Extremal Numbers for Directed Hypergraphs with Two Edges

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Abstract

Let a $2 \to 1$ directed hypergraph be a 3-uniform hypergraph where every edge has two tail vertices and one head vertex. For any such directed hypergraph F, let the nth extremal number of F be the maximum number of edges that any directed hypergraph on n vertices can have without containing a copy of F. In 2007, Langlois, Mubayi, Sloan, and Turán determined the exact extremal number for a particular directed hypergraph and found the extremal number up to asymptotic equivalence for a second directed hypergraph. Each of these forbidden graphs had exactly two edges. In this paper, we determine the exact extremal numbers for every $2 \to 1$ directed hypergraph that has exactly two edges.

1 Introduction

The combinatorial structure treated in this paper is a $2 \to 1$ directed hypergraph defined as follows.

Definition A $2 \to 1$ *directed hypergraph* is a pair $H = (V, E)$ where V is a finite set of *vertices* and the set of *edges* E is some subset of the set of all pointed 3-subsets of V. That is, each edge is three distinct elements of V with one marked as special. This special vertex can be thought of as the *head* vertex of the edge while the other two make up the *tail set* of the edge. If H is such that every 3-subset of V contains at most one edge of E, then we call H *oriented*. For a given H we will typically write its vertex and edge sets as $V(H)$ and $E(H)$. We will write an edge as $ab \to c$ when the underlying 3-set is $\{a, b, c\}$ and the head vertex is c.

For simplicity we will usually refer to $2 \to 1$ directed hypergraphs as *graphs* or sometimes as $(2 \to 1)$-*graphs* when needed to avoid confusion. This structure comes up as a particular instance of the model used to represent definite Horn formulas in the study of propositional logic and knowledge representation [1, 13]. Some combinatorial properties of this model were recently studied by Langlois, Mubayi, Sloan, and Turán in [10] and [9]. Before we can discuss their results we will need the following definitions.

Definition Given two graphs H and G, we call a function $\phi : V(H) \to V(G)$ a homomorphism if it preserves the edges of H:

$$ab \to c \in E(H) \implies \phi(a)\phi(b) \to \phi(c) \in E(G).$$

We will write $\phi : H \to G$ to indicate that ϕ is a homomorphism.

Definition Given a family $\mathcal{F}$ of graphs, we say that a graph G is $\mathcal{F}$*-free* if no injective homomorphism $\phi : F \to G$ exists for any $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$ we will write that G is F-free.

Definition Given a family $\mathcal{F}$ of graphs, let the *nth extremal number* $\text{ex}(n, \mathcal{F})$ denote the maximum number of edges that any $\mathcal{F}$-free graph on n vertices can have. Similarly, let the *nth oriented extremal number* $\text{ex}_o(n, \mathcal{F})$ be the maximum number of edges that any $\mathcal{F}$-free oriented graph on n vertices can have. Sometimes we will call the extremal number the *standard* extremal number or refer to the problem of determining the extremal number as the *standard version* of the problem to distinguish these concepts from their oriented counterparts. As before, if $\mathcal{F} = \{F\}$, then we will write $\text{ex}(n, F)$ or $\text{ex}_o(n, F)$ for simplicity.

Questions of finding extremal numbers for given forbidden graphs or hypergraphs are often called Turán-type extremal problems after Paul Turán due to his important early results and conjectures concerning forbidden complete r-graphs [14, 15, 16]. Turán problems for uniform hypergraphs make up a large and well-known area of research in combinatorics, and the questions are often surprisingly difficult to answer.

Extremal problems like this have also been considered for directed graphs and multigraphs (with bounded multiplicity) in [2] and [3] and for the more general directed multi-hypergraphs in [4]. In [3], Brown and Harary determined the extremal numbers for several types of specific directed graphs. In [2], Brown, Erdős, and Simonovits determined a general asymptotic structure of extremal sequences for every possible forbidden family of digraphs. These limiting structures turn out to be analogous to the familiar Turán graphs for simple 2-graphs.

The model of directed hypergraphs studied in [4] have r-uniform edges such that the vertices of each edge are given a linear ordering. However, there are many other ways that one could conceivably define a uniform directed hypergraph. The graph theoretic properties of a more general definition of a nonuniform directed hypergraph were studied by Gallo, Longo, Pallottino, and Nguyen in [7]. In that paper, a directed hyperedge was defined to be some subset of vertices with a partition into head vertices and tail vertices.

Recently in [5], this author tried to capture many of these possible definitions for "directed hypergraph" into one umbrella class of relational structures called generalized

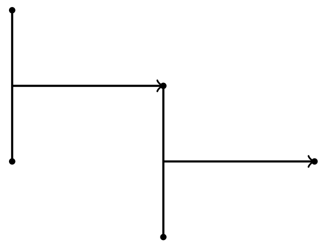

Figure 1: R_4

directed hypergraphs. The structures in this class include the uniform and simple versions of undirected hypergraphs, the totally directed hypergraphs studied in [4], the directed hypergraphs studied in [7], and the $2 \to 1$ model studied here and in [10, 9].

In [10, 9], the authors studied the extremal numbers for two small $(2 \to 1)$-graphs. They refer to these two graphs as the 4-resolvent and the 3-resolvent configurations after their relevance in propositional logic. Here, we will denote these graphs as R_4 and R_3 respectively and define them formally as

$$V(R_4) = \{a, b, c, d, e\} \text{ and } E(R_4) = \{ab \to c, cd \to e\}$$

and

$$V(R_3) = \{a, b, c, d\} \text{ and } E(R_3) = \{ab \to c, bc \to d\}.$$

2 The 4-resolvent graph R_4

In [9], the authors determined $\mathrm{ex}(n, R_4)$ exactly for sufficiently large n, and in [10] they determined the sequence $\mathrm{ex}(n, R_3)$ up to asymptotic equivalence. In these papers, the authors discuss a third graph with two edges which they call an Escher configuration because it calls to mind the famous M.C. Escher piece in which two hands draw each other. This graph is on four vertices $\{a, b, c, d\}$ and has edge set $\{ab \to c, cd \to b\}$. In this paper, we will denote this graph by E. These three graphs turn out to be the only three graphs with exactly two edges and more than three vertices for which the extremal numbers are cubic in n. They are also the only three with two edges on more than three vertices that do not satisfy the following definition.

Definition A graph H is *degenerate* if its vertices can be partitioned into three sets, $V(H) = T_1 \cup T_2 \cup K$ such that every edge of $E(H)$ is of the form $t_1t_2 \to k$ for some $t_1 \in T_1$, $t_2 \in T_2$, and $k \in K$.

An immediate consequence of a result shown in [5] is that the extremal numbers for a graph H are cubic in n if and only if H is not degenerate.

In our model of directed hypergraphs, there are nine different graphs with exactly two edges. Of these, four are not degenerate. One of these is the graph on three vertices with exactly two edges, $V = \{a, b, c\}$ and $E = \{ab \to c, ac \to b\}$. It is trivial to see that both the standard and oriented extremal numbers for this graph are $\binom{n}{3}$. The other three nondegenerate graphs are R_4, R_3, and E. We will determine both the standard and oriented extremal numbers for each of these graphs in Sections 2, 3, and 4 respectively.

Of the five degenerate graphs with exactly two edges, one has extremal numbers that are trivial to find. This is the graph with two independent edges, $V = \{a, b, c, d, e, f\}$ and $E = \{ab \to c, de \to f\}$. The extremal number for this graph comes directly from the known extremal number of the undirected 3-graph that consists of two independent edges - that is, the maximum number of edges in a 3-graph with edge intersection sizes never equal to zero. That extremal number is $\binom{n-1}{2}$ for sufficiently large n. Therefore, the oriented extremal number for two independent $2 \to 1$ edges is also $\binom{n-1}{2}$ and the standard extremal number is $3\binom{n-1}{2}$.

We will call the other four degenerate graphs with two edges I_0, I_1, H_1, and H_2 and define them as follows:

- $V(I_0) = \{a, b, c, d, x\}$ and $E(I_0) = \{ab \to x, cd \to x\}$
- $V(I_1) = \{a, b, c, d\}$ and $E(I_1) = \{ab \to c, ad \to c\}$
- $V(H_1) = \{a, b, c, d, x\}$ and $E(H_1) = \{ax \to b, cx \to d\}$
- $V(H_2) = \{a, b, c, d\}$ and $E(H_2) = \{ab \to c, ab \to d\}$

Here, the subscripts indicate the number of tail vertices common to both edges. The I graphs also share a head vertex while the H graphs do not. We will determine the oriented and extremal numbers for each of these graphs in Sections 5 – 8.

The proofs that follow rely heavily on the concept of a link graph. For undirected r-graphs, the link graph of a vertex is the $(r-1)$-graph induced on the remaining vertices such that each $(r-1)$-set is an $(r-1)$-edge if and only if that set together with the specified vertex makes an r-edge in the original r-graph [8]. In the directed hypergraph model here, there are a few ways that we could define the link graph of a vertex. We will need the following three definitions.

Definition Let $x \in V(H)$ for some graph H. The *tail link graph* of x T_x is the simple undirected 2-graph on the other $n-1$ vertices of $V(H)$ with edge set defined by all pairs of vertices that exist as tails pointing to x in some edge of H. That is, $V(T_x) = V(H) \setminus \{x\}$ and

$$E(T_x) = \{yz : yz \to x \in H\}.$$

The size of this set, $|T_x|$ will be called the *tail degree* of x. The degree of a particular vertex y in the tail link graph of x will be denoted $d_x(y)$.

Similarly, let D_x be the *directed link graph* of x on the remaining $n-1$ vertices of $V(H)$. That is, let $V(D_x) = V(H) \setminus \{x\}$ and

$$E(D_x) = \{y \to z : xy \to z \in E(H)\}.$$

Finally, let L_x denote the *total link graph* of x on the remaining $n-1$ vertices. That is, $V(L_x) = V(H) \setminus \{x\}$ and

$$E(L_x) = E(T_x) \cup E(D_x).$$

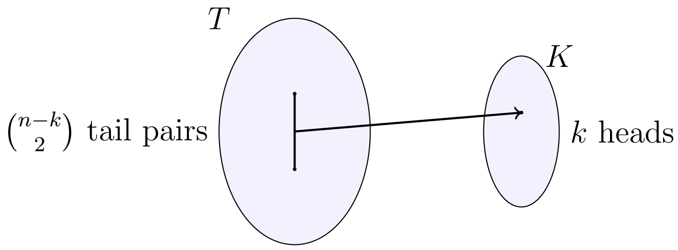

Figure 2: The lower bound construction for a graph with no R_4.

So L_x is a partially directed 2-graph.

The following notation will also be used when we want to count edges by tail sets.

Definition For any pair of vertices $x, y \in V(H)$ for some graph H let $t(x, y)$ denote the number of edges with tail set $\{x, y\}$. That is

$$t(x, y) = |\{v : xy \to v \in E(H)\}|.$$

In [10], the authors gave a simple construction for an R_4-free graph. Partition the vertices into sets T and K and take all possible edges with tail sets in T and head vertex in K. When there are n vertices, this construction gives $\binom{t}{2}(n-t)$ edges where $t = |T|$. This is optimized when $t = \lceil \frac{2n}{3} \rceil$. In [9], the authors showed that this number of edges is maximum for R_4-free graphs for sufficiently large n and that the construction is the unique extremal R_4-free graph.

We now give an alternate shorter proof that $\lfloor \frac{n}{3} \rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}$ is an upper bound on the extremal number for R_4 for sufficiently large n in both the standard and oriented versions of the problem. The proof also establishes the uniqueness of the construction.

Theorem 2.1. *For all $n \geqslant 29$,*

$$ex_o(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}$$

and for all $n \geqslant 56$,

$$ex(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}.$$

Moreover, in each case there is one unique extremal construction up to isomorphism when $n \equiv 0, 1$ mod 3 and exactly two when $n \equiv 2$ mod 3.

Proof. In either the standard or the oriented model, let H be an R_4-free graph on n vertices. Partition $V(H)$ into sets $T \cup K \cup B$ where T is the set of vertices that appear in tail sets of edges but never appear as the head of any edge, K is the set of vertices that do not belong to any tail set, and B is the set of vertices that appear as both heads and tails.

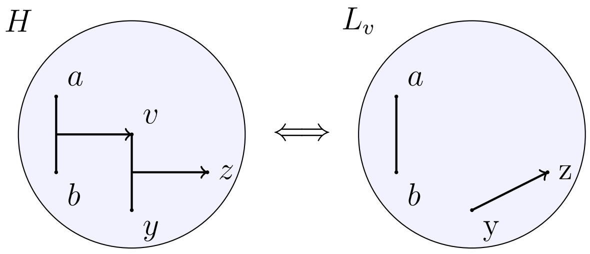

Figure 3: H contains a copy of R_4 if and only if the link graph of some vertex v contains a directed edge and an undirected edge that do not intersect.

If B is empty, then H is a subgraph of some R_4-free graph with the same structure as the lower bound construction. Therefore, H is either isomorphic to this construction or has strictly fewer edges. So assume that there exists some $v \in B$. The link graph L_v must contain at least one undirected edge and at least one directed edge. If any undirected edge is independent from any directed edge in L_v, then v would be the intersection vertex for an R_4 in H. Therefore, every directed edge in L_v is incident to every undirected edge.

We want to show that if $v \in B$, then $|E(L_v)| = O(n)$. Determining an upper bound on the number of edges in L_v is equivalent to determining an upper bound on the number of red and blue edges on $n-1$ vertices such that each red edge is incident to each blue edge and there is at least one edge of each color.

If we are working in the oriented model where multiple edges on the same triple are not allowed then no pair of vertices in L_v can hold more than one edge. If we are working in the standard model, then two vertices in this graph may have up to three edges between them, say two red and one blue.

First, we consider the oriented version. In this case we have at least one edge of each color and they must be incident. So let xy be blue and let yz be red. Then all other edges must be incident to x, y, or z. Moreover, any edge from x to the remaining $n-4$ vertices must be red since it is independent from yz and any edge from z to the remaining $n-4$ must be blue. Therefore, there are at most $2(n-4)$ edges from $\{x, y, z\}$ to the remaining $n-4$ vertices.

In the standard case our initial two red and blue edges may either be incident as before with xy blue and yz red or they might be incident in two vertices so that xy holds both a red and a blue edge. If none of the first type of incidence exists, then there can be at most 3 edges, all on xy.

So assume that the first type of incidence exists - xy is a blue edge and yz is a red edge. As before, all other edges must be incident to these three vertices such that any edge from x to the remaining $n-4$ vertices must be red, and any edge from z to these vertices must be blue. Edges from y may be either color.

However, note that if any vertex of the $n-4$ has a red edge from x, then none of the other vertices can have a blue edge from y or z. Similarly, any vertex with a blue edge from z means that no other vertices can have red edges from x or y. Therefore, if x has more than one red neighbor among the $n-4$ vertices, then there are at most $4(n-4)$

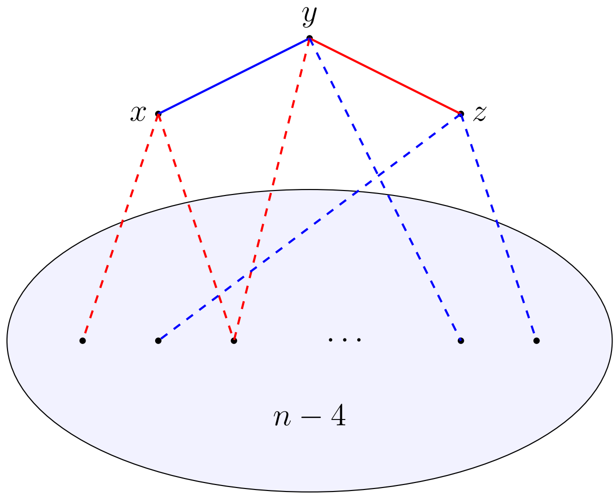

Figure 4: A simple graph on $n-1$ vertices with red and blue edges such that each red edge is incident to each blue edge and there is at least one blue edge, xy, and at least one red edge, yz, can have no edge contained in the remaining $n-4$ vertices. Moreover, only red edges can go from x to the remaining vertices and only blue edges can go from z to the remaining vertices.

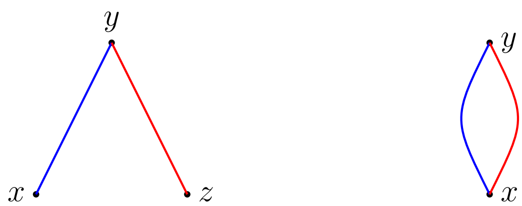

Figure 5: When two vertices are allowed to have up to two red edges and one blue edge, then an adjacent red and blue edge pair is either incident in one or two vertices.

edges between $\{x, y, z\}$ and the $n-4$ remaining vertices (since red edges have multiplicity up to 2). If z has more than one blue neighbor, then there are at most $2(n-4)$ edges between $\{x, y, z\}$ and the $n-4$ remaining vertices. Otherwise, x and z each have at most one neighbor among the $n-4$ vertices, and the best we can do is $3(n-4)$ edges, all from y. Therefore, there are at most $4(n-4)$ additional edges.

In either the standard or oriented versions of the problem, edges that do not contain vertices of B must have their tails in T and their heads in K. So there are at most

$$\left\lfloor \frac{n-b}{3} \right\rfloor \binom{\left\lceil \frac{2(n-b)}{3} \right\rceil}{2}$$

edges that do not intersect B where $b = |B|$. Hence,

$$|E(H)| < \left\lfloor \frac{n-b}{3} \right\rfloor \binom{\left\lceil \frac{2(n-b)}{3} \right\rceil}{2} + cnb$$

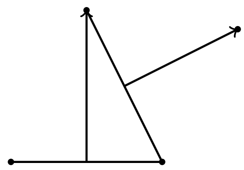

Figure 6: R_3

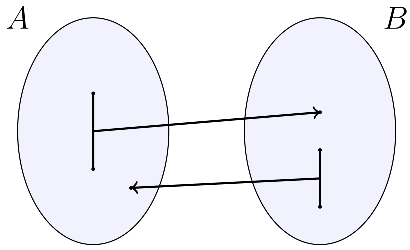

Figure 7: The unique R_3-free extremal construction.

where $c = 2$ in the oriented case and $c = 5$ in the standard case.

This expression is maximum on $b \in [0, n]$ only at the endpoint $b = 0$ for all $n \geqslant 29$ when $c = 2$ and for all $n \geqslant 56$ when $c = 4$.

Therefore, we can never do better than the lower bound construction. Moreover, since B must be empty to reach this bound, then the construction is unique when $n \equiv 0, 1 \bmod 3$. When $n \equiv 2 \bmod 3$, then

$$\left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2} = \left\lceil \frac{n}{3} \right\rceil \binom{\lfloor \frac{2n}{3} \rfloor}{2}$$

so there are exactly two non-isomorphic extremal constructions in that case. □

3 The 3-resolvent graph R_3

In [10], the authors gave a simple construction for an R_3-free graph. Partition the vertices into sets A and B and take all possible edges with a tail set in A and head vertex in B plus all possible edges with a tail set in B and head vertex in A. When there are n vertices, this construction gives $(n-a)\binom{a}{2} + a\binom{n-a}{2}$ edges where $a = |A|$. This is optimized when $a = \lceil \frac{n}{2} \rceil$. The authors showed that this number of edges is asymptotically equivalent to the sequence of extremal numbers for R_3-free graphs.

We show that in both the standard and the oriented versions of this problem that this construction is in fact the best that we can do. We will start with the oriented case since it is less technical.

3.1 The oriented version

Theorem 3.1. *For all n,*

$$ex_o(n, R_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Figure 8: Forbidden intersection types in L_x for any vertex x in an R_3-free graph.

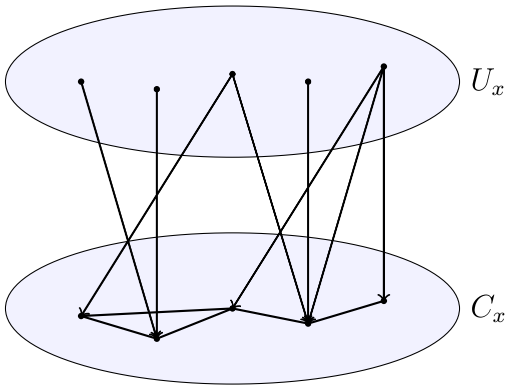

Figure 9: The structure of L_x for any x in an R_3-free graph

Moreover, there is one unique extremal R_3-free construction up to isomorphism for each n.

Proof. Let H be an R_3-free oriented graph on n vertices. Consider the total link graph, L_x, for some $x \in V(H)$. If

$$yz, z \to t \in E(L_x)$$

or if

$$y \to z, z \to t \in E(L_x),$$

then H is not R_3-free (See Figure 8).

Let $U_x \subseteq V(L_x)$ be the set of vertices that appear as the tail vertex of some directed edge in L_x. Then no edges of L_x can be contained entirely inside U_x - it is an independent set with respect to both directed and undirected edges. Moreover, all undirected edges of L_x must appear entirely within the complement, $C_x := V(L_x) \setminus U_x$. Hence, if we let $u_x = |U_x|$, then

$$2|E(H)| = \sum_{x \in V(H)} |D_x| \leqslant \sum_{x \in V(H)} u_x(n-1-u_x).$$

Each term of this sum is maximized when $u_x \in \left\{ \left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil \right\}$. Therefore, the result is immediate if n is even. The situation is slightly more complicated for odd n.

In this case,

$$u_x(n-1-u_x) \leqslant \left(\frac{n-1}{2} \right)^2$$

for each x. However, we need $u_x = \frac{n-1}{2}$ in order to attain this maximum value. This would mean that there are $\frac{n-1}{2}$ vertices in C_x, and so there are at most $\binom{\frac{n-1}{2}}{2}$ edges in T_x.

Therefore, if $u_x = \frac{n-1}{2}$ for each $x \in V(H)$, then

$$|E(H)| = \sum_{x \in V(H)} |T_x| < \frac{(n-2)(n-1)(n+1)}{8} = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Hence, we must assume that there exist some vertices for which $u_x \neq \frac{n-1}{2}$.

For each x let $i_x \in \{0, \ldots, \frac{n-1}{2}\}$ be the integer such that

$$u_x(n-1-u_x) = \left(\frac{n-1}{2} - i_x\right)\left(\frac{n-1}{2} + i_x\right).$$

Then,

$$|E(H)| \leqslant \frac{1}{2} \sum_{x \in V(H)} \left(\frac{n-1}{2} - i_x\right)\left(\frac{n-1}{2} + i_x\right) = \frac{n(n-1)^2}{8} - \frac{1}{2} \sum_{j=0}^{\frac{n-1}{2}} k_j j^2$$

where k_j is the number of vertices $x \in V(H)$ for which $i_x = j$.

Since the construction gives $\frac{(n-2)(n-1)(n+1)}{8}$ for odd n, then we are only interested in beating this. So set

$$\frac{(n-2)(n-1)(n+1)}{8} \leqslant \frac{n(n-1)^2}{8} - \frac{1}{2} \sum_{j=0}^{\frac{n-1}{2}} k_j j^2.$$

This gives

$$\sum_{j=0}^{\frac{n-1}{2}} k_j j^2 \leqslant \frac{n-1}{2}. \tag{1}$$

Since we can also find $|E(H)|$ by counting the number of undirected edges over the L_x, then we can upper bound the number of these by assuming $u_x = \frac{n-1}{2} - i_x$ for each x since this increases the size of C_x. This gives

$$|E(H)| \leqslant \sum_{x \in V(H)} \binom{\frac{n-1}{2} + i_x}{2} = \frac{n^3 - 4n^2 + 3n}{8} + \frac{1}{2} \sum_{j=0}^{\frac{n-1}{2}} j(n+j-2)k_j.$$

We can also set this greater than or equal to the known lower bound:

$$\frac{(n-2)(n-1)(n+1)}{8} \leqslant \frac{n^3 - 4n^2 + 3n}{8} + \frac{1}{2} \sum_{j=0}^{\frac{n-1}{2}} j(n+j-2)k_j$$

to get

$$\frac{(n-1)^2}{2} \leqslant \sum_{j=0}^{\frac{n-1}{2}} k_j j^2 + (n-2) \sum_{j=0}^{\frac{n-1}{2}} k_j j. \tag{2}$$

Subtracting (1) from (2) gives

$$\frac{(n-1)(n-2)}{2} \leqslant (n-2) \sum_{j=0}^{\frac{n-1}{2}} k_j j.$$

Therefore,

$$\sum_{j=0}^{\frac{n-1}{2}} k_j j^2 \leqslant \frac{n-1}{2} \leqslant \sum_{j=0}^{\frac{n-1}{2}} k_j j,$$

and so

$$0 \leqslant \sum_{j=0}^{\frac{n-1}{2}} k_j (j - j^2).$$

Since $j - j^2 < 0$ for any $j \geqslant 2$ and $j - j^2 = 0$ when $j = 0, 1$, then $k_j = 0$ for all $j \geqslant 2$.

Moreover, once all these are set to zero we get that

$$k_1 \leqslant \frac{n-1}{2} \leqslant k_1.$$

Therefore, $k_1 = \frac{n-1}{2}$ and so $k_0 = \frac{n+1}{2}$ since $\sum k_j = n$. This gives the desired upper bound.

Now we can show that the lower bound construction is the unique extremal example up to isomorphism. Let H be an extremal example on n vertices, and define a relation, $\sim$, on the vertices such that $x \sim y$ if and only if either $x = y$ or $y \in U_x$. This defines an equivalence relation on $V(H)$. Reflexivity and symmetry are both immediate. For transitivity note that the proof of the upper bound requires that every possible directed edge be taken from U_x to C_x for each $x \in V(H)$. Therefore, if we assume towards a contradiction that $y \in U_x$ and $z \in U_y$ but $z \notin U_x$, then $z \in C_x$. So $xy \to z \in E(H)$ which means $z \in C_y$, a contradiction.

When n is even there must be exactly two equivalence classes each of size $\frac{n}{2}$. Similarly, when n is odd there must be two equivalence classes of sizes $\frac{n-1}{2}$ and $\frac{n+1}{2}$. Therefore, the lower bound construction must be unique. □

3.2 The standard version

Theorem 3.2. *For all $n \geqslant 6$,*

$$ex(n, R_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Moreover, there is one unique extremal R_3-free construction up to isomorphism for each n.

Proof. Let H be an R_3-free graph on n vertices. Let $x \in V(H)$, and call any pair of vertices in L_x a multiedge if they contain more than one edge. Let $V(L_x) = U_x \cup C_x \cup M_x$ where M_x is the set of vertices that are incident to multiedges (that is, the minimal subset of

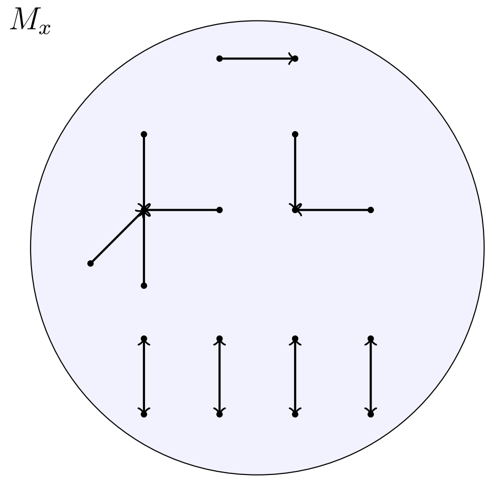

Figure 10: Example structure of M_x with 3 single directed edge stars and 4 double directed pairs.

vertices that contains all multiedges) and U_x and C_x are defined on the rest of the vertices as in Theorem 3.1. The goal is to show that if M_x is nonempty for any vertex x, then H has strictly fewer than the number of edges in the unique oriented construction given in Theorem 3.1. Therefore, that construction must be the unique extremal R_3 example for the standard problem as well.

There are three possibilities for multiedges in M_x: two oppositely directed edges, one directed edge and one undirected edge, and one undirected edge with two oppositely directed edges. If $y, z \in M_x$ have two directed edges between them, then neither y nor z is incident to any other edge in L_x since any incidence would create one of the two forbidden edge incidences of L_x as discussed in the previous theorem.

If y and z have only one directed edge (assume it is $y \to z$) and one undirected edge between them, then y cannot be incident to any more edges for the same reason as before, but z can be incident to undirected edges as well as directed edges with z at the head. This means that z may be the vertex of intersection of a star of these types of multiedges within M_x. Between any two such stars, the vertices of intersection may have an undirected edge between them, but no directed edges.

Therefore, the structure of the internal directed edges of M_x looks like Figure 10 with only the vertices of intersection of the single directed edge stars able to accept more edges from the rest of L_x. Directed edges from the rest of the graph to M_x must originate in U_x. Therefore, if M_x consists of d double directed edge pairs of vertices and k single directed stars with the ith star containing s_i vertices, then the total number of directed edges incident to vertices of M_x is at most

$$2d + \sum_{i=1}^{k} (s_i - 1 + u)$$

where u is the number of vertices in U_x.

If we assume that M_x is nonempty, then $|M_x| = m \geqslant 2$. The number of directed edges incident to or inside of M_x is at most $m + k(u-1)$. Therefore, for $u \geqslant 2$, the number of directed edges incident to vertices of M_x is maximized when the number of single directed edge stars is maximized. This is $\lfloor \frac{m}{2} \rfloor$ stars. Therefore, there are at most

$$\frac{m}{2}(u+1)$$

directed edges incident to vertices of M_x. Thus, if $|C_x| = c$, then L_x can have at most $uc + \frac{m}{2}(u+1)$ directed edges. And since $u \geqslant 2$, then

$$uc + \frac{m}{2}(u+1) < u(c+m).$$

So L_x has strictly less directed edges than a complete bipartite graph on the same number of vertices would. In Theorem 3.1 every L_x needed to be a complete bipartite graph in terms of the directed edges in order for the maximum number of edges to be obtained, and only in the case of odd n could some of these bipartitions be less than equal or almost equal. In those cases the parts could only have $\frac{n-1}{2} - 1$ and $\frac{n-1}{2} + 1$ vertices. Therefore, the only way that $u(c+m)$ could have more than this is if $u = c + m$ and so $u = \frac{n-1}{2}$.

We assume that $m \geqslant 2$ and $u \geqslant 2$, but if both are equal to 2, then $c = u - m = 0$ and $n = 4$, a contradiction since n is odd. Therefore, one of them must be strictly greater. So

$$uc + \frac{m}{2}(u+1) < (u-1)(u+1) = \left(\frac{n-1}{2} - 1\right)\left(\frac{n-1}{2} + 1\right).$$

This leaves only the cases where $u = 0$ and $u = 1$ which are both trivial.

So every link graph of H that contains a multiedge has strictly fewer than $(\frac{n-1}{2})^2 - 1$ directed edges. This is enough to prove that an extremal R_3-free graph on an even number of vertices must be oriented. However, if there are an odd number of vertices it is possible that there could be enough directed link graphs with the maximum $\left(\frac{n-1}{2}\right)^2$ directed edges to make up the deficit for the directed link graphs with strictly less than $\left(\frac{(n-3)(n+1)}{4}\right)$ due to multiedges.

In this case there would need to be at least $\frac{n+3}{2}$ vertices with directed link graphs that are complete bipartite graphs with parts of size $\frac{n-1}{2}$ each. Let S be the set of these vertices. For any $x, y \in S$ define the relation $x \sim y$ if and only if $y \in U_x$. As in the proof of Theorem 3.1, this turns out to be an equivalence relation. By the definition of S one equivalence class can hold at most $\frac{n+1}{2}$ vertices. So there must be two nonempty classes. Let these classes be A and B.

Given some $x, y \in A$, suppose there is some $z \notin S$ such that $z \in U_x$ and $z \notin U_y$. Then it follows that $z \in C_y$ and therefore there is an edge $xy \to z$ and an edge $xz \to w$ for some $w \in C_x$. Together these make a copy of R_3, a contradiction. Therefore, any z that is in U_x for some $x \in A$ is in U_y for all $y \in A$.

Let C be the set of vertices that are in every U_x for $x \in A$ but not in A itself. Since A is nonempty, there is at least one vertex $x \in A$, and by definition $|U_x| = \frac{n-1}{2}$. Therefore,

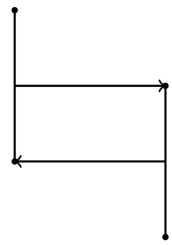

Figure 11: E

$|A| + |C| = \frac{n+1}{2}$. Similarly, let D be the set of vertices that are in every U_x for $x \in B$ but are not in B itself. By the same reasoning we get that $|B| + |D| = \frac{n+1}{2}$. Hence, $|A| + |B| + |C| + |D| = n + 1$. However, note that the sets A, B, C, and D are disjoint. So $|A| + |B| + |C| + |D| \leqslant n$, a contradiction. This is enough to show the result. □

4 The Escher graph E

In this section, we will prove the following result on the maximum number of edges of an E-free graph.

Theorem 4.1. *For all n,*

$$ex(n, E) = \binom{n}{3} + 2$$

and there are exactly two extremal construction up to isomorphism for each $n \geqslant 4$.

But first we will prove the easier oriented version of the problem. This result will be needed to prove Theorem 4.1.

4.1 The oriented version

Theorem 4.2. *For all n,*

$$ex_o(n, E) = \binom{n}{3}$$

and there is exactly one extremal construction up to isomorphism.

Proof. The upper bound here is trivial so we need only come up with an E-free construction that uses $\binom{n}{3}$ edges. Let H be the directed hypergraph defined on vertex set $V(H) = [n]$ and edge set,

$$E(H) = \{ab \to c : a < b < c\}.$$

That is take some linear ordering on the n vertices and for each triple direct the edge to the largest vertex. Then every triple has an edge and H contains no copy of E.

Now we will show that this construction is unique. Let H be an E-free graph on n vertices and $\binom{n}{3}$ edges. Define a relation on the vertices, $\prec$, where $x \prec y$ if and only if

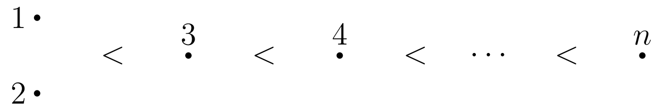

Figure 12: An “almost” linear ordering on the vertices of an E-free directed hypergraph.

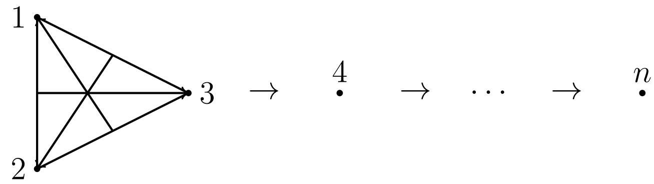

Figure 13: The first extremal construction, H_1, for an E-free directed hypergraph on n vertices.

there exists an edge in $E(H)$ with x in the tail and y as the head vertex. Then $\prec$ is a partial ordering of the vertices that is almost linear in that every pair of vertices are comparable except for the two smallest elements (see Figure 12). $\square$

We now shift our attention to the standard version of the problem where a triple of vertices can have more than one edge. Here, both of the lower bound constructions are similar to the unique extremal construction in the oriented version.

4.2 Two lower bound constructions for $\mathrm{ex}(n, E)$

The first construction is the same as the extremal construction in the oriented case but with two additional edges placed on the “smallest” triple. That is, let $H_1 = ([n], E_1)$ where

$$E_1 = \{ab \to c : a < b < c\} \cup \{13 \to 2, 23 \to 1\}.$$

See Figure 13.

Moreover, it is important to note that if an E-free graph with $\binom{n}{3} + 2$ edges has at least one edge on every vertex triple, then it must be isomorphic to H_1. This is because we can remove two edges to get an E-free subgraph where each triple has exactly one edge. Therefore, this must be the unique extremal construction established in Theorem 4.2. The only way to add two edges to this construction and avoid creating an Escher graph is to add the additional edges to the smallest triple under the ordering.

The second construction is also based on the oriented extremal construction. Let $H_2 = ([n], E_2)$ where

$$E_2 = (E_1 \setminus \{23 \to 4, 23 \to 1\}) \cup \{14 \to 2, 14 \to 3\}.$$

See Figure 14.

For the rest of this section we will show that any E-free graph is either isomorphic to one of these two constructions or has fewer than $\binom{n}{3} + 2$ edges. Roughly speaking, the

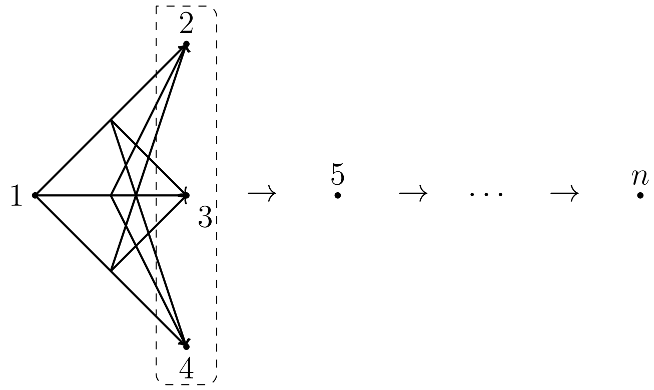

Figure 14: The second extremal construction, H_2, for an E-free graph on n vertices.

strategy is to take any E-free graph and show that we can add and remove edges to it so that we preserve E-freeness, remove most multiple edges from triples that had more than one, and never decrease the overall number of edges.

4.3 Add and Remove Edges

Let H be an E-free graph and represent its vertices as the disjoint union of three sets:

$$V(H) = D \cup R \cup T$$

where D (for 'Done') is the set of all vertices that have complete graphs on three or more vertices as tail link graphs, R (for 'Ready to change') is the set of vertices not in D that have at least three edges in their tail link graphs, and T is the set of all other vertices (those with 'Two or fewer edges in their tail link graphs').

The plan is now to remove and add edges in order make a new graph H' which is also E-free, has at least as many edges as H, and whose vertices make a disjoint union,

$$V(H') = D' \cup T'$$

where D' and T' are defined exactly the same as D and T except in terms of the vertices of H'.

That is, for each vertex $x \in R$, we will add all possible edges to complete T_x. This moves x from R to D. The edges removed will be all those that pointed from x to a vertex that points to x. This will destroy triples with more than one edge as we go. The following observation will ensure that this procedure only ever moves vertices from R to D, from R to T, from R to R, and from T to T. Since each step moves one vertex from R to D and ends when R is empty, then the procedure is finite. Here is the observation:

Lemma 4.3. *Let H be an E-free graph, and let $x, y \in V(H)$. If $d_x(y), d_y(x) > 0$, then $d_x(y) = d_y(x) = 1$. In other words, for any two vertices, x and y, if $d_y(x) \geqslant 2$, then $d_x(y) = 0$.*

Proof. Suppose not. Let $d_x(y), d_y(x) > 0$ and suppose $d_x(y) \geqslant 2$. Then there exist two distinct vertices, a and b such that

$$ay \to x, by \to x \in E(H).$$

There also exists a vertex c such that $xc \to y \in E(H)$. Since c must be distinct from either a or b if not both, then this yields an Escher graph. $\square$

Now, let us make the procedure slightly more formal: While there exist vertices in R, pick one, $x \in R$, and for each pair $a, b \in V(T_x)$, add the edge $ab \to x$ to $E(H)$ if it is not already an edge. Then, for each $a \in V(T_x)$, remove all edges of $E(H)$ of the form $xs \to a$ for any third vertex s.

Since there were at least three edges in T_x, then the added edges will move x from R to D. The removed edges, if any, will only affect vertices in R or in T since if xs is removed from T_a, then this implies that $a \in T_x$ and that $x \in T_a$ and so both had degree one in the other's tail link graph. Hence, $a \notin D$. Moreover, an affected vertex in R will either stay in R or move to T while an affected vertex in T will stay in T since it is only losing edges from its tail link graph.

At the end of this process D' will contain no triple of vertices with more than one edge. Therefore, the only such triples of vertices of H' will be entirely in T' or will consist of vertices from both T' and D'. We will show later that there cannot be too many of these triples. First, we need to show that after each step of this procedure, no Escher graph is created and at least as many edges are added to the graph as removed.

4.4 No copy of E is created and the number of edges can only increase

Fix a particular vertex $x \in R$ to move to D. Add and remove all of the designated edges. Suppose that we have created an Escher graph. Since the only added edges point to x, then the configuration must be of the form, $ab \to x, xc \to a$ for some distinct vertices, a, b, and c. Therefore, $a \in V(T_x)$ and so $xc \to a$ would have been removed in the process.

Now we will show that at least as many edges have been added to H as removed by induction on the number of independent edges in T_x. Start by assuming there are 0 independent edges in T_x and assume that there are k vertices in T_x that have degree one. Then at most k edges will be removed. If $k = 0$, then no edges are removed and there is a strict increase in the number of edges.

If $k = 1$, then let y_1 be the vertex with degree one and let y_2 be the vertex it is incident to. Since $d_x(y_2) \neq 1$ and $d_x(y_2) \geqslant 1$, then $d_x(y_2) \geqslant 2$. So there exists a third vertex, y_3, and similarly, $d_x(y_3) \geqslant 2$ but y_3 is not adjacent to y_1. Hence, there exists a fourth vertex, y_4. So at most one edge is removed and at least two edges are added, $y_1y_3 \to x$ and $y_1y_4 \to x$. Therefore, there is a strict increase in the number of edges.

If $k = 2$, then the fact that T_x has at least three edges means that there must be at least two additional vertices in T_x. Hence, at most two edges are removed but at least three are added. If $k \geqslant 3$, then at most k are removed but $\binom{k}{2}$ are added which nets

$$\binom{k}{2} - k = \frac{k(k-3)}{2} \geqslant 0$$

edges added.

Now, for the induction step, assume that T_x has $m > 0$ independent edges and that the process on a T_x with $m-1$ independent edges adds just as many edges as it removes. Let yz be an independent edge in T_x and let A be the set of vertices of T_x that are not y or z. Since T_x has at least three edges, then A contains at least three vertices. Therefore, the number of added edges is at least 6 between A and $\{y, z\}$. The number of edges removed from T_y and T_z together is at most 2. By assumption, the number of edges removed from the other tail link graphs of vertices in A is offset by the number of edges added inside A. Therefore, there is a strict increase in the number of edges.

To summarize, we have shown that H' is an E-free graph such that

$$|E(H)| \leqslant |E(H')|$$

and

$$V(H') = D' \cup T'$$

such that any triple of vertices of H' with more than one edge must intersect the set T'. We will now consider what is happening in T' by cases.

4.5 Case 1: $|T'| \geqslant 5$

Let $T' = \{x_1, x_2, \ldots, x_t\}$ for $t \geqslant 5$. For each x_i remove all edges of H' that have x_i as a head. By the definition of T' this will remove at most $2t$ edges from H'.

Next, add all edges to T' that follow the index ordering. That is, for each triple $\{x_i, x_j, x_k\}$ add the edge that points to the largest index, $x_i x_j \to x_k$ where $i < j < k$. This will add $\binom{t}{3}$ edges. The new graph has

$$\binom{t}{3} - 2t \geqslant 0$$

more edges than H'. Moreover, it is E-free and oriented. Therefore, $|E(H)| < \binom{n}{3}$.

4.6 Case 2: $|T'| \leqslant 4$ and there exists an $x \in T'$ such that T_x is two independent edges

Assume that some $x \in T'$ has a tail link graph T_x such that $ab, cd \in E(T_x)$ for four distinct vertices, $\{a, b, c, d\}$. If

$$d_a(x) = d_b(x) = d_c(x) = d_d(x) = 1,$$

then $a, b, c, d, x \in T'$, a contradiction of the assumption that $|T'| \leqslant 4$.

Therefore, we can add the edges

$$ac \to x, ad \to x, bc \to x, bd \to x$$

and remove any edges that point to a vertex from $\{a, b, c, d\}$ with x in the tail set. Because x has zero degree in at least one of those tail link graphs, then we have removed at most three edges and added four, a strict increase. We have also not created any triples of vertices with more than one edge or any Escher graphs.

We may now assume that $|T'| \leqslant 4$ and that the tail link graphs of vertices in T' are never two independent edges.

4.7 Case 3: $|T'| = 0, 1, 2$

First, note that if H' has a triple with more than one edge $\{x, y, z\}$ then at least two of its vertices must be in T' as a consequence of Lemma 4.3. Therefore, if $|T'| = 0, 1$, then H' is oriented and so

$$|E(H)| \leqslant |E(H')| \leqslant \binom{n}{3}.$$

Moreover, if $T' = \{x, y\}$ and H' is not oriented, then any vertex triple with more than one edge must have two edges of the form,

$$zx \to y, zy \to x$$

for some third vertex z. If there exist two such vertices $z_1 \neq z_2$ that satisfy this, then there would be an Escher graph. Hence, there is at most one vertex triple with more than one edge and it would have at most two edges. Therefore,

$$|E(H)| \leqslant |E(H')| \leqslant \binom{n}{3} + 1.$$

4.8 Case 4: $|T'| = 3$

First, suppose that there exists a triple $\{x, y, z\}$ with all three possible edges. Then $T' = \{x, y, z\}$. Since any triple with multiple edges must intersect T' in at least two vertices, then any additional such triple would make an Escher graph with one of the edges in T'. Therefore, H' has exactly one triple of vertices with all three edges on it and no others. So

$$|E(H) \leqslant |E(H')| \leqslant \binom{n}{3} + 2.$$

Moreover, to attain this number of edges, no triple of vertices can be empty of edges. In this case, H' must be isomorphic to the first construction H_1.

Next, assume that no triple of vertices has all three edges and let $T' = \{x, y, z\}$. Therefore, H' needs at least two triples of vertices that each hold two edges or else

$$|E(H)| \leqslant |E(H')| \leqslant \binom{n}{3} + 1$$

automatically. Suppose one of the multiedges is $\{x, y, z\}$ itself. Then without loss of generality let the edges be $xy \to z$ and $xz \to y$. The second triple with two edges must have its third vertex in D'. Call this vertex v. The vertex x cannot be in this second triple of vertices without creating an Escher graph. So the edges must be $vy \to z$ and $vz \to y$. But this also creates an Escher graph.

Therefore, neither of the two triples that hold two edges are contained entirely within T'. So without loss of generality they must be $vx \to y, vy \to x$ and $wy \to z, wz \to y$. If $v \neq w$, then $vx, wz \in T_y$, a contradiction to our assumption that T' contains no vertices with tail link graphs that are two independent edges. Hence, $v = w$.

Since $v \in D$, then T_v has at least three vertices. Moreover, since v is in the tail link graphs of each vertex of T', then none of these vertices can be in T_v. Remove all edges pointing to the vertices of T'. This is at most 6 edges. Add all possible edges with v as the head and a tail set among the set $V(T_v) \cup \{x, y, z\}$. This adds at least 12 new edges. The new graph is oriented and E-free. Therefore, $|E(H)| < \binom{n}{3}$.

4.9 Case 5: $|T'| = 4$

First, assume that there is some triple $\{x, y, z\}$ that contains all three possible edges. As before, there are no additional triples with more than one edge. So

$$|E(H)| \leqslant |E(H')| \leqslant \binom{n}{3} + 2.$$

The first construction H_1 is the unique extremal construction under this condition since all triples must be used at least once.

So assume that all triples with more than one edge have two edges each. Then we must have at least two. Assume that one of them is contained within $T' = \{a, b, c, d\}$. Without loss of generality let it be $ab \to c, ac \to b$. Since the second such triple intersects T' in at least two vertices, then it must intersect $\{a, b, c\}$ in at least one vertex.

If it intersects $\{a, b, c\}$ in two vertices, then without loss of generality (to avoid a copy of E) the second triple must be of the form $ab \to x, ax \to b$. Hence, $x \in T'$ so $x = d$.

But now there is no edge possible on $\{b, c, d\}$. Therefore, there must be a third such triple for H' to have $\binom{n}{3} + 2$ edges. This triple must be $ac \to d, ad \to c$. And the only way to actually make it to the maximum number of edges now must be to have an edge on every other triple.

Every triple of the form $\{b, c, s\}$ for $s \in D$ must have the edge $bc \to s$ since the other two options would create an Escher graph. Similarly, $bd \to s$ and $cd \to s$ are the only options for triples of the form $\{b, d, s\}$ and $\{c, d, s\}$ respectively. Next, any triple of the form $\{a, b, s\}$ must hold the edge $ab \to s$ since the other two edges create Escher graphs. Similarly, every triple of the forms $\{a, c, s\}$ and $\{a, d, s\}$ must hold the edges $ac \to s$ and $ad \to s$ respectively.

Since each triple contained in D holds exactly one edge, then the induced subgraph on D must be isomorphic to the oriented extremal example of an E-free graph on $n - 4$ vertices. Therefore, the entire graph H' must be isomorphic to the second extremal construction H_2 in order to attain $\binom{n}{3} + 2$ edges.

So assume that the second triple with two edges intersects $\{a, b, c\}$ in only one vertex. Then these edges must be $xa \to d, xd \to a$. This can be the only additional triple with two edges. So to make it to $\binom{n}{3} + 2$ edges we need each triple to have an edge. However, the edge for $\{a, b, d\}$ is forced to be $ad \to b$ and the edge for $\{b, c, d\}$ is forced to be

$bc \to d$. This makes an Escher graph. So

$$|E(H)| \leqslant |E(H')| \leqslant \binom{n}{3} + 1.$$

Now assume that no vertex triple with multiple edges is contained entirely within T', but assume that there are at least two such triples in H'. The only way that two triples could have distinct vertices in D' is if they were of the forms (without loss of generality), $xa \to b, xb \to a$, and $yc \to d, yd \to c$. Otherwise, the pairs of the two triples that are in T' would intersect resulting in either a copy of E (if both triples use the same pair) or a vertex in T' with two independent edges as a tail link graph.

So there must be exactly two such triples. Therefore, all other triples of vertices must contain exactly one edge in order to reach $\binom{n}{3} + 2$ edges overall. To avoid the forbidden subgraph this edge must be $ab \to c$ for the triple $\{a, b, c\}$ and $cd \to a$ for the triple $\{a, c, d\}$. But this is an Escher graph. Hence, not all triples may be used and so

$$|E(H)| \leqslant |E(H')| \leqslant \binom{n}{3} + 1.$$

Therefore, we may now assume for each multiedge triple that the vertex from D' is always x. First, assume that there are only two such triples. As before, if we assume that the only two such triples are $xa \to b, xb \to a$ and $xc \to d, xd \to c$, then there can be not be an edge on both $\{a, b, c\}$ and $\{a, c, d\}$. Hence, there would be a suboptimal number of edges overall.

On the other hand, if the only two such triples are adjacent in T', then they are, without loss of generality, $xa \to b, xb \to a$ and $xb \to c, xc \to a$. In this case, no edge can go on the triple $\{a, b, c\}$ at all and so there are at most $\binom{n}{3} + 1$ edges overall.

Therefore, we must assume there are at least three such triples that meet at x. If these three triples make a triangle in T', then they are $xa \to b, xb \to a$, $xb \to c, xc \to b$, and $xc \to a, xa \to c$. Again, there can be no edges on the triple $\{a, b, c\}$. Hence, every other triple must hold an edge to attain $\binom{n}{3} + 2$ edges overall.

On the triple $\{a, b, d\}$ this edge must be $ab \to d$ to avoid making a copy of E. Similarly, we must have the edges $ac \to d$ and $bc \to d$. But this means that $d \notin T'$, a contradiction.

On the other hand, if there are three triples of vertices with more than one edge on each that do not make a triangle in T' or if there are four or more such triples, then x is in the tail link graphs for each vertex in T'. Hence, none of these vertices may be in the tail link graph, T_x. However, $x \in D'$ so its tail link graph has at least three vertices. Remove all edges pointing to vertices of T' (at most 8). Add all edges pointing to x with tail sets in T' (6 new edges) and between T' and $V(T_x)$ (at least 12 new edges). So this adds at least ten edges to H' to create H''. H'' is oriented so

$$|E(H)| < |E(H'')| \leqslant \binom{n}{3}.$$

This exhausts all of the cases and establishes that

$$\mathrm{ex}(n, E) = \binom{n}{3} + 2$$

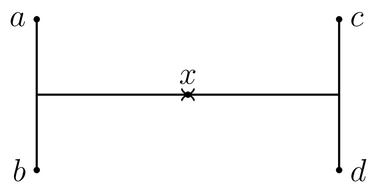

Figure 15: I_0

with exactly two extremal examples up to isomorphism.

5 Forbidden I_0

In this section I_0 will denote the forbidden graph where two edges intersect in exactly one vertex such that this vertex is the head of both edges. That is $V(I_0) = \{a, b, c, d, x\}$ and $E(I_0) = \{ab \to x, cd \to x\}$ (see Figure 15). In this section, we will prove the following result on the oriented extremal numbers of I_0.

Theorem 5.1. *For all $n \geqslant 9$,*

$$ex_o(n, I_0) = \begin{cases} n(n-3) + \frac{n}{3} & n \equiv 0 \textit{ mod 3} \\ n(n-3) + \frac{n-4}{3} & n \equiv 1 \textit{ mod 3} \\ n(n-3) + \frac{n-5}{3} & n \equiv 2 \textit{ mod 3} \end{cases}$$

with exactly one extremal example up to isomorphism when $3|n$, exactly 18 non-isomorphic extremal constructions when

$$n \equiv 1 \textit{ mod 3},$$

and exactly 32 constructions when

$$n \equiv 2 \textit{ mod 3}.$$

The proof for this is rather long. However, the standard version of the problem is much simpler so we will begin there.

Theorem 5.2. *For each $n \geqslant 5$,*

$$ex(n, I_0) = n(n-2)$$

and for each $n \geqslant 6$, there are exactly $(n-1)^n$ different labeled I_0-free graphs that attain this maximum number of edges.

Proof. Let H be I_0-free on $n \geqslant 5$ vertices. For any $x \in V(H)$, the tail link graph T_x cannot contain two independent edges (see Figure 16). Therefore, by the Erdős-Ko-Rado Theorem [6] the edge structure of T_x is either a triangle or a star with k edges all

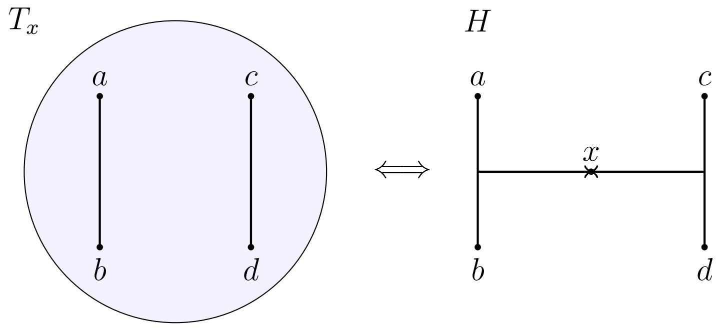

Figure 16: $ab, cd \in E(T_x)$ if and only if $ab \to x, cd \to x \in H$

intersecting in a common vertex for some $0 \leqslant k \leqslant n-2$. So each vertex $x \in V(H)$ is at the head of at most $n-2$ edges. Hence,

$$|E(H)| = \sum_{x \in V(H)} |E(T_x)| \leqslant n(n-2).$$

On the other hand, many different extremal constructions exist that give $n(n-2)$ edges on n vertices without the forbidden intersection. Let

$$f : [n] \to [n]$$

be any function such that $f(x) \neq x$ for any $x \in [n]$. Define H_f as the graph with vertex set $V(H_f) = [n]$ and edge set

$$E(H_f) = \bigcup_{x \in [n]} \{f(x)y \to x : y \in [n] \setminus \{x, f(x)\}\}.$$

Certainly each vertex x is at the head of $n-2$ edges and each of its tail sets contains $f(x)$ which prevents the forbidden subgraph. So $|E(H_f)| = n(n-2)$, and H_f is I_0-free for any such function f.

Moreover, there are $(n-1)^n$ different functions f that will make such a construction on $[n]$. So this gives us $(n-1)^n$ labeled extremal I_0-free graphs. Conversely, since any I_0-free graph with the maximum number of edges must have $n-2$ edges in T_x for each vertex x, then all tail link graphs must be $(n-2)$-stars for all $n \geqslant 6$. Therefore, these constructions give all possible extremal examples. □

The oriented version of this problem is less straight forward, but determining $\mathrm{ex}_o(n, I_0)$ also begins with the observation that every tail link graph of an I_0-free graph will either be a triangle, a star, or empty. Broadly speaking, as n gets large, it would make more sense for most, if not all, tail link graphs to be stars in order to fit as many edges into an I_0-free graph. This motivates the following auxiliary structure.

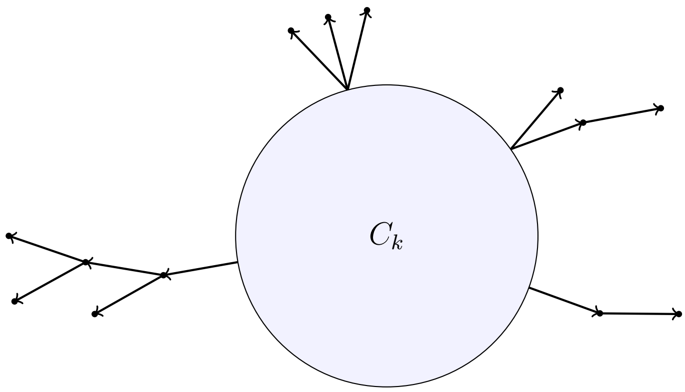

Figure 17: The structure of a connected component of the gate G

5.1 Gates

Let H be some I_0-free graph. For each $x \in V(H)$ for which T_x is a star (with at least one edge), let $g(x)$ denote the common vertex for the edges of T_x. We will refer to this vertex as the *gatekeeper* of x (in that it is the gatekeeper that any other vertex must pair with in order to "access" x). In the case where T_x contains only a single edge we may choose either of its vertices to serve as the gatekeeper. In this way, we have constructed a partial function, $g : V(H) \nrightarrow V(H)$.

Next, construct a directed 2-graph G on the vertex set $V(H)$ based on this partial function:

$$y \to x \in E(G) \iff y = g(x).$$

We will call this digraph the *gate* of H (or more properly, G is the gate of H *under* g since g is not necessarily unique).

The edge structure of any gate G is not difficult to determine. Since g is a partial function, then each vertex has in-degree at most one in G. Therefore, the structure of any connected component of G can be described as a directed cycle on k vertices, C_k, for $1 \leqslant k$ (where $k = 1$ implies a single vertex) unioned with k disjoint directed trees, each with its root vertex on this cycle (see Figure 17). We will refer to this kind of general structure as a *k-cycle with branches.*

Let

$$\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k$$

be the set of maximal connected components of a gate of H where, for each k, $\mathcal{C}_k$ is the set of maximal connected components that are k-cycles with branches. Note that

$$|E(H)| = \sum_{x \in V(H)} |T_x| = \sum_{C \in \mathcal{C}} \left(\sum_{x \in V(C)} |T_x| \right) = \sum_{k=1}^{n} \left(\sum_{C \in \mathcal{C}_k} \left(\sum_{x \in V(C)} |T_x| \right) \right).$$

The next section determines for each k an upper bound on

$$\sum_{x \in V(C)} |T_x|$$

as a function of the number of vertices, $|V(C)|$, for any $C \in \mathcal{C}_k$.

5.2 Bounding $\sum_{x \in V(C)} |T_x|$ for any connected component C of the gate

Loosely speaking, each gatekeeper edge of a connected component C represents at most $n-2$ edges of H. We will arrive at an upper bound on the sum $\sum_{x \in V(C)} |T_x|$ by adding this maximum for each edge of C, and then subtracting the number of triples of vertices that such a count has included more than once. This will happen for any triple of vertices which contain two or three gatekeeper edges. We make this observation formal in the following definition and lemma.

Definition Let G be some gate and let C be a maximal connected component of G. Let $P(C)$ be the set of $2 \to 1$ *possible edges* defined by

$$P(C) = \bigcup_{a \to b \in E(C)} \{av \to b : v \in V(H) \setminus \{a, b\}\}.$$

Lemma 5.3. *Let G be a gate, and let C be a maximal connected component of G. If a set of three distinct vertices $\{x, y, z\} \subseteq V(C)$ are spanned by two gatekeeper edges of G, then $P(C)$ contains at least two edges on these three vertices.*

Proof. Without loss of generality, the two spanning edges on $\{x, y, z\}$ are either of the form

$$x \to y \to z \text{ or } x \leftarrow y \to z.$$

In the former case, $P(C)$ contains the edges $xz \to y$ and $yx \to z$. In the latter case, $P(C)$ contains the edges $yz \to x$ and $yx \to z$. □

Now comes the main counting lemma.

Lemma 5.4. *Let H be an I_0-free graph on $n \geqslant 8$ vertices. Let G be a gate of H. Let C be a maximal connected component of G with m vertices. Then*

- $\sum_{x \in V(C)} |T_x| \leqslant m(n-3)$ *if $C \in \mathcal{C}_k$ for any $k \neq 3$ with equality possible only if $C = C_k$ for some $k \geqslant 4$,*
- $\sum_{x \in V(C)} |T_x| \leqslant m(n-3) + 1$ *if $C = C_3$, and*
- $\sum_{x \in V(C)} |T_x| \leqslant m(n-3)$ *for all other $C \in \mathcal{C}_3$ with equality possible only if C is a 3-cycle with exactly one nonempty directed path coming off of it.*

Proof. For convenience let

$$S = \sum_{x \in V(C)} |T_x|.$$

Note that for each $x \in V(C)$ with in-degree one, $ab \in T_x$ implies that $ab \to x \in P(C)$. Hence, if $C \notin \mathcal{C}_1$, then every edge counted in the sum S is in $P(C)$. Moreover, $|P(C)| = m(n-2)$.

If $C \in \mathcal{C}_k$ for $k \geqslant 4$, then by Lemma 5.3, each intersection of gatekeeper edges of C yields two edges on the same triple of vertices in $P(C)$. Conversely, since C contains no C_3, then each distinct triple of vertices contains at most two gatekeeper edges. Therefore, each triple contains at most two edges of $P(C)$. Hence,

$$S \leqslant m(n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2}$$

where $d_G(x)$ denotes the total number of vertices incident to x in the gate.

Since C has m edges, then $\sum_{x \in V(C)} d_G(x) = 2m$. So

$$S \leqslant m(n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2} \leqslant m(n-3)$$

by Jensen's Inequality. Moreover, equality happens if and only if $d_G(x) = d_G(y)$ for all $x, y \in V(C)$. Therefore, this inequality is strict for all $C \in \mathcal{C}_k$ unless $C = C_k$.

Similarly, if $C \in \mathcal{C}_2$, then $P(C)$ contains at least $\sum_{x \in V(C)} \binom{d_G(x)}{2}$ multiedges for the same reason as before. But here there are an additional $n-2$ edges counted for each triple containing the C_2. Also,

$$\sum_{x \in V(C)} d_G(x) = 2(m-1).$$

Hence,

$$S \leqslant m(n-2) - (n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2} \leqslant (m-1)(n-2) - m\binom{\frac{2(m-1)}{m}}{2}$$

by Jensen's Inequality. This is strictly less than $m(n-3)$.

In the acyclic case, Lemma 5.3 implies that the sum of all $|T_x|$ for each $x \in V(C)$ other than the root vertex is less than or equal to

$$(m-1)(n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2}.$$

The root vertex itself is the head vertex of at most 3 edges in H so Jensen's Inequality gives

$$S \leqslant (m-1)(n-2) - m\binom{\frac{2(m-1)}{m}}{2} + 3 < m(n-3)$$

for all $n \geqslant 8$.

Finally, if $C \in \mathcal{C}_3$, then each intersection of gatekeeper edges of C yields two edges on the same triple of vertices in $P(C)$. However, exactly one triple of vertices contains three gatekeeper edges and has three edges in $P(C)$. But the rest have at most two since there is only one triangle in C. Therefore, $\sum_{x \in V(C)} \binom{d_G(x)}{2}$ counts each triple of vertices that contain more than one gatekeeper edge exactly once except for the triple that makes up the C_3 which it counts three times. Since we must subtract off two edges in $P(C)$ on these three vertices to eliminate repeated triples, then we must subtract $\sum_{x \in V(C)} \binom{d_G(x)}{2} - 1$ from $|P(C)|$. Therefore,

$$S \leqslant m(n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2} + 1.$$

So by Jensen's Inequality,

$$S \leqslant m(n-3) + 1$$

with equality possible only if all of the degrees $d_G(x)$ are equal. This can only happen if $C = C_3$.

If we want to see for which $C \in \mathcal{C}_3$ the second best bound of $m(n-3)$ could be attained, then we need to set

$$\sum_{x \in V(C)} \binom{d_G(x)}{2} = m + 1.$$

Assume that the vertices are $x_1, \ldots, x_m$, and for each x_i let

$$d_i = d_G(x_i) - 2.$$

Then $\sum_{i=1}^m d_i = 0$ and a quick calculation shows that $\sum_{i=1}^m d_i^2 = 2$. Therefore, the only possibility is for some $d_i = 1$ and another to equal -1 and all the rest must be 0. This corresponds with one vertex degree equal to 3, another equal to 1, and all others equal to 2. The only way that this can happen in a C_3 with branches is to have exactly one branch, and that branch must be a directed path. □

This shows that the best we can hope for in terms of the average number of edges per vertex over any connected component of the gate is $n - 3 + \frac{1}{3}$, and this could be attained only in the case where the component is a directed triangle with no branches. Otherwise, the average number of edges of a component is at most $n-3$, and this is attainable only if the component is a directed triangle with a single directed path coming off of one of its vertices or a directed k-cycle with no branches for some $k \geqslant 4$.

This is enough for us to establish the upper bound for $\mathrm{ex}_o(n, I_0)$ and to characterize the necessary structure of the gate for any graph attaining this upper bound.

5.3 Upper Bound on $\mathrm{ex}_o(n, I_0)$

Let H be an I_0-free graph on $n \geqslant 9$ vertices. Let G be a gate of H. Let $\mathcal{C}$ be the set of maximal connected components of G and break $\mathcal{C}$ into three disjoint subsets based on the

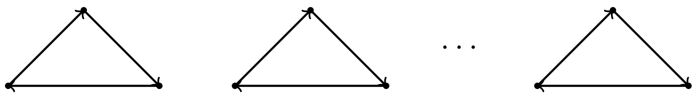

Figure 18: Structure of the gate for an extremal I_0-free graph when $n \equiv 0 \bmod 3$.

maximum average number of edges attainable for the components in each. That is, let

$$\mathcal{C} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

where $\mathcal{D}_1$ contains all components with maximum average number of edges per vertex strictly less than $n-3$: those components that are either acyclic, contain a C_2, contain a C_3 with nonempty branches that are more than just a single path, or contain a C_k for $k \geqslant 4$ with some nonempty branch; $\mathcal{D}_2$ is the set of all components with maximum number of edges per vertex of $n-3$: those that contain a directed C_3 and exactly one directed path or those that are a directed k-cycles for any $k \geqslant 4$ and no branches; and $\mathcal{D}_3$ is the set of components with a maximum average greater than $n-3$: the directed triangles.

For each i let d_i be the total number of vertices contained in the components of $\mathcal{D}_i$. Then

$$|E(H)| \leqslant d_3\left(n-3+\frac{1}{3}\right)+(n-d_3)(n-3)$$

with equality possible only if $d_1 = 0$. This is enough to prove the following.

Lemma 5.5. *Let H be an I_0-free graph on $n \geqslant 9$ vertices such that $n \equiv 0 \bmod 3$, then*

$$|E(H)| \leqslant n(n-3)+\frac{n}{3}.$$

Moreover, the only way for H to attain this maximum number of edges is if the gate of H is a disjoint union of directed triangles.

The next two lemmas give the maximum number when $n \equiv 1, 2 \bmod 3$. There is only slightly more to consider in these cases.

Lemma 5.6. *Let H be an I_0-free graph on $n \geqslant 9$ vertices such that $n \equiv 1 \bmod 3$, then*

$$|E(H)| \leqslant n(n-3)+\frac{n-4}{3}.$$

Moreover, the only way for H to attain this maximum number of edges is if the gate of H is a disjoint union of $\frac{n-4}{3}$ directed triangles together with either a directed C_4 or a 3-cycle with an extra edge.

Proof. Since $n \equiv 1 \bmod 3$, then $d_3 \leqslant n-1$. If $d_3 = n-1$, then the gate consists of $\frac{n-1}{3}$ disjoint directed triangles and one isolated vertex which means that

$$|E(H)| \leqslant (n-1)\left(n-3+\frac{1}{3}\right)+3.$$

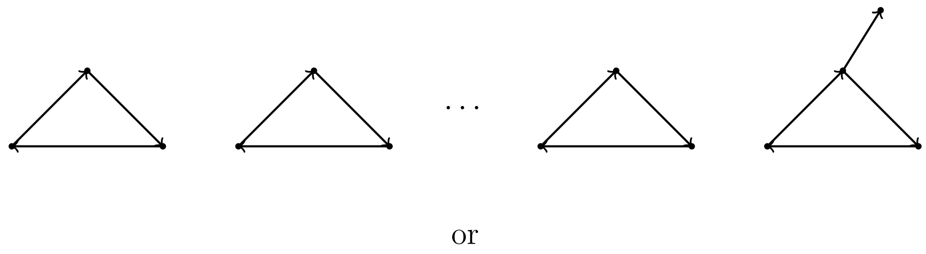

Figure 19: The only possible structures of the gate of an extremal I_0-free graph when $n \equiv 1 \bmod 3$.

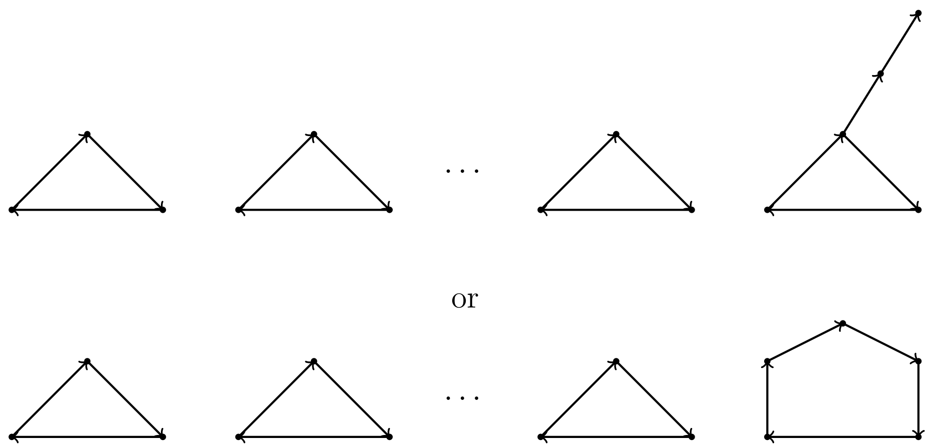

Figure 20: The only possible structures of the gate of an extremal I_0-free graph when $n \equiv 2 \bmod 3$.

If $d_3 \leqslant n-4$, then we can do better with

$$|E(H)| \leqslant (n-4)\left(n-3+\frac{1}{3}\right)+4(n-3)$$

only in the case of $\frac{n-4}{3}$ disjoint directed triangles and one component from $\mathcal{D}_2$ in the gate. Therefore,

$$|E(H)| \leqslant n(n-3)+\frac{n-4}{3}.$$

□

Lemma 5.7. *Let H be an I_0-free graph on $n \geqslant 11$ vertices such that $n \equiv 2 \bmod 3$, then*

$$|E(H)| \leqslant n(n-3)+\frac{n-5}{3}.$$

Moreover, the only way for H to attain this maximum number of edges is if the gate of H is a disjoint union of $\frac{n-5}{3}$ directed triangles together with either a directed C_5 or a 3-cycle with a directed path of two edges.

Proof. Since $n \equiv 2 \bmod 3$, then $d_3 \leqslant n-2$ and equality implies that G consists of $\frac{n-2}{3}$ disjoint directed triangles and two additional vertices that are either both isolated, contain one edge, or are a C_2 giving 6, $3+(n-2)$, or $n-2$ additional edges respectively. The best we can do when $d_3 = n-2$ is therefore,

$$|E(H)| \leqslant (n-2)\left(n-3+\frac{1}{3}\right)+(n+1).$$

Otherwise, $d_3 \leqslant n-5$ and the best we can do is

$$|E(H)| \leqslant (n-5)\left(n-3+\frac{1}{3}\right)+5(n-3).$$

This is better. Moreover, this will happen only when the five non-triangle vertices are in a component (or components) of G that give an average of $n-3$. So they must either make a C_5 or a directed triangle with one path. □

5.4 Lower bound constructions

The structure of the gates necessary to attain the maximum number of edges for a I_0-free graph determined in the previous section are also sufficient. Of these gates, none of them have acyclic components. Therefore, any graph that produces one of these gates has only vertices with stars for tail link graphs. This immediately implies that there is no I_0 in any graph that has such a gate.

Moreover, if H is a graph with a gate G that is one of these configurations, then

$$E(H) \subseteq \bigcup_{C \in \mathcal{C}} P(C)$$

where $\mathcal{C}$ is the set of maximal connected components of G. All that is left to do in order to construct an extremal example is to pick which edges of each $P(C)$ to delete in order to eliminate triples of vertices with more than one edge.

Lemma 5.8. *Let H be an I_0-free graph on $n \geqslant 9$ vertices such that $n \equiv 0 \bmod 3$, then*

$$|E(H)| \geqslant n(n-3)+\frac{n}{3}$$

and there is exactly one extremal construction up to isomorphism.

Proof. We know from Lemma 5.5 that the only way H can possibly attain $n(n-3)+\frac{n}{3}$ edges is if its gate is the disjoint union of $\frac{n}{3}$ directed triangles. Therefore, each $P(C_3)$ contains exactly one vertex triple with all three possible edges. So two of these must be deleted for each component in order to arrive at an extremal construction. The three choices for this deletion on each component are all isomorphic to each other. Therefore, there is exactly one extremal construction up to isomorphism. □

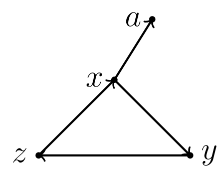

Figure 21: C_3 plus an edge

Figure 22: C_4 with 2 additional edges in opposite tail link graphs

Lemma 5.9. *Let H be an I_0-free graph on $n \geqslant 9$ vertices such that $n \equiv 1 \bmod 3$, then*

$$|E(H)| \geqslant n(n-3) + \frac{n-4}{3}$$

and there are exactly 18 extremal constructions up to isomorphism.

Proof. We know from Lemma 5.6 that if H has $n(n-3) + \frac{n-4}{3}$ edges, then its gate is the disjoint union of $\frac{n-4}{3}$ directed triangles with either a directed C_4 or a C_3 plus an edge on the remaining 4 vertices. As in the previous proof, there is only one choice up to isomorphism for which edges to delete from each $P(C_3)$. However, this will not be true of the last component on the remaining four vertices.

First, let's consider the case where the last component is a C_3 plus one edge. Call the vertices $\{x, y, z, a\}$ where $x \to y \to z \to x$ is the C_3 and $x \to a$ is the additional edge. First, note that we have the following three mutually exclusive choices for edges with head vertices in this component:

1. $xa \in T_y$ or $xy \in T_a$,

2. $za \in T_x$ or $xz \in T_a$, and

3. $zx \in T_y$, $yz \in T_x$, or $xy \in T_z$.

This gives 12 choices, and each choice is unique up to isomorphism.

Next consider the case of C_4. Each 3-subset of these four vertices holds two edges of $P(C)$ - one that points along the direction of the two gatekeeper edges and one that points the middle vertex of the two gatekeeper edges. For each triple one of these edges must be deleted to arrive at a legal oriented construction.

Each tail link graph must have at least $n-4$ edges, and combined they must contain four additional edges. Since each can have up to two more edges, then the distribution of these additional edges must be one of the following integer partitions of 4:

- 2, 2, 0, 0
- 2, 1, 1, 0

- 1, 1, 1, 1

There is only one choice up to isomorphism with a distribution of 2, 2, 0, 0. Each of the three ways to place 2, 1, 1, 0 around C_4 are possible but each distribution has only one way up to isomorphism. Finally, there are two ways up to isomorphism to put an extra edge into each tail link graph. So all together there are six nonisomorphic ways to distribute these extra edges to the C_4 tail link graphs. □

Lemma 5.10. *Let H be an I_0-free graph on $n \geqslant 9$ vertices such that $n \equiv 2 \mod 3$, then*

$$|E(H)| \geqslant n(n-3) + \frac{n-5}{3}$$

and there are exactly 32 extremal constructions up to isomorphism.

Proof. We can do the same kind of analysis when $n = 3k+2$ as in the previous proof. We know from Lemma 5.7 that the gate of any extremal construction must be all directed triangles together with either a directed C_5 or a directed triangle with a directed path of length two coming off of it (see Figure 20).

First, consider the C_5 case. Let the vertices be $\{x_0, \ldots, x_4\}$. For each gatekeeper edge, $x_i \to x_{i+1}$, every edge of the form $x_i v \to x_{i+1}$ must be an edge in H for any vertex

$$v \neq x_i, x_{i+1}, x_{i-1}, x_{i+2}.$$

Each gatekeeper edge can represent up to two additional edges of H, but again, every intersection of gatekeeper edges requires a mutually exclusive choice. Ultimately, we can add 5 additional edges so the extra edges must be distributed in one of the following ways:

- 2, 2, 1, 0, 0
- 2, 1, 1, 1, 0
- 1, 1, 1, 1, 1

There are 2 ways to get the first distribution up to isomorphism, 4 ways to get the second, and 2 ways to get the third. Therefore, there are 8 extremal constructions with this gate up to isomorphism.

Now consider the case of a directed triangle with a directed two path coming off of it. If we label the vertices as $\{x, y, z, a, b\}$ (see Figure 23), the mutually exclusive choices are

1. $ax \to y$ or $yx \to a$,
2. $az \to x$ or $zx \to a$,
3. $zx \to y$, $yz \to x$, or $xy \to z$, and
4. $xa \to b$ or $bx \to a$

This gives 24 ways of reaching the maximum, and each way is unique up to isomorphism. Therefore, there are 32 total distinct extremal graphs up to isomorphism. □

This establishes the main result of this section.

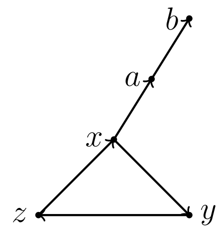

Figure 23: C_3 plus a 2-path

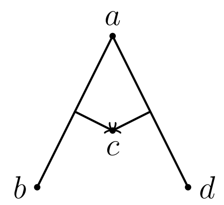

Figure 24: I_1

6 Forbidden I_1

In this section I_1 will denote the forbidden graph where two edges intersect in exactly two vertices such that one vertex is the head for both edges and the other is in the tail set of each edge. That is $V(I_1) = \{a, b, c, d\}$ and $E(I_1) = \{ab \to c, ad \to c\}$ (see Figure 24).

Theorem 6.1. *For all $n \geqslant 4$,*

$$ex(n, I_1) = ex_o(n, I_1) = n \left\lfloor \frac{n-1}{2} \right\rfloor$$

and there are

$$\left(\frac{(n-1)!}{2^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left\lfloor \frac{n-1}{2} \right\rfloor !} \right)^n$$

labeled graphs that attain this maximum in the standard case.

Proof. Let H be an I_1-free graph on n vertices. For any $x \in V(H)$, T_x is a simple undirected 2-graph on $n-1$ vertices such that no two edges are adjacent (this is true for either version of the problem). Therefore, the edges of T_x are a matching on at most $n-1$ vertices. So there are at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ edges in T_x for every $x \in V(H)$. Thus,

$$|E(H)| = \sum_{x \in V(H)} |T_x| \leqslant n \left\lfloor \frac{n-1}{2} \right\rfloor .$$

This shows the upper bound for both versions.

Now we want to find lower bound constructions. In the standard version of the problem there are many extremal constructions since for each vertex x, we may pick any maximum matching on the remaining $n-1$ vertices to serve as the edges of T_x. So

$$\mathrm{ex}(n, I_1) = n\left\lfloor \frac{n-1}{2} \right\rfloor.$$

Moreover, the number of labeled graphs that attain this maximum equals the number of ways to take a maximum matching to construct each tail link graph. For even k, the number of matchings on k vertices is

$$M_k = (k-1)M_{k-2}$$

since if we fix some vertex, then we can pick any of the remaining $k-1$ vertices to go with it and then take the number of matchings on the remaining $n-2$. Since $M_2 = 1$, then in general for even k,

$$M_k = \prod_{i=1}^{\frac{k}{2}} (2i-1).$$

If k is odd, then we can first select the vertex left out of the matching to get

$$M_k = kM_{k-1} = k \cdot \prod_{i=1}^{\frac{k-1}{2}} (2i-1) = \prod_{i=1}^{\frac{k+1}{2}} (2i-1).$$

Therefore, the number of labeled extremal I_1-free graphs on n vertices is

$$\left(\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2i-1) \right)^n = \left(\frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor !} \right)^n.$$

In the oriented version of the problem we need to be more careful with the construction. First, assume that n is even and define a graph H with vertex set $V(H) = \mathbb{Z}_n$ and edge set

$$E(H) = \bigcup_{i=0}^{n-1} \left\{ (i+2k)(i+2k+1) \to i : k = 1, 2, \ldots, \frac{n-2}{2} \right\}.$$

This construction creates a maximum matching for each tail link graph (with $i+1$ as the odd vertex out for each T_i). So H has the extremal number of edges and contains no I_1. Therefore, all we need to show is that it has no triple with more than one edge.

If H does contain such a triple, then there exist three integers in $\mathbb{Z}_n$ that can be represented as both $\{a, a+2k, a+2k+1\}$ and $\{b, b+2i, b+2i+1\}$ with $a \neq b$. Without loss of generality we can assume that $b = 0$. If $a + 2k = 0$, then $a + 2k + 1 = 1$, but 1 is not in any tail set that points to 0. Therefore, it must be the case that $a + 2k + 1 = 0$, but then $a + 2k = n - 1$. Therefore, the set is equal to $\{0, n-1, n-2\}$, and $a = n-2$, but $n-1$ does not point to $n-2$, a contradiction. Therefore, H can have no such triple.

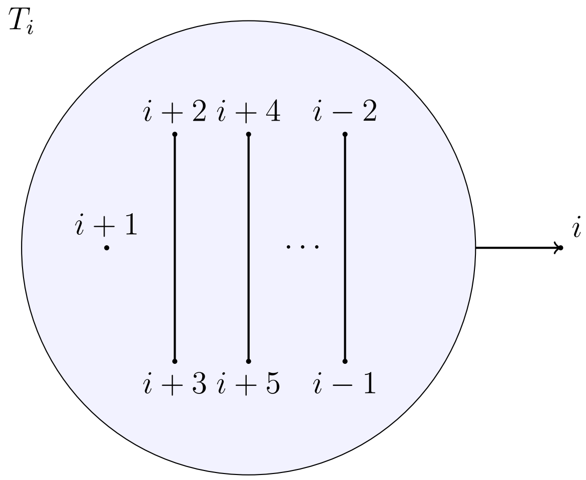

Figure 25: T_i in the oriented extremal construction for even n

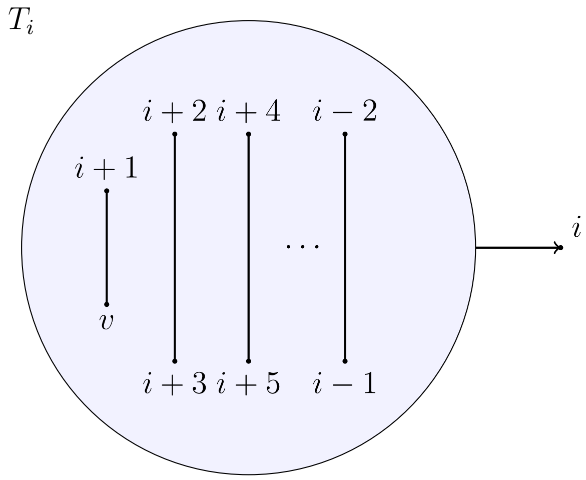

Figure 26: T_i in the oriented extremal construction on $n+1$ vertices for even n

Now, we consider odd $n+1$. Here, let $V(H) = \mathbb{Z}_n \cup \{v\}$ where v is a new vertex and use all of the edges from the even construction plus some new ones that all contain v. So $E(H) = E_{even} \cup E_{new} \cup E_v$ where

$$E_{even} = \bigcup_{i=0}^{n-1} \left\{ (i+2k)(i+2k+1) \to i : k = 1, 2, \dots, \frac{n-2}{2} \right\},$$

and

$$E_{new} = \{v(i+1) \to i : i = 0, 1, \dots, n-1\}.$$

Certainly, the construction has so far avoided the forbidden subgraph and given each of the first n vertices the maximum number of tails. Now E_v can be constructed as any set of $\frac{n}{2}$ disjoint pairs of elements from $\mathbb{Z}_n$ all pointing at v so that no pair consists of two

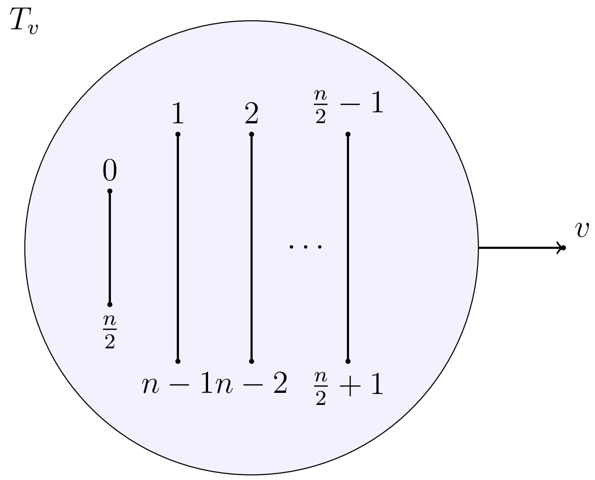

Figure 27: T_v in the oriented extremal construction on $n+1$ vertices for even n

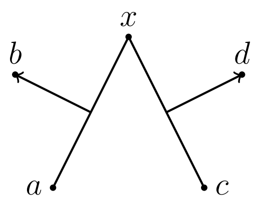

Figure 28: H_1

sequential numbers mod n. So any maximum matching of the n elements that observes this condition will do.

In particular, we can let

$$E_v = \left\{(i)(n-i) \to v : i = 1, \ldots, \frac{n}{2} - 1\right\} \cup \left\{(0)\left(\frac{n}{2}\right) \to v\right\}.$$

So

$$\mathrm{ex}_o(n, I_1) = n \left\lfloor \frac{n-1}{2} \right\rfloor. \qquad \square$$

7 Forbidden H_1

In this section H_1 will denote the forbidden graph where two edges intersect in exactly one vertex such that it is in the tail set of each edge. That is $V(H_1) = \{a, b, c, d, x\}$ and $E(H_1) = \{ax \to b, cx \to d\}$ (see Figure 28). First we will show the following result for the oriented version of the problem.

Theorem 7.1. *For all $n \geqslant 6$,*

$$ex_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

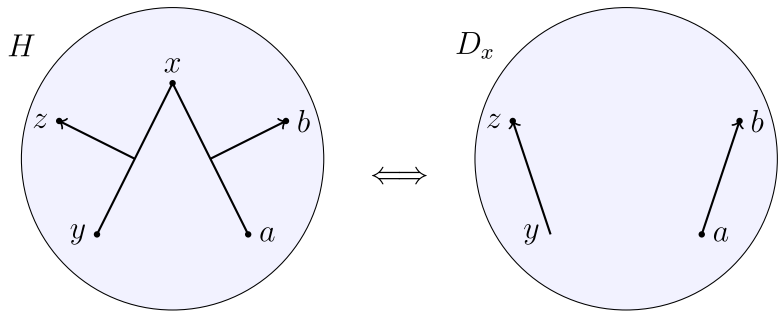

Figure 29: H has a copy of H_1 with intersection vertex x if and only if the directed link graph D_x has a pair of disjoint directed edges.

We will use this result to solve the standard version of the problem.

Theorem 7.2. *For all $n \geqslant 8$,*

$$ex(n, H_1) = \binom{n+1}{2} - 3.$$

Moreover, there is one unique extremal construction up to isomorphism for each n.

First, note that the proof of Theorem 7.1 is straightforward when n is even. To get a lower bound construction we can take a maximum matching on the n vertices and use each pair of this matching as the tail set to point at all $n-2$ other vertices. That is, let H be the graph with vertex set,

$$V(H) = \{x_1, \ldots, x_{\frac{n}{2}}, y_1, \ldots, y_{\frac{n}{2}}\}$$

and edge set,

$$E(H) = \bigcup_{i=1}^{\frac{n}{2}} \{x_i y_i \to z : z \in V(H) \setminus \{x_i, y_i\}\}.$$

To show that this is also an upper bound, let H be an H_1-free oriented graph on n vertices. Then for any $x \in V(H)$, the directed link graph D_x cannot have two independent edges (see Figure 29). Therefore, D_x is either empty, a triangle, or a star with at most $n-2$ edges. Since $n \geqslant 5$, then $|D_x| \leqslant n-2$ for each x. So

$$|E(H)| = \frac{1}{2} \sum_{x \in V(H)} |D_x| \leqslant \frac{1}{2} n(n-2).$$

Hence, we are finished for even n. However, this proof falls apart when n is odd. We will need a different strategy.

7.1 Counting edges by possible tail pairs

The basis of our strategy in getting an upper bound on $\mathrm{ex}_o(n, H_1)$ is to count the edges of an H_1-free graph H by its tail sets. That is,

$$|E(H)| = \sum_{\{x,y\} \in \binom{V(H)}{2}} t(x, y)$$

It is simple but important to note that if H is H_1-free, then any two pairs of vertices that each points to two or more other vertices must necessarily be disjoint.

Lemma 7.3. *Let H be a H_1-free oriented graph. If $x_1, x_2, y_1, y_2 \in V(H)$ so that*

$$t(x_1, y_1), t(x_2, y_2) \geqslant 2$$

and $\{x_1, y_1\} \neq \{x_2, y_2\}$, then $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$.

Proof. Suppose, towards a contradiction, that $x_1 = x_2 = x$ but $y_1 \neq y_2$. Since $t(x, y_1) \geqslant 2$, then there exists some vertex z_1 distinct from x, y_1, and y_2 such that

$$xy_1 \to z_1 \in E(H).$$

Similarly, since $t(x, y_2) \geqslant 2$, then there exists some vertex z_2 distinct from x, y_1, and y_2 such that

$$xy_2 \to z_2 \in E(H).$$

If $z_1 \neq z_2$, then this gives a copy of H_1.

So assume that they are the same vertex, $z_1 = z_2 = z$. Since $t(x, y_1) \geqslant 2$, then there is some second vertex that x and y_1 point to that is distinct from z. The only choice that would not create a copy of H_1 with the edge $xy_2 \to z$ is y_2. Similarly, since $t(x, y_2) \geqslant 2$, then there is some second vertex that x and y_2 point to that is distinct from z. The only choice that would not create a copy of H_1 with the edge $xy_1 \to z$ is y_1. So

$$xy_1 \to y_2, xy_2 \to y_1 \in E(H)$$

which contradicts the fact that H is oriented. □

Therefore, if we assume that H is H_1-free on n vertices, then we can split its vertices up into k disjoint pairs such that each serves as a tail set to at least two edges of H plus a set of $n - 2k$ vertices that belong to no such pair. That is,

$$V(H) = \{x_1, y_1, \ldots, x_k, y_k\} \cup R$$

so that $t(x_i, y_i) \geqslant 2$ for $i = 1, \ldots, k$ and $t(w, v) \leqslant 1$ for all other vertex pairs, $\{w, v\}$ (see Figure 30).

We now have two cases to consider. Either there are no such pairs ($k = 0$) or there is at least one ($k \geqslant 1$).

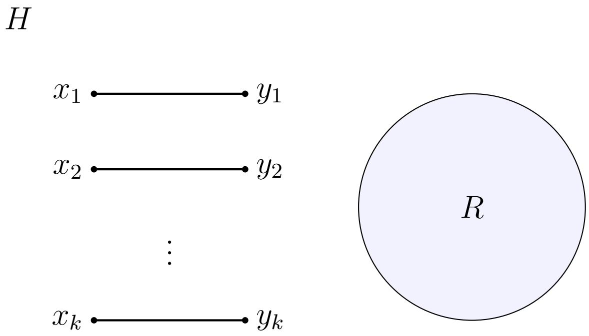

Figure 30: An H_1-free graph on n vertices breaks down into k disjoint pairs that each point to at least two other vertices plus a remainder set R with $n-2k$ vertices that belong to no such pair.

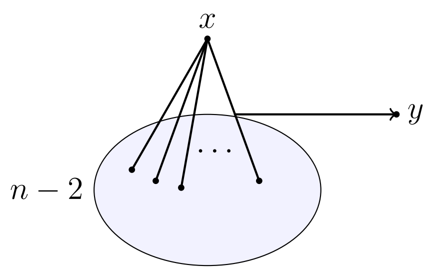

Figure 31: The special case configuration discussed in Lemma 7.4. Here, vertex x joins with every other element to point to vertex y.

7.2 No pair points to more than one vertex ($k = 0$)

Assume that $k = 0$. Then $t(x, y) \leqslant 1$ for every pair $\{x, y\} \in \binom{V(H)}{2}$. If $|D_x| \leqslant n - 3$ for all $x \in V(H)$, then

$$|E(H)| = \frac{1}{2} \sum_{x \in V(H)} |D_x| \leqslant \frac{1}{2} n(n-3) < \frac{1}{2}(n-1)(n-2)$$

and we are done. Otherwise, there exists some vertex x that belongs to $n - 2$ tail sets. Therefore, D_x is a star of directed edges with some common vertex of intersection y. Either $t(x, y) = 0$ or $t(x, y) = 1$.

If $t(x, y) = 0$, then all of the $n-2$ directed edges of D_x must point to y (see Figure 31). Such a configuration in H limits the number of edges to $\binom{n-1}{2}$ as proven in Lemma 7.4.

On the other hand, if $t(x, y) = 1$, then $xy \to z \in E(H)$ for some vertex z, and $xv \to y$ for all other vertices $v \neq x, y, z$. Such a configuration in H will limit the number of edges to $\binom{n-1}{2}$ as proven in Lemma 7.5.

Lemma 7.4. *Let H be an oriented graph on $n \geqslant 6$ vertices such that $t(x, y) \leqslant 1$ for each pair $\{x, y\} \in \binom{V(H)}{2}$. If H is H_1-free and contains vertices x and y such that $xv \to y \in$*

$E(H)$ for each $v \in V(H) \setminus \{x, y\}$, then

$$|E(H)| \leqslant \binom{n-1}{2}.$$

See Figure 31.

Proof. We want to show that there can be no more than $\binom{n-2}{2}$ additional edges in H other than the $n-2$ edges described in the statement of the lemma. This would give an upper bound on the total number of edges of

$$\binom{n-2}{2} + (n-2) = \binom{n-1}{2}.$$

First, note that every triple of the form $\{x, y, v\}$ already holds an edge. This implies that any additional edge cannot contain both x and y since H is oriented. On the other hand, if we were to add an edge, $vw \to u$, that excluded both x and y completely, then this new edge would create a copy of H_1 with the existing edge, $vx \to y$. Therefore, every additional edge must be on a triple of the form $\{v, w, x\}$ or $\{v, w, y\}$.

However, x is already in the maximum number of tails. So given any pair of non-$\{x, y\}$ vertices, $\{v, w\}$, the only possible additional edges are

$$vw \to x, vw \to y, yv \to w, \text{and } yw \to v.$$

The last three all appear on the triple, $\{v, w, y\}$, and are therefore mutually exclusive choices when it comes to adding them to the graph. The first two are also mutually exclusive choices since $t(v, w) \leqslant 1$.

So assume, towards a contradiction, that we could add $\binom{n-2}{2} + 1$ more edges to the existing configuration. Then some pair $\{v, w\}$ of non-$\{x, y\}$ vertices must be used twice. Without loss of generality, this means we must add the edges $vw \to x$ and $yv \to w$.

Now, let u be any of the remaining $n-4$ vertices. The possible edge $uv \to y$ would create a copy of H_1 with $vw \to x$, and the possible edge $uv \to x$ would create a copy of H_1 with $vy \to w$. Therefore, the pair $\{v, u\}$ cannot be a tail set for any edge.

We can also view the potential additional edges as two different types: those that have a tail set of two non-$\{x, y\}$ vertices and those that have a tail set that includes y. There were originally at most $\binom{n-2}{2}$ of the first type that we are allowed to add in total, one edge for every distinct pair. However, v can now no longer be in a tail set with any of the other $n-4$ vertices. So there are now at most $\binom{n-2}{2} - (n-4)$ edges of this first type left possible to add. Therefore, in order to add $\binom{n-2}{2} + 1$ edges over all, we will need at least $n-3$ of them to be of the second type - those that have y in the tail set.

Note that x must be an isolated vertex in the directed link graph D_y. Hence, there are at most $n-3$ tails containing y since otherwise the directed graph D_y would have $n-2$ edges among $n-2$ vertices. In this case, D_y would have two independent directed edges and so H would have a copy of H_1 with y as its intersection vertex. Moreover, D_y must be a star with a single vertex of intersection. Since $v \to w \in E(D_y)$, then this vertex of intersection must either be v or w.

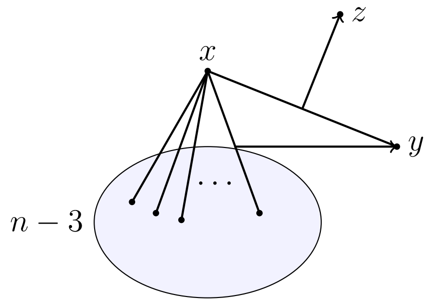

Figure 32: The special case configuration discussed in Lemma 7.5. Here, x joins with every vertex except z to point to y and then joins with y to point to z.

Hence, in order to add $\binom{n-2}{2}+1$ edges, we will need to have $\binom{n-2}{2}-(n-4)$ edges that have non-$\{x,y\}$ tail sets. Since the tail set, $\{v,w\}$, already points to x, then this implies that all such edges must also point to x. Otherwise, we would have some edge of the form $ab \to y$. If $a = w$ or $b = w$, then this would create a copy of H_1 with $vw \to x$. If both elements are distinct from w, then we would still need to point the pair wa either to x or to y. Either choice would create a copy of H_1.

Let u be one of the remaining vertices. Then u must be adjacent to a directed edge of D_y for there to be $n-3$ edges added with y in the tail set. If v is the vertex of intersection of D_y, then this edge must either be $u \to v$ or $v \to u$. Either yields a copy of H_1. Similarly, if w is the vertex of intersection of D_y, then either $wy \to u \in E(H)$ or $uy \to w \in E(H)$. Again, either of these yields a copy of H_1. Therefore, $\binom{n-2}{2}+1$ edges cannot be added to the existing configuration. □

Lemma 7.5. *Let H be an oriented graph on $n \geqslant 6$ vertices such that for each pair $x, y \in V(H)$, $t(x,y) \leqslant 1$. If H is H_1-free and contains vertices x, y, and z such that $xy \to z \in E(H)$ and $xv \to y \in E(H)$ for each $v \in V(H) \setminus \{x,y,z\}$ (see Figure 32), then*

$$|E(H)| \leqslant \binom{n-1}{2}.$$

Proof. Let $W = \{1, 2, \ldots, n-3\}$ be the set of non-$\{x,y,z\}$ vertices. Any additional edge to this graph must have a tail set of the form $\{i,j\}$, $\{i,y\}$, $\{i,z\}$, or $\{y,z\}$ for $i,j \in W$. An ij tail can only point to x or to y and there are $\binom{n-3}{2}$ pairs like this possible. An iy tail cannot point to x because H is oriented. It cannot point to j since that would create a copy of H_1 with $xy \to z$. Therefore, it could only point to z. An iz tail could not point to any j since this would create a copy of H_1 with the edge $ix \to y$. Therefore, it could only point to y or to x. And a yz tail could not point to x since H is oriented. Therefore, it could only point to some i.

Assume, towards a contradiction, that we can add

$$\binom{n-2}{2}+1 = \binom{n-3}{2}+(n-3)+1$$

edges to the existing configuration. Since we can add at most $\binom{n-3}{2}$ edges with tail sets made entirely of vertices from W, then we must have at least $n-2$ additional edges from the other possibilities.

For each $i \in W$ we could have

$$iy \to z, yz \to i, iz \to y, \text{and } iz \to x.$$

The first three of these are mutually exclusive choices since they are all on the same triple. Similarly, the last two are mutually exclusive choices since we are only allowing up to one edge per possible tail set.

Therefore, in order to add $n-2$ of these types of edges, two must use the same element of W. Given the mutually exclusive choices above this implies that there is some vertex $i \in W$ such that either $iz \to x, yi \to z \in E(H)$ or $iz \to x, yz \to i \in E(H)$.

In the first case, ij is no longer a possible tail for any edge for all $n-4$ remaining vertices $j \in W$. This is because $iz \to x$, $yi \to z$, and $ix \to y$ create a triangle in D_i. So any additional edge with i in the tail would give two independent edges in D_i and therefore a copy of H_1.

Hence, we can get at most $\binom{n-3}{2} - (n-4)$ edges with tails in W. This means that we will need $2(n-3)$ edges from the other possible edges to make up the difference if we want to add

$$\binom{n-3}{2} + (n-3) + 1$$

more edges.

Since each of the $n-3$ vertices from W can be in up to two of these additional edges, then $iz \to x$ would need to be an edge for every $i \in W$ and that $\{y, z, i\}$ also needs to hold one edge for every $i \in W$.

If $yz \to i$ is used once, then we get a copy of H_1 with $jz \to x$ for some other $j \in W$. Therefore, for all $i \in W$ we must have the edges $iy \to z$ and $iz \to x$. However, any pair $i, j \in W$ can now point to nothing since the only possibilities for such a tail were x or y to begin with and both of these options create copies of H_1. So in this case the most that we can add is

$$2(n-3) \leqslant \binom{n-3}{2} + (n-3)$$

for all $n \geqslant 6$.

In the other case we have added $iz \to x$ and $yz \to i$ for some i. Which means that $yz \to j$ is not allowed for any $j \neq i$ from W. Also, $jz \to y$ would make a copy of H_1 with $iz \to x$ and $jz \to x$ would make a copy of H_1 with $yz \to i$. Therefore, for all $j \neq i$ we can only add the edge $jy \to z$.

In order to add $\binom{n-3}{2} + n - 2$ edges, we will need all of these as well as all possible edges with tails in W. However, since $iz \to x$, all of the edges with tails completely in W must also point to x. Otherwise, some pair ab would point to y. If $a = i$ or $b = i$, then this would make a copy of H_1 with $iz \to x$. If $i \neq a, b$, then consider where the pair ai points. It must either point to x or to y, but either of these would create a copy of H_1.

So all pairs of W must point to x and for all $j \in W$ not equal to i we must have the edge $jy \to z$. But $jy \to z$ and $ij \to x$ create a copy of H_1, a contradiction. Hence, it is not possible to add more than $\binom{n-3}{2} + (n-3)$ edges to the configuration. Since the configuration already has $n-2$ edges, then there can be no more than $\binom{n-1}{2}$ edges total. $\square$

Together these two lemmas take care of the cases where all pairs of vertices point to at most one vertex in H.

7.3 At least one pair of vertices is the tail set to more than one edge of H ($k > 0$)

We return to our description of an H_1-free oriented graph as being made up of $k \geqslant 1$ vertex pairs that each serve as tail sets to strictly more than one edge plus a set R of the remaining $n-2k$ vertices,

$$V(H) = \{x_1, y_1, \dots, x_k, y_k\} \cup R$$

(see Figure 30). For each pair $\{x_i, y_i\}$ we want to prove the following upper bound,

$$t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \leqslant n - 2.$$

That is, the total number of edges that include either x_i or y_i or both in the tail set is at most $n-2$.

Now,

$$|E(H)| = \sum_{\{x,y\} \in \binom{V(H)}{2}} t(x,y) \leqslant \sum_{i=1}^{k} \left(t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \right) + \sum_{\{x,y\} \in \binom{R}{2}} t(x,y).$$

Note that each pair of vertices in R act as a tail set at most once so

$$\sum_{\{x,y\} \in \binom{R}{2}} t(x,y) \leqslant \binom{n-2k}{2}.$$

Therefore, proving the upper bound for each $\{x_i, y_i\}$ pair would imply that

$$|E(H)| \leqslant k(n-2) + \binom{n-2k}{2}.$$

Since

$$k(n-2) + \binom{n-2k}{2} = 2k^2 - (n+1)k + \binom{n}{2}$$

is a quadratic polynomial with positive leading coefficient in terms of k, then it is maximized at the endpoints. Here, that means at $k = 1$ and at $k = \lfloor \frac{n}{2} \rfloor$.

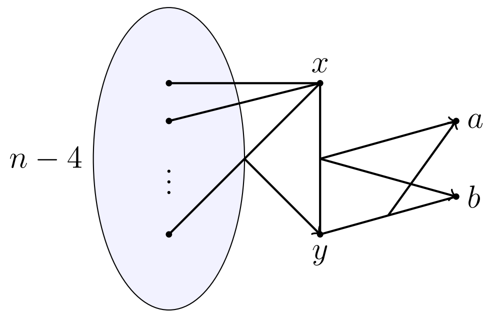

Figure 33: An H_1-free graph containing this configuration with have at most $\binom{n-1}{2}$ edges as shown in Lemma 7.6.

When n is odd, both of these values for k give the upper bound,

$$|E(H)| \leqslant \binom{n-1}{2}.$$

Only when n is even can we do better and get

$$|E(H)| \leqslant \frac{n(n-2)}{2}$$

in the case where $k = \frac{n}{2}$. In either case this give an upper bound of

$$|E(H)| \leqslant \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

So we need only prove that, in general,

$$t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \leqslant n - 2.$$

This is straightforward to show if $t(x_i, y_i) \geqslant 3$. However, when $t(x_i, y_i) = 2$ there is a case where it fails to hold. This is taken care of in the following lemma.

Lemma 7.6. *Let H be an oriented graph on $n \geqslant 6$ vertices. If H is H_1-free and contains vertices x, y, a, and b such that $\{x, y\}$ is the tail set to exactly 2 edges with*

$$xy \to a, xy \to b, yb \to a \in E(H),$$

and for each $v \in V(H) \setminus \{x, y, a, b\}$, $xv \to y$ (see Figure 33), then

$$|E(H)| \leqslant \binom{n-1}{2}.$$

Proof. First consider which pairs of vertices could possibly be a tail set to an edge in this graph. Let $W = \{1, \ldots, n-4\}$ be the set of vertices other than $\{x, y, a, b\}$. Then $\{i, j\}$ can be a tail set to $ij \to x$ and $ij \to y$ for any pair $i, j \in W$. Since $xy \to a$, then xi can point to nothing other than y. Similarly, xa and xb could only possibly point to b and a respectively, but either would create a copy of H_1 with $xi \to y$ for any $i \in W$. Also, by assumption xy can point to nothing else. Hence, x is in no additional tail sets.

Since $yb \to a$ and $xy \to a$, then ya cannot point to b or to x. It can also not point to any $i \in W$ since this would create a copy of H_1 with $xy \to b$. So y can be in no additional tails. The pair ab can point to anything aside from y since H is oriented, and ai can point to x or y for any $i \in W$ but not to b or another element of W since either would create a copy of H_1 with $xi \to y$. Similarly, bi can point to y for each $i \in W$ but not to x or to a or to another element of W since these would create a copy of H_1 with either $yb \to a$ or $xi \to y$.

Leaving aside the edges with tail sets completely in W for the moment, this means there are $4(n-4)+1$ possible edges remaining. There are $n-4$ each of types $ai \to x$, $ai \to y$, $bi \to y$, and $ab \to i$ plus one extra edge which is $ab \to x$.

Suppose we are able to use at least $2(n-4)+1$ of these edges. First, if one of them is $ab \to x$, then there could be none of the types $ai \to y$ or $bi \to y$. So all of the ones of type $ab \to i$ and $ai \to x$ would need to be used. But since $n \geqslant 6$, there are at least two vertices in W. So there would exist edges $ai \to x$ and $ab \to j$ with $i \neq j$, a copy of H_1. Therefore, $ab \to x$ cannot be used if we want to get more than $2(n-4)$ of these edges.

Hence, we need to use at least three types of edges from the four possible types. Since any of the types $ai \to x$, $ai \to y$, and $bi \to y$ eliminate the possibility of using any edge $ab \to j$ where $j \neq i$, then we can use at most one of this last type of edge. But since $n \geqslant 6$, then $2(n-4)+1 \geqslant 5$ which means one of the other types gets used at least twice. Regardless of which one it is, there can be nothing used from the $ab \to i$ types of edges.

Therefore, we must use $2(n-4)+1$ edges from only the first three types. So there must be a vertex i from W that belongs to three of these edges, say

$$ai \to y, bi \to y, \text{and } ai \to x.$$

But the edges $bi \to y$ and $ai \to x$ form an H_1, a contradiction. Thus, at most $2(n-4)$ of these kinds of edges can be used over all.

Now let us look at the edges with tail sets contained in W. We have seen that each ij can point to x or to y, but nothing so far has kept the pair from pointing to both. However, if some pair does point to both, then no other tail could use either of these vertices since this would create a copy of H_1. Therefore, if there are $1 \leqslant l$ such pairs, then there are at most $2l + \binom{n-4-2l}{2}$ edges with tails from W. If $n = 6$, then this gives exactly one such pair and only 7 edges overall. If $n \geqslant 7$, then $l \leqslant \frac{n-4}{2}$ implies that

$$2l \leqslant n-4 \leqslant \binom{n-4}{2} - \binom{n-4-2l}{2}.$$

Hence,

$$2l + \binom{n-4-2l}{2} \leqslant \binom{n-4}{2}.$$

So there are at most $\binom{n-1}{2}$ edges in H. □

7.4 First main result, $\mathrm{ex}_o(n, H_1) = \lfloor \frac{n}{2} \rfloor (n-2)$.

Now we can proceed with establishing the upper bound under the assumption that the configuration presented in Lemma 7.6 does not occur in our directed hypergraph. As we've seen, all that is necessary to show is that

$$t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \leqslant n - 2$$

for any pair of vertices $\{x_i, y_i\}$ that serves as the tail set to at least two edges.

So let $\{x, y\}$ be such a pair, and divide the rest of the vertices of H into two groups, those that are a head vertex to some edge with xy as the tail and those that are not. That is,

$$V(H) \setminus \{x, y\} = \{h_1, \dots, h_m\} \cup \{n_1, \dots, n_t\}$$

where for each $i = 1, \dots, m$, there exists an edge, $xy \to h_i \in E(H)$ and for each $j = 1, \dots, t$, $xy \to n_j \notin E(H)$ (note that $t(x, y) = m$ and that $m + t = n - 2$).

Now, consider an edge that contains either x or y in the tail but not both. Then the other tail vertex is either some h_i or some n_j. In the case of n_j, this edge must either be of the form $xn_j \to y$ or $yn_j \to x$ to avoid a copy of H_1 with both $xy \to h_1$ and $xy \to h_2$. Moreover, since H is oriented, there can be at most one. Hence,

$$\sum_{j=1}^{t} (t(x, n_j) + t(y, n_j)) \leqslant t.$$

Now consider a tail set that includes either x or y and some h_i. Without loss of generality, assume that xh_1 is the tail to some edge. Since $t(x, y) \geqslant 2$, there is some other vertex h_2 such that $xy \to h_2 \in E(H)$. In order to avoid a copy of H_1 with this edge, xh_1 must either point to y or to h_2. However, $xh_1 \to y \notin E(H)$ since this would give the triple $\{x, y, h_1\}$ more than one edge.

Therefore, $xh_1 \to h_2$ is the only option. However, if $t(x, y) \geqslant 3$, then this will create a copy of H_1 with $xy \to h_3$. So xh_i and yh_i cannot be tails to any edge. So

$$\sum_{i=1}^{m} (t(x, h_i) + t(y, h_i)) = 0.$$

Therefore,

$$\begin{aligned} & t(x, y) + \sum_{v \neq x, y} (t(x, v) + t(y, v)) \\ & = m + \sum_{j=1}^{t} (t(x, n_j) + t(y, n_j)) + \sum_{i=1}^{m} (t(x, h_i) + t(y, h_i)) \\ & \leqslant m + t \\ & = n - 2 \end{aligned}$$

when $t(x, y) \geqslant 3$.

The only other possibility is that $t(x, y) = 2$. So suppose this is the case and that the head vertices to xy are a and b. Without loss of generality, assume that $yb \to a \in E(H)$. Note that this precludes any edges of the form $yn_j \to x$. Similarly, if we added the edge $xa \to b$ or the edge $xb \to a$, then we could not add any edges of the form $xn_j \to y$ and so

$$\sum_{j=1}^{t} (t(x, n_j) + t(y, n_j)) = 0.$$

Moreover, $ya \to b$ would lead to more than one edge on the triple $\{y, a, b\}$. So

$$\sum_{i=1}^{m} (t(x, h_i) + t(y, h_i)) = 2$$

and in total we would have,

$$t(x, y) + \sum_{v \neq x,y} (t(x, v) + t(y, v)) = 4 \leqslant n - 2.$$

On the other hand, if xa and xb are not tails to any edge, then the only way we could get a sum of more than $n - 2$ is if $xn_j \to y \in E(H)$ for all $j = 1, \ldots, n - 4$. But this is exactly the configuration described in Lemma 7.6 which we have excluded.

Therefore,

$$t(x, y) + \sum_{v \neq x,y} (t(x, v) + t(y, v)) \leqslant n - 2$$

for any such pair, and this is enough to establish that

$$\mathrm{ex}_o(n, H_1) \leqslant \left\lfloor \frac{n}{2} \right\rfloor (n - 2).$$

Conversely, we have already considered an extremal construction in the case where n is even, and this same construction will work when n is odd. That is, take a maximum matching of the vertices (leaving one out) and use each matched pair as the tail set for all $n - 2$ possible edges.

Another construction that works for odd n that is not extremal for even n is to designate one vertex as the only head vertex and then make all $\binom{n-1}{2}$ pairs of the rest of the vertices tail sets.

Therefore,

$$\mathrm{ex}_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n - 2).$$

Also, note that the only way that any construction could have more than $\binom{n-1}{2}$ edges is if n is even *and* the vertices are partitioned into $\frac{n}{2}$ pairs such that each points to at least two other vertices. This fact comes directly from the requirement that $k = \frac{n}{2}$ in the optimization of

$$k(n - 2) + \binom{n - 2k}{2}$$

in order for the expression to be more than $\binom{n-1}{2}$.

7.5 Intersections of multiedge triples in the standard version

Now, let H be an H_1-free graph on n vertices under the standard version of the problem so that any triple of vertices can now have up to all three possible directed edges. If we let t_H be the number of triples of vertices of H that hold at least one edge, and we let m_H be the number of triples that hold at least two, then we have the following simple observation:

$$|E(H)| \leqslant t_H + 2m_H.$$

We start our path towards an upper bound on $|E(H)|$ by finding an upper bound on the number of multiedge triples, m_H. We will need to prove some facts about the multiedge triples of H. First, any triple which holds two edges of H might as well hold three.

Lemma 7.7. *Let H be an H_1-free graph such that some triple of vertices $\{x, y, z\}$ contains two edges. Define H' by $V(H') = V(H)$ and*

$$E(H') = E(H) \cup \{xy \to z, xz \to y, yz \to x\}.$$

Then H' is also H_1-free.

Proof. Suppose H' is not H_1-free. Since H is H_1-free and the two graphs differ by at most one edge, then they must differ by exactly one edge. Without loss of generality, say

$$\{xy \to z\} = E(H') \setminus E(H).$$

This edge must be responsible for creating the copy of H_1 in H'. So it must intersect another edge in exactly one vertex that is in the tail set of both.

Therefore, without loss of generality, there is an edge $xt \to s \in H$ where $\{s, t\} \cap \{y, z\} = \emptyset$. However, since $\{x, y, z\}$ already contained two edges of H, then $xz \to y \in H$. Since $xt \to s$ and $xz \to y$ make a copy of H_1, then H cannot be H_1-free, a contradiction. □

Next, we want to show that no two multiedge triples can intersect in exactly one vertex.

Lemma 7.8. *Let H be a H_1-free graph. If two vertex triples $\{x, y, z\}$ and $\{s, t, r\}$ each contain two or more edges of H, then*

$$|\{x, y, z\} \cap \{s, t, r\}| \neq 1.$$

Proof. Suppose

$$|\{x, y, z\} \cap \{s, t, r\}| = 1$$

By Lemma 7.7, since H is H_1-free, the graph created from H by adding all three possible edges on the triples $\{x, y, z\}$ and $\{s, t, r\}$ is also H_1-free. But if $x = r$ and x, y, z, s, and t are all distinct, then this graph contains $xy \to z$ and $xs \to t$ which is a copy of H_1, a contradiction. □

Therefore, we can use an upper bound on the number of undirected 3-uniform hyperedges such that no two intersect in exactly one vertex as an upper bound on the number of multiedge triples. Moreover, the extremal examples are easy to describe which will be important for finding the upper bound for $\mathrm{ex}(n, H_1)$ as well as for establishing the uniqueness of the lower bound construction.

Lemma 7.9. *Let H be a 3-uniform undirected hypergraph on n vertices such that no two edges intersect in exactly one vertex, then*

$$|E(H)| \leqslant \begin{cases} n & n \equiv 0 \ mod\ 4 \\ n-1 & n \equiv 1 \ mod\ 4 \\ n-2 & n \equiv 2,3 \ mod\ 4 \end{cases}$$

and H is the disjoint union of $K_4^{(3)}$s, $K_4^{(3)}$s minus an edge (K_4^-), and sets of edges that all share a common intersection of two vertices - a sunflower with a two vertex core.

Proof. Two edges of H are either disjoint or they intersect in two vertices. So connected components of H that have 1 or 2 edges are both sunflowers. A third edge can be added to a two-edge sunflower by either using the two common vertices to overlap with both edges in two or by using one common vertex and the two petal vertices. So a connected component of H with 3 edges is either a sunflower or a K_4^-.

The only way to connect a fourth edge to the three-edge sunflower is to make a four-edge sunflower, and this is true for a k-edge sunflower to a $(k+1)$-edge sunflower for all $k \geqslant 3$. The only way to add a fourth edge to the K_4^- is to make a $K_4^{(3)}$ and then no new edges may be connected to a $K_4^{(3)}$ without intersecting two of its edges in exactly one vertex each. Therefore, these are the only possible connected components of H.

A sunflower with k edges uses $k+2$ vertices, and a $K_4^{(3)}$ has four edges on 4 vertices. Therefore, if $n \equiv 0 \bmod 4$ we can get at most n edges with a disjoint collection of $K_4^{(3)}$s. Similarly, the best we can do when $n \equiv 1 \bmod 4$ is $n-1$ edges with a disjoint collection of $K_4^{(3)}$s plus one isolated vertex since any sunflower will automatically limit the number of edges to $n-2$. And if $n \equiv 2 \bmod 4$ or $n \equiv 3 \bmod 4$, then $n-2$ is the best that we can do. □

In general, the only way to actually have an H_1-free graph with n multiedge triples is if the multiedge triples form an undirected 3-uniform hypergraph of $\frac{n}{4}$ disjoint $K_4^{(3)}$ blocks when $n \equiv 0 \bmod 4$.

In this case there can be no additional directed edges in H since such an edge would either intersect one of these $K_4^{(3)}$s in one tail vertex which would create a copy of H_1 since this means it intersects three of the multiedge triples in exactly one tail vertex (we may assume that each multiedge has all three edges per Lemma 7.7) or it would intersect one of the $K_4^{(3)}$s in two tail vertices which means that it intersects two of the multiedge triples in exactly one tail vertex (see Figure 34).

So in this case, the number of total edges would be bound by

$$3n < \binom{n+1}{2} - 3$$

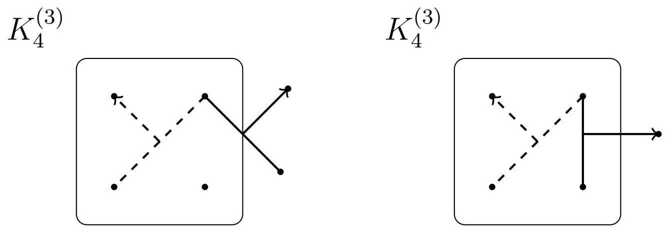

Figure 34: An edge that intersects a $K_4^{(3)}$ block of multiedge triples in one or two tail vertices will create a copy of H_1.

for all $n \geqslant 7$.

Next, the only way to have $n-1$ multiedge triples is to either have $\frac{n-1}{4}$ disjoint $K_4^{(3)}$ blocks when $n \equiv 1 \bmod 4$ or to have $\frac{n}{4}-1$ disjoint $K_4^{(3)}$ blocks with one K_4^- when $n \equiv 0 \bmod 4$. In the first case any additional edge must have at least one and perhaps two of its tail vertices in a single $K_4^{(3)}$ block of multiedge triples which we have already seen will create a copy of H_1. So there are at most

$$3(n-1) < 3n < \binom{n+1}{2} - 3$$

total edges in this case.

In the second case, any additional edge that has no tail vertices in a $K_4^{(3)}$ block must have both tail vertices in the K_4^-. If the head to such an edge were outside of the K_4^-, then the edge must intersect one of the three multiedge triples of the block in exactly one tail vertex since there are two triples that it intersects in one tail vertex each, one of which must be a multiedge triple. On the other hand, it could have its head vertex inside the K_4^-. In this case, the additional edge must lie on the triple without multiple edges. This is the only edge that can be added. So there are at most

$$3(n-1)+1 < 3n < \binom{n+1}{2} - 3$$

total edges in this case.

7.6 An H_1-free graph with $n-2$ multiedge triples

Now, the only ways to have exactly $n-2$ multiedge triples is either to have $\frac{n}{4}-2$ of the $K_4^{(3)}$ blocks plus two K_4^- blocks of multiedge triples when $n \equiv 0 \bmod 4$ or to have k of the $K_4^{(3)}$ blocks of multiedge triples plus a sunflower with $n-4k-2$ petals. The first case is suboptimal for the same reasons already considered. So let us consider the second case.

First, assume that $k=0$ and that we have $n-2$ multiedge triples that make a sunflower (see Figure 35). How many edges can we add? This structure already has all possible edges with 2 vertices in the core (or so we may assume by Lemma 7.7). On the other hand, if an additional edge has no vertices in the core, then it would intersect two multiedge triples in one tail vertex each which would create a copy of H_1.

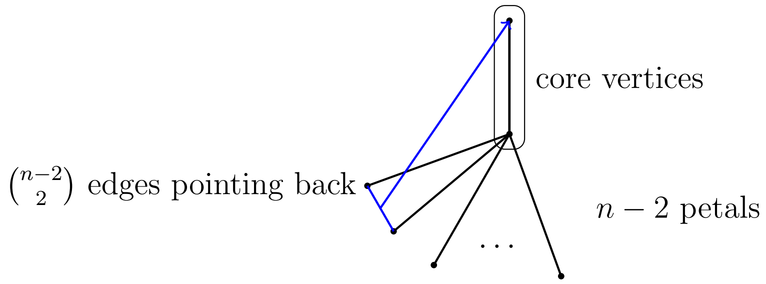

Figure 35: The unique extremal construction for an H_1-free graph has $\binom{n-2}{2} + 3(n-2)$ edges.

Therefore, any additional edge must include exactly one vertex from the core. If this vertex is in the tail set to the additional edge and the sunflower has at least three petals, then the additional edge intersects in exactly one tail vertex of the multiedge triples of the sunflower, a contradiction. Since we assume that $n \geqslant 6$, then the sunflower has at least three petals. Hence, any additional edge must intersect the core in only its head vertex.

If any two additional edges have different core vertices as the head, then either the tail sets of these edges must be exactly the same or completely disjoint to avoid a copy of H_1. Hence, pairs of petal vertices that point to both core vertices must be independent of all other tail sets. And all other petal vertices fall into disjoint sets as to whether they are in additional edges that point to the first core vertex or the second. The number of additional edges will be maximized if every pair of petal vertices point to the same core vertex. Moreover, this will give a total of

$$3(n-2) + \binom{n-2}{2} = \binom{n+1}{2} - 3$$

edges.

We will soon see that this is the best that we can do and that this construction, where the multiedge triples make a sunflower with $n-2$ petals with $\binom{n-2}{2}$ additional edges pointing from pairs of petal vertices to a single core vertex, is unique up to isomorphism.

First we will need to see that $k = 0$ is the number of $K_4^{(3)}$ multiedge triple blocks that optimizes the total number of edges. So suppose there are k such blocks and that the other $n - 4k$ vertices are in a sunflower. Then from prior considerations we know that any additional edge must have both tail vertices in this sunflower. If one of these tail vertices coincides with a petal vertex of the sunflower, then there will be a copy of H_1. Therefore, the tail vertices must coincide with the core and the only possibility for such an edge is to point out to a vertex in one of the k blocks.

Therefore, there are at most

$$3(4k) + 3(n-4k-2) + \binom{n-4k-2}{2} + 4k$$

edges in such a construction. Since this expression is quadratic in k with positive leading coefficient, then it must maximize at the endpoints, $k = 0$ or $k = \frac{n}{4}$, and we already know that $k = \frac{n}{4}$ is suboptimal. Therefore, if there are exactly $n-2$ multiedge triples, then they must form a sunflower with a two-vertex core and from there the only way to maximize the total number of edges is to add every possible edge with tail set among the petal vertices all pointing to the same head vertex in the core.

7.7 Fewer than $n-2$ multiedge triples

Now suppose that H has fewer than $n-2$ multiedge triples. If $t_H \leqslant \binom{n-1}{2}$, then

$$|E(H)| \leqslant t_H + 2m_H < \binom{n-1}{2} + 2(n-2) = \binom{n+1}{2} - 3.$$

So we must assume that $t_H > \binom{n-1}{2}$. Also, if $m_H = 0$, then we know that

$$|E(H)| \leqslant \mathrm{ex}_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n-2) < \binom{n+1}{2} - 3.$$

So assume that there is at least one multiedge triple, $\{x, y, z\}$. This triple has at least two edges. Assume without loss of generality that they are $xy \to z$ and $xz \to y$.

Let H' be an oriented graph arrived at by deleting edges from multiedge triples of H until each triple has at most one edge and every triple that had at least one edge in H still has at least one in H'. In other words, H' is any subgraph of H such that $t_{H'} = t_H$ and $m_{H'} = 0$. Without loss of generality, assume that

$$xy \to z \in E(H').$$

Since $t_{H'} > \binom{n-1}{2}$, then n must be even. Moreover, there is a matching on the vertices so that every matched pair $\{a, b\}$ points to at least two other vertices. That is, $t(a, b) \geqslant 2$.

Now consider the directed link graphs of the vertices. As stated before, these are either triangles or stars with a common vertex. However, if two or more of these link digraphs have three or fewer edges each (for instance, if they are triangles), then there are fewer edges than we are assuming since

$$|E(H')| = \frac{1}{2} \sum_{x \in V(H')} |D_x| \leqslant \frac{1}{2}\left(6 + (n-3)(n-2)\right) < \binom{n-1}{2}$$

for all $n \geqslant 8$. We will show that it must be the case that here at least two directed link graphs are restricted to at most three directed edges each, contradicting our assumptions about the number of edges in H.

First, note that $x \to z \in D_y$ and $y \to z \in D_x$. To avoid a contradiction, at least one of these two directed link graphs must have four or more edges. Without loss of generality, assume that it is D_y. Therefore, D_y is a star and not a triangle. So the additional three directed edges in D_y must either all be incident to z or to x.

If these directed edges are all incident to z, then y and z must be partners under the matching which means that x has another partner x' distinct from y and z. Since

$t(x,x') \geqslant 2$ in H', then x' must point to two vertices in D_x. Since D_x already has $y \to z$ and no two edges may be independent in any directed link graph, then x' must point to y and to z, forming a triangle.

Next, consider $D_{x'}$. We know that

$$x \to y, x \to z \in D_{x'}.$$

If there is an additional edge in $D_{x'}$ that does not complete this triangle then it is either of the form $x \to t$ or $t \to x$. If $x \to t \in D_{x'}$ then $x' \to t, y \to z \in D_x$, a contradiction. If $t \to x \in D_{x'}$, then $x' \to x \in D_t$. But since t has its own matched vertex, then there exists a distinct t' such that

$$t' \to x, t' \to x' \in D_{t'}.$$

So either $|D_{x'}| \leqslant 3$ or $|D_{t'}| \leqslant 3$. Either way, this gives us two directed link graphs that have at most three edges each. So $t_{H'} < \binom{n-1}{2}$.

Therefore, we must assume that the three additional edges in D_y are incident to x and that y and x are partners under the matching. So z has some other partner under the matching z' distinct from x and y. Now, delete the edge $xy \to z$ from H' and add $xz \to y$ to get a new directed hypergraph H''. It follows that H'' has no multiedge triples and is H_1-free since we still have a subgraph of H.

In adding $xz \to y$ we have added $x \to y$ to D_z. Since z' must point to two vertices in D_z, then this addition means that D_z is a triangle under H''. Hence, $|D_z| = 2$ under H'.

Now, the same argument as above applies to $D_{z'}$. The only way for $|D_{z'}| > 3$ would mean either $z \to a \in D_{z'}$ or $a \to z \in D_{z'}$ for some a distinct from x, y, z, and z'. The first case would mean that two independent directed edges, $z' \to a$ and $x \to y$ are in D_z, a contradiction. The second case would mean that $z' \to z \in D_a$. Since a has its own partner under the matching that must point to two vertices in D_a, then in this case, D_a is a triangle.

Therefore, $t_H > \binom{n-1}{2}$ and $m_H \geqslant 1$ cannot both be true in any H_1-free graph. This is enough to complete the result,

$$\mathrm{ex}(n, H_1) = \binom{n+1}{2} - 3.$$

This also exhausts the remaining cases in order to demonstrate that the extremal construction is unique.

8 Forbidden H_2

In this section H_2 will denote the forbidden graph where two edges intersect in exactly two vertices such that the set of intersection is the tail set to each edge. That is $V(H_2) = \{a,b,c,d\}$ and $E(H_2) = \{ab \to c, ab \to d\}$ (see Figure 36).

Theorem 8.1. *For all $n \geqslant 5$,*

$$ex(n, H_2) = ex_o(n, H_2) = \binom{n}{2}.$$

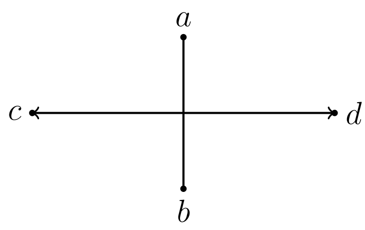

Figure 36: H_2

Moreover, there are $(n-2)^{\binom{n}{2}}$ *different labeled* H_2*-free graphs attaining this extremal number when in the standard version of the problem.*

Proof. Let H be H_2-free. Regardless of which version of the problem we are considering, each pair of vertices acts as the tail set to at most one directed edge. Therefore,

$$\mathrm{ex}(n, H_2), \mathrm{ex}_o(n, H_2) \leqslant \binom{n}{2}.$$

In the standard version of the problem any function, $f : \binom{[n]}{2} \to [n]$, that sends each pair of vertices to a distinct third vertex, $f(\{a,b\}) \notin \{a,b\}$, has an associated H_2-free construction H_f with $\binom{n}{2}$ edges. That is, for any such function, f, let $V(H_f) = [n]$ and

$$E(H_f) = \left\{ a, b \to f(\{a,b\}) : \{a,b\} \in \binom{[n]}{2} \right\}.$$

Since each pair of vertices acts as the tail set to exactly one directed edge, then H_f is H_2-free and has $\binom{n}{2}$ edges. So

$$\mathrm{ex}(n, H_2) = \binom{n}{2}.$$

Moreover, there are $(n-2)^{\binom{n}{2}}$ distinct functions from $\binom{[n]}{2}$ to $[n]$ such that no pair is mapped to one of its members. Therefore, there are $(n-2)^{\binom{n}{2}}$ labeled graphs that are H_2-free with $\binom{n}{2}$ edges.

In the oriented version of the problem lower bound constructions can be defined inductively on n.

First, let $n = 5$ and define G_5 as the oriented graph with vertex set

$$V(G_5) = \{0, 1, 2, 3, 4\}$$

and the following edges: $0,1 \to 2$; $1,3 \to 0$; $0,4 \to 1$; $0,2 \to 3$; $2,4 \to 0$; $0,3 \to 4$; $2,3 \to 1$; $1,2 \to 4$; $1,4 \to 3$; and $3,4 \to 2$.

Each pair of vertices of G_5 are in exactly one tail set, and each triple of vertices appear together in exactly one edge. Therefore, this construction is H_2-free with $\binom{5}{2}$ edges.

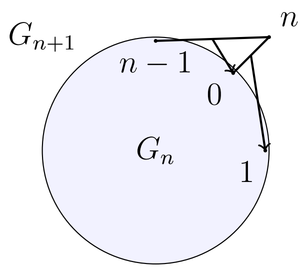

Figure 37: Inductive construction of H_2-free oriented graphs

Now, let $n \geqslant 5$, and define G_{n+1} by $V(G_{n+1}) = \{0, 1, \ldots, n\}$ and

$$E(G_{n+1}) = E(G_n) \cup \{ni \to (i+1) : i = 0, \ldots, n-1\}$$

where addition is taken modulo n.

Then G_{n+1} has n more edges than G_n. So $|E(G_{n+1})| = \binom{n+1}{2}$.

Any two new edges intersect in at most two vertices. Similarly, any new edge and any old edge also intersect in at most two vertices. Hence, at most one edge appears on a given triple of vertices. So G_{n+1} is oriented.

Moreover, all tail sets for the new edges are distinct from each other and from any tail sets for the edges of G_n. So G_{n+1} is H_2-free. Therefore,

$$\mathrm{ex}_o(n, H_2) = \binom{n}{2}. \qquad \square$$

9 Conclusion

There are many additional extremal questions that we can ask about $2 \to 1$ directed hypergraphs, and many ways that the model can be generalized. In this final section, we will briefly review several open questions that come up naturally in this work.

9.1 Extremal numbers for tournaments

In [3], Brown and Harary started studying extremal problems for directed 2-graphs by determining the extremal numbers for many "small" digraphs and for some more general types of digraphs such as tournaments - a digraph where every pair of vertices has exactly one directed edge. We could follow their plan of attack in studying this $2 \to 1$ model and also look for the extremal numbers of tournaments. Here, a tournament could be defined as a graph with exactly one directed edge on every three vertices. In particular, a transitive tournament might be an interesting place to begin. A transitive tournament is a tournament where the direction of each edge is based on an underlying linear ordering of the vertices as in the oriented lower bound construction of Theorem 4.2.

Denote the $2 \to 1$ transitive tournament on k vertices by TT_k. Since the "winning" vertex of the tournament will have a complete K_{k-1} as its tail link graph, then any H on

n vertices for which each T_x is K_{k-1}-free must be TT_k-free. Therefore,

$$n\left(\frac{n-1}{k-2}\right)^2\binom{k-2}{2} \leqslant \mathrm{ex}(n,TT_k), \mathrm{ex}_o(n,TT_k).$$

This also immediately shows that the transitive tournament on four vertices with the "bottom" edge removed has this extremal number exactly.

Theorem 9.1. *Let TT_4^- denote the graph with vertex set $V(TT_4^-)=\{a,b,c,d\}$ and edge set*

$$E(TT_4^-)=\{ab\to d, bc\to d, ac\to d\}.$$

Then

$$ex(n,TT_4^-)=n\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil.$$

Is it still true if we add an edge to $\{a,b,c\}$?

Conjecture 9.2. *Let TT_4 denote the graph with vertex set $V(TT_4)=\{a,b,c,d\}$ and edge set*

$$E(TT_4)=\{ab\to d, bc\to d, ac\to d, ab\to c\}.$$

Then

$$ex(n,TT_4)=n\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil.$$

9.2 Generalizing to $r\to 1$ directed hypergraphs

The $2\to 1$ directed hypergraph originally came to the author's attention as a way to model definite Horn clauses in propositional logic. Definite Horn clauses are more generally modeled by $r\to 1$ edges for any r. Therefore, it seems natural to ask about the extremal numbers for graphs with two $(r\to 1)$-edges. If we look at every $(r\to 1)$-graph with exactly two edges, then we see that these fall into four main types of graph. Let i be the number of vertices that belong to the tail set of both edges. Then let $I_r(i)$ denote the graph where both edges point to the same head vertex, let $H_r(i)$ denote the graph where the edges point to different head vertices neither of which are in the tail set of the other, let $R_r(i)$ denote the graph where the first edge points to a head vertex in the tail set of the second edge and the second edge points to a head not in the tail set of the first edge, and let $E_r(i)$ denote the graph where both edges point to heads in the tail sets of each other.

This extends the notation used in this paper. The degenerate cases are generalized to $I_r(i)$ and $H_r(i)$, and the nondegenerate cases generalize to $R_r(i)$ and $E_r(i)$. For example, the 3-resolvent R_3 is $R_2(1)$. The split between degenerate and nondegenerate is maintained in this way as well as shown in [5].

To what extent do the proofs presented in this paper extend to these graphs? Some translate immediately. For example, in the standard version of the problem it can easily be seen that

$$\mathrm{ex}(n,I_r(0))=n\binom{n-2}{r-1}$$

using Erdős-Ko-Rado [6] for the upper bound and the same basic construction for the lower bound that we used in proving the same result for I_0. More generally, we can get an upper bound of

$$\mathrm{ex}(n, I_r(i)) \leqslant n\binom{n-1}{r-1}$$

by applying the uniform Ray-Chaudhuri - Wilson Theorem [12] to the tail link graph of each vertex of an $I_r(i)$-free graph. We can get a general lower bound of

$$n\binom{n-i-2}{r-i-1} \leqslant \mathrm{ex}(n, I_r(i))$$

by constructing an $I_r(i)$-free graph in the following way: for each vertex x fix a set of $i+1$ vertices not including x, C_x, and then add every possible edge with x at the head and C_x in the tail set.

An easy lower bound construction for an $H_r(i)$-free graph is to fix a vertex x and take all possible edges that point to it giving

$$\binom{n-1}{r} \leqslant \mathrm{ex}(n, H_r(i)).$$

To get an upper bound also on the order of n^r note that we can extend the concept of the directed link graph to apply to more than one vertices. For instance, here let the directed link graph of a set of vertices A of cardinality i be the $(r-i) \to 1$ directed hypergraph on $n-i$ vertices, $V \setminus A$, for which every edge becomes an edge of the original $(r \to 1)$-graph when A is added to the tail set. In this case, no directed tail link graph for any set of i vertices can contain two independent directed edges. Therefore,

$$\mathrm{ex}(n, H_r(i)) \leqslant \frac{(r-i+1)\binom{n-i-1}{r-i}\binom{n}{i}}{\binom{r}{i}} = \frac{n(r-i+1)}{n-i}\binom{n-1}{r}.$$

It is easy to see that any $r \to 1$ transitive tournament on n vertices would be $E_r(i)$-free. This immediately solves the oriented version and gives a lower bound for the standard version:

$$\mathrm{ex}_o(n, E_r(i)) = \binom{n}{r+1}.$$

As in the first lower bound construction for E we can add r edges to the smallest $r+1$ vertices in the linear order given by the transitive tournament to get a few more edges in the standard case. Is this the best that we can do?

Conjecture 9.3.

$$ex(n, E_r(i)) = \binom{n}{r+1} + r.$$

For the generalized resolvent configurations, the lower bound constructions for R_3 and R_4 both generalize to the $r \to 1$ setting. When $i \geqslant 1$, then the construction that worked for R_3 gives the better lower bound. Split the vertices into two equal or almost equal parts and take all edges that point from an r-set in one to a vertex in the other. This gives

$$n\binom{\frac{n}{2}}{r} \leqslant \mathrm{ex}(n, R_r(i))$$

for $i \geqslant 1$. When $i = 0$, the same generalization of the construction for R_4 will produce an $R_r(0)$-free graph.

9.3 Differences between oriented and standard extremal numbers

It is interesting to look at the differences between the oriented and standard extremal problems for a given forbidden graph not only in their values but in the difficulty level of their proofs. For instance, the proof of the standard case of I_0 is quite easy while the proof of the oriented case took a lot of effort. For the Escher graph E the situation was reversed. What about the character of these two graphs determines that one version of the problem should be easy and the other difficult, and what is the difference between the two that swaps which version is which?

A more exact request is to ask for a characterization that determines the difference in the value. For instance, H_2, I_1, R_3, R_4, and the case of two completely overlapping edges each have oriented and standard numbers that are exactly the same while H_1 and I_0 each have differences that are linear in n, the Escher graph E has a constant difference, and the graph made up of two independent edges has a quadratic difference.

Of course, we get an immediate easy bound by observing that every non-oriented F-free graph contains an oriented F-free graph that can be arrived at by removing edges from each triple of vertices until only one remains. So

$$\mathrm{ex}(n, F) \leqslant 3\mathrm{ex}_o(n, F) \leqslant 3\mathrm{ex}(n, F)$$

for any forbidden graph F. The cases in this paper where the difference between the two numbers is zero shows that the upper bound is tight while the case of two independent edges shows that the lower bound is also tight.

But what causes the difference? Perhaps, it would be good to begin answering this question by narrowing the focus to nondegenerate graphs since in this paper almost every nondegenerate case had no difference in the values, and the only one that did had only constant difference. Will the difference always be at most constant or at least $o(n^3)$? No, any graph F that contains a triple with all three possible edges is certainly not degenerate, and the standard extremal number of F is at least twice as much as the oriented extremal number.

But what if we restrict ourselves further and only consider oriented nondegenerate forbidden graphs, then is

$$\mathrm{ex}(n, F) - \mathrm{ex}_o(n, F) = o(n^3)$$

for every oriented nondegenerate F? Between revisions of this paper, Dániel Gerbner and Balázs Keszegh produced an interesting counterexample to this claim as well. At this point it is unclear to the author what might be an appropriate characterization for graphs with small differences between extremal numbers.

9.4 General structural results

On a more general level we can ask about the structure of extremal $(2 \to 1)$-graphs. For instance, it was already shown in [10] that the 4-resolvent configuration R_4 has a stability result. Roughly speaking, R_4-free graphs with many edges differ only slightly from the given extremal construction. While we have shown that several of the extremal constructions in this paper are unique, we have not shown that any are stable.

Another avenue of research is to ask for canonical extremal structures. That is, for a forbidden graph F can we fix some constant r such that we can construct an F-free graph on n vertices such that the n vertices are partitioned into r parts and whether $xy \to z$ is an edge or not depends entirely on which parts x, y, and z are in? If we have a general r-part structure like this that is F-free for every n and the limit of the ratio of the number of edges given by the structure over $\mathrm{ex}(n, F)$ is one, then we call this a canonical F-free extremal structure. For instance, the Turán graphs are canonical extremal structures with respect to 2-graphs. Applying this idea to hypergraphs is already a major area of research (see [11]) so it seems likely that the question of whether every $(2 \to 1)$-graph has such a canonical extremal structure would be even more difficult.

Otherwise, many other topics in extremal hypergraph theory can be ported to the $2 \to 1$ model. For instance, in [5], this author showed that supersaturation holds for directed hypergraphs and proved some preliminary results on the existence of jumps and nonjumps. Other generalizations of this kind remain open as well.

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