Core partitions with distinct parts

Huan Xiong*

Institut de Recherche Mathématique Avancée
Université de Strasbourg
Strasbourg, France

xiong@math.unistra.fr

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Abstract

Simultaneous core partitions have attracted much attention since Anderson's work on the number of (t_1, t_2)-core partitions. In this paper we focus on simultaneous core partitions with distinct parts. The generating function of t-core partitions with distinct parts is obtained. We also prove results on the number, the largest size and the average size of $(t, t+1)$-core partitions with distinct parts. This gives a complete answer to a conjecture of Amdeberhan, which is partly and independently proved by Straub, Nath and Sellers, and Zaleski recently.

Keywords: simultaneous core partition; distinct part; hook length; largest size; average size

1 Introduction

The aim of this paper is to study simultaneous core partitions with distinct parts. Let us recall some basic definitions first. We refer the reader to [9, 16] for the basic knowledge on partitions. A *partition* is a finite nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers. Here λ_i $(1 \leqslant i \leqslant \ell)$ are called the parts of λ and $|\lambda| = \sum_{1\leqslant i\leqslant \ell} \lambda_i$ is the *size* of λ. A partition λ is usually identified with its *Young diagram*, which is a collection of left-justified rows with λ_i boxes in the i-th row. The *hook length* of the box $\square = (i, j)$ in the i-th row and j-th column of the Young diagram, denoted by $h(i, j)$, is the number of boxes exactly to the right, or exactly below, or the box itself. For example, Figure 1 shows the Young diagram and hook lengths of the partition $(5, 3, 3, 2, 1)$.

For positive integers $t_1, t_2, \ldots, t_m$, a partition is called a $(t_1, t_2, \ldots, t_m)$-*core partition* if none of its hook lengths belongs to $\{t_1, t_2, \ldots, t_m\}$. In particular, a partition is called a

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9	7	5	2	1
6	4	2		
5	3	1		
3	1			
1				

Figure 1: The Young diagram of the partition $(5, 3, 3, 2, 1)$ and the hook lengths.

t-core partition if none of its hook lengths equals t (see [7, 18]). For example, we can see from Figure 1 that $\lambda = (5, 3, 3, 2, 1)$ is a $(8, 10)$-core partition.

Many results have been obtained in the study of $(t_1, t_2, \ldots, t_m)$-core partitions. For $m = 2$, Anderson [3] showed that the number of (t_1, t_2)-core partitions is the *rational Catalan number* $\frac{1}{t_1+t_2}\binom{t_1+t_2}{t_1}$ when t_1 and t_2 are coprime to each other. Olsson and Stanton [14] found the largest size of such partitions, which is $\frac{(t_1^2-1)(t_2^2-1)}{24}$. Various results on the enumeration of (t_1, t_2)-core partitions are achieved in [4, 6, 8, 12, 17, 20]. A specific type of simultaneous core partitions, $(t, t+1, \ldots, t+p)$-core partitions, had been well studied. Results on the number, the largest size and the average size of such partitions can be found in [2, 10, 21, 24].

Much attention has been attracted to simultaneous core partitions with distinct parts since Amdeberhan's conjectures (see [**?**]) on this subject in 2015. The results on the enumeration of $(t, t+1)$, $(t, t+2)$ and $(t, nt \pm 1)$-core partitions with distinct parts can be found in several papers [10, 19, 22, 23, 25, 26] published since 2016. In this paper[1], we obtain the generating function of t-core partitions with distinct parts in Theorem 1.1. We also prove the results on the number, the largest size and the average size of $(t, t+1)$-core partitions with distinct parts in Theorem 1.2, which verify Amdeberhan's conjecture on such partitions. Notice that part of Theorem 1.2 was generalized independently by [11, 19, 22, 26]. In fact, Straub [19] and Nath-Sellers [11] found the number of $(t, nt-1)$ and $(t, nt+1)$-core partitions with distinct parts respectively. The largest sizes of the above two kinds of partitions were given by the author [22]. Zaleski [26] obtained the explicit expressions for the moments of the sizes of $(t, t+1)$-core partitions with distinct parts, which gave a generalization of Theorem 1.2(4). Furthermore, Zaleski and Zeilberger [25] obtained the moments of the sizes of $(2t+1, 2t+3)$-core partitions with distinct parts, whose number, largest size and average size were given by Yan, Qin, Jin and Zhou [23]. Our main results are stated next.

Theorem 1.1. *Suppose that $t \geqslant 2$. Let $cd_t(n)$ be the number of t-core partitions of size n with distinct parts. Then the generating function for such partition is*

$$\sum_{n\geqslant 0} cd_t(n)q^n = \sum_{(n_1,n_2,\cdots,n_{t-1})\in\mathcal{C}_t} q^{\sum_{i=1}^{t-1}\left(in_i+t\binom{n_i}{2}\right)-\binom{\sum_{i=1}^{t-1}n_i}{2}}, \tag{1.1}$$

[1] We mention that our paper is one of the earliest papers in the study of simultaneous core partitions with distinct parts (the first edition of this paper was available on arxiv since August 2015, which is cited by all six papers above).

where $\mathcal{C}_t = \{(x_1, x_2, \cdots, x_{t-1}) \in \mathbb{N}^{t-1} : x_i x_{i+1} = 0$ *for* $1 \leqslant i \leqslant t-2\}$. *In particular, when* $t = 2, 3, 4$, *we have*

$$\sum_{n\geqslant 0} cd_2(n)q^n = \sum_{n\geqslant 0} q^{\binom{n+1}{2}},$$

$$\sum_{n\geqslant 0} cd_3(n)q^n = \sum_{n\geqslant 1} q^{n^2} + \sum_{n\geqslant 0} q^{n(n+1)},$$

and

$$\sum_{n\geqslant 0} cd_4(n)q^n = \sum_{n\geqslant 1} q^{\frac{n(3n+1)}{2}} + \sum_{n\geqslant 0}\sum_{m\geqslant 0} q^{\frac{n(3n-1)+3m(m+1)-2mn}{2}}.$$

Theorem 1.2 (Amdeberhan's conjecture, see [1, 19]). *Let* $t \geqslant 2$ *be a positive integer and* $(F_i)_{i\geqslant 0} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ *be the Fibonacci numbers. For* $(t, t+1)$*-core partitions with distinct parts, we have the following results.*

(1) *The number of such partitions is* F_{t+1}.

(2) *The largest size of such partitions is* $\lfloor \frac{1}{3}\binom{t+1}{2} \rfloor$, *where* $\lfloor x \rfloor$ *is the largest integer not greater than* x.

(3) *The number of such partitions with the largest size is* 2 *if* $t \equiv 1 \pmod 3$ *and* 1 *otherwise.*

(4) *The total sum of the sizes of these partitions and the average size are, respectively, given by*

$$\sum_{\substack{i+j+k=t+1\\ i,j,k\geqslant 1}} F_iF_jF_k \quad \text{and} \quad \sum_{\substack{i+j+k=t+1\\ i,j,k\geqslant 1}} \frac{F_iF_jF_k}{F_{t+1}}.$$

2 The β-sets of core partitions

In this section, we study the properties of β-sets of t-core partitions and obtain the generating function for t-core partitions of size n with distinct parts.

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition whose corresponding Young diagram has ℓ rows. The β*-set* $\beta(\lambda)$ of λ is defined to be the set of *first-column hook lengths* in the Young diagram of λ (for example, see [14, 21]), i.e.,

$$\beta(\lambda) = \{h(i, 1) : 1 \leqslant i \leqslant \ell\}.$$

The following results are well-known and easy to prove.

Lemma 2.1 ([3, 5, 14, 21]). *(1) Suppose* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ *is a partition. Then* $\lambda_i = h(i, 1) - \ell + i$ *for* $1 \leqslant i \leqslant \ell$. *Thus the size of* λ *is* $|\lambda| = \sum_{x\in\beta(\lambda)} x - \binom{|\beta(\lambda)|}{2}$.

*(2) (**Abacus condition for t-core partitions.**) A partition* λ *is a* t*-core partition if and only if for any* $x \in \beta(\lambda)$ *with* $x \geqslant t$, *we always have* $x - t \in \beta(\lambda)$.

Remark 2.2. *An element* $x \in \beta(\lambda)$ *is called* t-maximal *if* $x + t \notin \beta(\lambda)$. *Lemma 2.1(2) implies that the* β*-set* $\beta(\lambda)$ *of a* t*-core partition* λ *is determined by all* t*-maximal elements in* $\beta(\lambda)$. *Thus there is a bijection* η *which sends each* t*-core partition* λ *to*

$$(n_1, n_2, \cdots, n_{t-1}) := (n_1(\lambda), n_2(\lambda), \cdots, n_{t-1}(\lambda)) \in \mathbb{N}^{t-1}$$

such that $t(n_i-1)+i$ is t-maximal in $\beta(\lambda)$ if $n_i \geqslant 1$; and $i \notin \beta(\lambda)$ if $n_i{=}0$ for $1 \leqslant i \leqslant t-1$. In this case, $\beta(\lambda) = \bigcup_{i=1}^{t-1} \bigcup_{j=0}^{n_i-1} \{jt+i\}$ and therefore $|\beta(\lambda)| = \sum_{i=1}^{t-1} n_i$.

Example 2.3. *Let $\lambda = (5,3,3,2,1)$ be a 8-core partition. Then $\beta(\lambda) = \{9,6,5,3,1\}$ and $\eta(\lambda) = (2,0,1,0,1,1,0)$.*

By Lemma 2.1(1), we have the following result.

Lemma 2.4. *The partition λ is a partition with distinct parts if and only if there does not exist $x, y \in \beta(\lambda)$ with $x - y = 1$.*

Proof. Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Then by Lemma 2.1(1), we have $\lambda_i = \lambda_{i+1}$ if and only if $h(i,1) - \ell + i = h(i+1,1) - \ell + i + 1$, which is equivalent to $h(i,1) - h(i+1,1) = 1$. This implies the claim. □

Let $[t] = \{x \in \mathbb{N} : 1 \leqslant x \leqslant t\}$ for every $t \geqslant 1$. We say that a subset B of $[t]$ is *nice* if $x - y \neq 1$ for any $x, y \in B$. Let $\mathcal{B}_t$ be the set of nice subsets of $[t-1]$ and $a_t = |\mathcal{B}_t|$ be the number of nice subsets of $[t-1]$. Recall that $\mathcal{C}_t = \{(x_1, x_2, \cdots, x_{t-1}) \in \mathbb{N}^{t-1} : x_i x_{i+1} = 0 \text{ for } 1 \leqslant i \leqslant t-2\}$ for every $t \geqslant 2$.

Example 2.5. *Let $t = 5$. The set of all nice subsets of $[4]$ are*

$$\mathcal{B}_5 = \{\{1\},\{2\},\{3\},\{4\},\{1,3\},\{1,4\},\{2,4\}\}.$$

Notice that $\mathcal{C}_5$ is determined by $\mathcal{B}_5$ in the following manner: $\mathcal{C}_5 = \{(x_1,0,0,0) \in \mathbb{N}^4\} \cup \{(0,x_2,0,0) \in \mathbb{N}^4\} \cup \{(0,0,x_3,0) \in \mathbb{N}^4\} \cup \{(0,0,0,x_4) \in \mathbb{N}^4\} \cup \{(x_1,0,x_3,0) \in \mathbb{N}^4\} \cup \{(x_1,0,0,x_4) \in \mathbb{N}^4\} \cup \{(0,x_2,0,x_4) \in \mathbb{N}^4\}$.

Let CD_t be the set of t-core partitions with distinct parts and $cd_t(n)$ be the number of partitions in CD_t with size n. Recall that the bijection η is defined in Remark 2.2.

Theorem 2.6. *The mapping η gives a bijection between the sets CD_t and $\mathcal{C}_t$. If $\eta(\lambda) = (n_1, n_2, \cdots, n_{t-1})$ for some t-core partition λ with distinct parts, then*

$$|\lambda| = \sum_{i=1}^{t-1} \left(i n_i + t\binom{n_i}{2} \right) - \binom{\sum_{i=1}^{t-1} n_i}{2}. \tag{2.1}$$

Proof. By Lemma 2.4 we know if λ is a t-core partition with distinct parts, then $\{i, i+1\} \not\subseteq \beta(\lambda)$ for $1 \leqslant i \leqslant t-2$. Also we know $i \in \beta(\lambda)$ iff $n_i(\lambda) \geqslant 1$. Then the bijection between CD_t and $\mathcal{C}_t$ is described in Remark 2.2. By the definition of η, we know $\eta(\lambda) = (n_1, n_2, \cdots, n_{t-1})$ means

$$\beta(\lambda) = \bigcup_{i=1}^{t-1} \bigcup_{j=0}^{n_i-1} \{jt+i\}.$$

Therefore by Lemma 2.1(1) we derive (2.1). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The formula (1.1) is a direct corollary of Theorem 2.6.

In particular, when $t=2$, we have $\mathcal{C}_2=\mathbb{N}$. Then by (1.1) we obtain

$$\sum_{n\geqslant 0} cd_2(n)q^n = \sum_{n_1\geqslant 0} q^{n_1+2\binom{n_1}{2}-\binom{n_1}{2}} = \sum_{n\geqslant 0} q^{\binom{n+1}{2}}.$$

When $t=3$, We have

$$\mathcal{C}_3 = \{(x_1,0)\in\mathbb{N}^2 : x_1\geqslant 1\}\cup\{(0,x_2)\in\mathbb{N}^2 : x_2\geqslant 0\}.$$

By (1.1) we obtain

$$\begin{aligned}\sum_{n\geqslant 0} cd_3(n)q^n &= \sum_{n_1\geqslant 1} q^{n_1+3\binom{n_1}{2}-\binom{n_1}{2}} + \sum_{n_2\geqslant 0} q^{2n_2+3\binom{n_2}{2}-\binom{n_2}{2}}\\ &= \sum_{n\geqslant 1} q^{n^2} + \sum_{n\geqslant 0} q^{n(n+1)}.\end{aligned}$$

When $t=4$, We have

$$\mathcal{C}_4 = \{(0,x_2,0)\in\mathbb{N}^2 : x_2\geqslant 1\}\cup\{(x_1,0,x_3)\in\mathbb{N}^2 : x_1\geqslant 0, x_3\geqslant 0\}.$$

By (1.1) we obtain

$$\begin{aligned}\sum_{n\geqslant 0} cd_4(n)q^n &= \sum_{n_2\geqslant 1} q^{2n_2+4\binom{n_2}{2}-\binom{n_2}{2}} + \sum_{n_1\geqslant 0}\sum_{n_3\geqslant 0} q^{n_1+3n_3+4\binom{n_1}{2}+4\binom{n_3}{2}-\binom{n_1+n_3}{2}}\\ &= \sum_{n\geqslant 1} q^{\frac{n(3n+1)}{2}} + \sum_{n\geqslant 0}\sum_{m\geqslant 0} q^{\frac{n(3n-1)+3m(m+1)-2mn}{2}}.\end{aligned}$$

□

3 $(t,t+1)$-core partitions with distinct parts

In this section we focus on $(t,t+1)$-core partitions with distinct parts. We have the following characterization for β-sets of $(t,t+1)$-core partitions.

Lemma 3.1. *Let $t\geqslant 2$ be a positive integer. Suppose that λ is a $(t,t+1)$-core partition. Then we have*

$$\beta(\lambda)\subseteq \bigcup_{1\leqslant k\leqslant t-1}\{x\in\mathbb{N} : (k-1)(t+1)+1\leqslant x\leqslant kt-1\}.$$

Proof. By Lemma 2.1(2) we have $at+b(t+1)\notin\beta(\lambda)$ for every nonnegative integers $a,b\in\mathbb{N}$, which means that

$$\beta(\lambda)\subseteq\mathbb{N}\setminus\{at+b(t+1) : a,b\in\mathbb{N}\}.$$

The above set is related to the Frobenius problem (see [15]). Notice that for any $k \in \mathbb{N}$, we have

$$\{at + b(t+1) : a, b \in \mathbb{N},\ a + b = k\} = \{x \in \mathbb{N} : kt \leqslant x \leqslant k(t+1)\}.$$

Therefore

$$\beta(\lambda) \subseteq \bigcup_{1 \leqslant k \leqslant t-1} \{x \in \mathbb{N} : (k-1)(t+1) + 1 \leqslant x \leqslant kt - 1\}. \qquad \square$$

Lemma 3.2. *Let $t \geqslant 2$ be a positive integer. Suppose that λ is a $(t, t+1)$-core partition with distinct parts. Then*

$$\beta(\lambda) \subseteq [t-1] = \{x \in \mathbb{N} : 1 \leqslant x \leqslant t-1\}.$$

Proof. By Lemma 2.1(2) we have $t, t+1 \notin \beta(\lambda)$ since $0 \notin \beta(\lambda)$. For $x \geqslant t+2$, if $x \in \beta(\lambda)$, by Lemma 2.1(2) we know $x - t, x - (t+1) \in \beta(\lambda)$. But by Lemma 2.4 this is impossible since λ is a partition with distinct parts. Then $x \notin \beta(\lambda)$ and thus $\beta(\lambda)$ is a subset of $[t-1]$. $\square$

Now we are ready to prove our main result Theorem 1.2.

Proof of Theorem 1.2. (1) By Lemmas 2.1(2), 2.4 and 3.2, a partition λ is a $(t, t+1)$-core partition with distinct parts if and only if $\beta(\lambda)$ is a nice subset of $[t-1]$. Thus the number of $(t, t+1)$-core partitions with distinct parts equals the number a_t of nice subsets of $[t-1]$, which is equal to F_{t+1} since it is well known that the Fibonacci number F_{n+2} counts subsets of $\{1, 2, \ldots, n\}$ not containing two consecutive elements (see [13]).

(2) Suppose that λ is a $(t, t+1)$-core partition with distinct parts such that $\beta(\lambda) = \{x_1, x_2, \ldots, x_k\}$. By (1) we already know $\beta(\lambda)$ is a nice subset of $[t-1]$. Thus

$$|\lambda| = \sum_{i=1}^{k} x_i - \binom{k}{2} \leqslant \sum_{i=1}^{k} (t+1-2i) - \binom{k}{2} = -\frac{3}{2}(k - \frac{2t+1}{6})^2 + \frac{(2t+1)^2}{24}.$$

When $t = 3n$ for some integer n, we obtain

$$|\lambda| \leqslant -\frac{3}{2}(k - \frac{6n+1}{6})^2 + \frac{(6n+1)^2}{24} \leqslant \frac{3n^2}{2} + \frac{n}{2}.$$

When $t = 3n+1$ for some integer n, we obtain

$$|\lambda| \leqslant -\frac{3}{2}(k - \frac{6n+3}{6})^2 + \frac{(6n+3)^2}{24} \leqslant \frac{3n^2}{2} + \frac{3n}{2}.$$

When $t = 3n+2$ for some integer n, we obtain

$$|\lambda| \leqslant -\frac{3}{2}(k - \frac{6n+5}{6})^2 + \frac{(6n+5)^2}{24} \leqslant \frac{3n^2}{2} + \frac{5n}{2} + 1.$$

Finally, in each case we always obtain

$$\lambda \leqslant \lfloor \frac{1}{3}\binom{t+1}{2} \rfloor.$$

(3) By (2) we know, if λ is a $(t, t+1)$-core partition with distinct parts which has the largest size, then its β-set must be $\beta(\lambda) = \{t-1, t-3, \ldots, t-(2k-1)\}$ for some integer k. When $t = 3n$ for some integer n, λ has the largest size $\lfloor \frac{1}{3}\binom{t+1}{2} \rfloor$ if and only if $k = n$; when $t = 3n+1$ for some integer n, λ has the largest size $\lfloor \frac{1}{3}\binom{t+1}{2} \rfloor$ if and only if $k = n$ or $n+1$; when $t = 3n+2$ for some integer n, λ has the largest size $\lfloor \frac{1}{3}\binom{t+1}{2} \rfloor$ if and only if $k = n+1$. Therefore we prove the claim.

(4) First we introduce some sequences. For every $t \geqslant 2$, let

$$b_t = \sum_{B \in \mathcal{B}_t} |B|, \quad c_t = \sum_{B \in \mathcal{B}_t} |B|^2,$$

$$d_t = \sum_{B \in \mathcal{B}_t} \sum_{x \in B} x, \quad e_t = d_t - \sum_{B \in \mathcal{B}_t} \binom{|B|}{2},$$

and

$$\phi_t = \sum_{\substack{i+j=t \\ i,j \geqslant 1}} F_i F_j, \quad \psi_t = \sum_{\substack{i+j+k=t \\ i,j,k \geqslant 1}} F_i F_j F_k.$$

Lemmas 2.1, 2.4 and 3.2 imply that e_t equals the total sum of the sizes of all $(t, t+1)$-core partitions with distinct parts. Thus we just need to show that $e_t = \psi_{t+1}$ for $t \geqslant 2$.

When $t \geqslant 4$, suppose that $B \in \mathcal{B}_t$. If $t-1 \in B$, then $t-2 \notin B$. Therefore

$$\begin{aligned} b_t &= \sum_{\substack{B \in \mathcal{B}_t \\ t-1 \notin B}} |B| + \sum_{\substack{B \in \mathcal{B}_t \\ t-1 \in B}} |B| = \sum_{B \in \mathcal{B}_{t-1}} |B| + \sum_{B \in \mathcal{B}_{t-2}} (|B|+1) \\ &= b_{t-1} + b_{t-2} + a_{t-2} = b_{t-1} + b_{t-2} + F_{t-1}. \end{aligned}$$

Similarly we have

$$\begin{aligned} c_t &= \sum_{\substack{B \in \mathcal{B}_t \\ t-1 \notin B}} |B|^2 + \sum_{\substack{B \in \mathcal{B}_t \\ t-1 \in B}} |B|^2 = \sum_{B \in \mathcal{B}_{t-1}} |B|^2 + \sum_{B \in \mathcal{B}_{t-2}} (|B|+1)^2 \\ &= c_{t-1} + c_{t-2} + 2b_{t-2} + F_{t-1} \end{aligned}$$

and

$$\begin{aligned} d_t &= \sum_{\substack{B \in \mathcal{B}_t \\ t-1 \notin B}} \sum_{x \in B} x + \sum_{\substack{B \in \mathcal{B}_t \\ t-1 \in B}} \sum_{x \in B} x = \sum_{B \in \mathcal{B}_{t-1}} \sum_{x \in B} x + \sum_{B \in \mathcal{B}_{t-2}} (t-1 + \sum_{x \in B} x) \\ &= d_{t-1} + d_{t-2} + (t-1)F_{t-1}. \end{aligned}$$

Notice that

$$e_t = d_t - \sum_{B \in \mathcal{B}_t} \binom{|B|}{2} = d_t - \frac{1}{2}(c_t - b_t),$$

which means that

$$e_t - e_{t-1} - e_{t-2} = (t-1)F_{t-1} - b_{t-2}.$$

Since $e_2 = \psi_3 = 1$, $e_3 = \psi_4 = 3$, to show that $e_t = \psi_{t+1}$ for $t \geqslant 2$, we just need to show that

$$\psi_{t+1} - \psi_t - \psi_{t-1} = (t-1)F_{t-1} - b_{t-2} \tag{3.1}$$

for $t \geqslant 4$.

Notice that $F_0 = 0,\ F_1 = 1$. We have

$$\begin{aligned}
\psi_{t+1} = \sum_{\substack{i+j+k=t+1\\ i,j,k\geqslant 1}} F_iF_jF_k &= \sum_{\substack{i+j+k=t+1\\ j,k\geqslant 1\\ i\geqslant 2}} F_iF_jF_k + \sum_{\substack{j+k=t\\ j,k\geqslant 1}} F_jF_k\\
&= \sum_{\substack{i+j+k=t+1\\ j,k\geqslant 1\\ i\geqslant 2}} F_{i-1}F_jF_k + \sum_{\substack{i+j+k=t+1\\ j,k\geqslant 1\\ i\geqslant 2}} F_{i-2}F_jF_k + \phi_t\\
&= \sum_{\substack{i'+j+k=t\\ i',j,k\geqslant 1}} F_{i'}F_jF_k + \sum_{\substack{i'+j+k=t-1\\ i',j,k\geqslant 1}} F_{i'}F_jF_k + \phi_t\\
&= \psi_t + \psi_{t-1} + \phi_t.
\end{aligned}$$

Thus (3.1) is equivalent to

$$(t-1)F_{t-1} - b_{t-2} = \phi_t. \tag{3.2}$$

Notice that (3.2) is true for $t = 4, 5$. Also we have

$$\begin{aligned}
&(t-1)F_{t-1} - b_{t-2} - \big((t-2)F_{t-2} - b_{t-3}\big) - \big((t-3)F_{t-3} - b_{t-4}\big)\\
&= (t-1)F_{t-1} - (t-2)F_{t-2} - (t-3)F_{t-3} - F_{t-3}\\
&= (t-1)F_{t-1} - (t-2)F_{t-1} = F_{t-1}
\end{aligned}$$

and

$$\begin{aligned}
\phi_t &= \sum_{\substack{i+j=t\\ i,j\geqslant 1}} F_iF_j = \sum_{\substack{i+j=t\\ j\geqslant 1\\ i\geqslant 2}} F_iF_j + F_{t-1} = \sum_{\substack{i+j=t\\ j\geqslant 1\\ i\geqslant 2}} F_{i-1}F_j + \sum_{\substack{i+j=t\\ j\geqslant 1\\ i\geqslant 2}} F_{i-2}F_j + F_{t-1}\\
&= \sum_{\substack{i'+j=t-1\\ i',j\geqslant 1}} F_{i'}F_j + \sum_{\substack{i'+j=t-2\\ i',j\geqslant 1}} F_{i'}F_j + F_{t-1} = \phi_{t-1} + \phi_{t-2} + F_{t-1}.
\end{aligned}$$

Now we obtain (3.2) is true for $t \geqslant 4$. This implies that $e_t = \psi_{t+1}$ for $t \geqslant 2$. Therefore the total sum of the sizes of all $(t, t+1)$-core partitions with distinct parts is

$$e_t = \psi_{t+1} = \sum_{\substack{i+j+k=t+1 \\ i,j,k \geqslant 1}} F_i F_j F_k.$$

Then by (1) the average size of these partitions is

$$\sum_{\substack{i+j+k=t+1 \\ i,j,k \geqslant 1}} \frac{F_i F_j F_k}{F_{t+1}}. \qquad \square$$

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