The r-matching sequencibility of complete graphs

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Abstract

Alspach [*Bull. Inst. Combin. Appl.*, 52 (2008), pp. 7–20] defined the maximal matching sequencibility of a graph G, denoted $ms(G)$, to be the largest integer s for which there is an ordering of the edges of G such that every s consecutive edges form a matching. Alspach also proved that $ms(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor$. Brualdi et al. [*Australas. J. Combin.*, 53 (2012), pp. 245–256] extended the definition to cyclic matching sequencibility of a graph G, denoted $cms(G)$, which allows cyclical orderings and proved that $cms(K_n) = \left\lfloor \frac{n-2}{2} \right\rfloor$.

In this paper, we generalise these definitions to require that every s consecutive edges form a subgraph where every vertex has degree at most $r \geqslant 1$, and we denote the maximum such number for a graph G by $ms_r(G)$ and $cms_r(G)$ for the non-cyclic and cyclic cases, respectively. We conjecture that $ms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor$ and $\left\lfloor \frac{rn-1}{2} \right\rfloor - 1 \leqslant cms_r(K_n) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$ and that both bounds are attained for some r and n. We prove these conjectured identities for the majority of cases, by defining and characterising selected decompositions of K_n. We also provide bounds on $ms_r(G)$ and $cms_r(G)$ as well as results on hypergraph analogues of $ms_r(G)$ and $cms_r(G)$.

Keywords: Graph; matching; edge ordering; matching sequencibility; graph decomposition; hypergraph

1 Introduction

The (maximal) *matching sequencibility* of a simple graph G, denoted $ms(G)$, is the largest integer s for which there exists an ordering of the edges of G so that every s consecutive edges form a matching. Alspach [1] determined $ms(K_n)$, as follows.

Theorem 1 (Alspach [1]). *For each integer $n \geqslant 3$,*

$$ms(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor .$$

Brualdi, Kiernan, Meyer and Schroeder [3] considered the *cyclic matching sequencibility* $cms(G)$ of a graph G, which is the natural analogue of the matching sequencibility for G when cyclic orders are allowed. They proved the cyclic analogue of Theorem 1, below.

Theorem 2 (Brualdi et al. [3])**.** *For each integer* $n \geqslant 4$,

$$cms(K_n) = \left\lfloor \frac{n-2}{2} \right\rfloor .$$

The aim of this paper is to extend Theorem 1 and Theorem 2 by generalising the notion of matching sequencibility. In particular, for a graph G, $ms_r(G)$ denotes the analogue of $ms(G)$ where consecutive edges form a subgraph whose vertices each has degree at most r. Similarly, $cms_r(G)$ is defined analogously to $ms_r(G)$ where we allow cyclic orderings of the edges of G.

Conjecture 3. Let $n \geqslant 3$ and $1 \leqslant r \leqslant n-2$ be integers. Then

$$ms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{rn-1}{2} \right\rfloor - 1 \leqslant cms_r(K_n) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor .$$

The main results include the three to follow which verify the conjecture in many cases. In each result we assume $n \geqslant 3$ and $1 \leqslant r \leqslant n-2$.

Theorem 4. *If* n *or* r *is even, or* n *is odd and either* $r \geqslant \frac{n-1}{2}$ *or* $\gcd(r, n-1) = 1$, *then*

$$ms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor .$$

Theorem 5. *If* n *is even, or* n *is odd and* $r = \frac{n-1}{2}$, *then*

$$cms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor .$$

Theorem 6. *If* n *is odd and* r *is even, then*

$$\left\lfloor \frac{rn-1}{2} \right\rfloor - 1 \leqslant cms_r(K_n) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor .$$

One might ask which of the above bounds holds for which values of r and n. We discuss this question at the end of the paper and prove the following theorem which is the fourth and final of our main results.

Theorem 7. *For odd integers* r *and* n,

$$cms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor \quad \textit{if and only if} \quad cms_{n-1-r}(K_n) = \left\lfloor \frac{(n-1-r)n-1}{2} \right\rfloor .$$

The paper is organised as follows. In Section 2 we generalise the methods of [1] and [3], expressed as Propositions 9–11. The propositions allow us to reduce the problem of determining $ms_r(G)$ and $cms_r(G)$ to ordering subgraphs of G which are partially r-sequenceable for a smaller value of r. However, the parities of n and r play a crucial role in the effectiveness of Propositions 9–11: the case when n is odd is trickier and more so when r is also odd.

Section 3 defines the *Walecki decomposition* [1] and other decompositions of K_n. These are central to the proofs of Theorems 4–6; those are presented in Sections 4–7. Section 8 presents the proof of Theorem 7 and, as part of that proof, we consider sequencibility when certain general conditions are placed on consecutive edges of orderings of graphs. Section 9 concludes the paper with a discussion on Conjecture 3 and related open problems, and we provide some recursive bounds on $ms_r(G)$ and $cms_r(G)$ for general graphs G as well as for the complete k-graph $\mathcal{K}_n^k$; see Proposition 33 and Theorem 35, respectively.

2 Preliminaries

In this paper, graphs will always be simple. A matching of a graph G is a subgraph M in which each vertex has degree 1. A graph G is $(\leqslant r)$*-regular* if each of its vertices has degree at most r. If every vertex has degree equal to r, then G is r*-regular*. In particular, a matching of a graph is a 1-regular subgraph. For an integer n, let $[n] := \{0, 1, \dots, n-1\}$, where $[0] = \emptyset$. An *ordering* or *labelling* of a graph $G = (V, E)$ is a bijective function $\ell : E \to [|E|]$. The image of e under ℓ is called the *label* of e. The edges $e_0, \dots, e_{s-1}$ are *consecutive* in ℓ if the labels of $e_0, \dots, e_{s-1}$ are consecutive integers. For an ordering ℓ of a graph G, we let $ms_r(\ell)$ denote the largest integer s for which every s consecutive edges of ℓ form a $(\leqslant r)$-regular subgraph of G. We define $ms_r(G)$ to be the maximum value of $ms_r(\ell)$ over all orderings ℓ of G. In particular, the special case $ms_1(G)$, which we also denote as $ms(G)$, is the same number as presented in the Introduction. The edges $e_0, \dots, e_{s-1}$ of a graph $G = (V, E)$ are *cyclically consecutive* in ℓ if the labels of $e_0, \dots, e_{s-1}$ are consecutive integers modulo $|E|$. We define $cms_r(\ell)$ and $cms_r(G)$ analogously to $ms_r(\ell)$ and $ms_r(G)$, respectively, where we allow cyclically consecutive edges. If G is a $(\leqslant r)$-regular graph, then, by definition, $cms_r(G) = ms_r(G) = |E(G)|$. So for the remainder of the paper, we only consider the more interesting case in which r is strictly less than the maximum degree of a vertex of G, denoted by $\Delta(G)$.

Lemma 8. *Let G be a graph on n vertices with $r < \Delta(G)$, then*

$$cms_r(G) \leqslant ms_r(G) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor .$$

Proof. If rn is odd, then a $(\leqslant r)$-regular graph on n vertices can have at most $\frac{rn-1}{2}$ edges and so $ms_r(G) \leqslant \lfloor \frac{rn-1}{2} \rfloor$. If rn is even, then a $(\leqslant r)$-regular graph on n vertices can have at most $\frac{rn}{2}$ edges. If ℓ is a labelling of G satisfying $ms_r(\ell) = \frac{rn}{2}$, then the edges $\ell^{-1}(0), \dots, \ell^{-1}(\frac{rn-2}{2})$ form a r-regular graph as do the edges $\ell^{-1}(1), \dots, \ell^{-1}(\frac{rn}{2})$. This means the edges $\ell^{-1}(1), \dots, \ell^{-1}(\frac{rn-2}{2})$, form a graph in which every vertex has degree r

except two which have degree $r-1$. Therefore, $\ell^{-1}(0) = \ell^{-1}(\frac{rn}{2})$, a contradiction. Thus, $ms_r(G) \leqslant \lfloor \frac{rn-1}{2} \rfloor$. The inequality $cms_r(G) \leqslant ms_r(G)$ is trivially true by definition. □

For disjoint graphs $G_0, \ldots, G_{a-1}$ on the same vertex set V, with labellings $\ell_0, \ldots, \ell_{a-1}$ respectively, let $\ell_0 \vee \cdots \vee \ell_{a-1}$ denote the ordering ℓ of $G = (V, \bigcup_{i=0}^{a-1} E(G_i))$ defined by $\ell(e_{ij}) = \ell_j(e_{ij}) + \sum_{l=0}^{j-1} |E(G_l)|$ where $e_{ij} \in E(G_j)$ for all i and j. Let s be an integer and G and G' be disjoint graphs on the same vertex set V and each having at least $s-1$ edges. Also, let G and G' have labellings ℓ and ℓ', respectively, and let G_s be the subgraph of $(V, E(G) \cup E(G'))$ that consists of the last $s-1$ edges of ℓ and the first $s-1$ edges of ℓ'. Then we will let $\ell \vee_s \ell'$ denote the ordering of G_s for which the edges of G_s appear in the same order as they do in $\ell \vee \ell'$. Now we define $ms_r(\ell, \ell')$ to be the largest integer s such that $\ell \vee_s \ell'$ has r-matching sequencibility s.

An *r-regular decomposition* of a graph G is a set of edge-disjoint r-regular subgraphs of G that partition the edge set of G. A *$(\leqslant r)$-regular graph decomposition* and a *matching decomposition* are defined analogously.

The main method used to prove Theorems 4–6 is to decompose K_n into regular parts (regular in the sense that every vertex has the same degree), then order the edges in each part, and concatenate the parts to obtain an ordering for K_n. The following propositions will facilitate this, under certain conditions. The propositions are given in more generality than we will require them, as they may be useful for other matching sequencibility problems. In each proposition, the subscripts of the orderings ℓ_i are taken modulo t: $\ell_{i+u} = \ell_{i'}$ exactly when $i' \equiv i + u \mod t$.

Proposition 9. *Let G be a graph that decomposes into matchings $M_0, \ldots, M_{t-1}$, each with n edges and orderings $\ell_0, \ldots, \ell_{t-1}$, respectively. Suppose, for some $\epsilon \in [n]$ and $r < \Delta(G)$, that $ms(\ell_i, \ell_{i+r}) \geqslant n-\epsilon$ for all $i \in [t-r]$. Then $ms_r(G) \geqslant rn-\epsilon$, and if $ms(\ell_i, \ell_{i+r}) \geqslant n-\epsilon$ for all $i \in [t]$, then $cms_r(G) \geqslant rn-\epsilon$.*

Proposition 10. *Let $r < \Delta(G)$ be even, set $u := \frac{r}{2}$, and let G be a graph that decomposes into $(\leqslant 2)$-regular graphs $R_0, \ldots, R_{t-1}$, each with n edges, and with orderings $\ell_0, \ldots, \ell_{t-1}$, respectively. Suppose, for some non-zero $\epsilon \in \left[\lceil \frac{n}{2} \rceil\right]$, that $ms_2(\ell_i, \ell_{i+u}) \geqslant n-\epsilon$ for all $i \in [t-u]$. Then $ms_r(G) \geqslant \frac{rn}{2} - \epsilon$, and if $ms_2(\ell_i, \ell_{i+u}) \geqslant n-\epsilon$ for all $i \in [t]$, then $cms_r(G) \geqslant \frac{rn}{2} - \epsilon$.*

Proposition 11. *Let $3 \leqslant r < \Delta(G)$ be odd, set $u := \frac{r-1}{2}$, and let G be a graph that decomposes into $(\leqslant 2)$-regular graphs $R_0, \ldots, R_{t-1}$, each with n edges, and with orderings $\ell_0, \ldots, \ell_{t-1}$, respectively. Suppose, for some non-zero $\epsilon \in \left[\lceil \frac{n}{2} \rceil\right]$, that $ms(\ell_i, \ell_{i+u+1}) \geqslant \lceil \frac{n}{2} \rceil - \epsilon$ for all $i \in [t-u-1]$ and $ms_3(\ell_i, \ell_{i+u}) \geqslant \lceil \frac{3n}{2} \rceil - \epsilon$ for all $i \in [t-u]$. Then $ms_r(G) \geqslant \lfloor \frac{rn+1}{2} \rfloor - \epsilon$, and if $ms(\ell_i, \ell_{i+u+1}) \geqslant \lceil \frac{n}{2} \rceil - \epsilon$ and $ms_3(\ell_i, \ell_{i+u}) \geqslant \lceil \frac{3n}{2} \rceil - \epsilon$ for all $i \in [t]$, then $cms_r(G) \geqslant \lfloor \frac{rn+1}{2} \rfloor - \epsilon$.*

For a graph G with ordering ℓ, $L_\ell(G)$ denotes the edges of G listed the same order as ℓ and say that ℓ *corresponds* to $L_\ell(G)$; i.e., if $e_0, \ldots, e_{k-1}$ is a list of the edges of G, then ℓ corresponds to that list if $\ell(e_i) = i$ for all $i \in [k]$. Also, for graphs $G_0, \ldots, G_{a-1}$ with labellings $\ell_0, \ldots, \ell_{a-1}$, respectively, $L_{\ell_0}(G_0) \vee \cdots \vee L_{\ell_{a-1}}(G_{a-1})$ denotes the lists of edges which the ordering $\ell_0 \vee \cdots \vee \ell_{a-1}$ corresponds to. The proofs of Proposition 9–11 are very similar, so we provide the proof of Proposition 11 and leave the details of the other two to the reader.

Proof of Proposition 11. The cyclic and non-cyclic cases are similar so we only show the cyclic case. Let ℓ be the ordering corresponding to $L_{\ell_0}(R_0) \vee \cdots \vee L_{\ell_{t-1}}(R_{t-1})$. Consider a set E of $\lfloor \frac{rn+1}{2} \rfloor - \epsilon$ consecutive edges of ℓ. The edges of E, in order, will always be of the form

$$\underbrace{e_1, \ldots, e_j}_{\text{edges in } R_i}, L_{\ell_{i+1}}(R_{i+1}) \vee \cdots \vee L_{\ell_{i+u+1-a}}(R_{i+u+1-a}), \underbrace{e_{j+1}, \ldots, e_{an-\lfloor \frac{n}{2} \rfloor - \epsilon}}_{\text{edges in } R_{i+u+2-a}},$$

for some $i \in [t]$, $j \in [n+1]$, and a. Without loss of generality, we can assume that $E \cap E(R_i)$ and $E \cap E(R_{i+u+2-a})$ are non-empty and so $0 < j < an - \lfloor \frac{n}{2} \rfloor - \epsilon$. There are $0 < an - \lfloor \frac{n}{2} \rfloor - \epsilon \leqslant 2n$ edges in $E \cap (E(R_i) \cup E(R_{i+u+2-a}))$. Therefore, $a = 1$ or $a = 2$.

If $a = 1$, then the first j edges and last $\lceil \frac{n}{2} \rceil - j - \epsilon$ edges of E are the last j edges of ℓ_i and the first $\lceil \frac{n}{2} \rceil - j - \epsilon$ edges of ℓ_{i+u+1}, respectively. Thus, the $j + \lceil \frac{n}{2} \rceil - j - \epsilon = \lceil \frac{n}{2} \rceil - \epsilon$ edges of $E \cap (E(R_i) \cup E(R_{i+u+1}))$ are consecutive in $\ell_i \vee_{\lceil \frac{n}{2} \rceil - \epsilon} \ell_{i+u+1}$ and, by assumption, form a matching. The remaining edges of E are from the $u = \frac{r-1}{2}$ ($\leqslant 2$)-regular graphs $R_{i+1}, \ldots, R_{i+u}$ and, hence, the edges of E form a ($\leqslant r$)-regular graph.

If $a = 2$, then the first j edges and last $\lceil \frac{3n}{2} \rceil - j - \epsilon$ edges of E are the last j edges of ℓ_i and the first $\lceil \frac{n}{2} \rceil - j - \epsilon$ edges of ℓ_{i+u}, respectively. Therefore, the $j + \lceil \frac{3n}{2} \rceil - j - \epsilon = \lceil \frac{3n}{2} \rceil - \epsilon$ edges of $E \cap (E(R_i) \cup E(R_{i+u}))$ are consecutive in $\ell_i \vee_{\lceil \frac{3n}{2} \rceil - \epsilon} \ell_{i+u}$ and, by assumption, form a ($\leqslant 3$)-regular graph. The remaining edges of E are from the $u - 1 = \frac{r-3}{2}$ ($\leqslant 2$)-regular graphs $R_{i+1}, \ldots, R_{i+u-1}$, and, thus, the edges of E form a ($\leqslant r$)-regular graph. $\square$

Remark 12. Proposition 11 holds for $r = 1$ by replacing the assumption $ms_3(\ell_i, \ell_{i+u}) = ms_3(\ell_i, \ell_i) \geqslant \lceil \frac{3n}{2} \rceil - \epsilon$ with $ms(\ell_i) \geqslant \lceil \frac{n}{2} \rceil - \epsilon$.

Note that Proposition 11 requires two conditions on the orderings $\ell_0, \ldots, \ell_{t-1}$, namely $ms(\ell_i, \ell_{i+u+1}) \geqslant \lceil \frac{n}{2} \rceil - \epsilon$ and $ms_3(\ell_i, \ell_{i+u}) \geqslant \lceil \frac{3n}{2} \rceil - \epsilon$ (for the relevant i), whereas Proposition 9 and Proposition 10 require only one condition. This makes Proposition 11 harder to use and largely explains why Theorems 4–6 cover more cases when r and n are not both odd. Also, the requirement of two conditions in Propositions 11 may suggest that the case when r and n are odd is inherently more difficult for K_n (or even any graph of odd order).

Here and throughout the paper, we will allow orderings to be defined on sets of integers; that is, a bijection $\alpha : A \to [|A|]$ on a set E of integers will also be considered an ordering. However, we will only use such orderings for re-indexing. The following auxiliary lemma guarantees the existence of an ordering of integers with particular useful properties. It will find repeated use in later sections, so it is given here for easy reference.

Lemma 13. *Let t and u be integers with $t > u$ and set $d := \gcd(u, t)$. Define $a_{i,j} := \left(i \pmod{\frac{t}{d}}\right) + j\frac{t}{d} \pmod{t}$ for all integers i and j. Then there exists an ordering α of $[t]$ with the property that $\alpha(a_{i+1,j}) = \alpha(a_{i,j}) + u \pmod{t}$ for all $i \in \left[\frac{t}{d}\right]$ and $j \in [d]$.*

Proof. We check that the function $\alpha : [t] \to [t]$ defined by $\alpha(a_{i,j}) = iu + j \pmod t$ for $i \in \left[\frac{t}{d}\right]$ and $j \in [d]$ will suffice. First, we will show that α is a bijection. Suppose that $iu + j \equiv i'u + j' \pmod t$, with $i, i' \in \left[\frac{t}{d}\right]$ and $j, j' \in [d]$. Then $(i - i')u \equiv j' - j \pmod t$. As d divides t and u, any multiple of u modulo t is also a multiple of d. Thus, $j - j'$ is a multiple of d, while $0 \leqslant |j - j'| \leqslant d - 1$. This is only possible if $j = j'$ and so $(i - i')u \equiv 0 \pmod t$. As $0 \leqslant |i' - i| \leqslant \frac{t}{d} - 1$ and $\operatorname{lcm}(t, u) = \frac{tu}{d}$, we must also have that $i = i'$. Thus, α is injective and so bijective; α is thus an ordering of $[t]$. For any $i \in \left[\frac{t}{d}\right]$ and $j \in [d]$,

$$\alpha(a_{i+1,j}) = (i+1)u + j \pmod t = iu + j + u \pmod t = \alpha(a_{i,j}) + u \pmod t\,.$$

Hence, α has the required properties. □

The function α in the proof can be used to show a non-cyclic version of the lemma:

Lemma 14. *Let $t > u$. Then there exists an ordering α of $[t]$ with the property that, if $\alpha(a) \leqslant t - u - 1$, then $\alpha(a+1) = \alpha(a) + u$.*

Example 15. Let $t = 10$ and $u = 4$; then $d = \gcd(4, 10) = 2$. The following table summarises the function α that is produced by Lemma 13.

x	0	1	2	3	4	5	6	7	8	9
$\alpha(x)$	0	4	8	2	6	1	5	9	3	7

3 Decompositions of the complete graph K_n

To prove Theorems 4–6, we will require matching decompositions of K_n when n is even and 2-regular decompositions of K_n when n is odd, so that we can apply the applicable proposition from Section 2. Here we present the required decompositions.

3.1 Decompositions of K_n for even n

Let $n = 2m$, $r \in [2m-1] - \{0\}$, $c \mid 2m - 1$ and $d = \frac{2m-1}{c}$. Let $V_{c,d} = \{v_\infty\} \cup \{v_{i,j} : i \in [c], j \in [d]\}$ be the vertex set of K_{2m}. We set $v_{i,j} := v_{i',j'}$, whenever $i' = i \pmod c$ and $j' = j \pmod d$. The following sets (with the singleton excluded) are given in [6]. For an integer x and odd integer y, let $P_{x,y} = \left\{\{x + l, x - l\} : l \in \left[\frac{y+1}{2}\right]\right\}$, where the elements of the members of $P_{x,y}$ are taken modulo y. We also note the following useful fact.

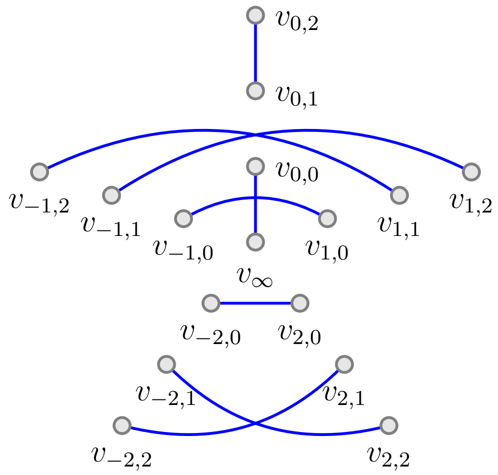
Remark 16. Each family $P_{x,y}$ forms a partition of $[y]$ into pairs and a singleton, and the set $\{P_{x,y} : x \in [y]\}$ partitions the pairs and singletons of $[y]$.

For $i \in [c]$ and $j \in [d]$, let $M_{i,j}$ be the matching of K_{2m} with edge set

$$\left\{\{v_\infty, v_{i,j}\}\right\} \cup \left\{\{v_{a_1,b_1}, v_{a_2,b_2}\} \;:\; \{a_1, a_2\} \in P_{i,c}, \{b_1, b_2\} \in P_{j,d} \text{ and } a_1 \neq i \text{ or } b_1 \neq j \;\right\}.$$

These are indeed matchings, as the edge incident to v_∞ in each $M_{i,j}$ is unique and a vertex v_{a_1,b_1}, which is not adjacent to v_∞ in $M_{i,j}$, is incident to edges $\{v_{a_1,b_1}, v_{a_2,b_2}\}$ with $\{a_1, a_2\} \in P_{i,c}$ and $\{b_1, b_2\} \in P_{j,d}$; also by the above remark, the choice for such an a_2 and b_2 is unique.

Example 17. When $c = 5$ and $d = 3$, $M_{0,0}$ is the following subgraph of K_{2m}.



Lemma 18. *The set* $\{M_{i,j} \,:\, i \in [c],\, j \in [d]\}$ *is a matching decomposition of* K_n.

Proof. Clearly, $M_{i,j}$ is the unique matching containing the edge $\{v_\infty, v_{i,j}\}$. For $a_1, a_2 \in [c]$ and $b_1, b_2 \in [d]$, with $a_1 \neq a_2$ or $b_1 \neq b_2$, the edge $\{v_{a_1,b_1}, v_{a_2,b_2}\}$ is in the matching $M_{i,j}$, for some $i \in [c]$ and $j \in [d]$ if $\{a_1, a_2\} \in P_{i,c}$ and $\{b_1, b_2\} \in P_{j,d}$. By Remark 16, such i and j, and therefore $M_{i,j}$ exist and are uniquely given. Thus, the edge $\{v_{a_1,b_1}, v_{a_2,b_2}\}$ occurs in exactly one matching. Hence, $\{M_{i,j} \,:\, i \in [c], j \in [d]\}$ is a matching decomposition of K_n. □

Note that this decomposition is the same for different values of c, just indexed differently. Indeed, the bijection $\tau_{c,d} : V_{c,d} \to V_{2m-1,1}$ defined by $\tau_{c,d}(v_\infty) = v_\infty$ and $\tau_{c,d}(v_{a,b}) = v_{ad+b,0}$ for $a \in [c]$ and $b \in [d]$ is an isomorphism, showing that the decomposition for a particular value of c is isomorphic to the decomposition for $c = 2m - 1$. Note that the Walecki decomposition (see [1]) decomposes K_{2m} into Hamiltonian cycles and a complete matching, from which the matching decomposition for $c = 2m - 1$, given above, can be easily obtained.

3.2 Decompositions of K_n when n is odd

Let $n = 2m + 1$. We will present two different decompositions for K_{2m+1}. The first is the Walecki decomposition [1] mentioned above. Let $V = \{\infty\} \cup \mathbb{Z}_{2m}$ be the vertex set of K_{2m+1}. Let H_0 be the Hamiltonian cycle $\infty, 0, 1, -1, 2, -2, \ldots, x, -x, \ldots, m-1, -(m-1), m$, as depicted in Figure 1.

Let σ be the permutation $\sigma = (\infty)(0 \;\; 1 \;\; \cdots \;\; 2m-2 \;\; 2m-1)$. Let $H_i = \sigma^i(H_0)$ for $i \in [m]$, where σ acts on the vertices of V. Alspach [1] proved the following lemma, and we give a similar proof for completeness.

Lemma 19. *The set* $\{H_0, \ldots, H_{m-1}\}$ *is a 2-regular decomposition of* K_{2m+1}.

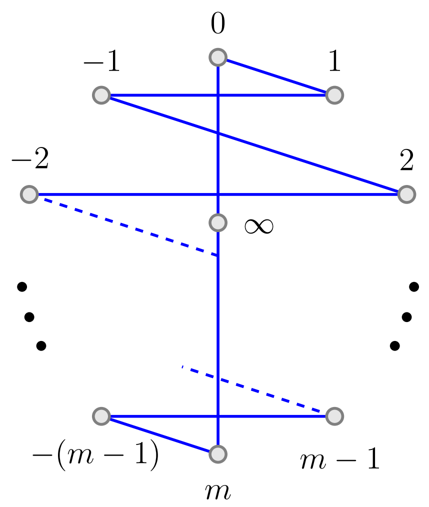


Figure 1: The Hamiltonian cycle H_0

Proof. As each H_i has $2m+1$ edges, we only need to show that the edges of $H_0, \dots, H_{m-1}$ are disjoint. Clearly the edges $\{\infty, i\}$ and $\{\infty, i+m\}$ are only present in H_i, for all $i \in [m]$. For the remaining edges, let the *length* of an edge $\{i, j\}$ with $i, j \neq \infty$ be $j - i$ mod $2m$ or $i - j$ mod $2m$, whichever lies in $[m+1] - \{0\}$.

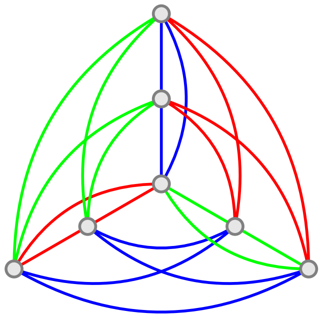
We check, for every fixed length $l \in [m+1] - \{0\}$, that the edges of length l in $H_0, \dots, H_{m-1}$ are distinct. Note that the edges of H_i that are not incident to ∞ are $\{i+x, i-x\}$ for $x \in [m] - \{0\}$, and $\{i+x, i-x+1\}$ for $x \in [m+1] - \{0\}$. The edges of even length $l < m$ in H_i are $\left\{i + \frac{l}{2}, i - \frac{l}{2}\right\}$ and $\left\{i + m - \frac{l}{2}, i - m + \frac{l}{2}\right\}$, and neither edge is an edge of H_j for $j \neq i$. The edges of odd length $l < m$ in H_i are $\left\{i + \frac{l+1}{2}, i - \frac{l+1}{2} + 1\right\}$ and $\left\{i + m - \frac{l-1}{2}, i - m + \frac{l-1}{2} + 1\right\}$, and neither edge is an edge of H_j for $j \neq i$. If m is even, then the edge of length m in H_i is $\left\{i + \frac{m}{2}, i - \frac{m}{2}\right\}$, and is only an edge of H_i. If m is odd, then the edge of length m in H_i is $\left\{i + \frac{m+1}{2}, i - \frac{m+1}{2} + 1\right\}$, and is only an edge of H_i. Therefore, the edges of every length $l \in [m+1] - \{0\}$ in $H_0, \dots, H_{m-1}$ are distinct. □

The 2-regular decomposition $\{H_0, \dots, H_{m-1}\}$ of K_{2m+1} has a particular disadvantage relevant to us. If ℓ_0 is an ordering of H_0, then the permutation σ induces orderings $\ell_1, \dots, \ell_{m-1}, \ell_m$ of $H_1, \dots, H_{m-1}, H_m = H_0$, respectively. However, $\ell_m \neq \ell_0$. Therefore, the 2-regular decomposition $\{H_0, \dots, H_{m-1}\}$ is not ideal for constructing cyclic orderings in this fashion.

The second decomposition overcomes this problem but does not exist for all n. Recall that $n = 2m+1$ and let m be odd. Let $V_{m,2} = \{v_\infty\} \cup \{v_{i,j} \ : \ i \in [m], j \in [2]\}$ be the vertex set of K_{2m+1}. For an integer x and an odd integer y, let $P_{x,y}$ be as defined in Subsection 3.1. Let R_i be the subgraph of K_{2m+1} with edge set

$$\left\{\{v_\infty, v_{i,0}\}, \{v_\infty, v_{i,1}\}, \{v_{i,0}, v_{i,1}\}\right\} \cup \left\{\{v_{a_1,b_1}, v_{a_2,b_2}\} \ : \ \{a_1, a_2\} \in P_{i,m}, a_1 \neq i, b_1, b_2 \in [2]\right\}.$$

Example 20. The following graph depicts R_0, R_1 and R_2 for K_7.



Clearly R_0, R_1 and R_2 form a 2-regular decomposition of K_7. Now we will show that the same is true in general.

Lemma 21. *For odd m, $\{R_0, \ldots, R_{m-1}\}$ is a 2-regular graph decomposition of K_{2m+1}.*

Proof. Clearly, the edges $\{v_\infty, v_{i,0}\}, \{v_\infty, v_{i,1}\}$ and $\{v_{i,0}, v_{i,1}\}$ are present only in R_i. The edge $\{v_{a_1,b_1}, v_{a_2,b_2}\}$ with $a_1 \neq a_2$ is present in R_i for i such that $\{a_1, a_2\} \in P_{i,m}$. By Remark 16, such an i exists and is unique. Thus, every edge is in a unique R_i for $i \in [m]$. □

4 Proof of Theorems 4 and 5 for when n is even

Write $n = 2m$ and let $r \in [n-1] - \{0\}$. Set $d := \gcd(r, 2m-1)$ and $c := \frac{2m-1}{d}$, and as in Subsection 3.1, define $V_{c,d}$ to be the vertex set of K_{2m}. Also, let $M_{i,j}$ be the matchings defined in Subsection 3.1 for $i \in [c]$ and $j \in [d]$. Define $\ell_{i,j}$ to be the following ordering of $M_{i,j}$:

$$\begin{aligned}
\ell_{i,j}(\{v_\infty, v_{i,j}\}) &= 0\,, \\
\ell_{i,j}(\{v_{i+x,j}, v_{i-x,j}\}) &= x \quad \text{for} \quad x \in \left[\frac{c+1}{2}\right] - \{0\}\,, \\
\ell_{i,j}(\{v_{i+2x,j+y}, v_{i-2x,j-y}\}) &= (y-1)c + \frac{c+1}{2} + \left(x + i\frac{c-1}{2} \pmod{c}\right) \\
&\quad \text{for} \quad x \in [c],\ y \in \left[\frac{d+1}{2}\right] - \{0\}\,.
\end{aligned}$$

Example 22. When $c = 5$ and $d = 3$, the matching $M_{0,0}$ is labelled as in Figure 2.

Note that the matchings $M_{i,0}$ are obtained by rotating the above graph but the orderings $\ell_{i,0}$ are not. We will use the following notation. Let $V'_{j,0} = \{v_\infty\} \cup \{v_{z,j} \ : \ z \in [c]\}$ for $j \in [d]$ and let $V'_{j,y} = \{v_{z,j\pm y} \ : \ z \in [c]\}$ for $j \in [d]$ and $y \in \left[\frac{d+1}{2}\right] - \{0\}$. Clearly, $V'_{j,0}, \ldots, V'_{j,\frac{d-1}{2}}$ partition $V_{c,d}$ for all $j \in [d]$. The crucial component of the proof of Theorems 4 and 5 for when n is even is the following lemma which allows us to apply Proposition 9.

Lemma 23. *For all $i \in [c]$ and $j \in [d]$, $ms(\ell_{i,j}, \ell_{i+1,j}) \geqslant m - 1$.*

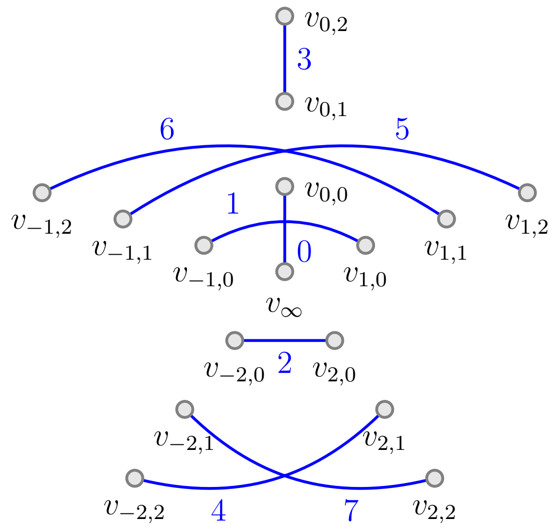


Figure 2: The matching $M_{0,0}$

Proof. Consider a set of $m-1$ consecutive edges E in $\ell = \ell_{i,j} \vee_{m-1} \ell_{i+1,j}$. As $M_{i,j}$ and $M_{i+1,j}$ are matchings, two edges incident to a common vertex in E cannot both be from $M_{i+1,j}$ or both be from $M_{i,j}$. The edges of E are the edges labelled $m-l, \ldots, m-1$ by $\ell_{i,j}$ and the edges labelled $0, \ldots, m-l-2$ by $\ell_{i+1,j}$, for some $l \in [m-1] - \{0\}$. Thus, for a vertex v to be incident to two edges in E, v must be incident to e_1 in $M_{i,j}$ and e_2 in $M_{i+1,j}$ which satisfy $\ell_{i,j}(e_1) \geqslant m-l$ and $\ell_{i+1,j}(e_2) \leqslant m-l-2$. Therefore, $\ell_{i,j}(e_1) - \ell_{i+1,j}(e_2) \geqslant m-l-(m-l-2) = 2$. Hence, to check that E forms a matching, it suffices to show, for all vertices $v \in V_{c,d}$, that if v is incident to e_1 in $M_{i,j}$ and e_2 in $M_{i+1,j}$, then $\ell_{i,j}(e_1) - \ell_{i+1,j}(e_2) < 2$. Let $v \in V'_{j,y}$ for some $y \in [\frac{d+1}{2}]$.

First, suppose that $y = 0$. If e_1 in $M_{i,j}$ and e_2 in $M_{i+1,j}$ are both incident to ∞, then $\ell_{i,j}(e_1) - \ell_{i+1,j}(e_2) = 0 - 0 = 0 < 2$. We therefore only need to check the remaining vertices in $V'_{j,0}$. Let $v = v_{i+x,j}$ for some $x \in [\frac{c+1}{2}]$. Then v is incident to the edge labelled x by $\ell_{i,j}$. Let $x' \in [\frac{c+1}{2}]$ be the integer such that v is either $v_{i+1+x',j}$ or $v_{i+1-x',j}$. In either case, v is incident to the edge labelled x' by $\ell_{i+1,j}$. Therefore, it suffices to show that $x - x' < 2$. If $v = v_{i+1+x',j}$, then $i + x \equiv i + 1 + x' \pmod{c}$ and so $x \equiv x' + 1 \pmod{c}$. As $x, x' \in [\frac{c+1}{2}]$, it follows that $x = x' + 1$, and so $x - x' = 1 < 2$. Otherwise, $v = v_{i+1-x',j}$, implying that $i + x \equiv i + 1 - x' \pmod{c}$, and so $x + x' \equiv 1 \pmod{c}$. As $x, x' \in [\frac{c+1}{2}]$, it follows that $\{x, x'\} = \{0, 1\}$. Therefore, $x - x'$ is 1 or -1 and in particular less than 2. The case in which $v = v_{i-x,j}$ for some $x \in [\frac{c+1}{2}]$ can be treated in a similar fashion and is left to the reader.

Now suppose that $y \neq 0$. Let $v = v_{i+2x,j+y}$ for some $x \in [c]$. Let e_1 and e_2 be the edges incident to v in $M_{i,j}$ and $M_{i+1,j}$, respectively. Let $v = v_{i+1+2x',j+y}$ for some $x' \in [c]$. Then $i + 2x \equiv i + 1 + 2x' \pmod{c}$. As $\gcd(2, c) = 1$, this reduces to

$$x' \equiv x + \frac{c-1}{2} \pmod{c}. \tag{1}$$

We will now check that $\ell_{i,j}(e_1)-\ell_{i+1,j}(e_2) < 2$. The labels of e_1 and e_2 are, respectively,

$$(y-1)c+\frac{c+1}{2}+\left(x+i\frac{c-1}{2} \pmod c\right) \text{ and } (y-1)c+\frac{c+1}{2}+\left(x'+(i+1)\frac{c-1}{2} \pmod c\right).$$

By (1), the difference between these two labels is

$$\left(x+i\frac{c-1}{2} \pmod c\right)-\left(x+i\frac{c-1}{2}+(1+1)\frac{c-1}{2} \pmod c\right).$$

The right term is $c-1$ more than the left term before taking modulo c. So the difference is either 1 or $-(c-1)$ and in particular less than 2. Hence, $\ell_{i,j}(e_1)-\ell_{i+1,j}(e_2) < 2$. The case in which $v = v_{i-2x,j-y}$ is similar and therefore omitted. Thus, E forms a matching. □

Proof of Theorems 4 and 5 when n is even. Theorem 4 when n is even follows from Theorem 5 when n is even, so we only prove the latter. Let α and $a_{i,j}$ be as defined in Lemma 13 for $u = r$ and $t = 2m-1$. Let $M'_{\alpha(a_{i,j})} = M_{i,j}$ and $\ell_{\alpha(a_{i,j})} = \ell_{i,j}$ for all $i \in [c]$ and $j \in [d]$. If $x = \alpha(a_{i,j})$, then, by Lemmas 13 and 23, $ms(\ell_x, \ell_{x+r}) = ms(\ell_{i,j}, \ell_{i+1,j}) \geqslant m-1$ (where $x+r$ in ℓ_{x+r} is taken modulo $2m-1$). Thus, Proposition 9 yields $cms_r(K_{2m}) \geqslant rm-1$, using the matchings $M'_0, \dots, M'_{2m-1}$ ordered by $\ell'_0, \dots, \ell'_{2m-1}$, respectively. The reverse inequality, $cms_r(K_{2m}) \leqslant rm-1$, follows from Lemma 8. This completes the proof. □

5 Proof of Theorem 4 for when n is odd and $\gcd(r, n-1) = 1$

Let $n = 2m+1$ and $r \in [2m] - \{0\}$ be an integer such that $\gcd(r, 2m) = 1$. Also, let V_{2m} be the vertex set of K_{2m+1}, and H_i be the Hamiltonian cycles from Subsection 3.2 for $i \in [m]$. Let ℓ_i be the ordering of H_i defined as follows:

$$\begin{aligned} \ell_i(\{\infty, i\}) &= 0\,, \\ \ell_i(\{\infty, i+m\}) &= m\,, \\ \ell_i(\{i+rx, i-rx\}) &= x \quad \text{for non-zero } x \in [m]\,, \\ \ell_i\left(\left\{i+rx+\frac{r+1}{2}, i-rx-\frac{r-1}{2}\right\}\right) &= m+x+1 \quad \text{for} \quad x \in [m]\,. \end{aligned}$$

This is indeed valid as the edge $\{i+a, i-a\}$ has label $ar^{-1} \pmod m$ for $a \in [m]-\{0\}$ and the edge $\{i+a, i-a+1\}$ has the label $m+1+\left(r^{-1}\left(a-\frac{r+1}{2}\right) \pmod m\right)$ for $a \in [m]$. Also, it is clear that the first m edges of ℓ_i form a matching as do the last m edges of ℓ_i.

Example 24. When $n = 11$ and $r = 3$, the Hamiltonian cycle H_0 is labelled as in Figure 3.

Lemma 25. *Let r be odd and set $u := \frac{r-1}{2}$. Then $ms_3(\ell_i, \ell_{i+u}) \geqslant 3m+1$ for all $i \in [m-u]$ and $ms(\ell_i, \ell_{i+u+1}) \geqslant m$ for all $i \in [m-u-1]$.*

Proof. First, we will show that if $i \in [m-u]$, then $ms_3(\ell_i, \ell_{i+u}) \geqslant 3m+1$. Let $\ell = \ell_i \vee_{3m+1} \ell_{i+u}$ and consider a set E of $3m+1$ consecutive edges in ℓ. A vertex has degree

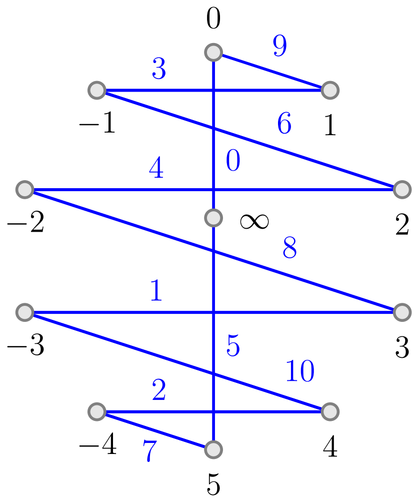


Figure 3: The Hamiltonian cycle H_0

greater than 3 in E, only if it has degree two in both $E \cap E(H_i)$ and $E \cap E(H_j)$. In particular, the result follows immediately if there are m or fewer edges in either $E \cap E(H_i)$ or $E \cap E(H_{i+u})$, as the first m edges of ℓ_a form a matching as do the last m edges of ℓ_a for all $a \in [m]$. So, suppose that there are $2m - l$ edges in $E \cap E(H_i)$ and so $m + 1 + l$ edges in $E \cap E(H_{i+u})$ for some $l \in [m]$. Let W_1 be the vertices of degree at most 1 in $E \cap E(H_i)$ and W_2 be the vertices of degree 2 in $E \cap E(H_{i+u})$. To show that E forms a $(\leqslant 3)$-regular graph, it suffices to check that $W_2 \subseteq W_1$.

The first m edges of ℓ_{i+u} form a matching in which $i + u + m$ is the only isolated vertex. Therefore, the vertices W_2 are those incident to edges with labels between m and $m + l$, excluding $i + u + m$. In particular, $W_2 = \{\infty\} \cup \{i + \frac{r-1}{2} + rx + \frac{r+1}{2}, i + \frac{r-1}{2} - rx - \frac{r-1}{2} \ : \ x \in [l]\}$, where $[0] = \emptyset$. So $W_2 = \{\infty\} \cup \{i + r(x + 1), i - rx \ : \ x \in [l]\} = \{\infty\} \cup (\{i + rx, i - rx \ : \ x \in [l + 1]\} - \{i - rl\})$, by re-indexing. Similarly, the vertices in W_1 are those incident to an edge in $E(H_i) - E$; these edges are labelled between 0 and l by ℓ_i. Thus, $W_1 = \{\infty\} \cup \{i + rx', i - rx' \ : \ x' \in [l + 1]\}$. Comparing W_1 and W_2 shows that $W_2 \subseteq W_1$.

Finally, we will show that if $i \in [m-u-1]$, then $ms(\ell_i, \ell_{i+u+1}) \geqslant m$. Let $\ell = \ell_i \vee_m \ell_{i+u+1}$ and consider a set E of m consecutive edges in ℓ. Suppose that there are $m - l$ edges in $E \cap E(H_i)$, and so l edges in $E \cap E(H_{i+u+1})$ for some non-zero $l \in [m]$. The first m edges of ℓ_a form a matching as do the last m edges of ℓ_a for $a \in [m]$. Thus, E does not form a matching only if a vertex is incident to an edge in $E \cap E(H_i)$ and an edge in $E \cap E(H_{i+u+1})$. Let W_0 be the vertices incident to no edges in $E \cap E(H_i)$ and W_1 be the vertices incident to an edge in $E \cap E(H_{i+u+1})$. To prove that E forms a matching, it suffices to show that $W_1 \subseteq W_0$.

The last m edges of ℓ_i form a matching in which ∞ is the only vertex not incident to an edge. Thus, the members of W_0 are the vertices incident to an edge with a label in between $m+1$ and $m+l$, along with ∞. Hence, $W_0 = \{\infty\} \cup \{i + rx + \frac{r+1}{2}, i - rx - \frac{r-1}{2} \ :$

$x \in [l]\}$. The members of W_1 are the vertices incident to one of the first l edges of ℓ_{i+u+1}, so $W_1 = \{\infty, i + \frac{r+1}{2}\} \cup \{i + rx + \frac{r+1}{2}, i - rx + \frac{r+1}{2} \ : \ x \in [l] - \{0\}\}$. Therefore, $W_1 = \{\infty\} \cup \left(\{i + rx + \frac{r+1}{2}, i - rx - \frac{r-1}{2} \ : \ x \in [l]\} - \{i - r(l-1) - \frac{r-1}{2}\}\right) \subseteq W_0$. This completes the proof. □

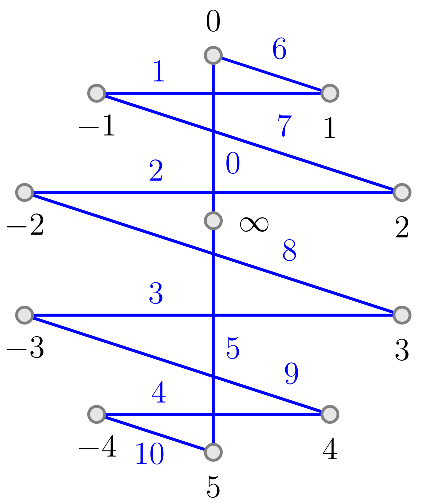
Proof of Theorem 4 when n is odd and $\gcd(r, n-1) = 1$. By Lemma 25 and Proposition 11, using $H_0, \ldots, H_{m-1}$ with labellings $\ell_0, \ldots, \ell_{m-1}$, respectively, we have that $ms_r(K_n) \geqslant \frac{rn-1}{2}$. The reverse inequality follows from Lemma 8. □

6 Proof of Theorem 6 and the remaining cases of Theorem 4

Let $n = 2m+1$, $r \in [2m] - \{0\}$ and let V_{2m} and H_i be as defined in Section 3.2. Also, let ℓ_i be the labelling of H_i defined as follows:

$$\begin{aligned} \ell_i(\{\infty, i\}) &= 0\,, \\ \ell_i(\{\infty, i+m\}) &= m\,, \\ \ell_i(\{i+x, i-x\}) &= x && \text{for } x \in [m] - \{0\}\,, \\ \ell_i(\{i+x, i-x+1\}) &= m+x && \text{for } x \in [m+1] - \{0\}\,. \end{aligned}$$

Example 26. When $n = 11$, the Hamiltonian cycle H_0 is labelled as follows:



The other H_i's and their labellings are obtained by rotating the above graph. Therefore, ℓ_i is just the labelling that takes alternating edges of the Hamilton cycle H_i starting from $\{\infty, i\}$. Thus, it is clear that $ms(\ell_i) = m$ for $i \in [m]$.

6.1 Proof of Theorem 4 for when n is odd and r is even

We will require the following lemma to apply Proposition 10.

Lemma 27. *For distinct* $i, j \in [m]$, $ms_2(\ell_i, \ell_j) \geqslant 2m + 1 - |j - i|$.

Proof. Let $s = 2m+1-|j-i|$ and consider a set E of s consecutive edges of $\ell = \ell_i \vee_s \ell_j$. As $ms(\ell_a) \geqslant m$ for all $a \in [m]$, the last m edges of ℓ_i form a matching as do the first m edges of ℓ_j. Thus, if both $E \cap E(H_i)$ and $E \cap E(H_j)$ contain fewer than $m+1$ edges, then $E \cap E(H_i)$ and $E \cap E(H_j)$ both form matchings. Hence, no vertex would have degree greater than 2 in E. So, consider a set of s consecutive edges E in ℓ where either $E \cap E(H_i)$ or $E \cap E(H_j)$ contains at least $m+1$ edges. Let $a \in \{i,j\}$ be the integer for which $|E \cap E(H_a)| \geqslant m+1$. Let W_2 be the vertices of degree 2 in $E \cap E(H_a)$ and W_1 be the vertices of degree 1 in $E \cap E(H_{a'})$, where $a' \in \{i,j\}$ and $a' \neq a$. As $ms(\ell_{a'}) \geqslant m$, $E \cap E(H_{a'})$ forms a matching. Thus, there exists a vertex incident to more than two edges in E only if $W_1 \cap W_2 \neq \emptyset$. Hence, to show that E forms a $(\leqslant 2)$-regular graph, it suffices to prove that $W_1 \cap W_2 = \emptyset$.

If $a = i$, then $E \cap E(H_i)$ has $s-l$ edges and $E \cap E(H_j)$ has l edges for some non-zero $l \in [s-m]$. The last m edges of ℓ_i form a matching in which ∞ is the only vertex not incident to an edge. Therefore, the vertices in W_2 are the vertices incident to an edge labelled x by ℓ_i such that $l + |j-i| \leqslant x \leqslant m$, excluding ∞. In particular, W_2 contains $i+m$ and $i+x$ and $i-x$ for $l+|j-i| \leqslant x \leqslant m-1$. The vertices in W_1 are the vertices incident to any of the first l edges of ℓ_j; hence, W_1 contains ∞, j, and $j+x'$ and $j-x'$ for $x' \in [l] - \{0\}$. Then W_2 and W_1 can be expressed (modulo $2m$) as

$$W_2 = \{i+l+|j-i|, \ldots, i+(m-1), i+m, i-(m-1), \ldots, i-l-|j-i|\}$$
$$\text{and} \quad W_1 = \{\infty, j-l+1, \ldots, j-1, j, j+1, \ldots, j+l-1\}\,.$$

As $i - |j-i| < j+1$ and $i + |j-i| > j-1$, $W_1 \cap W_2 = \emptyset$. The case in which $a = j$ is similar and we omit the details. Thus, the s consecutive edges of E form a $(\leqslant 2)$-regular graph. □

Proof of Theorem 4 when n is odd and r is even. Let α satisfy the properties in Lemma 14 with $u = \frac{r}{2}$ and $t = m$. Let $H'_{\alpha(i)} = H_i$ and $\ell'_{\alpha(i)} = \ell_i$ for all $i \in [m]$. If $x = \alpha(i) \in [m-u]$, then $ms_2(\ell'_x, \ell'_{x+u}) = ms_2(\ell_i, \ell_{i+1}) \geqslant 2m$, by Lemmas 14 and 27. Thus, applying Proposition 10 to $H'_0, \ldots, H'_{m-1}$ with orderings $\ell'_0, \ldots, \ell'_{m-1}$, respectively, yields the inequality $ms_r(K_n) \geqslant \frac{rn}{2} - 1$. The reverse inequality, $ms_r(K_n) \leqslant \frac{rn}{2} - 1$, follows from Lemma 8. □

6.2 Proof of Theorem 4 for when n and r are odd and $r \geqslant \frac{n-1}{2}$

We will require the following two lemmas.

Lemma 28. *For a fixed $\frac{t}{2} \leqslant u \leqslant t-1$ there exists an ordering α_u of $[t]$ such that*

(1) $\alpha_u(i+u) = \alpha_u(i) - 1 \quad$ for all $i \in [t-u]$;

(2) $\alpha_u(i+u+1) = \alpha_u(i) + 1 \quad$ for all $i \in [t-u-1]$.

Proof. We show that the ordering $\alpha_u : [t] \to [t]$, defined delow, will suffice:

$$\alpha_u(i) := \begin{cases} 2i+1 & \text{if } 0 \leqslant i \leqslant t-u-1 \\ i+(t-u) & \text{if } t-u \leqslant i \leqslant u-1 \\ 2(i-u) & \text{if } u \leqslant i \leqslant t-1\,. \end{cases}$$

It is easy to check that α_u is injective and thus bijective; α_u is thus an ordering of $[t]$. For each integer $i \in [t-u]$, $\alpha_u(i+u) = 2(i+u-u) = 2i = \alpha_u(i) - 1$. Similarly for $i \in [t-u-1]$, $\alpha_u(i+u+1) = 2(i+u+1-u) = 2i+2 = \alpha_u(i)+1$. □

Lemma 29. *If $i < j$, then $ms(\ell_i, \ell_j) \geqslant m+1-(j-i)$ and $ms_3(\ell_i, \ell_j) \geqslant 3m+1-(j-i)$. If $i > j$, $ms(\ell_i, \ell_j) \geqslant m-(i-j)$ and $ms_3(\ell_i, \ell_j) \geqslant 3m+2-(i-j)$.*

Proof. Let $\ell = \ell_i \vee_s \ell_j$, where s is yet to be specified. First, let $s = 3m+2-|j-i|-\epsilon$ where $\epsilon = 0$ if $i > j$ and $\epsilon = 1$ otherwise. We want to show that $ms_3(\ell) \geqslant s$. Consider a set of s consecutive edges E of ℓ. For each $a \in [m]$, the first m edges of ℓ_a form a matching, as do the last m edges of ℓ_a, since $ms(\ell_a) \geqslant m$. Thus, if either $E \cap E(H_i)$ or $E \cap E(H_j)$ contain fewer than $m+1$ edges, then the degree of a vertex cannot be more than 3 in E. So, it suffices to assume that $|E \cap E(H_i)| \geqslant m+1$ and $|E \cap E(H_j)| \geqslant m+1$. Suppose that there are $s-(m+l)$ edges in $E \cap E(H_i)$, and so $m+l$ edges in $E \cap E(H_j)$ for some non-zero $l \in [s-2m]$.

Let W_a be the vertices incident to two edges in $E \cap E(H_a)$ for $a = i, j$. To show that E forms a ($\leqslant 3$)-regular graph, it suffices to prove that $W_i \cap W_j = \emptyset$, as H_i and H_j are 2-regular. The last m edges of ℓ_i form a matching that covers every vertex except ∞. Therefore, W_i contains the vertices that are incident to one of the edges with labels between $|j-i|+l+\epsilon-1$ and m, apart from ∞. Thus, the vertices in W_i are $i+m$ and $i+x$ and $i-x$ for $|j-i|+l+\epsilon-1 \leqslant x \leqslant m-1$. Similarly, the first m edges of ℓ_j form a matching that covers all vertices except $j+m$. Hence, W_j contains the vertices that are incident to one of the edges with labels between m and $m+l-1$, except $j+m$. Therefore, the vertices in W_j are ∞ and $j+x'$ and $j-x'+1$ for $x \in [l]-\{0\}$. So $W_i \cap W_j$ is clearly empty when $l=1$. In the remaining cases, we can then express W_i and W_j as follows modulo $2m$:

$$\begin{aligned} W_i &= \{i+|j-i|+l+\epsilon-1, \ldots, i+(m-1), i+m, \\ &\qquad i-(m-1), \ldots, i-|j-i|-l-\epsilon+1\} \\ &= \begin{cases} \{2i-j+l-1, 2i-j+l, \ldots, j-l+1\} & \text{if } i > j \\ \{j+l, j+l+1, \ldots, 2i-j-l\} & \text{if } i < j \end{cases} \\ \text{and} \quad W_j &= \{\infty, j-l+2, \ldots, j-1, j, j+1, \ldots, j+l-1\}. \end{aligned}$$

We see that $W_i \cap W_j = \emptyset$. Thus, the s consecutive edges of E form a ($\leqslant 3$)-regular graph.

Now set $\ell := \ell_i \vee_s \ell_j$ with $s = m+1-|j-i|-\epsilon$ where $\epsilon = 0$ if $i < j$ and $\epsilon = 1$ otherwise. We want to show that $ms(\ell) \geqslant s$. Consider a set of s consecutive edges E of ℓ. Suppose that there are $s-l$ edges in $E \cap E(H_i)$ and so l edges in $E \cap E(H_j)$ for some non-zero $l \in [s]$. Let W_a be the vertices incident to an edge in $E \cap E(H_a)$ for $a = i, j$. As $ms(\ell_b) \geqslant m$ for all $b \in [m]$, the last m edges of ℓ_i form a matching as do the first m edges of ℓ_j. In particular, $E \cap E(H_i)$ and $E \cap E(H_j)$ are matchings. Thus, to show that E forms a matching, it suffices to prove that $W_i \cap W_j = \emptyset$. The vertices in W_i and W_j are the vertices incident to one of the last $s-l$ edges of ℓ_i and one of the first l edges of ℓ_j, respectively. Hence, the vertices in W_i are $i+x$ and $i-x+1$ for $l+|j-i|+\epsilon \leqslant x \leqslant m$, while the vertices in W_j are ∞, j, and $j+x'$ and $j-x'$ for $x' \in [l]-\{0\}$. We can then express W_i and W_j as follows modulo $2m$:

$$\begin{aligned} W_i &= \{i+l+|j-i|+\epsilon, \ldots, i+(m-1), i+m, \\ &\qquad i-m+1, \ldots, i-|j-i|-\epsilon-l+1\} \\ &= \begin{cases} \{j+l, j+l+1, \ldots, 2i-j-l+1\} & \text{if } i<j \\ \{2i-j+l+1, 2i-j+l+2, \ldots, j-l\} & \text{if } i>j \end{cases} \\ \text{and} \quad W_j &= \{\infty, j-l+1, \ldots, j-1, j, j+1, \ldots, j+l-1\}\,. \end{aligned}$$

We see that $W_i \cap W_j = \emptyset$. Thus, the s consecutive edges in E form a matching. □

Proof of Theorem 4 for when n and r are odd and $r \geqslant \frac{n-1}{2}$. Theorem 5 implies the result for $r = \frac{n-1}{2}$. So, suppose that $r \geqslant \frac{n+1}{2}$. Let α_u be a labelling with the properties given in Lemma 28 for $u = \frac{r-1}{2}$ and $t = m = \frac{n-1}{2}$. Set $H'_i := H_{\alpha_u(i)}$ and $\ell'_i := \ell_{\alpha_u(i)}$ for each $i \in [m]$. By Lemma 28, $\ell'_{i+u} = \ell_{\alpha_u(i+u)} = \ell_{\alpha_u(i)-1}$ for each $i \in [m-u]$ and $\ell'_{i+u+1} = \ell_{\alpha_u(i+u+1)} = \ell_{\alpha_u(i)+1}$ for each $i \in [m-u-1]$. Thus, $ms_3(\ell'_i, \ell'_{i+u}) \geqslant 3m+1$ for each $i \in [m-u]$ and $ms(\ell'_i, \ell'_{i+u+1}) \geqslant m$ for each $i \in [m-u-1]$, by Lemma 29. Hence, $ms_r(K_n) \geqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$ follows from Proposition 11, using the decomposition $H'_0, \ldots, H'_{m-1}$ of K_n with orderings $\ell'_0, \ldots, \ell'_{m-1}$, respectively. Lemma 8 implies that $ms_r(K_n) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$, completing the proof. □

6.3 Proof of Theorem 6

To prove Theorem 6, we will need the following ordering of the integers in $\{l, l+1 \ldots, l+t-1\}$. An ordering $\alpha : A \to [|A|]$ corresponds to a list $a_0, a_1 \ldots, a_{k-1}$ of the integers of A if $\alpha(a_i) = i$ for all i. Let $t \in [m+1]-\{0\}$ and $l \in [m-t+1]$, and let $\alpha_{l,t}$ be the ordering corresponding to

$$\begin{aligned} &l, l+2, \ldots, l+t-2, l+t-1, l+t-3, \ldots, l+1 \quad \text{if } t \text{ is even} \\ \text{and} \qquad &l, l+2, \ldots, l+t-1, l+t-2, l+t-4, \ldots, l+1 \quad \text{if } t \text{ is odd}\,. \end{aligned}$$

Proof of Theorem 6. Set $d := \gcd(\frac{r}{2}, m)$ and $c := \frac{m}{d}$. Let α and $a_{i,j}$ be defined as in Lemma 13 with $u = \frac{r}{2}$ and $t = m$. Let $H'_{\alpha(a_{i,j})} := H_{\alpha^{-1}_{jc,c}(i)}$ and $\ell'_{\alpha(a_{i,j})} := \ell_{\alpha^{-1}_{jc,c}(i)}$ for all $i \in [c]$ and $j \in [d]$. This is indeed well-defined, as α is a bijection of $[m]$ and $\alpha_{jc,c}$ are bijections of the disjoint sets $\{jc, jc+1, \ldots, jc+c-1\}$ for $j \in [d]$. By Lemma 13, $\alpha(a_{i+1,j}) = \alpha(a_{i,j}) + u \pmod m$ for all $i \in [c]$ and $j \in [d]$. By definition, it is clear that $|\alpha^{-1}_{l,t}(i+1) - \alpha^{-1}_{l,t}(i)| \leqslant 2$ for all $i \in [t]$ where $i+1$ is reduced modulo t. Thus for $x = \alpha(a_{i,j})$, Lemma 27 implies that

$$ms_2(\ell'_x, \ell'_{x+u}) = ms_2\left(\ell'_{\alpha(a_{i,j})}, \ell'_{\alpha(a_{i+1,j})}\right) = ms_2\left(\ell_{\alpha^{-1}_{jc,c}(i)}, \ell_{\alpha^{-1}_{jc,c}(i+1)}\right) \geqslant 2m-1\,.$$

By applying Proposition 10 to $H'_0, \ldots, H'_{m-1}$ with orderings $\ell'_0, \ldots, \ell'_{m-1}$, respectively, we see that $cms_r(K_n) \geqslant \frac{rn}{2} - 2 = \left\lfloor \frac{rn-1}{2} \right\rfloor - 1$. By Lemma 8, $cms_r(K_n) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$. □

7 Proof of Theorem 5 for when $r = \frac{n-1}{2}$

We will first prove the case when r is even.

Proof of Theorem 5 when $r = \frac{n-1}{2}$ for even r. Let H_i be the Hamiltonian cycle defined in Subsection 3.2. Also, let ℓ_i be the ordering of H_i defined in Section 6. Let α and $a_{i,j}$ be as defined in Lemma 13 for $u = \frac{r}{2}$ and $t = m$. Let $H'_{\alpha(a_{i,j})} := H_{a_{i,j}}$ and $\ell'_{\alpha(a_{i,j})} := \ell_{a_{i,j}}$ for $i \in [2]$ and $j \in [\frac{r}{2}] = [\frac{m}{2}]$. As $\gcd(r, m) = \frac{m}{2}$, it follows that $\frac{t}{u} = 2$. Thus, $|a_{i+1,j} - a_{i,j}| = 1$ for all i, j. Lemmas 13 and 27 therefore imply that, for $x = \alpha(a_{i,j})$,

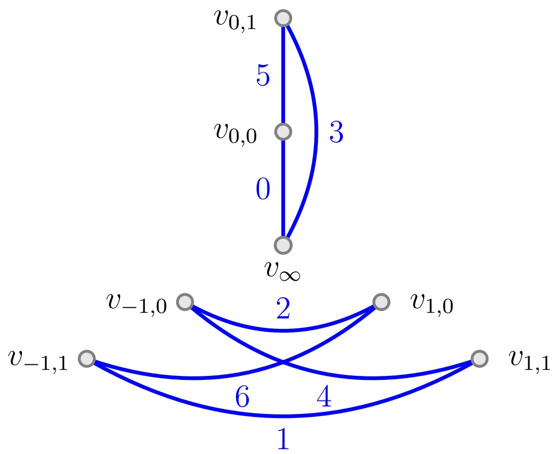
$$ms_2(\ell'_x, \ell'_{x+u}) = ms_2\left(\ell'_{\alpha(a_{i,j})}, \ell'_{\alpha(a_{i+1,j})}\right) = ms_2\left(\ell_{a_{i,j}}, \ell_{a_{i+1,j}}\right) \geqslant 2m$$

for all i, j. By applying Proposition 10 to $H'_0, \ldots, H'_{m-1}$ ordered by $\ell'_0, \ldots, \ell'_{m-1}$, respectively, we see that $ms_r(K_n) \geqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$. The reverse inequality follows from Lemma 8, completing the proof. □

Now let $r = m$ be odd and let R_i be the 2-regular graph defined in Section 3.2 for $i \in [m]$. Let ℓ_i be the ordering of R_i defined as follows:

$$\begin{aligned} \ell_i(\{v_\infty, v_{i,0}\}) &= 0\,, \\ \ell_i(\{v_\infty, v_{i,1}\}) &= m\,, \\ \ell_i(\{v_{i+2x,x}, v_{i-2x,x}\}) &= x \qquad \text{for } x \in [m] - \{0\}\,, \\ \ell_i(\{v_{i+2x-1,x}, v_{i-(2x-1),x+1}\}) &= m + x \quad \text{for } x \in [m+1] - \{0\}\,. \end{aligned}$$

Example 30. When $n = 7$, R_0 is ordered as



For each i, set $R_i := R_{i'}$ and $\ell_i := \ell_{i'}$ where $i' \equiv i \pmod{m}$. It is easy to check that ℓ_i is indeed a valid ordering of R_i and that the first m edges of ℓ_i form a matching as do the last m edges of ℓ_i.

Lemma 31. *For all i, $ms_3(\ell_i, \ell_{i+1}) \geqslant 3m + 1$ and $ms(\ell_i, \ell_{i-1}) \geqslant m$.*

Proof. First, we will show that $ms_3(\ell_i, \ell_{i+1}) \geqslant 3m+1$ for all i. Set $\ell := \ell_i \vee_{3m+1} \ell_{i+1}$ and consider a set of $3m+1$ consecutive edges E of ℓ. A vertex v has degree more than 3 in E only if v has degree two in both $E \cap E(R_i)$ and $E \cap E(R_{i+1})$. The first m edges of ℓ_j form a matching as do the last m edges of ℓ_j for all j. Therefore, if there are m or fewer edges in either $E \cap E(R_i)$ or $E \cap E(R_{i+1})$, then E forms a ($\leqslant 3$)-regular graph. So, suppose that there are $2m+1-l$ edges in $E \cap E(R_i)$ and so $m+l$ edges in $E \cap E(R_{i+1})$ for some non-zero $l \in [m+1]$. Let W_2 be the vertices of degree 2 in $E \cap E(R_{i+1})$ and W_1 be the vertices of degree at most 1 in $E \cap E(R_i)$. To show that E forms a ($\leqslant 3$)-regular graph, it suffices to show that $W_2 \subseteq W_1$.

As R_i is 2-regular and the first m edges of ℓ_i form a matching, W_1 contains the vertices incident to one of the first l edges of ℓ_i. Thus, $W_1 = \{v_\infty\} \cup \{v_{,i+2x,x}, v_{i-2x,x} \ : \ x \in [l]\}$. The first m edges of ℓ_{i+1} form a matching which covers every vertex except $v_{i,1}$. Therefore, W_2 the vertices incident to an edge with label between m and $m+l-1$, excluding $v_{i,1}$: $W_2 = \{v_\infty\} \cup \left\{v_{i+1+2x'-1,x'}, v_{i+1-(2x'-1),x'+1} \ : \ x' \in [l] - \{0\}\right\}$. By simplifying and re-indexing, we see that $W_2 = \{v_\infty\} \cup \left(\{v_{i+2x',x'}, v_{i-2x',x'} \ : \ x' \in [l]\} - \{v_{i-2(l-1),l-1}\}\right) \subseteq W_1$. Therefore, E forms a ($\leqslant 3$)-regular graph and $ms_3(\ell_i, \ell_{i+1}) \geqslant 3m+1$.

Finally, we show that $ms(\ell_i, \ell_{i-1}) \geqslant m$ for all i. Set $\ell := \ell_i \vee_m \ell_{i-1}$ and consider a set of m consecutive edges E of ℓ. Suppose that there are $m-l$ edges in $E \cap E(R_i)$ and thus l edges in $E \cap E(R_{i-1})$ for some non-zero $l \in [m]$. Let W_1 be the vertices incident to an edge in $E \cap E(R_{i-1})$ and W_0 be the vertices not incident to any edge in $E \cap E(R_i)$. The last m edges of ℓ_i form a matching as do the first m edges of ℓ_{i-1}. Thus, a vertex v is incident to 2 or more edges of E only if $v \in W_1$ and v is incident to an edge in $E \cap E(R_i)$. Therefore, it suffices to show that $W_1 \subseteq W_0$.

The last m edges of ℓ_i form a matching in which v_∞ is the only isolated vertex. Therefore, W_0 contains the vertex v_∞ along with the vertices incident to an edge with label between $m+1$ and $m+l$; that is, $W_0 = \{v_\infty\} \cup \left\{v_{i+2x-1,x}, v_{i-(2x-1),x+1} \ : \ x \in [l+1] - \{0\}\right\}$. By re-indexing, we see that $W_0 = \{v_\infty, v_{i-1,0}, v_{i-1+2l,l}\} \cup \{v_{i-1+2x,x}, v_{i-1-2x,x} : x \in [l] - \{0\}\}$. Now, W_1 contains the vertices that are incident to one of the first l edges of ℓ_{i-1}. In other words, $W_1 = \{v_\infty, v_{i-1,0}\} \cup \{v_{i-1+2x',x'}, v_{i-1-2x',x'} \ : \ x' \in [l] - \{0\}\} \subseteq W_0$. Hence, E forms a matching and $ms(\ell_i, \ell_{i-1}) \geqslant m$, as required. □

Proof of Theorem 5 when $r = \frac{n-1}{2}$ for odd r. Set $u := \frac{r-1}{2} = \frac{m-1}{2}$ and let $\beta \ : \ [m] \to [m]$ be the function defined by $\beta(i) := iu^{-1} \pmod{m}$ for all $i \in [m]$. The function β is clearly a bijection. Set $R'_x := R_{\beta(x)}$ and $\ell'_x := \ell_{\beta(x)}$ for $x \in [m]$. For any i, $\beta(i+u) \equiv (i+u)u^{-1} \equiv \beta(i) + 1 \pmod{m}$. Also, as $u^{-1} \equiv -2 \pmod{m}$, we see that

$$\beta(i+u+1) \equiv -2(i+u+1) \equiv -2i - 2u - 2 \equiv -2i - 1 \equiv \beta(i) - 1 \pmod{m}.$$

Lemma 31 thus implies that, for any $x \in [m]$, $ms(\ell'_x, \ell'_{x+u+1}) = ms(\ell_{\beta(x)}, \ell_{\beta(x)-1}) \geqslant m$ and $ms_3(\ell'_x, \ell'_{x+u}) = ms(\ell_{\beta(x)}, \ell_{\beta(x)+1}) \geqslant 3m+1$. By applying Proposition 11 to $R'_0, \ldots, R'_{m-1}$ ordered by $\ell'_0, \ldots, \ell'_{m-1}$, respectively, we see that $cms_r(K_n) \geqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$. By Lemma 8, $cms_r(K_n) \leqslant \left\lfloor \frac{rn-1}{2} \right\rfloor$, and the result follows. □

8 General conditions and the proof of Theorem 7

In the process of proving Theorem 7, we develop some notions of sequencibility where an arbitrary condition is placed on the subgraphs formed by consecutive edges. We express such a condition by letting $\mathcal{C}$ be an arbitrary family of graphs on a fixed set of vertices V with some fixed vertex labelling.

A ordering ℓ of some graph is *cyclically* $(s, \mathcal{C})$*-sequenceable* if all s cyclically consecutive edges in ℓ form a graph in $\mathcal{C}$. A graph G is *cyclically* $(s, \mathcal{C})$*-sequenceable* if there exists a cyclically $(s, \mathcal{C})$-sequenceable ordering ℓ of G. Note that s is not maximised here: for an arbitrary set of conditions $\mathcal{C}$, maximising s may be trivial or otherwise not of interest. For a graph $G = (V, E)$, let

$$\mathcal{C}^{\complement_G} := \{(V, E(C)\Delta E(G)) \;:\; C \in \mathcal{C}\},$$

where $E(C)\Delta E(G)$ is the symmetric difference of $E(C)$ and $E(G)$.

Lemma 32. *Let $\mathcal{C}$ be a set of conditions on vertex-labelled graphs; let G be a graph, and let s be an integer. Then for an ordering ℓ of G, ℓ is cyclically $(s, \mathcal{C})$-sequenceable if and only if ℓ is cyclically $(|E(G)| - s, \mathcal{C}^{\complement_G})$-sequenceable.*

Proof. Let $k = |E(G)|$ and let ℓ be an ordering of G which is cyclically $(s, \mathcal{C})$-sequenceable. Also, let e_i be the edge of G labelled i by ℓ. Consider a set of $(k-s)$-cyclically consecutive edges E in ℓ, namely $e_j, e_{j+1}, \dots, e_{j+k-s-1}$ for some $j \in [k]$, where the subscripts are taken modulo k. The edges of G not in E are $e_{j+k-s}, e_{j+k-s+1}, \dots, e_{j-1}$, and they are in this order in ℓ. By assumption, the s edges of $E(G) - E$ form a graph in $\mathcal{C}$. Thus, the $(k-s)$-cyclically consecutive edges $e_j, e_{j+1}, \dots, e_{j+k-s-1}$ must form a member of $\mathcal{C}^{\complement_G}$. Hence, ℓ is cyclically $(k-s, \mathcal{C}^{\complement_G})$-sequenceable. If ℓ is cyclically $(|E(G)| - s, \mathcal{C}^{\complement_G})$-sequenceable, then it follows, from the above argument and the identities $(\mathcal{C}^{\complement_G})^{\complement_G} = \mathcal{C}$ and $k - (k-s) = s$, that ℓ is cyclically $(s, \mathcal{C})$-sequenceable. $\square$

Proof of Theorem 7. Let $s_a = \left\lfloor \frac{an-1}{2} \right\rfloor$ for each $a \in [n-1]$. Also, let $\mathcal{C}_r$ be the set of all vertex-labelled $(\leqslant r)$-regular graphs on n vertices. Suppose that $cms_r(\ell) = s_r$ and, in particular, suppose that ℓ is cyclically $(s_r, \mathcal{C}_r)$-sequenceable for some ordering of K_n. Then, by Lemma 32, ℓ is $\left(\frac{n(n-1)}{2} - s_r, \mathcal{C}_r^{\complement_{K_n}}\right)$-sequenceable. Set $s' := \frac{n(n-1)}{2} - s_r = s_{n-1-r} + 1$. Then $\mathcal{C}_r^{\complement_{K_n}}$ is the family of all vertex-labelled subgraphs of K_n whose vertices each have degree at least $n-1-r$. The minimum number of edges in a member of $\mathcal{C}_r^{\complement_{K_n}}$ is s'. Also, any member of $\mathcal{C}_r^{\complement_{K_n}}$ with s' edges must be a graph in which each vertex has degree $n-1-r$ except one vertex which has degree $n-r$.

Consider $s'+1$ cyclically consecutive edges $e_0, \dots, e_{s'}$ in ℓ. Since ℓ is $(s', \mathcal{C}_r^{\complement_{K_n}})$-sequenceable, the edges of each of $E_0 := \{e_0, \dots, e_{s'-1}\}$ and $E_1 := \{e_1, \dots, e_{s'}\}$ form a member of $\mathcal{C}_r^{\complement_{K_n}}$. Also as $s' < \frac{n(n-1)}{2}$, $e_0 \neq e_{s'}$. I claim that this ensures that the edges $E' := \{e_1, \dots, e_{s'-1}\}$ form a $(\leqslant n-1-r)$-regular graph. Assume otherwise; then some vertex v is incident to at least $n-r$ of the edges in E'. Let v_0 and v_1 be the endpoints of e_0. The edges of E_0 must form a graph in $\mathcal{C}_r^{C_{K_n}}$, i.e., a graph whose vertices each has degree $n-1-r$ except one which has degree $n-r$. As v is incident to at least $n-r$ of the edges in $E' \subseteq E_0$, v must be incident to exactly $n-r$ of the edges in E_0. In particular, v must be distinct from v_0 and v_1. So, v_0 and v_1 are each incident to $n-1-r$ of the edges in E_0. However, this means that v_0 and v_1 are each incident to $n-2-r$ of the edges in E'. Thus, either v_0 or v_1 is incident to only $n-2-r$ of the edges in E_1, since $e_{s'} \neq e_0$. Therefore, the graph formed by the edges of E_1 is not in $\mathcal{C}_r^{\complement K_n}$, a contradiction.

Any set E' of $s'-1 = s_{n-1-r}$ cyclically consecutive edges in ℓ is a consecutive subsequence of some $s'+1$ cyclically consecutive edges in ℓ of the form $e_0 \vee L_\ell(E') \vee e_{s'}$. Thus, by the above argument, every vertex must have degree at most $n-1-r$ in E'. Hence, $cms(\ell) \geqslant s_{n-1-r}$ and, therefore, $cms_{n-1-r}(K_n) \geqslant s_{n-1-r}$. By Lemma 8, $cms_{n-1-r}(K_n) = s_{n-1-r}$. The reverse direction, namely that $cms_{n-1-r}(K_n) = s_{n-1-r}$ implies $cms_r(K_n) = s_r$, follows by applying the above argument with r replaced by $n-1-r$. □

Note that a similar result to Lemma 32 could be can proved for non-cyclic sequences. However, the notion of sequencibility would have to be generalised to allow partially cyclical sequences. A result analogous to Theorem 7 follows from a similar proof, but there is not an equivalence between $ms_r(K_n) = \frac{rn-1}{2}$ and $ms_{n-1-r}(K_n) = \frac{(n-1-r)n-1}{2}$ for odd r and n.

9 Concluding remarks

When r is even and n is odd, we expect that $cms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor$. Theorem 5 confirms this for even $r = \frac{n-1}{2}$. By computer search, we were able to find the following two orderings for K_7 that show that $cms_2(K_7) = 6 = \left\lfloor \frac{2n-1}{2} \right\rfloor$ and $cms_4(K_7) = 13 = \left\lfloor \frac{4n-1}{2} \right\rfloor$, respectively. The orderings are represented by the sequence of edges that has corresponding ordering value sequence $0, \ldots, 20$.

$$\begin{aligned}
&\{\infty,0\},\{1,2\},\{3,-2\},\{3,-1\},\{1,-1\},\{\infty,-2\},\{0,2\},\\
&\{0,1\},\{2,3\},\{\infty,-1\},\{-2,-1\},\{1,3\},\{\infty,2\},\{0,-2\},\\
&\{0,3\},\{\infty,1\},\{2,-1\},\{1,-2\},\{\infty,3\},\{0,-1\},\{2,-2\}\,;
\end{aligned}$$

$$\begin{aligned}
&\{\infty,0\},\{\infty,1\},\{0,2\},\{1,3\},\{2,-2\},\{0,-1\},\{-1,-2\},\\
&\{\infty,3\},\{1,2\},\{\infty,-2\},\{0,3\},\{1,-1\},\{2,3\},\{0,-2\},\\
&\{\infty,-1\},\{0,1\},\{\infty,2\},\{3,-2\},\{2,-1\},\{1,-2\},\{3,-1\}\,.
\end{aligned}$$

We also found an ordering for K_9, showing that $cms_2(K_9) = 8 = \left\lfloor \frac{2n-1}{2} \right\rfloor$, as given below.

$$\begin{aligned}
&\{\infty,0\},\{\infty,1\},\{0,2\},\{1,3\},\{2,4\},\{3,-3\},\{4,-2\},\{-1,-3\},\{\infty,-2\}\\
&\{0,1\},\{\infty,2\},\{1,-1\},\{2,3\},\{0,-3\},\{3,4\},\{-3,-2\},\{\infty,4\},\{0,-1\},\\
&\{1,-2\},\{2,-1\},\{\infty,3\},\{2,-3\},\{0,4\},\{1,-3\},\{3,-2\},\{4,-1\},\{0,-2\},\\
&\{\infty,-1\},\{1,2\},\{\infty,-3\},\{0,3\},\{1,4\},\{2,-2\},\{3,-1\},\{4,-3\},\{-1,-2\}\,.
\end{aligned}$$

When r and n are both odd, Theorem 5 implies that $cms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor$ for $r = \frac{n-1}{2}$. If there are other cases for which $cms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor$ with r and n odd, then, by Theorem 7, the condition for which $cms(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor$ holds must be invariant under replacing r with $n-1-r$.

Proposition 33. *For a graph G and integers r_1, r_2,*

$$ms_{r_1 r_2}(G) \geqslant r_2\, ms_{r_1}(G) \quad \text{and} \quad cms_{r_1 r_2}(G) \geqslant r_2\, cms_{r_1}(G)\,.$$

Note that this proposition and Theorems 1 and 2 together imply that $ms_r(K_n) \geqslant r\frac{n-1}{2}$ and $cms_r(K_n) \geqslant r\frac{n-3}{2}$, respectively, when n is odd.

Proof. Let $s = cms_{r_1}(G)$ and let ℓ be a labelling of G for which $cms_{r_1}(\ell) = cms_{r_1}(G)$. Any set of $r_2 s$ cyclically consecutive edges E of ℓ are just r_2 sets of s cyclically consecutive edges of ℓ and in each set, every vertex has degree at most r_1. Thus, every vertex has degree at most $r_1 r_2$ in E and, hence, $cms_{r_1 r_2}(G) \geqslant cms_{r_1 r_2}(\ell) \geqslant r_2 s$. The non-cyclic case is similar. □

A hypergraph is a pair (V, E) where V is a set and E is a family of subsets of V. A k-graph is a hypergraph (V, E) for which each member $e \in E$ has cardinality $|e| = k$. For instance, each graph is a 2-graph. The notion of matching sequencibility naturally extends to hypergraphs, as do Proposition 33 and the propositions of Section 2, using analogous proofs. For example the natural hypergraph analogue of Proposition 9 is as follows.

Proposition 34. *Let $\mathcal{H}$ be a hypergraph that decomposes into matchings $\mathcal{M}_0, \ldots, \mathcal{M}_{t-1}$, each with n edges and orderings $\ell_0, \ldots, \ell_{t-1}$, respectively. Suppose, for some $\epsilon \in [n]$ and $r < \Delta(\mathcal{H})$, that $ms(\ell_i, \ell_{i+r}) \geqslant n - \epsilon$ for all $i \in [t-r]$. Then $ms_r(\mathcal{H}) \geqslant rn - \epsilon$, and if $ms(\ell_i, \ell_{i+r}) \geqslant n - \epsilon$ for all $i \in [t]$, then $cms_r(\mathcal{H}) \geqslant rn - \epsilon$.*

The natural analogue of K_n for k-graphs is the complete k-graph on n vertices, denoted $\mathcal{K}_n^k$, whose edges are all the vertex subsets of size k. Katona [4] proved that $cms(\mathcal{K}_n^k) \geqslant \left\lfloor \frac{n}{k^2} \right\rfloor$ for sufficiently large n, under the assumption that a particular conjecture holds. Katona [4] also conjectured that $cms(\mathcal{K}_n^k) \geqslant \left\lfloor \frac{n}{k} \right\rfloor - 1$. By similar reasoning to Lemma 8, $ms_r(\mathcal{G}) \leqslant \left\lfloor \frac{rn-1}{k} \right\rfloor$ for any k-graph $\mathcal{G}$, which is not a ($\leqslant r$)-regular k-graph. This leads us to conjecture that $ms_r(\mathcal{K}_n^k) = \left\lfloor \frac{rn-1}{k} \right\rfloor$ and $\left\lfloor \frac{rn-1}{k} \right\rfloor - 1 \leqslant cms_r(\mathcal{K}_n^k) \leqslant \left\lfloor \frac{rn-1}{k} \right\rfloor$ for all r, n, and k, and to expect that $cms_r(\mathcal{K}_n^k)$ can attain both bounds.

We prove a result similar to Katona's, Theorem 35 below, for the special case in which $k \mid n$.

Theorem 35. *Let $k \mid n$, a be the largest integer such that $\frac{n}{k} - (a-1)k > 0$ and b be the largest integer such that $\frac{n}{k} - (b-1)(k+1) > 0$. Then, for $r < \Delta(\mathcal{K}_n^k)$,*

$$ms_r(\mathcal{K}_n^k) \geqslant (r-1)\frac{n}{k} + a \qquad \text{and} \qquad cms_r(\mathcal{K}_n^k) \geqslant (r-1)\frac{n}{k} + b\,.$$

To prove this theorem, we use Baranyai's Theorem [2] which states that if $k \mid n$, then $\mathcal{K}_n^k$ has a complete matching decomposition. Note that in such a decomposition, each matching has size $\frac{n}{k}$.

Proof of Theorem 35. By Baranyai's Theorem, there is a matching decomposition of $\mathcal{K}_n^k$, say, $\mathcal{M}_0, \ldots, \mathcal{M}_{N-1}$ where $N = \frac{k}{n}\binom{n}{k}$. Set $d := \gcd(r, N)$ and $c := \frac{N}{d}$, and arbitrarily re-index the matching decomposition as $\mathcal{M}_{i,j}$ for $i \in [c]$ and $j \in [d]$. We construct orderings $\ell_{i,j}$ of $\mathcal{M}_{i,j}$ for $i \in [c]$ and $j \in [d]$ such that $ms(\ell_{i,j}, \ell_{i+1,j}) \geqslant b$ for all $i \in [c]$. Choose an arbitrary ordering $\ell_{0,j}$ of $\mathcal{M}_{0,j}$. Suppose, by induction on i for a fixed j, that $\ell_{0,j}, \ldots, \ell_{i,j}$ have been constructed, that the first l edges of $\ell_{i+1,j}$ are $e_0, \ldots, e_{l-1}$, and that any b consecutive edges of $L_{\ell_{i,j}}(\mathcal{M}_{i,j}), e_0, \ldots, e_{l-1}$ form a matching. Let $e'_0, \ldots, e'_{b-2}$ be the last $b-1$ edges of $\ell_{i,j}$. If $l \leqslant b-2$, then there are at least $\frac{n}{k} - (b-l-1)k - l > 0$ edges in $\mathcal{M}_{i+1,j} - \{e_0, \ldots, e_{l-1}\}$ which do not share a common vertex with any of the edges $e'_l, \ldots, e'_{b-2}$. Thus, we can choose an edge e_l in $\mathcal{M}_{i+1,j} - \{e_0, \ldots, e_{l-1}\}$ such that $e'_l, \ldots, e'_{b-2}, e_0, \ldots, e_{l-1}, e_l$ forms a matching and let $\ell_{i+1,j}(e_l) = l$. If $l > b-2$, then for an arbitrary edge e_l in $\mathcal{M}_{i+,j} - \{e_0, \ldots, e_{l-1}\}$, let $\ell_{i+1,j}(e_l) = l$. Now let $e_0, \ldots, e_{b-2}$ be the last $b-1$ edges of $\ell_{0,j}$. We are free to permute the ordering $\ell_{0,j}$ and maintain the identity $ms(\ell_{0,j}, \ell_{1,j}) \geqslant b$ so long as the labels of $e_0, \ldots, e_{b-2}$ remain unchanged. Therefore, we can apply a similar argument to the above to permute $\ell_{0,j}$ in such a way that ensures that $ms(\ell_{c-1,j}, \ell_{0,j}) \geqslant b$, since $\frac{n}{k} - (b-1) - (b-l-1)k - l > 0$ for all $l \in [b-1]$. Set $\mathcal{M}'_{\alpha(a_{i,j})} := \mathcal{M}_{i,j}$ and $\ell'_{\alpha(a_{i,j})} := \ell_{i,j}$ where α and $a_{i,j}$ are defined as in Lemma 13 for $u = r$ and $t = N$. Proposition 34 and Lemma 13 yields the second inequality using the matchings $\mathcal{M}'_0, \ldots, \mathcal{M}'_{N-1}$ ordered by $\ell'_0, \ldots, \ell'_{N-1}$, respectively. The non-cyclic case is similar and, therefore, omitted. $\square$

Kühn and Osthus [5] offer an alternate decomposition of $\mathcal{K}_n^k$ than those given by Baranyai's Theorem, into *Berge cycles* which broadly generalise cycles in graphs. Their decomposition would however not be likely to be useful for proving a matching sequencibility result.

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