

3 Motzkin numbers modulo $2, 4$

Recall that $\omega_2(n) = a$ if $n = (2\alpha+1)2^a$. Note that $\omega_2(0)$ is not defined. The following properties are easy to check and will be frequently used without mentioning.

Lemma 7. *For nonnegative integer α we have*

$$\omega_2(2\alpha+1) = 0, \qquad \omega_2(2\alpha) = \omega_2(\alpha) + 1, \qquad \alpha\omega_2(\alpha) \equiv_2 0;$$

$$\omega_2(\alpha!) = \sum_{i=1}^{\alpha} \omega_2(i), \qquad \omega_2((2\alpha+1)!) = \omega_2((2\alpha)!) = \omega_2(\alpha!) + \alpha.$$

Proof. The first, second and fourth formulas follow easily by definition. The third formula follows from the first two formulas by discussing the parity of α. Finally,

$$\omega_2((2\alpha+1)!) = \omega_2((2\alpha)!) = \omega_2((2\alpha)!!)) = \omega_2(\alpha!) + \alpha,$$

where in the second equality, we removed all the odd factors to get $(2\alpha)!! = 2^\alpha \alpha!$. $\square$

3.1 Motzkin numbers modulo 2

Proposition 8. *We have*

$$M(2k+1) \equiv_2 M(2k) \equiv_2 \omega_2(2k+2).$$

In particular $M(4s) \equiv_2 M(4s+1) \equiv_2 1$.

Proof. We apply Theorem 5 and Lemma 6 and follow the notations there. Clearly, we have $f(k) \equiv_2 0$ and $g(k) \equiv_2 0$. Thus we have

$$\begin{aligned} M(4s) &\equiv_2 M(0) = 1 = \omega_2(4s+2), \\ M(4s+2) &\equiv_2 M(2\beta - 2) + \sum_{i=0}^{a} \big(M(\beta 2^{i+2} - 4)\big) = a + 2 = \omega_2(4s+4), \end{aligned}$$

where in the second equation, $s + 1 = \beta 2^a$ for some odd number β and $a \geqslant 0$. The proposition then follows. ☐

3.2 Motzkin numbers modulo 4

Lemma 9. *We have the following characterization of Motzkin numbers modulo* 4.

$$\begin{aligned} M(4s) &\equiv_4 1 + 2\omega_2(s!) \equiv_4 1 + 2L(s) + 2s, \\ M(4s+1) &\equiv_4 M(4s), \\ M(4s+2) &\equiv_4 \begin{cases} 2\alpha + 2, & s{=}(2\alpha+1)2^{2a} - 1,\ a \geqslant 0, \\ 2\alpha + 2L(\alpha) + 3, & s{=}(2\alpha+1)2^{2a+1} - 1,\ a \geqslant 0. \end{cases} \\ M(4s+3) &\equiv_4 -M(4s+2) + 2, \end{aligned}$$

where $L(r) = \sum_{i=1}^{r-1} M(2i)$.

Consequently, $M(n) \equiv_4 0$ *if and only if* $n = (4i+1)4^{j+1} - 1$ *or* $n = (4i+3)4^{j+1} - 2$ *for some nonnegative integers* i *and* j*. That is, Conjecture 1 holds true.*

Proof. We first show that the second part follows from the first part. Clearly, $M(n) \equiv_4 0$ if and only if either i) $M(n) = M(4r+2) \equiv_4 2\alpha + 2 \equiv_4 0$ for $r = (2\alpha+1)4^a - 1$. Hence, $\alpha = 2i + 1$ for some i and $n = (4i+3)4^{a+1} - 2$; Or ii) $M(n) = M(4r+3) \equiv_4 -M(4r+2) + 2 \equiv_4 2\alpha \equiv_4 0$ for $r = (2\alpha+1)4^a - 1$. Hence, $\alpha = 2i$ for some i and $n = (4i+1)4^{a+1} - 1$.

Now we prove the first part by Theorem 5 and Lemma 6. First, we have

$$f(k) \equiv_4 0 \quad \text{and} \quad g(k) \equiv_4 -2k(2k+1)M(2k-1) \equiv_4 2k\omega_2(2k) \equiv_4 2k,$$

where we have used Proposition 8. Thus, the recurrence reduces to

$$M(2k+2) \equiv_4 M(2k) + (-1)^k M(k), \tag{10}$$

$$M(2k+1) \equiv_4 (2k+1)M(2k) + 2k. \tag{11}$$

Clearly, the odd case reduces to the even case by (11).

For $M(4s)$ we have

$$\begin{aligned} M(4s+4) - M(4s) &\equiv_4 -M(2s+1) + M(2s) \equiv_4 -2sM(2s) - 2s \\ &\equiv_4 2\chi(s = 2\alpha+1)(1 + \omega_2(\alpha+1) + 1) \equiv_4 2\omega_2(s+1), \end{aligned}$$

which is equivalent to $M(4s) \equiv_4 1 + 2\omega_2(s!) \equiv_4 1 + 2L(s) + 2s$.

For $M(4s+2)$, we write $s = \beta 2^a - 1$ for a unique odd number β. We have

$$\begin{aligned} M(4s+2) &\equiv_4 M(2\beta-2) + \sum_{i=0}^{a} M(\beta 2^{i+2} - 4) \\ &\equiv_4 M(2\beta-2) + \sum_{i=0}^{a}(1 + 2L(\beta 2^i - 1) + 2(\beta 2^i - 1)) \\ &\equiv_4 \chi(\beta = 2\alpha+1)1 + 2L(\alpha) + 2\alpha - (a+1) + \sum_{i=0}^{a}(2i + 2L(\alpha) + 2^{i+1}) \\ &\equiv_4 2(a+2)L(\alpha) + 2\alpha - a + a(a+1) + 2 \\ &\equiv_4 2(a+2)L(\alpha) + 2\alpha + a^2 - 2 \\ &\equiv_4 \begin{cases} 2\alpha + 2, & a \text{ is even}, \\ 2\alpha + 2L(\alpha) + 3, & a \text{ is odd}. \end{cases} \end{aligned}$$

This completes the proof. □

Indeed since $L(s)$ appears in computations modulo 8,we summarize its properties as follows.

Lemma 10. *Let* $L(s) = \sum_{i=0}^{s-1} M(2i)$, *with* $L(0) = 0$. *Then*

$$L(2s) \equiv_2 L(s), \qquad L(2s+1) \equiv_2 1 + L(s), \qquad L(s) = h_2(s!) + s,$$

$$L(2s) \equiv_4 1 - (-1)^s + L(s), \qquad L(2s+1) \equiv_4 1 - L(s).$$

Proof. The modulo 2 result is obvious since $L(s) \equiv_2 \sum_{i=0}^{s-1} \omega_2(2i+2) = \omega_2(s!) + s$.

For the modulo 4 result, we have, by definition,

$$\begin{aligned} L(2s) &\equiv_4 \sum_{i=0}^{2s-1} M(2i) = \sum_{i=0}^{s-1}(M(4i+2) + M(4i)) \\ \text{(by (10))} &\equiv_4 \sum_{i=0}^{s-1}(2M(4i) + M(2i)) \\ &\equiv_4 \sum_{i=0}^{s-1}(2 + M(2i)) \\ &\equiv_4 2s + L(s) \end{aligned}$$

$$\equiv_4 1-(-1)^s+L(s).$$

By the above formula and Lemma 9, we have

$$L(2s+1)=L(2s)+M(4s)=2s+L(s)+1+2s+2L(s)=1-L(s).$$

This completes the proof. □

Let $[n]_2=n_kn_{k-1}\cdots n_1n_0$ be the binary expansion of $n\geqslant 1$. Then $n=n_k2^k+\cdots+n_1\cdot 2+n_0$. Denote by $\|n\|=n_k+\cdots+n_1+n_0$, the sum of the binary digits of n. A 0-run of $[n]_2$ is a maximal 0-subword $n_in_{i+1}\cdots n_j$ for some $0\leqslant i<j\leqslant k$, such that $n_{j+1}=1$ and $n_{i-1}\neq 0$ (including the case $i=0$). Denote by $\mathcal{Z}(n)$ the number of 0-runs of $[n]_2$. We have the following explicit result.

Proposition 11. *We have*

$$L(n)\equiv_2 \|n\|, \quad \textit{and } L(n)\equiv_4 2\mathcal{Z}(n)+\chi(\|n\|\equiv_2 1).$$

Proof. The modulo 2 case is straightforward by Lemma 10.

For the modulo 4 case, we proceed by induction on n. The proposition clearly holds for the base case $n=1$. Assume it holds for all numbers smaller than n. We show that it holds for n by considering the following two cases.

Case 1: If $n=2s+1$, then $[n]_2$ is obtained from $[s]_2$ by adding a 1 at the end. By Lemma 10 and the induction hypothesis for s, we have

$$L(2s+1)\equiv_4 1-L(s)\equiv_4 1-2\mathcal{Z}(s)-\chi(\|s\|\equiv_2 1)\equiv_4 2\mathcal{Z}(s)+1-\chi(\|s\|\equiv_2 1),$$

which clearly equals to $2\mathcal{Z}(n)+\chi(\|n\|\equiv_2 1)$.

Case 2: If $n=2s$, then $[n]_2$ is obtained from $[s]_2$ by adding a 0 at the end. i) If s is odd, then by Lemma 10 and the induction hypothesis for s, we have

$$L(2s)\equiv_4 1-(-1)^s+L(s)=2+L(s)=2(\mathcal{Z}(s)+1)+\chi(\|s\|\equiv_2 1),$$

which clearly equals to $2\mathcal{Z}(n)+\chi(\|n\|\equiv_2 1)$. ii) Similarly, if s is even, then

$$L(2s)\equiv_4 1-(-1)^s+L(s)=L(s)=2\mathcal{Z}(s)+\chi(\|s\|\equiv_2 1).$$

This also equals to $2\mathcal{Z}(n)+\chi(\|n\|\equiv_2 1)$. □

We remark that the sequence $L(n) \mod 2$ turns out to be the Thue-Morse sequence. See [1] for a survey on the Thue-Morse sequence.

4 Motzkin numbers modulo 8

Lemma 12. *The recursion from Theorem 5 reduces modulo 8 to*

$$\begin{aligned} M(2k+2)-M(2k) &\equiv_8 (-1)^k M(k)+f(k), \ \textit{where } f(k)=4\chi(k\equiv_4 3)\omega_2((k+1)/2), \\ M(2k+1) &\equiv_8 (2k+1)M(2k)+g(k), \\ &\qquad\qquad \textit{where } g(k)=\chi(k=2\alpha+1)(4\alpha-2M(4\alpha)). \end{aligned}$$

Proof. By Theorem 5, we have

$$f(k)\equiv_8 4\left(\binom{k+1}{2}-(-1)^k k\right)M(k-1)-4\binom{k}{2}M(k-2).$$

i) When $k=2\alpha$, we have

$$\begin{aligned} f(2\alpha) &\equiv_8 4(\alpha-2\alpha)M(2\alpha-1)-4\alpha M(2\alpha-2) \\ &\equiv_8 4\alpha\ \omega_2(2\alpha)-4\alpha\ \omega_2(2\alpha)\equiv_8 0. \end{aligned}$$

ii) When $k=2\alpha+1$, we have

$$\begin{aligned} f(2\alpha+1) &\equiv_8 4(\alpha+1+2\alpha+1)M(2\alpha)-4\alpha M(2\alpha-1) \\ &\equiv_8 4\alpha\ \omega_2(2\alpha+2)-4\alpha\ \omega_2(2\alpha) \\ &\equiv_8 4\chi(\alpha\equiv_2 1)\omega_2(\alpha+1)\equiv_8 4\chi(k\equiv_4 3)\omega_2((k+1)/2). \end{aligned}$$

We also have

$$g(k)\equiv_8 -2\binom{2k+1}{2}M(2k-1)+4\binom{2k+1}{3}M(2k-2).$$

i) When $k=2\alpha$, we have

$$g(2\alpha)\equiv_8 -4\alpha M(4\alpha-1)\equiv_8 4\alpha\ \omega_2(4\alpha)\equiv_8 0.$$

ii) When $k=2\alpha+1$, we have

$$\begin{aligned} g(2\alpha+1) &\equiv_8 -2(2\alpha+3)M(4\alpha+1)+4M(4\alpha) \\ &\equiv_8 4\alpha M(4\alpha)-2M(4\alpha) \\ &\equiv_8 4\alpha-2M(4\alpha). \end{aligned}$$

This completes the proof. □

Now we are ready to prove Theorem 3, which, by Proposition 11, can be restated as Propositions 13 and 14 blow.

Proposition 13. *We have*

$$M(4s)\equiv_8 \begin{cases} 1-2L(\alpha)+4\alpha, & s=2\alpha, \\ 1-2L(\alpha), & s=2\alpha+1. \end{cases} \tag{12}$$

Proof. We apply Lemmas 6 and 12 to obtain

$$\begin{aligned} M(4s+4)-M(4s) &\equiv_8 f(2s)+f(2s+1)-2sM(2s)-g(s) \\ &\equiv_8 -2sM(2s)+\chi(s=2\alpha+1)(4\omega_2(2\alpha+2)-4\alpha+2M(4\alpha)). \end{aligned}$$

i) When $s=2\alpha$, we have

$$M(4s+4)-M(4s)\equiv_8 -4\alpha M(4\alpha)\equiv_8 4\alpha\ \omega_2(4\alpha+2)\equiv_8 4\alpha.$$

ii) When $s=2\alpha+1$, we have

$$\begin{aligned} M(4s+4)-M(4s) &\equiv_8 -2(2\alpha+1)M(4\alpha+2)+(4\omega_2(2\alpha+2)-4\alpha+2M(4\alpha)) \\ &\equiv_8 -2(M(4\alpha+2)-M(4\alpha))-4\alpha\ \omega_2(4\alpha+4)+4\omega_2(2\alpha+2)+4\alpha \\ \text{(by (10))} &\equiv_8 -2(M(2\alpha))+4(\alpha+1)\omega_2((\alpha+1))+4(\alpha+1) \\ &\equiv_8 -2M(2\alpha)+4(\alpha+1), \end{aligned}$$

where the last step is easily checked by considering the parity of α.

Finally, let $M'(4s)$ be defined by the right hand side of (12). Then $M'(0)=1$ and

$$\begin{aligned} M'(8\alpha+4)-M'(8\alpha) &\equiv_8 4\alpha, \\ M'(8\alpha+8)-M'(8\alpha+4) &\equiv_8 1-2L(\alpha+1)+4(\alpha+1)-1+2L(\alpha) \\ &\equiv_8 4(\alpha+1)-2M(2\alpha). \end{aligned}$$

Thus $M(4s)=M'(4s)$ and the proposition follows. □

The next results relies on Proposition 13.

Proposition 14. *We have*

$$M(4s+1)\equiv_8 \begin{cases} 1-2L(\alpha)+4\alpha, & s=2\alpha, \\ 1-2L(\alpha)+4, & s=2\alpha+1. \end{cases}$$

$$M(4s+2)\equiv_8 \begin{cases} 4, & s=(4\alpha+3)2^{2j}-1, \\ 2-4L(\alpha), & s=(4\alpha+1)2^{2j}-1, \\ -1+2L(\alpha), & s=(4\alpha+3)2^{2j+1}-1, \\ 3+2L(\alpha)+4\alpha, & s=(4\alpha+1)2^{2j+1}-1. \end{cases}$$

$$M(4s+3)\equiv_8 \begin{cases} -2+4L(\alpha), & s=(4\alpha+3)2^{2j}-1, \\ 4, & s=(4\alpha+1)2^{2j}-1, \\ -1+2L(\alpha), & s=(4\alpha+3)2^{2j+1}-1, \\ -1+2L(\alpha)+4\alpha, & s=(4\alpha+1)2^{2j+1}-1. \end{cases}$$

Proof. By Lemma 12, the odd case is reduced to the even case.

For $M(4s+1)$, we have

$$\begin{aligned} M(4s+1) &\equiv_8 (4s+1)M(4s) \\ &\equiv_8 4s+M(4s) \end{aligned}$$

$$\equiv_8 \begin{cases} 1-2L(\alpha)+4\alpha, & s=2\alpha,\\ 1-2L(\alpha)+4, & s=2\alpha+1. \end{cases}$$

For $M(4s+2)$, let β be odd. We simplify (9') using Lemma 12 and (12).

$$\begin{aligned} M(\beta 2^{a+2}-2)-M(\beta 2^{a+1}-2) &\equiv_8 M((2\alpha+1)2^{a+2}-4)+f((2\alpha+1)2^{a+1}-2)\\ &\equiv_8 \begin{cases} 1-2L((\beta-1)/2)+2(\beta-1) & a=0,\\ 1-2L(\beta 2^{a-1}-1) & a>0. \end{cases} \end{aligned} \tag{13}$$

Lemma 10 gives $L(2s+1)+L(s)\equiv_4 1$. Thus we have

$$M(\beta 2^{a+3}-2)-M(\beta 2^{a+1}-2)\equiv_8 2-2\left(L(\beta 2^{a-1}-1)+L(\beta 2^{a}-1)\right)\equiv_8 0, \qquad a>0.$$

This reduces $M(\beta 2^{a+1}-2)$ to the $a=0$ and $a=1$ case.

Moreover, setting $a=1$ in (13) gives

$$M(8\beta-2)\equiv_8 M(4\beta-2)+1-2L(\beta-1);$$

Setting $a=0$ in (13) gives

$$M(4\beta-2)\equiv_8 M(2\beta-2)+1-2L((\beta-1)/2)+2(\beta-1).$$

i) When $\beta=4\alpha+1$, we have

$$\begin{aligned} M((4\alpha+1)2^{2a+2}-2)\equiv_8 M(4(4\alpha+1)-2) &\equiv_8 M(8\alpha)+1-2L(2\alpha)\\ &\equiv_8 1-2L(\alpha)+4\alpha+1-2(2\alpha+L(\alpha))\\ &\equiv_8 2-4L(\alpha). \end{aligned}$$

Consequently,

$$\begin{aligned} M((4\alpha+1)2^{2a+3}-2)\equiv_8 M(8(4\alpha+1)-2) &\equiv_8 2-4L(\alpha)+1-2L(4\alpha)\\ &\equiv_8 3-4L(\alpha)-2(4\alpha+2\alpha+L(\alpha))\\ &\equiv_8 3+2L(\alpha)+4\alpha. \end{aligned}$$

ii) When $\beta=4\alpha+3$, we obtain

$$\begin{aligned} M((4\alpha+3)2^{2a+2}-2)\equiv_8 M(4(4\alpha+3)-2) &\equiv_8 M(8\alpha+4)+1-2L(2\alpha+1)+4(2\alpha+1)\\ &= 1-2L(\alpha)+1-2(1-L(\alpha))+4\\ &= 4. \end{aligned}$$

Consequently,

$$\begin{aligned} M((4\alpha+3)2^{2a+3}-2)\equiv_8 M(8(4\alpha+3)-2) &\equiv_8 4+1-2L(4\alpha+2)\\ &\equiv_8 5-2(4\alpha+2)+L(2\alpha+1)\\ &\equiv_8 1-2(1-L(\alpha)) \end{aligned}$$

$$\equiv_8 -1 + 2L(\alpha).$$

Finally, we compute $M(4s+3)$. By Lemma 12, we have

$$\begin{aligned} M(4s+3) &\equiv_8 (4s+3)M(4s+2) + g(2s+1) \\ &\equiv_8 -M(4s+2) + 4s - 2M(4s) \\ &\equiv_8 -M(4s+2) + 4s - 2(1+2s+2L(s)) \\ &\equiv_8 -M(4s+2) - 2 - 4L(s). \end{aligned}$$

i) When $\beta = 4\alpha+1$, we obtain

$$\begin{aligned} M((4\alpha+1)2^{2a+2}-1) &\equiv_8 -M((4\alpha+1)2^{2a+2}-2) - 2 - 4L((4\alpha+1)2^{2a}-1) \\ &\equiv_8 -2 + 4L(\alpha) - 2 - 4L(\alpha) \\ &\equiv_8 4. \end{aligned}$$

In the same way,

$$\begin{aligned} M((4\alpha+1)2^{2a+3}-1) &\equiv_8 -M((4\alpha+1)2^{2a+3}-2) - 2 - 4L((4\alpha+1)2^{2a+1}-1) \\ &\equiv_8 -3 - 2L(\alpha) - 4\alpha - 2 - 4L(\alpha) + 4 \\ &\equiv_8 -1 + 4\alpha + 2L(\alpha). \end{aligned}$$

ii) When $\beta = 4\alpha+3$, we have

$$\begin{aligned} M((4\alpha+3)2^{2a+2}-1) &\equiv_8 -M((4\alpha+3)2^{2a+2}-2) - 2 - 4L((4\alpha+3)2^{2a}-1) \\ &\equiv_8 -4 - 2 - 4L(\alpha) + 4 \\ &\equiv_8 -2 + 4L(\alpha). \end{aligned}$$

In the same way,

$$\begin{aligned} M((4\alpha+3)2^{2a+3}-1) &\equiv_8 -M((4\alpha+3)2^{2a+3}-2) - 2 - 4L((4\alpha+3)2^{2a+1}-1) \\ &\equiv_8 1 - 2L(\alpha) - 2 - 4L(\alpha) \\ &\equiv_8 -1 + 2L(\alpha). \end{aligned}$$

□

References

[1] J.-P. Allouche, J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in: C. Ding, T. Helleseth, H. Niederreiter (Eds.), Sequences and Their Applications, Proceedings of SETA'98, Springer, New York, 1999, pp. 1–16.

[2] K. S. Davis and W. A. Webb, Lucas' theorem for prime powers, European J. Combin. 11 (1990), 229–233.

[3] K. S. Davis and W. A. Webb, Pascal's triangle modulo 4, Fibonacci Quart. 29 (1991), 79–83.

[4] E. Deutsch and B. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, J. Number Theory 117 (2006), 191–215.

[5] L. E. Dickson, History of the Theory of Numbers, Vol. I, Chelsea, 1919 (Chapter XI).

[6] S.-P. Eu, S.-C. Liu, and Y.-N. Yeh, Catalan and Motzkin numbers modulo 4 and 8, European J. Combin. 29 (2008), 1449–1466.

[7] S.-P. Eu, S.-C. Liu,and Y.-N. Yeh, On the congruences of some combinatorial numbers, Stud. Appl. Math. 116 (2006), 135–144.

[8] I. M. Gessel, Some congruences for Apéry numbers, J. Number Theory 14 (1982), 362–368.

[9] A. Granville, Zaphod Beeblebrox's brain and the fifty-ninth row of Pascal's Triangle, Amer. Math. Monthly 99 (1992), 318–381.

[10] J. G. Huard, B. K. Spearman, and K. S. Williams, Pascal's triangle (mod 8), European J. Combin. 19 (1998), 45–62.

[11] M. Kauers, C. Krattenthaler, and T.W. Muüller, A method for determining the mod-2^k behaviour of recursive sequences, with applications to subgroup counting, Electron. J. Combin. 18(2) (2011), #P37.

[12] M. Klazar and F. Luca, On integrality and perioudicity of the Motzkin numbers, Aequationes Math. 69 (2005), 68–75.

[13] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, J. Reine Angew. Math. 44 (1852), 93–146.

[14] S.-C. Liu and J. C.-C. Yeh, Catalan numbers modulo 2^k, J. Integer Sequences 13 (2010), Art. 10.5.4, 26 pp.

[15] E. Lucas, Sur les congruences des nombres eulériens et des coeffcients différentiels des fonctions trigonométriques suivant un module premier, Bull. Soc. Math. France, 6 (1878), 49–54.

[16] Y. Mimura, Congruence properties of Apery numbers, J. Number Theory 16 (1983), 138–146.

[17] A. Postnikov and B. Sagan, Note: What power of two divides a weighted Catalan number, J. Combin. Theory Ser. A 114 (2007), 970–977.

[18] R. P. Stanley, Catalan addendum. New problems for Enumerative Combinatorics. Vol. 2, available at http://math.mit.edu/~rstan/ec/catadd.pdf.

[19] R. Sulanke and G. Xin, Hankel determinants for some common lattice paths, Adv. in Appl. Math., 40 (2008), 149–167.

[20] S. Wolfram, A New Kind of Science, Wolfram Media, 2002.

[21] Guoce Xin and Jing-Feng Xu, A short approach to Catalan numbers modulo 2^r, Electron. J. Combin. 18(1) (2011), #P177.

[22] D. Zeilberger, The method of creative telescoping, J. Symbolic Comput. 11 (1991), 195–204.

Addendum added April 3, 2018.

The main result of this paper has been also obtained, in a slightly different formulation, by Christian Krattenthaler and Thomas Müller in Theorem 11 of "Motzkin numbers and related sequences modulo powers of 2" (arXiv:1608.05657) using completely different methods.