# Periods of Ehrhart coefficients of rational polytopes

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**Abstract**

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope whose vertices have rational coordinates. By a seminal result of E. Ehrhart, the number of integer lattice points in the $k$th dilate of $\mathcal{P}$ ($k$ a positive integer) is a quasi-polynomial function of $k$ — that is, a "polynomial" in which the coefficients are themselves periodic functions of $k$. It is an open problem to determine which quasi-polynomials are the Ehrhart quasi-polynomials of rational polytopes. As partial progress on this problem, we construct families of polytopes in which the periods of the coefficient functions take on various prescribed values.

## 1 Introduction

Let $\mathcal{P} \subseteq \mathbb{R}^n$ ($n \geqslant 2$) be a *convex*[1] *rational polytope* — that is, the convex hull of finitely many points in $\mathbb{Q}^n$. By a famous theorem of Ehrhart [8], the number of integer lattice points in positive integer dilates $k\mathcal{P}$ of $\mathcal{P}$ is given by a quasi-polynomial function of $k$. In particular, there exist *coefficient functions* $c_{\mathcal{P},i} \colon \mathbb{Z} \to \mathbb{Q}$ with finite periods such that

$$|k\mathcal{P} \cap \mathbb{Z}^n| = \sum_{i=0}^{n} c_{\mathcal{P},i}(k)k^i, \qquad \text{for } k \in \mathbb{Z}_{\geqslant 1}. \tag{1}$$

The function $\mathrm{ehr}_{\mathcal{P}} \colon \mathbb{Z} \to \mathbb{Z}$ defined by $\mathrm{ehr}_{\mathcal{P}}(x) \coloneqq \sum_{i=0}^{n} c_{\mathcal{P},i}(x)x^i$ is the *Ehrhart quasi-polynomial* of $\mathcal{P}$. (We refer the reader to [2, 13, 21] for introductions to Ehrhart theory.)

The motivation for this paper is the problem of characterizing the Ehrhart quasi-polynomials of rational polytopes. It is well known that, if $\mathcal{P}$ is an *integral* polytope

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[1]We will also consider nonconvex polytopes, as defined below.

(meaning that all vertices are in $\mathbb{Z}^n$), then the coefficients $c_{\mathcal{P},i}$ are constants and $\mathrm{ehr}_{\mathcal{P}}$ is simply a polynomial. Already in this case, the question of which polynomials are Ehrhart polynomials is difficult. Beginning with the pioneering work of Stanley [19, 20], Betke & McMullen [4], and Hibi [14], many inequalities have been shown to be satisfied by the coefficients of Ehrhart polynomials of integral polytopes. (For recent work in this area, see [1, 6, 7, 10, 11, 17, 23, 22] and references therein.) Indeed, in the 2-dimensional case, a 1976 result of Scott [18] completely characterizes the Ehrhart polynomials of convex integral polygons. Nonetheless, a complete characterization is not yet known for the Ehrhart polynomials of convex integral polytopes in dimension 3 or higher.

Much less is known about the characterization of Ehrhart quasi-polynomials in the nonintegral case. For example, even in dimension 2, we do not know which polynomials are the Ehrhart polynomials of nonintegral convex polygons [15][2].

In this paper, we approach the problem of characterizing Ehrhart quasi-polynomials by focusing on the possible periods of the coefficient functions $c_{\mathcal{P},i}$ appearing in equation (1). Define the *period sequence* of $\mathcal{P}$ to be $(p_0, p_1, \dots, p_n)$, where $p_i$ is the (minimum) period of $c_{\mathcal{P},i}$. Our question is thus: *What are the possible period sequences of rational polytopes?*

If the interior of $\mathcal{P}$ is nonempty, then the leading coefficient function $c_{\mathcal{P},n}$ is a constant equal to the volume of $\mathcal{P}$, and so $p_n = 1$. More generally, McMullen [16] showed that each coefficient period $p_i$ is bounded by the corresponding *$i$-index* of $\mathcal{P}$. The $i$-index is the least positive integer $m_i$ such that every $i$-dimensional face of the dilate $m_i\mathcal{P}$ contains an integer lattice point in its affine span. We call $(m_0, \dots, m_n)$ the *index sequence* of $\mathcal{P}$. McMullen proved the following.

**Theorem 1.1** (McMullen [16, Theorem 6]). *Let $\mathcal{P}$ be an $n$-dimensional rational polytope with period sequence $(p_0, \dots, p_n)$ and index sequence $(m_0, \dots, m_n)$. Then $p_i$ divides $m_i$ for $0 \leqslant i \leqslant n$. In particular, $p_i \leqslant m_i$.*

We will refer to the inequalities $p_i \leqslant m_i$ in Theorem 1.1 as *McMullen's bounds*. It is easy to see that the indices $m_i$ of a rational polytope satisfy the divisibility relations $m_n \mid m_{n-1} \mid \cdots \mid m_0$ and hence that $m_n \leqslant m_{n-1} \leqslant \cdots \leqslant m_0$. Beck, Sam, and Woods [3] showed that McMullen's bounds are always tight in the $i = n-1$ and $i = n$ cases. It is also shown in [3] that, given any positive integers $m_n \mid m_{n-1} \mid \cdots \mid m_0$, there exists a polytope with $i$-index $m_i$ for $0 \leqslant i \leqslant n$. Moreover, all of McMullen's bounds are tight for this polytope. This construction establishes the following.

**Theorem 1.2** (Beck et al. [3]). *Let positive integers $p_{n-1} \mid p_{n-2} \mid \cdots \mid p_0$ be given. Then there exists an $n$-dimensional convex polytope in $\mathbb{R}^n$ with period sequence $(p_0, \dots, p_{n-1}, 1)$.*

The period sequences realized by Theorem 1.2 must satisfy the inequalities

$$p_0 \geqslant \cdots \geqslant p_{n-1}.$$

[2] Herrmann [12] characterizes the Ehrhart quasi-polynomials of half-integral not-necessarily-convex polygons.

In the following sections, we extend the set of known period sequences by constructing rational polytopes in which the period sequences do not satisfy these inequalities. Alternatively, our constructions may be thought of as examples in which a particular one of McMullen's bounds is arbitrarily far from tight. Our first main result is the following, which is proved in Section 3.

**Theorem 1.3.** *Let a positive integer $p$ be given. Then there exists an $n$-dimensional convex rational polytope in $\mathbb{R}^n$ with period sequence* $(1, p, 1, \ldots, 1)$.

In Theorem 1.3, we achieve *convex* polytopes. In other cases, we are unable to find convex constructions and must consider nonconvex rational polytopes. In general, we call a topological ball $\mathcal{B}$ in $\mathbb{R}^n$ a *rational polytope* (not necessarily convex) if $\mathcal{B}$ is a union $\bigcup_{i\in I} \mathcal{P}_i$ of a finite family $\{\mathcal{P}_i : i \in I\}$ of convex rational polytopes, all with the same affine span, in which every nonempty intersection $\mathcal{P}_i \cap \mathcal{P}_j$, $i \neq j$, is a common facet of $\mathcal{P}_i$ and $\mathcal{P}_j$.

Our second main result, proved in Section 4, is the construction of *non*convex polytopes with period sequences of the form $(1, \ldots, 1, p, 1)$.

**Theorem 1.4.** *Let a positive integer $p$ be given. Then there exists an $n$-dimensional nonconvex polytope in $\mathbb{R}^n$ with period sequence* $(1, \ldots, 1, p, 1)$*, provided that either* $3 \leqslant n \leqslant 11$ *or* $n = 13$.

## 2 Building blocks

In this section, we fix notation and recall results that will be used in the constructions below. We also establish Theorems 1.3 and 1.4 in the dimension $n = 2$ case.

We are interested in the period sequences of quasi-polynomials. This period sequence is invariant under addition of polynomials. Thus, it will be convenient to consider quasi-polynomials $f(x)$ and $g(x)$ to be *equivalent* when $f(x) - g(x)$ is a polynomial. In this case, we write $f(x) \equiv g(x)$. In particular, if $f(x) \equiv g(x)$, then $f(x)$ and $g(x)$ have the same period sequence. The chief convenience of this notation is that, if $\mathcal{Q} \cup \mathcal{R}$ is a union of rational polytopes $\mathcal{Q}$ and $\mathcal{R}$ such that $\mathcal{Q} \cap \mathcal{R}$ is integral, then $\mathrm{ehr}_{\mathcal{Q}\cup\mathcal{R}}(x) \equiv \mathrm{ehr}_{\mathcal{Q}}(x) + \mathrm{ehr}_{\mathcal{R}}(x)$.

The constructions in the following sections depend on certain 2-dimensional polygons studied in [15]. Let $p$ be a positive integer and set $q := p^2 - p + 1$. (Typically, $p$ will be the desired period of a coefficient function in the Ehrhart quasi-polynomial of a rational polytope.) Let $\ell \subseteq \mathbb{R}$ be the closed segment $[-\frac{1}{p}, 0]$. Then the Ehrhart quasi-polynomial of $\ell$ has the form

$$\mathrm{ehr}_\ell(x) = \tfrac{1}{p}x + c_{\ell,0}(x),$$

where $c_{\ell,0}$ has (minimum) period $p$. Let $P \subseteq \mathbb{R}^2$ be the convex pentagon with vertices $\mathbf{u}^+$, $\mathbf{u}^-$, $\mathbf{v}^+$, $\mathbf{v}^-$, $\mathbf{w}$, where

$$\mathbf{u}^{\pm} := \pm q\mathbf{e}_1, \qquad \mathbf{v}^{\pm} := \pm(q-1)\mathbf{e}_1 + \mathbf{e}_2, \qquad \mathbf{w} := \frac{q}{p}\mathbf{e}_2.$$

(We write $\mathbf{e}_i$ for the $i$th standard basis vector.) A key fact, proved in [15], is that the Ehrhart quasi-polynomials of $P$ and $\ell$ are "complements" of each other in the sense that the periodic parts of their coefficients cancel when the quasi-polynomials are added together[3]. That is,

$$\operatorname{ehr}_P(x) \equiv -\operatorname{ehr}_\ell(x). \tag{2}$$

As a warm-up for the following sections, we recall how $P$ and $\ell$ were used in [15] to construct a polygon with period sequence $(1, p, 1)$. Let $R \subseteq \mathbb{R}^2$ be the rectangle $[-q, q] \times \ell$, and consider the convex heptagon $H := \operatorname{Conv}(R \cup P) \subseteq \mathbb{R}^2$. Note that $R$ and $P$ are rational polygons whose intersection is the lattice segment with endpoints $\mathbf{u}^{\pm}$. Hence we can use equivalence (2) to compute that

$$\begin{aligned}\operatorname{ehr}_H(x) &\equiv (2qx+1)\operatorname{ehr}_\ell(x) + \operatorname{ehr}_P(x)\\ &\equiv (2qx+1)\operatorname{ehr}_\ell(x) - \operatorname{ehr}_\ell(x)\\ &= 2qx\operatorname{ehr}_\ell(x)\\ &\equiv 2q\, c_{\ell,0}(x)\, x.\end{aligned}$$

That is, $H$ has period sequence $(1, p, 1)$. This establishes the $n = 2$ cases of both Theorem 1.3 and Theorem 1.4.

## 3 Convex polytopes with period sequence $(1, p, 1, \ldots, 1)$

Let a positive integer $p \geqslant 1$ be given. Recall that we set $q := p^2 - p + 1$. We now prove Theorem 1.3 by constructing a convex rational polytope $H_n \subseteq \mathbb{R}^n$ with period sequence $(1, p, 1, \ldots, 1)$. Since the $n = 2$ case was established in the previous section, we assume that $n \geqslant 3$.

A useful fact about equivalence (2) is that it continues to hold when we take $i$-fold pyramids over both $P$ and $\ell$. More precisely, let $\mathcal{Q} \subseteq \mathbb{R}^d$ be a polytope, and let $\mathcal{Q}'$ be the embedded copy of $\mathcal{Q}$ in $\mathbb{R}^{d+1}$ defined by $\mathcal{Q}' := \{(\mathbf{x}, 0) \in \mathbb{R}^{d+1} : \mathbf{x} \in \mathcal{Q}\}$. Fix a point $\mathbf{a} \in \mathbb{Z}^{d+1}$ with final coordinate equal to 1. Then $\operatorname{Conv}(\mathcal{Q}' \cup \{\mathbf{a}\})$ is a (1-fold) *pyramid* over $\mathcal{Q}$. By induction, for $i \geqslant 2$, define an *$i$-fold pyramid* over $\mathcal{Q}$ to be a pyramid over an $(i-1)$-fold pyramid over $\mathcal{Q}$.

**Proposition 3.1.** *Let $P$ and $\ell$ be the pentagon and line segment defined in the previous section, and let $\Delta(P)$ and $\Delta(\ell)$ be $i$-fold pyramids over $P$ and $\ell$, respectively. Then*

$$\operatorname{ehr}_{\Delta(P)}(x) \equiv -\operatorname{ehr}_{\Delta(\ell)}(x). \tag{3}$$

*Proof.* The *Ehrhart series* $\operatorname{Ehr}_\mathcal{Q}(t)$ of a rational polytope $\mathcal{Q}$ is the generating function of $\operatorname{ehr}_\mathcal{Q}(x)$. That is, $\operatorname{Ehr}_\mathcal{Q}(t)$ is the formal power series

$$\operatorname{Ehr}_\mathcal{Q}(t) := \sum_{k=0}^{\infty} \operatorname{ehr}_\mathcal{Q}(k) t^k.$$

[3] In [15], the pentagon $P$ was reflected about the diagonal $x = y$.

It is well known that $\mathrm{Ehr}_{\mathcal{Q}}(t)$ is a rational function in $t$. Furthermore, if $\Delta(\mathcal{Q})$ is an $i$-fold pyramid over $\mathcal{Q}$, then

$$\mathrm{Ehr}_{\Delta(\mathcal{Q})}(t) = \frac{1}{(1-t)^i} \mathrm{Ehr}_{\mathcal{Q}}(t).$$

Given generating functions $F(t)$ and $G(t)$ of quasi-polynomials $f(x)$ and $g(x)$, respectively, we write $F(x) \equiv G(x)$ if $f(x) \equiv g(x)$. Hence, equivalence (2) implies that $\mathrm{Ehr}_P(t) \equiv -\mathrm{Ehr}_\ell(t)$, and so

$$\mathrm{Ehr}_{\Delta(P)}(t) = \frac{1}{(1-t)^i} \mathrm{Ehr}_P(t) \equiv -\frac{1}{(1-t)^i} \mathrm{Ehr}_\ell(t) = -\mathrm{Ehr}_{\Delta(\ell)}(t).$$

Equivalence (3) follows by comparing coefficients of the series. □

One example of an $i$-fold pyramid that we will have occasion to use is the simplex $S_n \subseteq \mathbb{R}^{n-1}$ given by

$$S_n \coloneqq \mathrm{Conv}\left\{\mathbf{0}, -\tfrac{1}{p}\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}\right\},$$

which is an $(n-2)$-fold pyramid over $\ell$. It is known that the period sequence of $S_n$ is $(p, 1, \ldots, 1)$. Indeed, up to a lattice-preserving transformation, $S_n$ is among the polytopes constructed by Beck et al. [3] to prove Theorem 1.2.

We also construct an $(n-2)$-fold pyramid over the pentagon $P$ as follows. Write $P' \subseteq \mathbb{R}^n$ for the embedded copy of $P$ defined by $P' \coloneqq \{(\mathbf{x}, 0, \ldots, 0) \in \mathbb{R}^n : \mathbf{x} \in P\}$. We set $P_n$ to be the pyramid

$$P_n \coloneqq \mathrm{Conv}\left(P' \cup \{\mathbf{e}_3, \ldots, \mathbf{e}_n\}\right).$$

Note that, by Proposition 3.1,

$$\mathrm{ehr}_{S_n}(x) \equiv -\mathrm{ehr}_{P_n}(x). \tag{4}$$

Let $W_n \subseteq \mathbb{R}^n$ be the translated prism over the simplex $S_n$ defined by

$$W_n \coloneqq ([-q, q] \times S_n) - q\mathbf{e}_2,$$

where $[-q, q] \subseteq \mathbb{R}$ is a closed segment. (The reason for the translation by $-q\mathbf{e}_2$ is that it will make the convex hull below easy to analyze.) From the construction of $W_n$, it follows that

$$\mathrm{ehr}_{W_n}(x) = (2qx + 1)\,\mathrm{ehr}_{S_n}(x).$$

We can now construct a convex polytope $H_n \subseteq \mathbb{R}^n$ which, we will show, has the period sequence $(1, p, 1, \ldots, 1)$. Let

$$H_n \coloneqq \mathrm{Conv}\left(W_n \cup P_n\right).$$

(See Figure 1 for the case where $n = 3$ and $p = 2$ case.) That $H_n$ has the desired period sequence is a direct consequence of the following lemma.

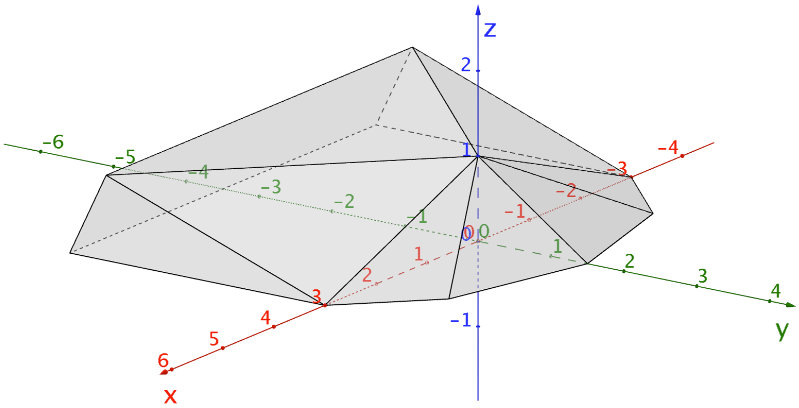


Figure 1: The polytope $H_3$ in the case where $p = 2$.

**Lemma 3.2.** *The polytope $H_n$ defined above is a union of the form*

$$H_n = W_n \cup M_n \cup P_n,$$

*where $W_n \cap M_n$, $M_n$, and $M_n \cap P_n$ are lattice polytopes.*

*Proof.* Note that $W_n$, respectively $P_n$, has a facet perpendicular to the $\mathbf{e}_2$-axis. Write $F_W$, respectively $F_P$, for this facet. That is,

$$\begin{aligned} F_W &= \text{Conv}\left\{\pm q\mathbf{e}_1, \pm q\mathbf{e}_1 + \mathbf{e}_3, \ldots, \pm q\mathbf{e}_1 + \mathbf{e}_n\right\} - q\mathbf{e}_2, \\ F_P &= \text{Conv}\left\{\pm q\mathbf{e}_1, \mathbf{e}_3, \ldots, \mathbf{e}_n\right\}. \end{aligned}$$

Let $M_n := \text{Conv}(F_W \cup F_P)$. To prove that $H_n = W_n \cup M_n \cup P_n$, it suffices to prove the following two statements:

1. For each facet $F \neq F_W$ of $W_n$, $P_n$ lies on the same side of the hyperplane supporting $F$ as $W_n$ does.

2. For each facet $F \neq F_P$ of $P_n$, $W_n$ lies on the same side of the hyperplane supporting $F$ as $P_n$ does.

In other words, excepting $F_W$ and $F_P$, no facet of $W_n$ is visible from a vertex of $P_n$ and vice versa [9, Section 22.3.1].

To prove statement (1), recall that $W_n$ is a prism over a simplex. From this, the required facet-defining inequalities are easily determined and shown to be satisfied by the vertices of $P_n$. To prove statement (2), note that $P_n$ is a pyramid over $P_{n-1}$. Hence, every facet of $P_n$ is either the "base" copy of $P_{n-1}$ or a pyramid over a facet of $P_{n-1}$ with apex $\mathbf{e}_n$. Thus, the required facet-defining inequalities are again easily determined by induction and shown to be satisfied by the vertices of $W_n$. □

It is now straightforward to complete the proof of Theorem 1.3. In particular, we prove the following:

**Theorem 3.3.** *Let a positive integer $p$ be given. Then the convex rational polytope $H_n \subseteq \mathbb{R}^n$ constructed above has period sequence $(1, p, 1, \dots, 1)$.*

*Proof.* Apply equivalence (4) to compute that $\operatorname{ehr}_{H_n}(x)$ satisfies

$$\operatorname{ehr}_{H_n}(x) \equiv \operatorname{ehr}_{W_n}(x) + \operatorname{ehr}_{P_n}(x) \equiv (2qx+1)\operatorname{ehr}_{S_n}(x) - \operatorname{ehr}_{S_n}(x) = 2qx \operatorname{ehr}_{S_n}(x).$$

As noted at the beginning of this section, $S_n$ has period sequence $(p, 1, \dots, 1)$. Therefore, $H_n$ has period sequence $(1, p, 1, \dots, 1)$, as desired. □

## 4 Nonconvex polytopes with period sequence $(1, \dots, 1, p, 1)$

We again fix a positive integer $p \geqslant 1$. The main result of this section is the proof of Theorem 1.4. In particular, we construct $n$-dimensional *non*convex rational polytopes with period sequence $(1, \dots, 1, p, 1)$, when $3 \leqslant n \leqslant 11$ or $n = 13$.

The reason for the constraint on the dimension $n$ in Theorem 1.4 is that our construction depends upon the existence of a solution to a particular system of Diophantine equations in $n-1$ variables, namely, the so-called *ideal Prouhet–Tarry–Escott* (PTE) problem. More precisely, we require integers $s_1, \dots, s_{n-1} > 0$ and $t_1, \dots, t_{n-2} > t_{n-1} = 0$ such that

$$p_k(s_1, \dots, s_{n-1}) = p_k(t_1, \dots, t_{n-1}) \qquad \text{for } 0 \leqslant k \leqslant n-2, \tag{5}$$

where $p_k(\mathbf{x})$ is the power-sum symmetric function of degree $k$ in $n-1$ variables.

Such solutions to system (5) are known to exist when the number of variables is between 2 and 10 (inclusive) or is 12 [5, Chapter 11][4]. No solution in 11 variables is known. Wright [24] conjectures that solutions exist for every number of variables $\geqslant 2$. However, Borwein [5, p. 87] gives a heuristic argument suggesting that this would be surprising.

Theorem 1.4 above is a corollary of the following theorem, which constructs a nonconvex polytope with period sequence $(1, \dots, 1, p, 1)$ in every dimension $n$ such that a suitable solution to system (5) exists.

**Theorem 4.1.** *Let a positive integer $p$ be given. Let $n \geqslant 3$ be such that there are integers $s_1, \dots, s_{n-1} > 0$ and $t_1, \dots, t_{n-2} > t_{n-1} = 0$ solving system (5) above. Then there exists an $n$-dimensional polytope $B_n \subseteq \mathbb{R}^n$ with period sequence $(1, \dots, 1, p, 1)$.*

[4] An integer solution to system (5) remains a solution after a constant integer is added to all of the values $s_i$, $t_j$. Therefore, it suffices to find an integer solution to (5) in which the minimum value among the $s_i$, $t_j$ appears only once. This condition is satisfied by the solutions listed on [5, p. 87] to the PTE problem.

*Proof.* Let $\ell = [-\frac{1}{p}, 0] \subseteq \mathbb{R}$, and let $P \subseteq \mathbb{R}^2$ be the pentagon defined in Section 2. Let $B_n \subseteq \mathbb{R}^n$ be the rational polytope defined as follows:

$$B_n \quad := \quad \left( \bigtimes_{i=1}^{n-1} [0, s_i] \right) \times \ell \quad \cup \quad \left( \bigtimes_{j=1}^{n-2} [0, t_j] \right) \times P.$$

Hence, $B_n$ is a union of two rational polytopes whose intersection is an integer polytope. Thus,

$$\operatorname{ehr}_{B_n}(x) \equiv \left( \prod_{i=1}^{n-1} (s_i x + 1) \right) \operatorname{ehr}_\ell(x) + \left( \prod_{j=1}^{n-2} (t_j x + 1) \right) \operatorname{ehr}_P(x).$$

Since $\operatorname{ehr}_P(x) \equiv -\operatorname{ehr}_\ell(x)$, it follows that

$$\operatorname{ehr}_{B_n}(x) \equiv \left( \prod_{i=1}^{n-1} (s_i x + 1) - \prod_{j=1}^{n-2} (t_j x + 1) \right) \operatorname{ehr}_\ell(x).$$

We now exploit the fact that the $s_i$ and $t_j$ solve system (5). Newton's identities relating the power-sum symmetric functions to the elementary symmetric functions imply that the $s_i$ and $t_j$ also solve the system

$$e_k(s_1, \ldots, s_{n-1}) = e_k(t_1, \ldots, t_{n-1}) \qquad \text{for } 0 \leqslant k \leqslant n-2,$$

where $e_k(\mathbf{x})$ is the elementary symmetric function of degree $k$ in $n-1$ variables. Hence,

$$\prod_{i=1}^{n-1} (s_i x + 1) - \prod_{j=1}^{n-2} (t_j x + 1) = s_1 \cdots s_{n-1} x^{n-1}.$$

Therefore, $\operatorname{ehr}_{B_n}(x) \equiv s_1 \cdots s_{n-1} x^{n-1} \operatorname{ehr}_\ell(x) \equiv s_1 \cdots s_{n-1} c_{\ell,0}(x) x^{n-1}$. That is, all coefficient functions of $\operatorname{ehr}_{B_n}(x)$ are constant except for the coefficient of $x^{n-1}$, which has period $p$, as desired. □

## References


[1] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, and R. P. Stanley, *Coefficients and roots of Ehrhart polynomials*, Integer points in polyhedra—geometry, number theory, algebra, optimization, Contemp. Math., vol. 374, Amer. Math. Soc., Providence, RI, 2005, pp. 15–36, arXiv:math/0402148.

[2] M. Beck and S. Robins, *Computing the continuous discretely*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Integer-point enumeration in polyhedra.

[3] M. Beck, S. V. Sam, and K. M. Woods, *Maximal periods of (Ehrhart) quasi-polynomials*, J. Combin. Theory Ser. A **115** (2008), no. 3, 517–525, arXiv:math/0702242.

[4] U. Betke and P. McMullen, *Lattice points in lattice polytopes*, Monatsh. Math. **99** (1985), no. 4, 253–265.

[5] P. Borwein, *Computational excursions in analysis and number theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 10, Springer-Verlag, New York, 2002.

[6] B. Braun, *Norm bounds for Ehrhart polynomial roots*, Discrete Comput. Geom. **39** (2008), no. 1-3, 191–193, arXiv:math/0602464.

[7] F. Breuer, *Ehrhart $f^*$-coefficients of polytopal complexes are non-negative integers*, Electron. J. Combin. **19** (2012), no. 4, Paper 16, 22, arXiv:1202.2652.

[8] E. Ehrhart, *Sur les polyèdres rationnels homothétiques à $n$ dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618.

[9] J. E. Goodman and J. O'Rourke (eds.), *Handbook of discrete and computational geometry*, second ed., Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2004.

[10] C. Haase, B. Nill, and S. Payne, *Cayley decompositions of lattice polytopes and upper bounds for $h^*$-polynomials*, J. Reine Angew. Math. **637** (2009), 207–216, arXiv:0804.3667.

[11] M. Henk and M. Tagami, *Lower bounds on the coefficients of Ehrhart polynomials*, European J. Combin. **30** (2009), no. 1, 70–83, arXiv:0710.2665.

[12] A. J. Herrmann, *Classification of Ehrhart quasi-polynomials of half-integral polygons*, Master's thesis, San Francisco State University, August 2010.

[13] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw Publications, Glebe, N.S.W., Australia, 1992.

[14] ———, *A lower bound theorem for Ehrhart polynomials of convex polytopes*, Adv. Math. **105** (1994), no. 2, 162–165.

[15] T. B. McAllister and M. Moriarity, *Ehrhart quasi-period collapse in rational polygons*, J. Combin. Theory Ser. A **150** (2017), 377–385, arXiv:1509.03680.

[16] P. McMullen, *Lattice invariant valuations on rational polytopes*, Arch. Math. (Basel) **31** (1978/79), no. 5, 509–516.

[17] J. Pfeifle, *Gale duality bounds for roots of polynomials with nonnegative coefficients*, J. Combin. Theory Ser. A **117** (2010), no. 3, 248–271, arXiv:0707.3010.

[18] P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc. **15** (1976), no. 3, 395–399.

[19] R. P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342, Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978).

[20] ———, *On the Hilbert function of a graded Cohen-Macaulay domain*, J. Pure Appl. Algebra **73** (1991), no. 3, 307–314.

[21] ———, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[22] A. Stapledon, *Inequalities and Ehrhart $\delta$-vectors*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5615–5626, arXiv:0801.0873.

[23] A. Stapledon, *Additive number theory and inequalities in Ehrhart theory*, Int. Math. Res. Not. IMRN (2016), no. 5, 1497–1540, arXiv:0904.3035.

[24] E. M. Wright, *An Easier Waring's Problem*, J. London Math. Soc. **S1-9** (1934), no. 4, 267.