Induced 2-degenerate Subgraphs of Triangle-free Planar Graphs

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Abstract

A graph is *k-degenerate* if every subgraph has minimum degree at most k. We provide lower bounds on the size of a maximum induced 2-degenerate subgraph in a triangle-free planar graph. We denote the size of a maximum induced 2-degenerate subgraph of a graph G by $\alpha_2(G)$. We prove that if G is a connected triangle-free planar graph with n vertices and m edges, then $\alpha_2(G) \geqslant \frac{6n-m-1}{5}$. By Euler's Formula, this implies $\alpha_2(G) \geqslant \frac{4}{5}n$. We also prove that if G is a triangle-free planar graph on n vertices with at most n_3 vertices of degree at most three, then $\alpha_2(G) \geqslant \frac{7}{8}n - 18n_3$.

1 Introduction

A graph is k-degenerate if every nonempty subgraph has a vertex of degree at most k. The *degeneracy* of a graph is the smallest k for which it is k-degenerate, and it is one less than the *coloring number*. It is well-known that planar graphs are 5-degenerate and that triangle-free planar graphs are 3-degenerate. The problem of bounding the size of an induced subgraph of smaller degeneracy has attracted a lot of attention. In this paper we are interested in lower bounding the size of maximum induced 2-degenerate subgraphs in triangle-free planar graphs. In particular, we conjecture the following.

Conjecture 1.1. *Every triangle-free planar graph contains an induced 2-degenerate subgraph on at least $\frac{7}{8}$ of its vertices.*

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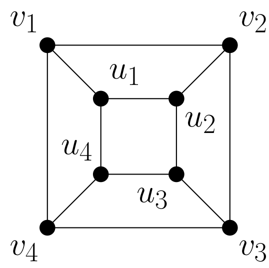

Figure 1: The cube.

Conjecture 1.1, if true, would be tight for the *cube*, which is the unique 3-regular triangle-free planar graph on 8 vertices (see Figure 1). For an infinite class of tight graphs, if G is a planar triangle-free graph whose vertex set can be partitioned into parts each inducing a subgraph isomorphic to the cube, then G does not contain an induced 2-degenerate subgraph on more than $\frac{7}{8}|V(G)|$ vertices.

Towards Conjecture 1.1, we prove the following weaker bound.

Theorem 1.2. *Every triangle-free planar graph contains an induced 2-degenerate subgraph on at least $\frac{4}{5}$ of its vertices.*

We believe the argument we use can be strengthened to give a bound $\frac{5}{6}$, however the technical issues are substantial and since we do not see this as a viable way to prove Conjecture 1.1 in full, we prefer to present the easier argument giving the bound $\frac{4}{5}$.

Triangle-free planar graphs have average degree less than 4, and thus they must contain some vertices of degree at most three. Nevertheless, they may contain only a small number of such vertices—there exist arbitrarily large triangle-free planar graphs of minimum degree three that contain only 8 vertices of degree three. It is natural to believe that 2-degenerate induced subgraphs are harder to find in graphs with larger vertex degrees, and thus one might wonder whether a counterexample to Conjecture 1.1 could not be found among planar triangle-free graphs with almost all vertices of degree at least four. This is a false intuition—such graphs are very close to being 4-regular grids, and their regular structure makes it possible to find large 2-degenerate induced subgraphs. To support this counterargument, we prove the following approximate form of Conjecture 1.1 for graphs with small numbers of vertices of degree at most three.

Theorem 1.3. *If G is a triangle-free planar graph on n vertices with n_3 vertices of degree at most three, then G contains an induced 2-degenerate subgraph on at least $\frac{7}{8}n - 18n_3$ vertices.*

Theorems 1.2 and 1.3 are corollaries of more technical results.

Definition 1.4. We say a graph is *difficult* if it is connected, every block is either a vertex, an edge, or isomorphic to the cube, and any two blocks isomorphic to the cube are vertex-disjoint.

We actually prove the following, which easily implies Theorem 1.2 since, by Euler's formula, a triangle-free planar graph G on at least three vertices satisfies $|E(G)| \leqslant 2|V(G)| - 4$.

Theorem 1.5. *If G is a triangle-free planar graph on n vertices with m edges and λ difficult components, then G contains an induced 2-degenerate subgraph on at least*

$$\frac{6n - m - \lambda}{5}$$

vertices.

The proof of Theorem 1.5 is the subject of Section 2.

Definition 1.6. If G is a plane graph, we let $f_3(G)$ denote the minimum size of a set of faces such that every vertex in G of degree at most three is incident to at least one of them.

We actually prove the following, which easily implies Theorem 1.3.

Theorem 1.7. *If G is a triangle-free plane graph on n vertices, then either G is 2-degenerate or G contains an induced 2-degenerate subgraph on at least*

$$\frac{7}{8}n - 18\left(f_3(G) - 2\right)$$

vertices.

The proof of Theorem 1.7 is the subject of Section 3.

Let us discuss some related results. To simplify notation, for a graph G we let $\alpha_k(G)$ denote the size of a maximum induced subgraph that is k-degenerate. Alon, Kahn, and Seymour [4] proved in 1987 a general bound on $\alpha_k(G)$ based on the degree sequence of G. They derive as a corollary that if G is a graph on n vertices of average degree $d \geqslant 2k$, then $\alpha_k(G) \geqslant \frac{k+1}{d+1}n$. Since triangle-free planar graphs have average degree at most four, this implies that if G is triangle-free and planar then $\alpha_2(G) \geqslant \frac{3}{5}n$. Our Theorem 1.2 improves upon this bound.

For the remainder of this section, let G be a planar graph on n vertices.

Note that a graph is 0-degenerate if and only if it is an independent set. The famous Four Color Theorem, the first proof of which was announced by Appel and Haken [5] in 1976, implies that $\alpha_0(G) \geqslant \frac{1}{4}n$. In the same year, Albertson [2] proved the weaker result that $\alpha_0(G) \geqslant \frac{2}{9}n$, which was improved to $\alpha_0(G) \geqslant \frac{3}{13}n$ by Cranston and Rabern [7]; the constant factor $\frac{3}{13}$ is the best known to date without using the Four Color Theorem. The factor $\frac{1}{4}$ is easily seen to be best possible by considering copies of K_4.

If additionally G is triangle-free, a classical theorem of Grőtzsch [9] says that G is 3-colorable, and therefore $\alpha_0(G) \geqslant \frac{n}{3}$. In fact, Steinberg and Tovey [15] proved that $\alpha_0(G) \geqslant \frac{n+1}{3}$, and a construction of Jones [10] implies this is best possible. Dvořák and Mnich [8] proved that there exists $\varepsilon > 0$ such that if G has girth at least five, then $\alpha_0(G) \geqslant \frac{n}{3-\varepsilon}$.

Note that a graph is 1-degenerate if and only if it contains no cycles. In 1979, Albertson and Berman [3] conjectured that every planar graph contains an induced forest on at least half of its vertices, i.e. $\alpha_1(G) \geqslant \frac{1}{2}n$. The best known bound for $\alpha_1(G)$ for planar graphs is $\frac{2}{5}n$, which follows from a classic result of Borodin [6] that planar graphs are acyclically 5-colorable.

Akiyama and Watanabe [1] conjectured in 1987 that if additionally G is bipartite then $\alpha_1(G) \geqslant \frac{5n}{8}$, and this may also be true if G is only triangle-free. The best known bound when G is bipartite is $\alpha_1(G) \geqslant \lceil \frac{4n+3}{7} \rceil$, which was proved by Wan, Xie, and Yu [16]. The best known bound when G is triangle-free is $\alpha_1(G) \geqslant \frac{5}{9}n$, which was proved by Le [13] in 2016. Kelly and Liu [11] proved that if G has girth at least five, then $\alpha_1(G) \geqslant \frac{2}{3}n$.

Kierstead, Oum, Qi, and Zhu [12] proved that if G is a planar graph on n vertices then $\alpha_3(G) \geqslant \frac{5}{7}n$, but the proof is yet to appear. A bound for $\alpha_3(G)$ of $\frac{5}{6}n$ may be possible, which is achieved by both the octahedron and the icosahedron.

In 2015, Lukot'ka, Mazák, and Zhu [14] studied α_4 for planar graphs. They proved that $\alpha_4(G) \geqslant \frac{8}{9}n$. A bound for $\alpha_4(G)$ of $\frac{11}{12}n$ may be possible, which is achieved by the icosahedron.

So far, bounds on $\alpha_2(G)$ for planar graphs have not been studied. However, as Lukot'ka, Mazák, and Zhu [14] pointed out, it is easy to see that every planar graph contains an induced outerplanar subgraph on at least half of its vertices. Since outerplanar graphs are 2-degenerate, this implies $\alpha_2(G) \geqslant \frac{1}{2}n$. Nevertheless, a bound of $\alpha_2(G) \geqslant \frac{2}{3}n$ may be possible, which is achieved by the octahedron. If G has girth at least five, $\alpha_2(G)$ may be as large as $\frac{19}{20}n$, which is achieved by the dodecahedron.

2 Proof of Theorem 1.5

In this section we prove Theorem 1.5. First we prove some properties of a hypothetical minimal counterexample (i.e., a plane triangle-free graph G with the smallest number n of vertices such that $\alpha_2(G) < \frac{6n-m-\lambda}{5}$, where $m = |E(G)|$ and λ is the number of difficult components of G).

2.1 Preliminaries

Lemma 2.1. *A minimal counterexample G to Theorem 1.5 is connected and has no difficult components.*

Proof. Note that the union of induced 2-degenerate subgraphs from each component of G is an induced 2-degenerate subgraph of G. Thus if G is not connected, then one of its components is a smaller counterexample, a contradiction.

Now suppose for a contradiction that G has a difficult component. Since G is connected, G is difficult. Note that G is not a single vertex, or else G is not a counterexample. Suppose that G contains a vertex x of degree 1; in this case, note that $G - x$ is a difficult graph. Since G is a minimal counterexample, there exists a set $S \subseteq V(G-x)$ that induces a 2-degenerate subgraph of size at least

$$\frac{6|V(G-x)|-|E(G-x)|-1}{5}=\frac{6|V(G)|-|E(G)|-1}{5}-1.$$

But then $S\cup\{x\}$ induces a 2-degenerate subgraph in G, contradicting that G is a minimal counterexample.

Therefore, G has minimum degree at least 2. Note that G is not a cube, or else G is not a counterexample. Since G is difficult, we conclude that G is not 2-connected and any end-block of G is a cube. Let X be the vertex set of an end-block of G, and observe that $G-X$ is a difficult graph. Since G is a minimal counterexample, there exists a set $S\subseteq V(G-X)$ that induces a 2-degenerate subgraph of size at least

$$\frac{6|V(G-X)|-|E(G-X)|-1}{5}=\frac{6|V(G)|-|E(G)|-1}{5}-7.$$

But then for any $v\in X$, $S\cup X\setminus\{v\}$ induces a 2-degenerate subgraph in G, contradicting that G is a minimal counterexample. □

We will often make use of the following induction lemma.

Lemma 2.2. *Let G be a minimal counterexample to Theorem 1.5, and let $X\subseteq V(G)$. If every induced 2-degenerate subgraph of $G-X$ can be extended to one of G by adding A vertices, then*

$$\lambda'\geqslant 5A-6|X|+|E(G)|-|E(G-X)|+1,$$

where λ' is the number of difficult components of $G-X$.

Proof. Let $S\subseteq V(G-X)$ induce a maximum 2-degenerate subgraph in $G-X$. Since G is a minimal counterexample,

$$|S|\geqslant\frac{6(|V(G)|-|X|)-|E(G-X)|-\lambda'}{5}. \tag{1}$$

Note that G has no difficult components by Lemma 2.1. Since S can be extended to induce a 2-degenerate subgraph in G by adding A vertices of X,

$$|S|+A<\frac{6|V(G)|-|E(G)|}{5}. \tag{2}$$

Combining (1) and (2) yields $\lambda'>5A-6|X|+|E(G)|-|E(G-X)|$, which gives the desired inequality since both sides are integers. □

Lemma 2.3. *A minimal counterexample G to Theorem 1.5 has no subgraph isomorphic to the cube that has fewer than six edges leaving.*

Proof. Let $X=\{v_1,v_2,v_3,v_4,u_1,u_2,u_3,u_4\}$ induce a cube in G where $v_1v_2v_3v_4v_1$ and $u_1u_2u_3u_4u_1$ are 4-cycles and v_i is adjacent to u_i for each $i\in\{1,2,3,4\}$, as in Figure 1. Suppose for a contradiction that $|E(X,V(G-X))|\leqslant 5$. Let S induce a 2-degenerate subgraph in $G-X$. First, we claim that there is some vertex $v \in X$ such that $S \cup X \setminus \{v\}$ induces a 2-degenerate subgraph in G.

If v_1 has at least three neighbors not in X, then $S \cup X \setminus \{v_1\}$ induces a 2-degenerate subgraph in G: Since $G[S]$ is 2-degenerate, it suffices to verify that for every non-empty $X' \subseteq X \setminus \{v_1\}$, there exists a vertex $x \in X'$ with at most two neighbors in $S \cup X'$. Since the cube is 3-edge-connected, there are at least three edges with one end in X' and the other end in $X \setminus X'$. Since there are at most five edges leaving X and at least three of them are incident with v_1, at most two such edges are incident with vertices of X'. Consequently, $\sum_{x\in X'} \deg_{G[X'\cup S]}(x) \leqslant 3|X'| - 3 + 2$, and thus X' indeed contains a vertex whose degree in $G[X' \cup S]$ is less than three.

By symmetry, we may assume no vertex in X has more than two neighbors not in X. If v_1 has two neighbors not in X, an analogous argument using the fact that the only 3-edge-cuts in the cube are the neighborhoods of vertices shows that $S \cup X \setminus \{v_1\}$ induces a 2-degenerate subgraph in G, unless each of u_1, v_2, and v_4 has a neighbor not in X. However, in that case it is easy to verify that $S \cup X \setminus \{u_1\}$ induces a 2-degenerate subgraph in G.

Hence, we may assume that each vertex of X has at most one neighbor not in X. Let $Z \subseteq X$ be a set of size exactly 5 containing all vertices of X with a neighbor outside of X. If Z contains all vertices of a face of the cube, then by symmetry we can assume that $Z = \{v_1, v_2, v_3, v_4, u_1\}$, and $S \cup X \setminus \{v_2\}$ induces a 2-degenerate subgraph in G. Otherwise, we have $|Z \cap \{v_1, v_2, v_3, v_4\}| \leqslant 3$ and $|Z \cap \{u_1, u_2, u_3, u_4\}| \leqslant 3$, and since $|Z| = 5$, by symmetry we can assume that $|Z \cap \{v_1, v_2, v_3, v_4\}| = 2$ and $v_1 \in Z$. However, then $S \cup X \setminus \{v_1\}$ induces a 2-degenerate subgraph in G.

This confirms that every set inducing a 2-degenerate subgraph of $G - X$ can be extended to a set inducing a 2-degenerate subgraph of G by the addition of 7 vertices. Let λ' be the number of difficult components of $G - X$. By Lemma 2.2, $\lambda' \geqslant |E(X, V(G-X))|$. Since G is connected, it follows that $G - X$ consists of exactly $|E(X, V(G-X))|$ difficult components, each connected by exactly one edge to the cube induced by X. But then G is a difficult graph, contradicting Lemma 2.1. □

Lemma 2.4. *A minimal counterexample G to Theorem 1.5 has minimum degree at least three.*

Proof. Suppose not. Let $v \in V(G)$ be a vertex of degree at most two. Note that v has degree at least one by Lemma 2.1. Note also that any induced 2-degenerate subgraph of $G - v$ can be extended to one of G by adding v. By Lemma 2.2, if $G - v$ has λ' difficult components, then $\lambda' \geqslant \deg(v)$. But then G is a difficult graph, contradicting Lemma 2.1. □

2.2 Reducing vertices of degree three

A cycle C in a plane graph is *separating* if both the interior and the exterior of C contain at least one vertex. The main result of this subsection is the following lemma.

Lemma 2.5. *A minimal counterexample G to Theorem 1.5 contains no vertex of degree three that is not contained in a separating cycle of length four or five.*

For the remainder of this subsection, let G be a minimal counterexample to Theorem 1.5, and suppose $v \in V(G)$ is a vertex of degree three that is not contained in a separating cycle of length four or five. Recall that a minimal counterexample is a plane graph, so G has a fixed embedding.

Claim 2.6. *The vertex v has no neighbors of degree at least five.*

Proof. Suppose for a contradiction v has a neighbor u of degree at least five, and let $X = \{u, v\}$. Note that any induced 2-degenerate subgraph of $G - X$ can be extended to one of G including v. By Lemma 2.2, the number of difficult components of $G - X$ is positive.

Let D be a difficult component of $G - X$. First, suppose D contains a vertex of degree at most one. By Lemma 2.4, this vertex is adjacent to u and v, contradicting that G is triangle-free. Therefore D has an end-block B isomorphic to the cube. Since G is triangle-free and planar, u has at most two neighbors in B, and u and v do not both have two neighbors in B. Hence $|E(X, V(B))| \leqslant 3$, so B has at most four edges leaving, contradicting Lemma 2.3. □

Claim 2.7. *The vertex v has no neighbors of degree three.*

Proof. Let u_1, u_2, and u_3 be the neighbors of v, and suppose for a contradiction that u_1 has degree three.

First, let us consider the case u_2 has degree at least four (and thus exactly four by Claim 2.6). Note that any induced 2-degenerate subgraph of $G - \{u_1, u_2, v\}$ can be extended to one of G including v and u_1. By Lemma 2.2, the number of difficult components of $G - \{u_1, u_2, v\}$ is positive.

Let D be a difficult component of $G - \{u_1, u_2, v\}$. Note that each leaf of D is adjacent to u_1 and u_2 and not adjacent to v by Lemma 2.4, since G is triangle-free. Now if D has at least two leaves, then v is contained in a separating cycle of length four, a contradiction. Note also that D is not an isolated vertex. Hence, D contains an end-block B isomorphic to the cube. If D contains another end-block, then we can choose B among the end-blocks isomorphic to the cube so that B has at most five edges leaving, contradicting Lemma 2.3. Therefore D is isomorphic to the cube. By Lemma 2.3, every neighbor of u_1, u_2, and v is in D, contradicting that G is planar and triangle-free.

Therefore we may assume u_2 and symmetrically u_3 have degree three. Note that any induced 2-degenerate subgraph of $G - \{u_1, u_2, u_3, v\}$ can be extended to one of G including u_1, u_2, and u_3. By Lemma 2.2, the number of difficult components of $G - \{u_1, u_2, u_3, v\}$ is positive.

Let D be a difficult component of $G - \{u_1, u_2, u_3, v\}$. First, suppose D is a tree. If D is an isolated vertex, this vertex is adjacent to u_1, u_2, and u_3 by Lemma 2.4, but then v is contained in a separating cycle of length four, a contradiction. Note that D is not an edge, or else it is contained in a triangle with one of u_1, u_2, or u_3, by Lemma 2.4. Similarly, D

is not a path, or else G contains a triangle or a vertex of degree at most two. Therefore D has at least three leaves. Since G has minimum degree three and $\{u_1, u_2, u_3, v\}$ has only six edges leaving, D is isomorphic to $K_{1,3}$. In this case, G is isomorphic to the cube, a contradiction.

Therefore we may assume D is not a tree, so D contains a block isomorphic to the cube. Let B be a block in D isomorphic to the cube with the fewest edges leaving. If D contains an endblock different from B, then at most five edges are leaving B, contradicting Lemma 2.3. Therefore D is isomorphic to the cube and all six edges leaving $\{u_1, u_2, u_3, v\}$ end in D, contradicting that G is planar and triangle-free. □

Claim 2.8. *The vertex v is not contained in a cycle of length four that contains another vertex of degree three.*

Proof. Suppose for a contradiction that u_1 and u_2 are neighbors of v with a common neighbor w of degree three that is distinct from v, and let $X = \{u_1, u_2, v, w\}$. By Claims 2.6 and 2.7, u_1 and u_2 have degree four. Note that any induced 2-degenerate subgraph of $G - X$ can be extended to one of G including $X \setminus \{u_1\}$. By Lemma 2.2, if λ' is the number of difficult components of $G - X$, then $\lambda' \geqslant 2$.

Let D_1 and D_2 be difficult components of $G - X$. Since there are only six edges leaving X, we may assume without loss of generality that $|E(X, V(D_1))| \leqslant 3$. Note that D_1 is not an isolated vertex by Lemma 2.4 since G is triangle-free. If D_1 contains a leaf, then it is adjacent to either both u_1 and u_2 or both v and w by Lemma 2.4. In either case, v is contained in a separating cycle of length four, a contradiction. Therefore D_1 contains an end-block isomorphic to the cube, contradicting Lemma 2.3. □

Claim 2.9. *Every edge incident with v is contained in a cycle of length four.*

Proof. Suppose for a contradiction u is a neighbor of v such that the edge uv is not contained in a cycle of length four. Let G' be the graph obtained from G by contracting the edge uv into a new vertex, say w, and observe that G' is planar and triangle-free.

Let $S \subseteq V(G')$ induce a maximum-size induced 2-degenerate subgraph of G'. We claim that G contains an induced 2-degenerate subgraph on at least $|S| + 1$ vertices. If $w \notin S$, then $S \cup \{v\}$ induces a 2-degenerate subgraph of G on at least $|S| + 1$ vertices, as claimed. Therefore we may assume $w \in S$. It suffices to show $S \setminus \{w\} \cup \{u, v\}$ induces a 2-degenerate subgraph in G. Given $S' \subseteq S \setminus \{w\} \cup \{u, v\}$, we will show $G[S']$ contains a vertex of degree at most two. If $S' \cap \{u, v\} = \varnothing$, then $G[S']$ equals $G'[S']$, which contains a vertex of degree at most two, as desired. Therefore we may assume $S' \cap \{u, v\} \neq \varnothing$. Note that $G'[S' \cup \{w\} \setminus \{u, v\}]$ contains a vertex x of degree at most two. If $x \neq w$, then since G is triangle-free, x is not adjacent to both u and v, and thus x has degree at most two in $G[S']$, as desired. So we may assume w has degree at most two in $G'[S' \cup \{w\} \setminus \{u, v\}]$. Now at least one of u and v has degree at most two in $G[S']$, as desired.

Since G is a minimal counterexample and $|V(G')| < |V(G)|$, we have

$$|S| \geqslant \frac{6|V(G')| - |E(G')| - \lambda'}{5} = \frac{6|V(G)| - |E(G)| - \lambda'}{5} - 1,$$

where λ' is the number of difficult components of G'. Furthermore, G contains an induced 2-degenerate subgraph on at least $|S|+1$ vertices as argued, and thus

$$|S|+1 < \frac{6|V(G)|-|E(G)|}{5}.$$

It follows that $\lambda' > 0$. Since G' is connected, G' is difficult. By Lemmas 2.3 and 2.4, G' cannot have an endblock not containing w, and thus G' is isomorpic to the cube. But then either u or v has degree at most two in G, which is a contradiction. $\square$

We can now prove Lemma 2.5.

Proof of Lemma 2.5. Suppose for a contradiction that G contains such a vertex v. By Claim 2.9, the vertex v has a neighbor u such that the edge uv is contained in two cycles of length four. Let x_1 and x_2 denote the other neighbors of v. Since uv is contained in two cycles of length four, for $i \in \{1,2\}$, u and x_i have a common neighbor y_i that is distinct from v. By Claims 2.6 and 2.7, u, x_1, and x_2 have degree four. Since v is not contained in a separating cycle of length four, $y_1 \neq y_2$, x_1 and y_2 are not adjacent, and x_2 and y_1 are not adjacent. By Claim 2.9, y_1 and y_2 have degree at least four. Let $X = \{v, u, x_1, x_1, y_1, y_2\}$, and note that $|E(G)| - |E(G-X)| = 8 + \deg(y_1) + \deg(y_2)$. Note also that any induced 2-degenerate subgraph of $G-X$ can be extended to one of G by adding u, v, x_1, and x_2. By Lemma 2.2, if λ' is the number of difficult components of $G-X$, $\lambda' \geqslant \deg(y_1)+\deg(y_2)-7 \geqslant 1$. Let D be a difficult component of $G-X$ such that the number of edges between D and X is minimum. Note that if $\deg(y_1) \geqslant 5$ or $\deg(y_2) \geqslant 5$, then $|E(V(D),X)| \leqslant 5$. Otherwise, $|E(V(D),X)| \leqslant 9$.

Since G is triangle-free and v is not contained in a separating cycle of length at most 5, each vertex of D has at most two neighbors in X, and if it has two, these neighbors are either $\{x_1, x_2\}$ or $\{y_1, y_2\}$. By Claim 2.8, if z is a leaf of D, we conclude that z is adjacent to y_1 and y_2. By planarity, D has at most two leaves. Furthermore, if D had two leaves, then all edges between D and X would be incident with y_1 and y_2, and by planarity and absence of triangles, we would conclude that G contains a vertex of degree two or a cube subgraph with at most four edges leaving, which is a contradiction. Hence, D has an end-block B isomorphic to the cube. Label the vertices of B according to Figure 1. By Lemma 2.3, D has at most one end-block isomorphic to the cube. Hence, either $D = B$, or D has precisely two end-blocks, one of which is a leaf and one of which is B.

Suppose $\deg(y_1) \geqslant 5$ or $\deg(y_2) \geqslant 5$. Then there are at most 5 edges between X and D. By Lemma 2.3, $B \neq D$, so D has at least two end-blocks. Therefore there are at most 3 edges between B and X, so there are at most 4 edges leaving B, contradicting Lemma 2.3. Hence, $\deg(y_1) = \deg(y_2) = 4$.

By planarity, all edges between B and X are contained in one face of B. Since G is triangle-free and v is not contained in a separating 4-cycle, there are at most 3 edges between B and $\{x_1, x_2\}$. If D has a leaf, then as we observed before, the leaf is adjacent to y_1 and y_2, and by planarity, all edges between B and X are incident with either $\{y_1, y_2, u\}$ or $\{x_1, x_2, y_1, y_2\}$. By Lemma 2.3, the former is not possible, and in the latter case, there are 3 edges between B and $\{x_1, x_2\}$, both y_1 and y_2 have a neighbor in B, and D consists

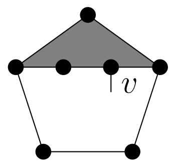

Figure 2: A vertex $v \in V(H) \setminus V(C)$ of degree three.

of B and the leaf. However, this is not possible, since G is triangle-free. Consequently, D is isomorphic to the cube.

Let us now consider the case that u has a neighbor in $V(D)$. We may assume without loss of generality that u is adjacent to v_1. Since v is not in a separating cycle of length at most five, x_1 and x_2 are not adjacent to v_1, v_2, or v_4. Therefore x_1 and x_2 each have at most one neighbor in $V(D)$. By Lemma 2.3, one of y_1 and y_2 has two neighbors in $V(D)$, and we may assume without loss of generality it is y_1. Since G is planar and triangle-free, y_1 is adjacent to v_2 and v_4, and v_3 is not adjacent to a vertex in X. Therefore x_1 and x_2 have no neighbors in $V(D)$, so $|E(V(D), X)| \leqslant 5$, a contradiction.

Hence, we may assume u has no neighbor in $V(D)$. By Lemma 2.3, at least two of the vertices $\{x_1, y_1, x_2, y_2\}$ have two neighbors in $V(D)$. Suppose x_1 has two neighbors in $V(D)$. Then y_1 and y_2 have at most one, since x_1 does not have a common neighbor with y_1 or y_2. Therefore x_2 has two neighbors in $V(D)$. Then y_1 and y_2 have no neighbors in $V(D)$, contradicting Lemma 2.3. Therefore we may assume by symmetry that y_1 and y_2 have two neighbors in $V(D)$. Then x_1 and x_2 have no neighbors in $V(D)$, again contradicting Lemma 2.3. ☐

2.3 Discharging

In this section, we use discharging to prove the following.

Lemma 2.10. *Every triangle-free plane graph with minimum degree three contains a vertex of degree three that is not contained in a separating cycle of length four or five.*

For the remainder of this subsection, suppose G is a counterexample to Lemma 2.10. We assume G is connected, or else we consider a component of G. Since G is planar and triangle-free, it contains a vertex of degree at most three, and thus G contains a separating cycle of length at most five. We choose a separating cycle C of length at most five in G so that the interior of C contains the minimum number of vertices, and we let H be the subgraph of G induced by the vertices in C and its interior. Note that C has no chords since G is triangle-free. By the choice of C, we have the following.

Claim 2.11. *The only separating cycle of G of length at most five belonging to H is C.*

Now we need the following claim about vertices of degree three in the interior of H (see Figure 2).

Claim 2.12. *If some vertex $v \in V(H) \setminus V(C)$ has degree three, then $|V(C)| = 5$, and v has precisely one neighbor in $V(C)$ and is incident to a face of length five whose boundary intersects C in a subpath with three vertices.*

Proof. Suppose $v \in V(H) \setminus V(C)$ has degree three. Since G is a counterexample, v is contained in a separating cycle C' in G of length four or five. By Claim 2.11, C' is not contained in H, and since C is chordless, C' contains a vertex not in $V(H)$. Since C' has length at most five, v has at least one neighbor in $V(C)$. By Claim 2.11, v has at most one neighbor in $V(C)$. Hence v has precisely one neighbor in $V(C)$, as desired. Note that $V(C) \cap V(C')$ is a pair of nonadjacent vertices, or else G contains a triangle. If v is not incident to a face of length five containing three vertices of C, or if $|V(C)| = 4$, then H contains a separating cycle of length at most five containing v, contradicting Claim 2.11. □

Proof of Lemma 2.10. For each $v \in V(H) \setminus V(C)$, let $ch(v) = \deg(v) - 4$, for each $v \in V(C)$, let $ch(v) = \deg(v) - 2$, and for each face f, let $ch(f) = |f| - 4$. Note that by Euler's formula, if $F(H)$ denotes the set of faces of H,

$$\sum_{v \in V(H)} ch(v) + \sum_{f \in F(H)} ch(f) = 4(|E(H)| - |V(H)| - |F(H)|) + 2|V(C)| = -8 + 2|V(C)|.$$

Now we redistribute the charges in the following way, and we denote the final charge ch_*. For each $v \in V(C)$, if $u \in V(H) \setminus V(C)$ has degree three and is adjacent to v, let v send one unit of charge to u. Note that by Claim 2.12, for each $v \in V(H)$, $ch_*(v) \geqslant 0$. Note also that for each $f \in F(H)$, $ch_*(f) \geqslant 0$. The sum of charges is unchanged, i.e., it is $-8 + 2|V(C)|$.

First, suppose $|V(C)| = 4$, and thus the sum of the charges is 0. Note that every vertex and face has precisely zero final charge, so every face has length precisely four. By Claim 2.12, every vertex $v \in V(H) \setminus V(C)$ has degree precisely four. Therefore every vertex in C has degree precisely two. Since C is separating, G is not connected, a contradiction.

Therefore we may assume $|V(C)| = 5$, so the sum of the charges is 2. Note that the outer face f has final charge $ch_*(f) = 1$. Since G has an even number of odd-length faces, it follows that G has another face f' of length 5 and final charge 1, and all other faces and vertices have zero final charge. In particular, all faces of H distinct from f and f' have length 4 and each vertex in $V(C)$ is adjacent only to vertices of degree three in $V(H) \setminus V(C)$. Using Claim 2.12, we conclude there are more than two vertices of $V(H) \setminus V(C)$ with a neighbor in C and at least two faces of length at least five in the interior of C, a contradiction. □

Now the proof of Theorem 1.5 follows easily from Lemmas 2.4, 2.5, and 2.10.

3 Proof of Theorem 1.7

For the remainder of this section, let G be a counterexample to Theorem 1.7 such that $f_3(G)$ is minimum, and subject to that, $|V(G)|$ is minimum, and let F be a set of $f_3(G)$ faces of G such that every vertex in G of degree at most three is incident to at least one of them.

3.1 Preliminaries

Lemma 3.1. *The graph G has minimum degree three.*

Proof. Suppose not. Since G is planar and triangle-free, G has minimum degree at most three. Therefore we may assume G contains a vertex v of degree at most two. By assumption, there is a face in F incident with v. Therefore $f_3(G-v) \leqslant f_3(G)$. Note that $G-v$ is not 2-degenerate or else G is. By the minimality of G, there exists $S \subseteq V(G-v)$ of size at least $\frac{7}{8}(|V(G)|-1) - 18\,(f_3(G)-2)$ such that $G[S]$ is 2-degenerate. Now $S \cup \{v\}$ induces a 2-degenerate subgraph of G on at least $\frac{7}{8}|V(G)| - 18\,(f_3(G)-2)$ vertices, contradicting that G is a counterexample. $\square$

Lemma 3.2. *If H is a triangle-free plane graph of minimum degree at least two such that $f_3(H) = 1$, then H has at least four vertices of degree two.*

Proof. Let f' be a face of H incident to all the vertices in H of degree at most three. We use a simple discharging argument. For each vertex v, assign initial charge $ch(v) = \deg(v) - 4$, and for each face f, assign initial charge $ch(f) = |f| - 4$. Now let f' send one unit of charge to each vertex v of degree at most three incident with f', and denote the final charge ch_*. By Euler's formula, the sum of the charges is -8. However, $ch_*(f') \geqslant -4$, and every other face has nonnegative final charge. Therefore the vertices have total final charge at most -4. Every vertex of degree at least three has nonnegative final charge, and every vertex v of degree two has final charge -1. Therefore H contains at least four vertices of degree two, as desired. $\square$

Lemmas 3.1 and 3.2 imply that $f_3(G) > 1$. A *cylindrical grid* is the Cartesian product of a path and a cycle.

Lemma 3.3. *If H is a triangle-free plane graph such that $f_3(H) = 2$, then either H has minimum degree at most two, or H is a cylindrical grid.*

Proof. Let H be a triangle-free plane graph of minimum degree three such that $f_3(H) = 2$. It suffices to show that H is a cylindrical grid. Let f_1 and f_2 be faces of H such that every vertex of degree at most three is incident to either f_1 or f_2. Again we use a simple discharging argument. For each vertex v, assign initial charge $ch(v) = \deg(v) - 4$, and for each face f, assign initial charge $ch(f) = |f| - 4$. Now for $i \in \{1, 2\}$, let f_i send one unit of charge to each vertex v incident to f_i, and denote the final charge ch_*. By Euler's formula, the sum of the charges is -8. However, $ch_*(f_1), ch_*(f_2) \geqslant -4$, and every other face and every vertex has nonnegative final charge. It follows that $ch_*(f_1) = ch_*(f_2) = -4$, and that every other face and every vertex has precisely zero final charge. Therefore the boundaries of f_1 and f_2 are disjoint, and every vertex incident with either f_1 or f_2 has degree three. Every other vertex has degree four, and every face that is not f_1 or f_2 has length four. It is easy to see that the only graphs with these properties are cylindrical grids, as desired. $\square$

Lemma 3.4. *A triangle-free cylindrical grid on n vertices contains an induced 2-degenerate subgraph on at least $\frac{7}{8}n$ vertices.*

Proof. Let H be a triangle-free cylindrical grid on n vertices. The vertices of H can be partitioned into k sets that induce cycles $C_1, \dots, C_k$ of equal length such that for each $i \in \{2, \dots, k-1\}$, every vertex in C_i has a unique neighbor in C_{i-1} and in C_{i+1}. Let X be any set of vertices containing precisely one vertex in C_{2i} for each $i \in \{1, \dots, \lfloor k/2 \rfloor\}$. Note that $H - X$ is an induced 2-degenerate subgraph on at least $\frac{7}{8}n$ vertices, as desired. $\square$

By Lemmas 3.3 and 3.4, we have $f_3(G) > 2$.

Definition 3.5. We say a subset of the plane is G-*normal* if it intersects G only in vertices. If f and f' are faces of G, we define $d(f, f')$ to be the smallest number of vertices contained in a G-normal curve with one end in f and the other end in f'. If f is a face of G and v is a vertex of G, we define $d(f, v)$ to be the minimum of $d(f, f')$ over all faces f' incident with v.

Lemma 3.6. *Let P be a G-normal connected subset of the plane that intersects a face in F or its boundary. Let X be the set of vertices of G contained in P. Suppose that H_1 and H_2 are disjoint induced subgraphs of $G - X$ such that $G - X = H_1 \cup H_2$. If $f_3(H_1) \geqslant 2$ and $f_3(H_2) \geqslant 2$, then $|X| \geqslant 21$.*

Proof. Note that there is a face of $G - X$ containing P in its interior, and any vertex of $G - X$ of degree at most three that has degree at least four in G is incident with this face. Therefore $f_3(H_1) + f_3(H_2) \leqslant f_3(G) + 1$. By the minimality of G, for each $i \in \{1, 2\}$, there exists $S_i \subseteq V(H_i)$ of size at least $\frac{7}{8}|V(H_i)| - 18(f_3(H_i) - 2)$ such that $G[S_i]$ is 2-degenerate. But $G[S_1 \cup S_2]$ is 2-degenerate, and

$$\begin{aligned} |S_1 \cup S_2| &\geqslant \frac{7}{8}\left(|V(G)| - |X|\right) - 18(f_3(H_1) + f_3(H_2) - 4) \\ &\geqslant \frac{7}{8}|V(G)| - 18(f_3(G) - 2) - \frac{7}{8}|X| + 18. \end{aligned}$$

Since G is a counterexample, $\frac{7}{8}|X| > 18$, so $|X| \geqslant 21$, as desired. $\square$

Note that Lemma 3.6 together with Lemmas 3.1 and 3.2 imply that G is connected.

Lemma 3.7. *All distinct faces $f, f' \in F$ satisfy $d(f, f') \geqslant 21$.*

Proof. Suppose not. Then there is a set X of at most 20 vertices such that f and f' are contained in the same face of $G - X$. Therefore $f_3(G - X) \leqslant f_3(G) - 1$.

Let $n = |V(G)|$. Recall that $f_3(G) \geqslant 3$, and thus $n > 20$, as otherwise the empty subgraph satisfies the requirements of Theorem 1.7. Note that $G - X$ is not 2-degenerate or else $G - X$ is an induced 2-degenerate subgraph on at least $n - 20 \geqslant \frac{7}{8}n - 18(f_3(G) - 2)$ vertices, contradicting that G is a counterexample. So by the minimality of G, there exists $S \subseteq V(G - X)$ of size at least $\frac{7}{8}(|V(G)| - |X|) - 18\left(f_3(G - X) - 2\right) \geqslant \frac{7}{8}|V(G)| - 18\left(f_3(G) - 2\right)$ such that $G[S]$ is 2-degenerate, contradicting that G is a counterexample. $\square$

Lemma 3.8. *For each $f \in F$ and $k \in \{0, \dots, 9\}$, if $C_k = \{v \in V(G) : d(f, v) = k\}$, then C_k induces a cycle in G. Furthermore, every vertex in C_k has at most one neighbor u satisfying $d(f, u) < k$.*

Proof. We assume without loss of generality that f is the outer face of G. We use induction on k. In the base case, C_0 is the set of vertices incident with f. We prove this case as a special case of the inductive step.

By induction, we assume that for each $k' < k$, $C_{k'}$ induces a cycle in G and each vertex $C_{k'}$ has at most one neighbor in $C_{k'-1}$. Let $H = G - \bigcup_{k'=0}^{k-1} C_{k'}$. Note that C_k is the set of vertices incident with the outer face of H. By Lemma 3.7, if $k > 0$ then every vertex of C_k has degree at least four in G.

First we show that every $v \in C_k$ has at most one neighbor in C_{k-1}. Here the base case is trivial, so we may assume $k > 0$. Suppose for a contradiction that a vertex $v \in C_k$ has two neighbors v_1 and v_2 in C_{k-1}. Let P_1 and P_2 be the two paths in the cycle $G[C_{k-1}]$ with ends v_1 and v_2. Since G is triangle-free, P_1 and P_2 have length at least two. For $i \in \{1, 2\}$, note that the subgraph of G drawn in the closure of the interior of the cycle $P_i + v_1 v v_2$ has minimum degree at least two and at most three vertices (v_1, v, and v_2) of degree two. Therefore by Lemma 3.2, it contains a face $f_i \in F$.

For $i \in \{1, 2\}$, there exists a simple G-normal curve A_i from v_i to f containing exactly one vertex from $C_{k'}$ for each $k' < k$. Let X consist of the vertices on A_1 and A_2 together with v, and note that $|X| \leqslant 19$. Let $G - X = H_1 \cup H_2$, where f_1 is a face of H_1 and f_2 is a face of H_2—neither f_1 nor f_2 is incident with a vertex of X by Lemma 3.7, and for the same reason the vertices in H_i incident with f_i have degree at least three for $i \in \{1, 2\}$. By Lemma 3.2, for $i \in \{1, 2\}$ we have either $f_3(H_i) \geqslant 2$ or H_i contains vertices of degree at most two. In the latter case, the vertices of degree at most two in H_i are incident with the outer face, and thus $f_3(H_i) \geqslant 2$. This contradicts Lemma 3.6. Therefore every vertex of C_k has at most one neighbor in C_{k-1}, as claimed. Note that this implies every vertex of C_k has degree at least three in H.

Now we claim that H is connected and C_k does not contain a cut-vertex of H. Suppose not. Then H contains at least two end-blocks B_1 and B_2. Note that B_1 and B_2 have minimum degree at least two and at most one vertex of degree two. Therefore by Lemma 3.2, $f_3(B_1), f_3(B_2) \geqslant 2$. But there is a connected G-normal subset of the plane intersecting G in a set of vertices X containing only one vertex of H and at most two vertices from each $C_{k'}$ for $k' < k$ such that $B_1 - X$ and $B_2 - X$ are in different components of $G - X$. Note that $|X| \leqslant 19$. By Lemma 3.7, $f_3(B_1 - X), f_3(B_2 - X) \geqslant 2$, contradicting Lemma 3.6. Hence H is connected, and C_k does not contain a cut-vertex of H, as claimed.

Since C_k does not contain a cut-vertex of H, the outer face of H is bounded by a cycle, say C. Now if C_k does not induce a cycle in G, then there is a chord of C, say uv. Let P_1 and P_2 be paths in C with ends at u and v such that $C = P_1 \cup P_2$. For $i \in \{1, 2\}$, let H_i be the graph induced by G on the vertices in $P_i \cup uv$ and its interior. Since H_i has minimum degree two and at most two vertices of degree two, by Lemma 3.2, $f_3(H_i) \geqslant 2$. But there is a connected G-normal subset of the plane containing u, v, and intersecting G in a set of vertices X containing at most two vertices from each $C_{k'}$ for $k' \leqslant k$. Note that $|X| \leqslant 20$. By Lemma 3.7, $f_3(H_1 - X), f_3(H_2 - X) \geqslant 2$, contradicting Lemma 3.6. □

Consider a face $f \in F$ and for $k \in \{0, \dots, 9\}$, let C_k be the cycle induced by $\{v \in V(G) : d(f, v) = k\}$ according to Lemma 3.8. For $k \in \{0, \dots, 8\}$ and $v \in V(C_k)$, let $n(v)$ denote the number of neighbors of v in C_{k+1} (note that $n(v) \geqslant 1$) and $n(f, k) = \sum_{v \in V(C_k)}(n(v) - 1)$. Let $g(f, k)$ be the sum of $|f'| - 4$ over all faces f' such that $d(f, f') = k+1$, i.e., the faces between cycles C_k and C_{k+1}. Let $b_k = 3$ if $k = 0$ and $b_k = 4$ otherwise, and let $c(f, k) = \sum_{v \in V(C_k)}(\deg(v) - b_k)$. Let us also define $n(f, -1) = g(f, -1) = 0$. Observe that

$$|C_{k+1}| = |C_k| + 2n(f, k) + g(f, k),$$

and

$$n(f, k) = n(f, k - 1) + g(f, k - 1) + c(f, k).$$

Consequently,

$$n(f, k) = \sum_{k'=0}^{k} c(f, k') + \sum_{k'=0}^{k-1} g(f, k').$$

The following lemma will be crucial.

Lemma 3.9. *For every $f \in F$,*

$$8 \sum_{k=0}^{9} n(f, k) \geqslant 249.$$

First we need the following claims.

Claim 3.10. *Every face $f \in F$ satisfies $n(f, 1) \geqslant 2$.*

Proof. Suppose that $n(f, 1) \leqslant 1$. For $k \in \{0, 1\}$, let C_k denote the cycle induced by $\{v \in V(G) : d(f, v) = k\}$ according to Lemma 3.8. If $n(f, 1) = 0$, then $c(f, 0) = c(f, 1) = 0$ and $g(f, 0) = 0$, i.e., $H = G[V(C_0 \cup C_1)]$ is a cylindrical grid and all vertices of C_1 have degree 4 in G. In this case, let v be an arbitrary vertex of C_1. If $n(f, 1) = 1$, then $c(f, 0) + g(f, 0) + c(f, 1) = 1$, so one of the following holds (see Figure 3):

- $c(f, 0) = 1$, so there is a vertex $v' \in V(C_0)$ of degree four and a vertex $v'' \in V(C_1)$ of degree two in H; we let v be any vertex of C_1 that is not v'' and is not adjacent to v'. Note that every vertex of $V(C_0) \setminus \{v'\}$ has degree three, and every vertex of C_1 has degree four in G. Or,

- $g(f, 0) = 1$, so there is a face of H of length five incident with a vertex $v' \in V(C_1)$ of degree two in H; we let v be any vertex of C_1 other than v'. Note that every vertex of C_0 has degree three and every vertex of C_1 has degree four in G. Or,

- $c(f, 1) = 1$, so H is a cylindrical grid and exactly one vertex of C_1 has degree five; we let v be this vertex.

Let $X = V(C_1 \cup C_2)$. Note that $f_3(G - X) \leqslant f_3(G)$, and by the minimality of G, there exists $S \subseteq V(G - X)$ inducing a 2-degenerate subgraph such that $|S| \geqslant \frac{7}{8}|V(G - X)| - 18\,(f_3(G - X) - 2) \geqslant \frac{7}{8}|V(G)| - 18\,(f_3(G) - 2) - (|X| - 1)$. But then $S \cup (X \setminus \{v\})$ induces a 2-degenerate subgraph of G, contradicting the assumption that G is a counterexample. $\square$

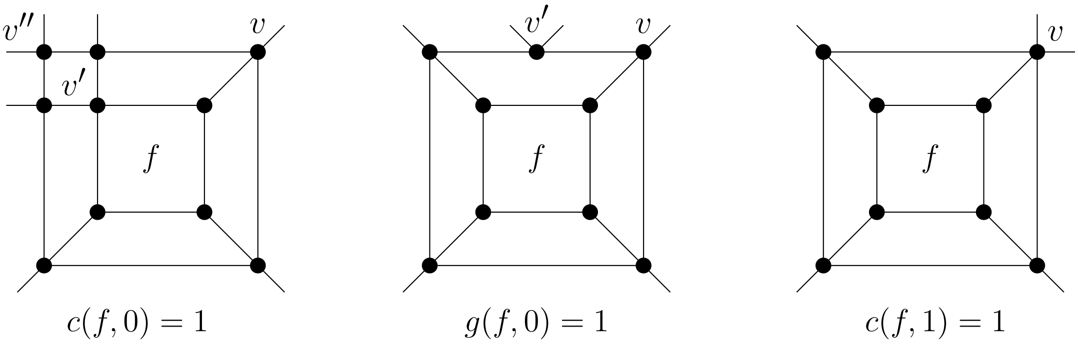

Figure 3: $n(f,1)=1$, when $|f|=4$.

Claim 3.11. *Let $f \in F$ and for $k \in \{0,\dots,9\}$, let C_k be the cycle induced by $\{v \in V(G) : d(f,v)=k\}$ according to Lemma 3.8. Then*

$$|C_0 \cup \ldots \cup C_9| \geqslant 184.$$

Proof. Claim 3.10 implies that $n(f,k) \geqslant 2$ for $k \in \{1,\dots,9\}$, and thus $|C_{k+1}| \geqslant |C_k|+4$ for $k \in \{1,\dots,8\}$. Since G is triangle-free, we have $|C_0| \geqslant 4$, and we conclude that

$$|C_0 \cup \ldots \cup C_9| \geqslant 10|C_0| + 4(1+2+\ldots+8) \geqslant 184. \qquad \square$$

Now we can prove Lemma 3.9.

Proof of Lemma 3.9. For $k \in \{0,\dots,9\}$, let C_k denote the cycle induced by $\{v \in V(G) : d(f,v)=k\}$ according to Lemma 3.8. Let $X = \bigcup_{k=0}^{9} V(C_k)$. By Claim 3.11, $|X| \geqslant 184$. For $k \geqslant 1$, let R_k be a smallest subset of $V(C_k)$ such that $\sum_{v \in V(C_k)\setminus R_k}(n(v)-1) \leqslant 1$. By Claim 3.10 and the monotonicity of $n(f,k)$, we have $n(f,k) \geqslant 2$, and thus R_k is non-empty. Note that $|R_k| \leqslant n(f,k)-1$. For $k=0$, let R_0 be defined in the same way if $n(f,0) \geqslant 2$, and let R_0 consist of an arbitrary vertex of C_0 otherwise. Let $R = \bigcup_{k=0}^{9} R_k$.

By Lemma 3.7, we have $f_3(G-X) \leqslant f_3(G)$, and by the minimality of G, there exists $S \subseteq V(G-X)$ inducing a 2-degenerate subgraph such that $|S| \geqslant \frac{7}{8}|V(G-X)| - 18\,(f_3(G-X)-2) \geqslant \frac{7}{8}|V(G)| - 18\,(f_3(G)-2) - \frac{7}{8}|X|$. We claim that $S \cup (X \setminus R)$ induces a 2-degenerate subgraph of G. Indeed, it suffices to show that for every non-empty $X' \subseteq X \setminus R$, the graph $G[S \cup X']$ has a vertex of degree two. Let k be the minimum index such that $X' \cap V(C_k) \neq \varnothing$. Note that by the choice of R_k, $C_k[X'] - R_k$ is a union of paths containing at most one vertex with more than one neighbor in C_{k+1}, and if there is such a vertex, it has exactly two neighbors in C_{k+1}. Consequently, one of the endvertices of these paths has degree at most two in $G[S \cup X']$.

Since G is a counterexample, we conclude that $|X \setminus R| < \frac{7}{8}|X|$, and thus $8|R|-1 \geqslant |X| \geqslant 184$. Since $|R| \leqslant 2 + \sum_{k=0}^{9}(n(f,k)-1) \leqslant -8 + \sum_{k=0}^{9} n(f,k)$, the inequality

$$8\sum_{k=0}^{9} n(f,k) \geqslant 249$$

follows. $\square$

3.2 Discharging

In this subsection we use discharging to complete the proof of Theorem 1.7.

Proof of Theorem 1.7. For each $v \in V(G)$, let $ch(v) = \deg(v) - 4$, and for each face f of G, let $ch(f) = |f| - 4$. Now we redistribute the charges according to the following rules and denote the final charge by ch_*.

1. Every face $f \in F$ sends 1 unit of charge to every vertex incident with f.
2. Afterwards, every face $f' \notin F$ and every vertex $v \in V(G)$ such that $d(f, f') \leqslant 9$ or $d(f, v) \leqslant 9$ for some face $f \in F$ sends all of its charge to f.

Observe that every vertex and every face not in F sends its charge to at most one face of F by Lemma 3.7. Clearly, all vertices and all faces not in F have non-negative final charge. By Euler's formula the sum of the charges is -8, so there exists some face $f \in F$ with negative charge.

By Lemma 3.8, for each $k \in \{0, \ldots, 9\}$, the vertices $v \in V(G)$ such that $d(f, v) = k$ induce a cycle in G, say C_k. Note that after the first discharging rule is applied, f has charge -4, and since $ch_*(f) \leqslant -1$, at most three units of charge are sent to f according to the second rule. Note that f receives precisely $c(f, k)$ total charge from vertices of C_k and precisely $g(f, k)$ total charge from faces between C_k and C_{k+1}. Hence, we have

$$3 \geqslant \sum_{k'=0}^{9} c(f, k') + \sum_{k'=0}^{8} g(f, k') \geqslant n(f, k)$$

for every $k \in \{0, \ldots, 9\}$. Therefore,

$$\sum_{k=0}^{9} n(f, k) \leqslant 30.$$

However, this contradicts Lemma 3.9, finishing the proof. $\square$

Let us remark that the constant 18 in the statement of Theorem 1.7 can be improved. In particular, one could extend the case analysis of Claim 3.10 to fully describe larger neighborhoods of the face, likely obtaining enough charge in a much smaller number of layers than 10 needed in our argument (at the expense of making the proof somewhat longer and harder to read).

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