Connected order ideals and $\boldsymbol{P}$-partitions

Ben P. Zhou*

Center for Combinatorics, LPMC
Nankai University
Tianjin 300071, P.R. China

benpzhou@163.com

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Abstract

Given a finite poset P, we associate a simple graph denoted by G_P with all connected order ideals of P as vertices, and two vertices are adjacent if and only if they have nonempty intersection and are incomparable with respect to set inclusion. We establish a bijection between the set of maximum independent sets of G_P and the set of P-forests, introduced by Féray and Reiner in their study of the fundamental generating function $F_P(\mathbf{x})$ associated with P-partitions. Based on this bijection, in the cases when P is naturally labeled we show that $F_P(\mathbf{x})$ can factorise, such that each factor is a summation of rational functions determined by maximum independent sets of a connected component of G_P. This approach enables us to give an alternative proof for Féray and Reiner's nice formula of $F_P(\mathbf{x})$ for the case of P being a naturally labeled forest with duplications. Another consequence of our result is a product formula to compute the number of linear extensions of P.

Keywords: P-partition; P-forest; linear extension; connected order ideal; maximum independent set

1 Introduction

Throughout this paper, we shall assume that P is a poset on $\{1, 2, \ldots, n\}$. We use $\leqslant_P$ to denote the order relation on P to distinguish from the natural order $\leqslant$ on integers. We say that P is naturally labeled if $i < j$ whenever $i <_P j$. A P-partition is a map f from P to the set $\mathbb{N}$ of nonnegative integers such that

(1) if $i <_P j$, then $f(i) \geqslant f(j)$;

(2) if $i <_P j$ and $i > j$, then $f(i) > f(j)$.

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For more information on P-partitions, we refer the reader to the book [9] of Stanley or the recent survey paper [5] of Gessel. Let $\mathscr{A}(P)$ denote the set of P-partitions. The fundamental generating function $F_P(\mathbf{x})$ associated with P-partitions is defined as

$$F_P(\mathbf{x}) = \sum_{f\in\mathscr{A}(P)} \mathbf{x}^f = \sum_{f\in\mathscr{A}(P)} x_1^{f(1)} x_2^{f(2)} \cdots x_n^{f(n)}.$$

One of the most important problems in the theory of P-partitions is to determine explicit expressions for $F_P(\mathbf{x})$. The main objective of this paper is to show that for any naturally labeled finite poset P, the generating function $F_P(\mathbf{x})$ can factorise.

Let us first review some background. The first explicit expression for $F_P(\mathbf{x})$ was given by Stanley [8]. Recall that a linear extension of P is a permutation $w = w_1w_2\cdots w_n$ on $\{1,2,\ldots,n\}$ such that $i<j$ whenever $w_i <_P w_j$. Let $\mathcal{L}(P)$ be the set of linear extensions of P. For a permutation w, write

$$\mathrm{Des}(w) = \{i \,|\, 1 \leqslant i \leqslant n-1, w_i > w_{i+1}\}$$

for the descent set of w. Stanley [8] showed that

$$F_P(\mathbf{x}) = \sum_{w\in\mathcal{L}(P)} \frac{\prod_{i\in\mathrm{Des}(w)} x_{w_1}x_{w_2}\cdots x_{w_i}}{\prod_{j=1}^{n}\left(1 - x_{w_1}x_{w_2}\cdots x_{w_j}\right)}. \tag{1}$$

Boussicault, Féray, Lascoux and Reiner [2] obtained a similar formula for $F_P(\mathbf{x})$ when P is a forest, namely, every element of P is covered by at most one other element. We say that j is the parent of i, if i is covered by j in P. Björner and Wachs [1] defined the descent set of a forest P as

$$\mathrm{Des}(P) = \{i \,|\, \text{ if } j \text{ is the parent of } i, \text{ then } i > j\}. \tag{2}$$

Thus, if $i \in \mathrm{Des}(P)$, then there exists a node $j \in P$ such that $i <_P j$ but $i > j$. In particular, when a forest P is naturally labeled, the descent set $\mathrm{Des}(P)$ is empty. For a forest P, Boussicault, Féray, Lascoux, and Reiner's formula is stated as

$$F_P(\mathbf{x}) = \frac{\prod_{i\in\mathrm{Des}(P)}\prod_{k\leqslant_P i}\ x_k}{\prod_{j=1}^{n}\left(1 - \prod_{\ell\leqslant_P j}\ x_\ell\right)}. \tag{3}$$

Furthermore, Féray and Reiner [4] obtained a nice formula for $F_P(\mathbf{x})$ when P is a naturally labeled forest with duplications, whose definition is given below. Recall that an order ideal of P is a subset J such that if $i \in J$ and $j \leqslant_P i$, then $j \in J$. Throughout the rest of this paper, we will use J to represent an order ideal of P. An order ideal J is connected if the Hasse diagram of J is a connected graph. A poset P is called a forest with duplications if for any connected order ideal J_a of P, there exists at most one other connected order ideal J_b such that J_a and J_b intersect nontrivially, namely,

$$J_a \cap J_b \neq \varnothing, \quad J_a \not\subset J_b \quad \text{and} \quad J_b \not\subset J_a.$$

We would like to point out that a naturally labeled forest must be a naturally labeled forest with duplications, while the Hasse diagram of a naturally labeled forest with duplications needs not to be a forest. Let $\mathcal{J}_{conn}(P)$ be the set of connected order ideals of P. For a naturally labeled forest with duplications, Féray and Reiner [4] proved that

$$F_P(\mathbf{x}) = \frac{\prod_{\{J_a,J_b\}\in\Pi(P)} \left(1 - \prod_{i\in J_a} x_i \prod_{j\in J_b} x_j\right)}{\prod_{J\in\mathcal{J}_{conn}(P)} \left(1 - \prod_{k\in J} x_k\right)}, \tag{4}$$

where $\Pi(P)$ consists of all pairs $\{J_a, J_b\}$ of connected order ideals that intersect nontrivially. Note that when P is a naturally labeled forest (with no duplication), both $\mathrm{Des}(P)$ and $\Pi(P)$ are empty, and each connected order ideal J of P must equal to $\{\ell \mid \ell \leqslant_P j\}$ for some $j \in \{1, 2, \ldots, n\}$ and vice versa, and hence formula (4) coincides with formula (3) in this special case.

For any poset P, Féray and Reiner [4] introduced the notion of P-forests and obtained a decomposition of the set $\mathcal{L}(P)$ in terms of linear extensions of P-forests. Recall that a P-forest F is a forest on $\{1, 2, \ldots, n\}$ such that for any node i, the subtree rooted at i is a connected order ideal of P, and that for any two incomparable nodes i and j in the poset F, the union of the subtrees rooted at i and j is a disconnected order ideal of P. Let $\mathscr{F}(P)$ stand for the set of P-forests. For example, for the poset P in Figure 1 there are three P-forests F_1, F_2 and F_3.

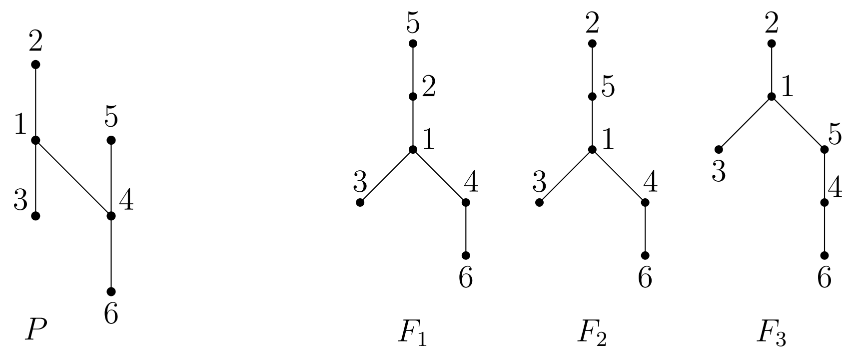


Figure 1: A poset P and the corresponding P-forests.

Féray and Reiner [4] showed that

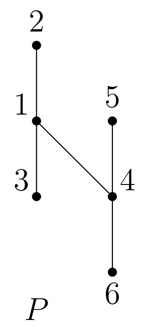
$$\mathcal{L}(P) = \biguplus_{F\in\mathscr{F}(P)} \mathcal{L}(F), \tag{5}$$

which was implied in [4, Proposition 11.7]. As was remarked by Féray and Reiner, the decomposition in (5) also appeared in the work of Postnikov [6] and Posnikov, Reiner and Williams [7]. Combining (1), (3) and (5), one readily sees that

$$F_P(\mathbf{x}) = \sum_{F\in\mathscr{F}(P)} \frac{\prod_{i\in \mathrm{Des}(F)} \prod_{k\leqslant_F i} x_k}{\prod_{j=1}^{n} \left(1 - \prod_{\ell\leqslant_F j} x_\ell\right)}. \tag{6}$$

Note that both (1) and (6) are summation formulas for $F_P(\mathbf{x})$. However, the expression of $F_P(\mathbf{x})$ factored nicely for certain posets, as shown in (3) and (4). Thus it is desirable to ask that for more general posets P whether $F_P(\mathbf{x})$ can factorise. In this paper, we show that $F_P(\mathbf{x})$ can factorise for any naturally labeled poset P.

Before stating our result, let us first introduce some definitions and notations. In the following we always assume that P is a poset on $\{1, 2, \ldots, n\}$. For any graph G, we use $V(G)$ to denote the set of vertices of G. We associate to P a simple graph denoted by G_P with the set $\mathcal{J}_{conn}(P)$ of connected order ideals of P as $V(G_P)$, and two vertices are adjacent if they intersect nontrivially. For example, if P is the poset given in Figure 1, then G_P is as illustrated in Figure 2, where we use $\Lambda_i^P = \{k \mid k \leqslant_P i\}$ to denote the principal order ideal of P generated by i, and adopt the notation $\Lambda_{i,j}^P = \Lambda_i^P \cup \Lambda_j^P$.



$$\begin{aligned}
\Lambda_1^P &= \{\mathbf{1}, 3, 4, 6\}\\
\Lambda_2^P &= \{1, \mathbf{2}, 3, 4, 6\}\\
\Lambda_3^P &= \{\mathbf{3}\}\\
\Lambda_4^P &= \{\mathbf{4}, 6\}\\
\Lambda_5^P &= \{4, \mathbf{5}, 6\}\\
\Lambda_6^P &= \{\mathbf{6}\}\\
\Lambda_{1,5}^P &= \{\mathbf{1}, 3, 4, \mathbf{5}, 6\}\\
\Lambda_{2,5}^P &= \{1, \mathbf{2}, 3, 4, \mathbf{5}, 6\}
\end{aligned}$$

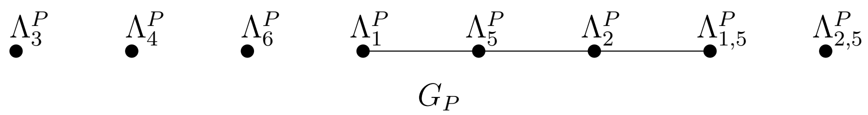


Figure 2: Connected order ideals of P and the graph G_P.

The first result of this paper is a bijection between the set of P-forests and the set of maximum independent sets of G_P. Recall that an independent set of a graph is a subset of vertices such that no two vertices of the subset are adjacent. A maximum independent set of a graph is an independent set that of largest possible size. For any graph G, we use $\mathscr{M}(G)$ to denote the set of maximum independent sets of G. We have the following result.

Theorem 1. *There exists a bijection between the set $\mathscr{F}(P)$ of P-forests and the set $\mathscr{M}(G_P)$ of maximum independent sets of G_P.*

The proof of this result will be given in Section 2, where we establish a bijection Φ from $\mathscr{F}(P)$ to $\mathscr{M}(G_P)$. Let Ψ be the inverse map of Φ. In view of the fact that $\Psi(M)$ is a forest, for a maximum independent set M of G_P, we can define the descent set $\mathrm{Des}(M)$ of M as the descent set $\mathrm{Des}(\Psi(M))$, namely,

$$\mathrm{Des}(M) = \mathrm{Des}(\Psi(M)), \tag{7}$$

where $\mathrm{Des}(\Psi(M))$ is given by (2). Suppose the graph G_P has h connected components, say $C_1, C_2, \ldots, C_h$. As usual, we use $V(C_r)$ to denote the vertex set of C_r for $1 \leqslant r \leqslant h$, respectively. It is clear that each maximum independent set of G_P is a disjoint union of maximum independent sets of G_P's connected components. Let $\mathscr{M}(C_r)$ denote the set of maximum independent sets of C_r for each $1 \leqslant r \leqslant h$, respectively. Given a $M_r \in \mathscr{M}(C_r)$, we shall further define a descent set for M_r as illustrated below. Let M be a maximum independent set of G_P such that $M \cap V(C_r) = M_r$. For any $J \in M$, let

$$\mu(M, J) = \bigcup_{J' \in M,\ J' \subset J} J'. \tag{8}$$

Define $\mathrm{Des}(M_r, M)$ and $\overline{\mathrm{Des}}(M_r, M)$ as

$$\begin{aligned} \mathrm{Des}(M_r, M) &= \big\{ i \in \mathrm{Des}(M) \mid \{i\} = J \setminus \mu(M, J) \text{ for some } J \in M_r \big\}, \\ \overline{\mathrm{Des}}(M_r, M) &= \big\{ J \in M_r \mid J \setminus \mu(M, J) = \{i\} \text{ for some } i \in \mathrm{Des}(M_r, M) \big\}. \end{aligned}$$

It is remarkable that $\mathrm{Des}(M_r, M)$ and $\overline{\mathrm{Des}}(M_r, M)$ are irrelevant to the choice of M when the poset P is naturally labeled. Precisely, we have the following result.

Theorem 2. *Suppose that P is a naturally labeled poset and G_P has connected components $C_1, C_2, \ldots, C_h$. Let M_r be a maximum independent set of C_r for some $1 \leqslant r \leqslant h$. Then for any two maximum independent sets M^1, M^2 of G_P satisfying $M^1 \cap V(C_r) = M^2 \cap V(C_r) = M_r$, we have*

$$\begin{aligned} \mathrm{Des}(M_r, M^1) &= \mathrm{Des}(M_r, M^2), \\ \overline{\mathrm{Des}}(M_r, M^1) &= \overline{\mathrm{Des}}(M_r, M^2). \end{aligned} \tag{9}$$

Therefore, for a naturally labeled poset P and a given $M_r \in \mathscr{M}(C_r)$, we can introduce the notation of $\mathrm{Des}(M_r)$ and $\overline{\mathrm{Des}}(M_r)$, which are respectively defined by

$$\begin{aligned} \mathrm{Des}(M_r) &= \mathrm{Des}(M_r, M), \\ \overline{\mathrm{Des}}(M_r) &= \overline{\mathrm{Des}}(M_r, M), \end{aligned} \tag{10}$$

where M is some maximum independent set of G_P such that $M \cap V(C_r) = M_r$.

The main result of this paper is as follows.

Theorem 3. *If P is a naturally labeled poset, and the graph G_P has h connected components C_1, C_2,...,C_h. Then we have*

$$F_P(\mathbf{x}) = \prod_{r=1}^{h} \sum_{M_r \in \mathscr{M}(C_r)} \frac{\prod_{J \in \overline{\mathrm{Des}}(M_r)} \prod_{k \in J} x_k}{\prod_{J \in M_r} (1 - \prod_{j \in J} x_j)}. \tag{11}$$

This paper is organized as follows. In Section 2, we shall give a proof of Theorem 1. In Section 3, we shall prove Theorems 2 and 3. Based on Theorem 3, we provide an alternative proof for Féray and Reiner's formula (4). In Section 4, Theorem 3 will be used to derive the generating function of major index of linear extensions of P, as well as to count the number of linear extensions of P.

2 The bijection Φ between $\mathscr{F}(P)$ and $\mathscr{M}(G_P)$

The aim of this section is to give a proof of Theorem 1. To this end, we shall establish a bijection Φ from $\mathscr{F}(P)$ to $\mathscr{M}(G_P)$ as mentioned before.

To give a description of the map Φ, we first note some properties of $\mathscr{F}(P)$ and $\mathscr{M}(G_P)$. Given $M \in \mathscr{M}(G_P)$ and $J \in M$, let

$$\begin{aligned} U(M,J) &= \{J' \in M \mid J' \subset J\}, \\ U_{max}(M,J) &= \big\{J_a \in U(M,J) \mid J_a \not\subset J_b \text{ for any } J_b \in U(M,J)\big\}. \end{aligned} \tag{12}$$

Recall that the set $\mu(M,J)$ is defined in (8), which is also an order ideal of P. Thus

$$\mu(M,J) = \bigcup_{J' \in U(M,J)} J' = \bigcup_{J' \in U_{max}(M,J)} J'. \tag{13}$$

The following assertion will be used in the future proofs.

Lemma 4. *For any $M \in \mathscr{M}(G_P)$ and $J \in M$, the intersection of any two elements of $U_{max}(M,J)$ is empty.*

Proof. Let $J_1, J_2 \in U_{max}(M,J)$. Because $U_{max}(M,J) \subset M$ and M is an independent set of G_P, it follows that J_1 and J_2 are not adjacent in G_P. Recall that for any two vertices $J_1, J_2 \in \mathscr{J}_{conn}(P)$ of G_P, J_1 and J_2 are not adjacent in G_P if and only if

$$J_1 \cap J_2 = \varnothing, \quad \text{or } J_1 \subset J_2, \quad \text{or } J_2 \subset J_1.$$

On the other hand, by the definition of $U_{max}(M,J)$, there is neither $J_1 \subset J_2$ nor $J_2 \subset J_1$. Hence $J_a \cap J_b = \varnothing$. □

Given a P-forest $F \in \mathscr{F}(P)$, let $\Lambda_i^F = \{j \mid j \leqslant_F i\}$ denote the principal order ideal of F generated by i. By definition of P-forest, each Λ_i^F is a connected order ideal of P, although Λ_i^F is not necessarily a principal order ideal of P. Then by the definition of G_P, each Λ_i^F is a vertex of G_P. Moreover, we have the following result.

Lemma 5. *For any P-forest $F \in \mathscr{F}(P)$, the principal order ideals $\Lambda_1^F, \Lambda_2^F, \ldots, \Lambda_n^F$ form a maximum independent set of G_P.*

Proof. We first show that $\{\Lambda_1^F, \Lambda_2^F, \ldots, \Lambda_n^F\}$ is an independent set of G_P, that is, for any two nodes i, j of F, the principal order ideals Λ_i^F and Λ_j^F are not adjacent in G_P. There are two cases to consider.

(1) The vertices i and j are incomparable in F. Since F is a forest, it is clear that $\Lambda_i^F \cap \Lambda_j^F = \varnothing$. This implies that Λ_i^F and Λ_j^F are not adjacent in G_P.

(2) The vertices i and j are comparable in F. If $i <_F j$, then $\Lambda_i^F \subset \Lambda_j^F$; If $j <_F i$, then $\Lambda_j^F \subset \Lambda_i^F$. In both circumstances, Λ_i^F and Λ_j^F are not adjacent in G_P.

We proceed to show that the independent set $\{\Lambda_1^F, \Lambda_2^F, \ldots, \Lambda_n^F\}$ is of the largest possible size. To this end, it is enough to verify that $|M| \leqslant n$ for any independent set M of G_P. Assume that $M = \{J_1, J_2, \ldots, J_k\}$ is an independent set of G_P, which means that J_i is a connected order ideal of P, and J_i, J_j are not adjacent in G_P for any $1 \leqslant i < j \leqslant k$. We further assume that the subscript satisfies $r < s$ whenever $J_r \subset J_s$. In fact, this can be achieved as follows. Consider M as a poset ordered by set inclusion. Then choose a subscript such that $J_1 J_2 \cdots J_k$ is a linear extension of M. Such a subscript satisfies the condition that $r < s$ whenever $J_r \subset J_s$.

For $1 \leqslant s \leqslant k$, let

$$I_s = \bigcup_{1 \leqslant r \leqslant s} J_r.$$

It is clear that $I_{s-1} \subseteq I_s$ for any $1 < s \leqslant k$. We claim that

$$\varnothing \neq I_1 \subset I_2 \subset \cdots \subset I_k \subseteq \{1, 2, \ldots, n\}, \tag{14}$$

which implies that $|M| = k \leqslant n$.

Suppose to the contrary that $I_s = I_{s-1}$ for some $1 < s \leqslant k$. Thus,

$$J_s \subseteq I_s = I_{s-1} = \bigcup_{1 \leqslant r \leqslant s-1} J_r. \tag{15}$$

The set $U(M, J_s)$ is defined as

$$U(M, J_s) = \{J' \mid J' \in M, J \subset J_s\} = \{J_r \mid 1 \leqslant r \leqslant s-1,\ J_r \subset J_s\}.$$

Clearly,

$$\mu(M, J_s) = \bigcup_{J' \in U(M, J_s)} J' \subseteq J_s. \tag{16}$$

Notice that for any $1 \leqslant r \leqslant s-1$, if J_r does not belong to $U(M, J_s)$, then $J_r \cap J_s = \varnothing$, since otherwise J_r and J_s intersect nontrivially, contradicting the assumption that M is an independent set of G_P. In view of relation (15), we have

$$J_s \subseteq \bigcup_{J' \in U(M, J_s)} J' = \mu(M, J_s),$$

which together with (13) and (16), leads to

$$J_s = \mu(M, J_s) = \bigcup_{J' \in U_{max}(M, J_s)} J'.$$

If $U_{max}(M, J_s)$ has only one element, say, $U_{max}(M, J_s) = \{J_r\}$ for some $1 \leqslant r \leqslant s-1$, then $J_s = J_r$, which is contrary to $J_r \subset J_s$. Next we may assume that $U_{max}(M, J_s)$ has more than one element. By Lemma 4, the intersection of any two elements of $U_{max}(M, J_s)$ is empty. Thus J_s is the union of some (at least two) nonintersecting connected order ideals, which can not be connected. This contradicts the fact that J_s is a connected order ideal. It follows that $I_{s-1} \subset I_s$ for each $1 < s \leqslant k$, as desired. □

By the above lemma, we can define a map $\Phi:\ \mathscr{F}(P) \longrightarrow \mathscr{M}(G_P)$ by letting

$$\Phi(F) = \{\Lambda_1^F, \Lambda_2^F, \ldots, \Lambda_n^F\}$$

for any $F \in \mathscr{F}(P)$. In order to show that Φ is a bijection, we shall construct the inverse map of Φ, denoted by Ψ. To give a description of Ψ, we need the following lemma.

Lemma 6. *Given $M \in \mathscr{M}(G_P)$ and $J \in M$, there exists a unique j such that*

$$J \setminus \mu(M, J) = \{j\}, \tag{17}$$

where $\mu(M,J)$ is given in (8). Moreover, j is a maximal element of J with respect to the order $\leqslant_P$, and

$$J_r \setminus \mu(M, J_r) \neq J_s \setminus \mu(M, J_s) \tag{18}$$

for any distinct $J_r, J_s \in M$.

Proof. By Lemma 5, we see that each maximum independent set of G_P should contain n vertices. Suppose that $M = \{J_1, J_2, \ldots, J_n\}$. As in the proof of Lemma 5, we may assume that

$$r < s \text{ whenever } J_r \subset J_s. \tag{19}$$

For $1 \leqslant s \leqslant n$, let

$$I_s = \bigcup_{1 \leqslant r \leqslant s} J_r.$$

By (14), we see that

$$\varnothing \neq I_1 \subset I_2 \subset \cdots \subset I_n \subseteq \{1, 2, \ldots, n\}. \tag{20}$$

Therefore, if setting $I_0 = \varnothing$, we obtain that for $1 \leqslant s \leqslant n$,

$$|I_s \setminus I_{s-1}| = 1. \tag{21}$$

Let $J = J_s$ for some $1 \leqslant s \leqslant n$. In view of (8) and (19), we get that

$$\mu(M, J_s) = \bigcup_{J' \in M,\ J' \subset J_s} J' = \bigcup_{1 \leqslant r \leqslant s-1, J_r \subset J_s} J_r \subseteq I_{s-1}.$$

Thus we have

$$J \setminus \mu(M, J) = J_s \setminus \mu(M, J_s) = J_s \setminus I_{s-1} = I_s \setminus I_{s-1}, \tag{22}$$

where the second equality follows from the fact that for any $1 \leqslant r \leqslant s-1$, either $J_r \subset J_s$ or $J_r \cap J_s = \varnothing$. In view of (21) and (22), we arrive at (17) and (18).

It remains to show that the unique element j of $J_s \setminus \mu(M, J_s)$ is a maximal element of J_s with respect to the order $\leqslant_P$. Suppose that j is not a maximal element of J_s. Then there exists a maximal element i of J_s such that $j <_P i$. By (17) and $j \neq i$, we see that $i \in \mu(M, J_s)$. Therefore, there exists some $J' \subset J_s$ of and $J' \in M$ such that $i \in J'$. Since J' is an order ideal of P, we get $j \in J' \subseteq \mu(M, J_s)$, contradicting with the fact $j \notin \mu(M, J_s)$. □

For any $M \in \mathscr{M}(G_P)$, it follows from (17) and (18) that

$$\{1, 2, \ldots, n\} = \biguplus_{J \in M} J \setminus \mu(M, J).$$

Let F_M be the poset on $\{1, 2, \ldots, n\}$ such that $i <_{F_M} j$ if and only if $J_a \subset J_b$, where J_a and J_b are the two connected order ideals in M satisfies $J_a \setminus \mu(M, J_a) = \{i\}$, $J_b \setminus \mu(M, J_b) = \{j\}$. The following result show an important property for principal order ideals of the poset F_M.

Lemma 7. *Given $M \in \mathscr{M}(G_P)$, let F_M be the poset defined as above. Then for any $1 \leqslant j \leqslant n$ we have $\Lambda_j^{F_M} = \{i \mid i \leqslant_{F_M} j\} = J$, where $J \in M$ satisfying $J \setminus \mu(M, J) = \{j\}$ as in Lemma 6.*

Proof. We use the principle of Noetherian induction.

If j is a minimal element of F_M with respect to the order $\leqslant_{F_M}$, then J is also a minimal element of M when M is regarded as a poset ordered by set inclusion. Hence $\Lambda_j^{F_M} = \{j\}$ and there exists no $J' \in M$ such that $J' \subset J$, which yields that $\mu(M, J) = \varnothing$. So $J = \{j\} \cup \mu(M, J) = \{j\}$, and then $\Lambda_j^{F_M} = J$.

Suppose that j is not a minimal element of F_M (with respect to the order $\leqslant_{F_M}$) and $\Lambda_i^{F_M} = J'$ holds for any $i <_{F_M} j$, where $J' \setminus \mu(M, J') = \{i\}$. The construction of F_M tells us that $i <_{F_M} j$ if and only if $J' \subset J$. Since $\Lambda_i^{F_M} \subset \Lambda_j^{F_M}$ holds for each $i <_{F_M} j$, we have

$$\Lambda_j^{F_M} = \{i \mid i \leqslant_{F_M} j\} = \{j\} \cup \left(\bigcup_{i <_{F_M} j} \Lambda_i^{F_M} \right).$$

Then by the induction hypothesis, we get that

$$\Lambda_j^{F_M} = \{j\} \cup \left(\bigcup_{J' \in M,\ J' \subset J} J' \right) = \{j\} \cup \mu(M, J) = J. \qquad \square$$

We proceed to examine more structure of F_M, and obtain the following result.

Lemma 8. *For any $M \in \mathscr{M}(G_P)$, the poset F_M is a P-forest.*

Proof. We first show that F_M is a forest. Suppose otherwise that F_M is not a forest. Then there exists an element i in F_M such that i is covered by at least two elements of F_M, say j, k. Thus j and k must be incomparable with respect to the order $\leqslant_{F_M}$. (Recall that in a poset P, we say that an element u is covered by an element v if $u <_P v$ and there is no element w such that $u <_P w <_P v$.) By Lemma 6, there exist $J_a, J_b, J_c \in M$ such that $J_a \setminus \mu(M, J_a) = \{i\}$, $J_b \setminus \mu(M, J_b) = \{j\}$ and $J_c \setminus \mu(M, J_c) = \{k\}$. By the construction of F_M, we see that $J_a \subset J_b$, $J_a \subset J_c$ and J_b, J_c are incomparable in M with respect to the set inclusion order. Hence, $J_b \not\subset J_c$, $J_c \not\subset J_b$ and $(J_b \cap J_c) \supseteq J_a \neq \varnothing$. This implies that J_b and J_c are adjacent in the graph G_P, contradicting the fact that M is an independent set.

We proceed to show that F_M is a P-forest. By Lemma 7, for each element i of F_M, the subtree $\Lambda_i^{F_M} = \{j \mid j \leqslant_{F_M} i\}$ of F_M rooted at i is a connected order ideal of P. To verify that F_M is a P-forest, we still need to check that for $1 \leqslant i, j \leqslant n$, if i and j are incomparable in F_M, then the union $\Lambda_i^{F_M} \cup \Lambda_j^{F_M}$ is a disconnected order ideal of P. By Lemma 6, assume that J_a and J_b are the connected order ideals in M such that $J_a \setminus \mu(M, J_a) = \{i\}$ and $J_b \setminus \mu(M, J_b) = \{j\}$. By Lemma 7, we have $J_a = \Lambda_i^{F_M}$ and $J_b = \Lambda_j^{F_M}$. Since i and j are incomparable in F_M, we obtain that $J_a \not\subset J_b$ and $J_b \not\subset J_a$. On the other hand, J_a and J_b are not adjacent in the graph G_P. This allows us to conclude that $J_a \cap J_b = \varnothing$. Therefore, as an order ideal of P, the union $J_a \cup J_b$ is disconnected, so is the union $\Lambda_i^{F_M} \cup \Lambda_j^{F_M}$. Hence F_M is a P-forest. □

With the above lemma, we can define the inverse map of Φ, denoted by $\Psi : \mathscr{M}(G_P) \to \mathscr{F}(P)$, by letting

$$\Psi(M) = F_M$$

for any $M \in \mathscr{M}(G_P)$.

Now we are in a position to give a proof of Theorem 1.

Proof of Theorem 1. We first prove that $\Psi(\Phi(F)) = F$ for any P-forest F and $\Phi(\Psi(M)) = M$ for any maximum independent set M of G_P. The proof of the former statement will be given below, and the proof of the latter will be omitted here. Given a P-forest F, by definition, the image of F under the map Φ is $\Phi(F) = \{\Lambda_1^F, \ldots, \Lambda_n^F\}$, which is a maximum independent set of G_P by Lemma 5. Of course, we have $\Lambda_i^F \subset \Lambda_j^F$ if and only if $i <_F j$. For each $1 \leqslant i \leqslant n$ let $J_i = \Lambda_i^F$ and then denote $M = \{J_1, J_2, \ldots, J_n\}$. We proceed to show that $\Psi(M) = F_M = F$. Note that both F_M and F are posets on $\{1, 2, \ldots, n\}$. It remains to show that $i <_{F_m} j$ if and only if $i <_F j$ for any $i, j \in \{1, 2, \ldots, n\}$. Recall that for $1 \leqslant i \leqslant n$ the principal order ideal Λ_i^F is the subtree of F rooted at i. Hence

$$J_i \setminus \mu(M, J_i) = \Lambda_i^F \setminus \left(\bigcup_{j <_F i} \Lambda_j^F \right) = \{i\}$$

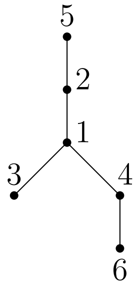
holds for each $1 \leqslant i \leqslant n$. By the construction of F_M, we know that $i <_{F_M} j$ if and only if $J_i \subset J_j$. On the other hand, in the given P-forest F, $i <_F j$ if and only if $\Lambda_i^F \subset \Lambda_j^F$. Since $J_i = \Lambda_i^F$ for each $1 \leqslant i \leqslant n$, it follows that $i <_{F_M} j$ if and only if $i <_F j$. Thus $F_M = F$, as desired.

Because $\Psi(\Phi(F)) = F$ for any P-forest F, the map Φ is one-to-one. Moreover, since the map Ψ is applicable to any maximum independent set M of G_P, the quality $\Phi(\Psi(M)) = M$ ensures that Φ is onto. Then Φ is bijective. □

We take the poset P in Figure 1 as an example to illustrate Theorem 1 and its proof. There are there P-forests F_1, F_2 and F_3 as shown in Figure 1. The graph G_P, as shown in Figure 2, has three maximum independent sets:

$$\begin{aligned} M^1 &= \{\Lambda_3^P, \Lambda_4^P, \Lambda_6^P, \Lambda_1^P, \Lambda_2^P, \Lambda_{2,5}^P\}, \\ M^2 &= \{\Lambda_3^P, \Lambda_4^P, \Lambda_6^P, \Lambda_1^P, \Lambda_{1,5}^P, \Lambda_{2,5}^P\}, \\ M^3 &= \{\Lambda_3^P, \Lambda_4^P, \Lambda_6^P, \Lambda_5^P, \Lambda_{1,5}^P, \Lambda_{2,5}^P\}. \end{aligned}$$

The principal order ideals of F_1 is as shown in Figure 3.



F_1

$$\begin{aligned} \Lambda_1^F &= \{1,3,4,6\} \\ \Lambda_2^F &= \{1,2,3,4,6\} \\ \Lambda_3^F &= \{3\} \\ \Lambda_4^F &= \{4,6\} \\ \Lambda_5^F &= \{1,2,3,4,5,6\} \\ \Lambda_6^F &= \{6\} \end{aligned}$$

Figure 3: The P-forest F_1 and its principal order ideals.

By the construction of Φ, we have

$$\begin{aligned} \Phi(F_1) &= \{\Lambda_1^{F_1}, \Lambda_2^{F_1}, \ldots, \Lambda_6^{F_1}\} \\ &= \big\{\{1,3,4,6\},\{1,2,3,4,6\},\{3\},\{4,6\},\{1,2,3,4,5,6\},\{6\}\big\}, \end{aligned}$$

which coincides with M^1. One can also verify that $\Phi(F_2) = M^2$ and $\Phi(F_3) = M^3$.

On the other hand, for the maximum independent set M^1, if we set $J_1 = \Lambda_1^P = \{1,3,4,6\}$, $J_2 = \Lambda_2^P = \{1,2,3,4,6\}$, $J_3 = \Lambda_3^P = \{3\}$, $J_4 = \Lambda_4^P = \{4,6\}$, $J_5 = \Lambda_{2,5}^P = \{1,2,3,4,5,6\}$, $J_6 = \Lambda_6^P = \{6\}$, then it is straightforward to verify that $J_i \backslash \mu(M^1, J_i) = \{i\}$ for $1 \leqslant i \leqslant 6$. And then, by definition, in the P-forest F_{M^1} there is $2 <_{F_{M^1}} 5$, $1 <_{F_{M^1}} 2$, $3 <_{F_{M^1}} 1$, $4 <_{F_{M^1}} 1$, $6 <_{F_{M^1}} 4$. One readily sees that $F_{M^1} = F_1$. Similarly, one can verify that $F_{M^2} = F_2$ and $F_{M^3} = F_3$.

3 $F_P(\mathrm{x})$ for naturally labeled P

The main objective of this section is to prove Theorems 2 and 3. The proofs are based on some properties of certain subgraphs of G_P. Although we require that the poset P in Theorems 2 and 3 be naturally labeled, these properties of G_P are valid for any finite poset P.

To begin with, let us first introduce some notations. For an order ideal J of P, let $gs(J)$ denote the set of maximal elements of J with respect to the order $\leqslant_P$, namely,

$$gs(J) = \{i \in J \mid \text{ there exists no } j \in J \text{ such that } i <_P j\}.$$

This set is also called the generating set of J. Clearly, when $gs(J) = \{i_1, i_2, \ldots, i_k\}$, we have $J = \Lambda_{i_1}^P \cup \Lambda_{i_2}^P \cup \cdots \cup \Lambda_{i_k}^P$. Let χ_J be the subgraph of G_P induced by the vertex subset $\{\Lambda_{i_1}^P, \Lambda_{i_2}^P, \ldots, \Lambda_{i_k}^P\}$. We have the following assertion.

Lemma 9. *For any connected order ideal J of P, the graph χ_J is connected.*

Proof. Assume that $gs(J) = \{i_1, i_2, \dots, i_k\}$. The proof is immediate if $k = 1$. In the following we shall assume that $k \geqslant 2$. Define

$$\mathrm{Conn}(i_1) = \big\{i_r \in gs(J) \,|\, \text{there is a path in } \chi_J \text{ connecting } \Lambda^P_{i_1} \text{ and } \Lambda^P_{i_r}\big\}.$$

Note that i_1 is always contained in $\mathrm{Conn}(i_1)$. It is enough to show that $\mathrm{Conn}(i_1) = gs(J)$. Otherwise, suppose that $\mathrm{Conn}(i_1) \neq gs(J)$. Let

$$I_1 = \bigcup_{j \in \mathrm{Conn}(i_1)} \Lambda^P_j \quad \text{and} \quad I_2 = \bigcup_{j \in gs(J) \setminus \mathrm{Conn}(i_1)} \Lambda^P_j.$$

Then both I_1 and I_2 are nonempty subsets of J satisfying that $I_1 \cup I_2 = J$, and both I_1 and I_2 are order ideals of P. Since J is a connected order ideal of P, it follows that $I_1 \cap I_2 \neq \varnothing$. Thus there exists some $u \in \mathrm{Conn}(i_1)$ and some $v \in gs(J) \setminus \mathrm{Conn}(i_1)$ such that $\Lambda^P_u \cap \Lambda^P_v \neq \varnothing$. Since both u and v are maximal elements in the connected order ideal J, we must have $\Lambda^P_u \not\subset \Lambda^P_v$ and $\Lambda^P_v \not\subset \Lambda^P_u$. This means that Λ^P_u and Λ^P_v are adjacent, implying that $v \in \mathrm{Conn}(i_1)$. This leads to a contradiction. □

We also need the following lemma.

Lemma 10. *Let J be a connected order ideal of P, and let C be any connected subgraph of G_P. Assume that J is not adjacent to any vertex of C. If there exists a vertex J_a of C such that $J_a \subset J$, then $J_b \subset J$ for any vertex J_b of C.*

Proof. We first consider the case when J_a and J_b are adjacent. In this case, J_b and J_a intersect nontrivially, and so we have $\varnothing \neq (J_a \cap J_b)$. On the other hand, since $J_a \subset J$, we obtain that

$$\varnothing \neq (J_a \cap J_b) \subset (J \cap J_b). \tag{23}$$

Combining (23) and the hypothesis that the vertices J_b and J are not adjacent, we get that $J_b \subset J$ or $J \subset J_b$. If $J \subset J_b$, then $J_a \subset J \subset J_b$, which is impossible because J_a and J_b intersect nontrivially. Hence we have $J_b \subset J$.

We now consider the case when J_a is not adjacent to J_b. Since C is connected, there exists a sequence $(J_0 = J_a, J_1, \dots, J_k = J_b)$ $(k \geqslant 2)$ of vertices of C such that J_i is adjacent to J_{i-1} for $1 \leqslant i \leqslant k$. By the above argument, J_1 is contained in J. Therefore, by a simple recursion we get that $J_b \subset J$. □

For example, let P be the poset given in Figure 4. The graph G_P is illustrated in Figure 5, where we adopt the notation $\Lambda^P_{i,j} = \Lambda^P_i \cup \Lambda^P_j$ and $\Lambda^P_{i,j,k} = \Lambda^P_i \cup \Lambda^P_j \cup \Lambda^P_k$. The graph G_P has totally 13 connected components, and among them there are four connected components C_1, C_2, C_3, C_4 which have more than one vertex.

- To illustrate the assertion of Lemma 9, for example, let $J = \Lambda^P_{4,5,6}$, then we have $gs(J) = \{4, 5, 6\}$. One can verify that the subgraph χ_J of G_P induced by the vertex subset $\{\Lambda^P_4, \Lambda^P_5, \Lambda^P_6\}$ is indeed connected.

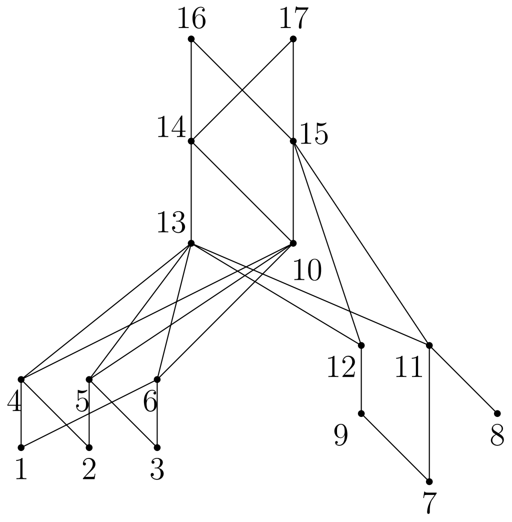


Figure 4: A naturally labeled poset P.

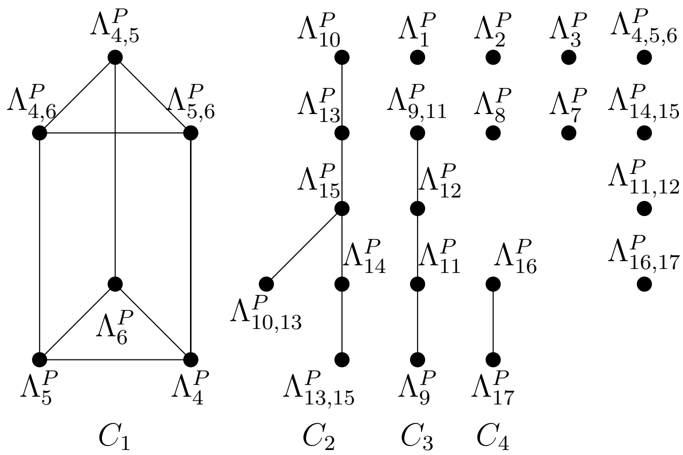


Figure 5: The graph G_P associated to the poset P in Figure 4.

- To illustrate the assertion of Lemma 10, for example, we let $J = \Lambda^P_{10}$, and let C be the connected component C_1 of G_P, then $\Lambda^P_5 \subset J$. In this case we see that $J' \subset \Lambda^P_{10}$ for any $J' \in V(C_1)$.

Now we turn to study a special subgraph of G_P, which is induced by the principal order ideals of P. This graph also plays an important role in our future proofs. Recall that the set of principal order ideals of P consists of $\Lambda^P_1, \Lambda^P_2, \ldots, \Lambda^P_n$. Let H_P be the subgraph of G_P induced by the vertex subset $\{\Lambda^P_1, \Lambda^P_2, \ldots, \Lambda^P_n\}$. For example, for the poset P and the graph G_P as illustrated in Figures 4 and 5, the graph H_P is as shown in Figure 6. It follows from Lemma 9 that for a given connected order ideal J the induced subgraph χ_J

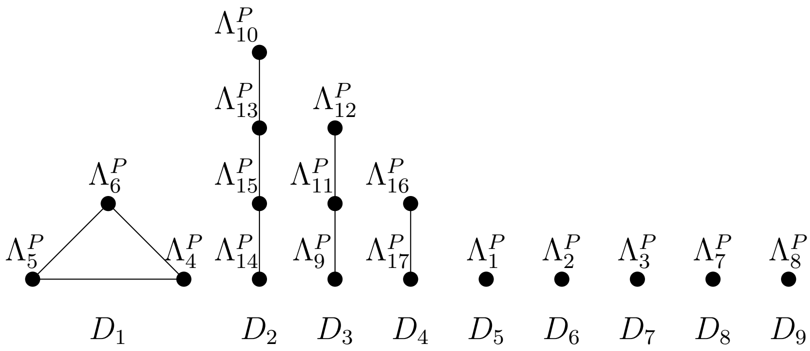


Figure 6: The subgraph H_P induced on G_P by principal order ideals.

must be a subgraph of certain connected component of H_P, where χ_J is defined as before Lemma 9. The graph H_P admits the following interesting properties.

Lemma 11. *Suppose that H_P has connected components $D_1, D_2, \ldots, D_\ell$. We have the following two assertions.*

(1) *Let $1 \leqslant r < s \leqslant \ell$, and let J_a, J_b be two connected order ideals of P. If χ_{J_a} is a subgraph of D_r while χ_{J_b} is a subgraph of D_s, then J_a and J_b are not adjacent in G_P.*

(2) *Given a connected order ideal J, suppose that χ_J is a subgraph of the connected component D_r of H_P, and hence $J \subseteq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$. If $J \neq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$, then there exists some $\Lambda_j^P \in V(D_r)$ such that J and Λ_j^P are adjacent in G_P.*

Proof. Let us first prove assertion (1). Suppose to the contrary that J_a and J_b are adjacent in the graph G_P. Then $J_a \cap J_b \neq \varnothing$. Since

$$J_a = \bigcup_{i \in gs(J_a)} \Lambda_i^P, \quad J_b = \bigcup_{j \in gs(J_b)} \Lambda_j^P,$$

there exist some $i \in gs(J_a)$ and $j \in gs(J_b)$ such that $\Lambda_i^P \cap \Lambda_j^P \neq \varnothing$. Notice that Λ_i^P is a vertex of the connected component D_r and Λ_j^P is a vertex of the connected component D_s, so Λ_i^P and Λ_j^P are not adjacent in the graph H_P. Since the graph H_P is a vertex induced subgraph of G_P, the order ideals Λ_i^P and Λ_j^P are also not adjacent in the graph G_P, hence they intersect trivially. Because $\Lambda_i^P \cap \Lambda_j^P \neq \varnothing$, we must have $\Lambda_i^P \subset \Lambda_j^P$ or $\Lambda_j^P \subset \Lambda_i^P$. If $\Lambda_i^P \subset \Lambda_j^P$, by Lemmas 9 and 10 we obtain that for any $k \in gs(J_a)$, there is $\Lambda_k^P \subset \Lambda_j^P$. Then,

$$J_a = \bigcup_{k \in gs(J_a)} \Lambda_k^P \subset \Lambda_j^P \subseteq J_b,$$

which implies that J_a and J_b are not adjacent in the graph G_P. If $\Lambda_j^P \subset \Lambda_i^P$, we can use a similar argument to deduce that J_a and J_b are not adjacent in the graph G_P. In both cases, we are led to a contradiction.

We proceed to prove assertion (2). Recall that $V(D_r)$ denotes the set of vertices of D_r. Assume that $gs(J) = \{i_1, \ldots, i_k\}$. Since $J \subseteq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$ but $J \neq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$, there exists some $\Lambda_j^P \in V(D_r)$ such that $\Lambda_j^P \nsubseteq J$. Let

$$\begin{aligned} V_1 &= \{\Lambda_i^P \in V(D_r) \mid \Lambda_i^P \subseteq J\}, \\ V_2 &= \{\Lambda_j^P \in V(D_r) \mid \Lambda_j^P \nsubseteq J\}. \end{aligned}$$

Clearly, we have $V_1 \cup V_2 = V(D_r)$ and $V_2 \neq \varnothing$. Since χ_J is a subgraph of D_r, we see that $V_1 \neq \varnothing$. Because D_r is a connected component of H_P, there exist some $\Lambda_i^P \in V_1$ and $\Lambda_j^P \in V_2$ such that Λ_i^P and Λ_j^P are adjacent in the graph H_P. Since H_P is a vertex induced subgraph of G_P, the vertices Λ_i^P and Λ_j^P are also adjacent in G_P, which means that Λ_i^P and Λ_j^P intersect nontrivially, namely

$$\Lambda_i^P \cap \Lambda_j^P \neq \varnothing, \quad \Lambda_i^P \not\subset \Lambda_j^P, \text{ and } \Lambda_j^P \not\subset \Lambda_i^P.$$

In view of that $\Lambda_i^P \subseteq J$ and $\Lambda_j^P \in V_2$, we get $J \neq \Lambda_j^P$ and

$$J \cap \Lambda_j^P \neq \varnothing, \quad J \not\subset \Lambda_j^P, \text{ and } \Lambda_j^P \not\subset J.$$

Hence J is adjacent to Λ_j^P, as desired. □

With the above lemma, we can further obtain another property of G_P.

Lemma 12. *Let C_r be a connected component of G_P with vertex set $V(C_r)$. Let J be a connected order ideal with the graph χ_J as defined as above. We have the following two assertions:*

(1) *Let J_r^{max} denote the set $\bigcup_{J' \in V(C_r)} J'$. Then J_r^{max} is an isolated vertex of the graph G_P.*

(2) *If χ_J is a subgraph of C_r, and $J \neq J_r^{max}$, then J is a vertex of C_r.*

Proof. Let us first prove assertion (1). It is clearly true when $|V(C_r)| = 1$. Suppose $|V(C_r)| \geqslant 2$. We first prove that J_r^{max} is a connected order ideal. Let V be a set of connected order ideals and assume V satisfies the condition:

$$V \subseteq V(C_r) \text{ and } \bigcup_{J \in V} J \text{ is a connected order ideal.} \qquad (*)$$

We claim that if V satisfies (*) and is of the largest possible size, then V must be equal to $V(C_r)$. Otherwise, suppose $V \subset V(C_r)$ but $V \neq V(C_r)$. Since C_r is a connected graph and $|V(C_r)| \geqslant 2$, there exist some $J_a \in V$ and $J_b \in (V(C_r) \setminus V)$ such that J_a and J_b are adjacent in G_P. Hence $J_a \cap J_b \neq \varnothing$, and then $(\bigcup_{J \in V} J) \cap J_b \neq \varnothing$. It follows that the set $V' = V \cup \{J_b\}$ also satisfies the condition (*), and $|V'| = |V| + 1$, contradicting the assumption that V is of the largest possible size.

We mow prove that J_r^{max} is not adjacent to any other vertex of G_P. For a $J \in \mathcal{J}_{conn}(P)$, if $J \in V(C_r)$, then $J \subset J_r^{max}$ and so J and J_r^{max} are not adjacent in G_P. If $J \notin V(C_r)$, namely, J is not adjacent to any vertex of C_r, we need to consider three cases:

(i) There exists some $J_a \in V(C)$ such that $J_a \subset J$. Then by Lemma 10 we obtain that $J_b \subset J$ for any other $J_b \in V(C_r)$. Hence $J_r^{max} \subset J$, and it follows that J and J_r^{max} are not adjacent in G_P;

(ii) There exists some $J_a \in V(C)$ such that $J \subset J_a$. Then $J \subset J_r^{max}$, and as a consequence, J and J_r^{max} are also not adjacent in G_P;

(iii) $J \cap J_a = \varnothing$ for any $J_a \in V(C_r)$. Then $J_r^{max} \cap J = \varnothing$ and, again, $\widetilde{J}$ and J are not adjacent in G_P.

Hence we conclude that J_r^{max} is an isolated vertex of the graph G_P.

To prove assertion (2), we first analyse some general properties of G_P. Suppose the graph H_P has ℓ connected components $D_1, D_2, \ldots, D_\ell$. Lemma 9 tells us that for any connected order ideal J', the graph $\chi_{J'}$ is connected, and that it must be a subgraph of D_k for some $1 \leqslant k \leqslant \ell$. For each $1 \leqslant k \leqslant \ell$, let

$$\mathcal{J}_{conn}^k(P) = \{J \in \mathcal{J}_{conn}(P) \mid \text{ the graph } \chi_J \text{ is a subgraph of } D_k\}.$$

In particular, if $J' = \Lambda_i^P \in V(D_k)$ is a principal order ideal, then the graph $\chi_{J'}$ has only one vertex Λ_i^P, thus $\chi_{J'}$ is of course a subgraph of D_k. It follows that $V(D_k) \subseteq \mathcal{J}_{conn}^k(P)$ for each $1 \leqslant k \leqslant \ell$. It is clear that

$$\mathcal{J}_{conn}(P) = \mathcal{J}_{conn}^1(P) \uplus \mathcal{J}_{conn}^2(P) \uplus \cdots \uplus \mathcal{J}_{conn}^\ell(P).$$

For each $1 \leqslant k \leqslant \ell$, let C_k be the connected component of G_P such that D_k is a subgraph of C_k (it turns out that for each D_k, there exists a unique C_k such that D_k is a subgraph of C_k). We proceed to show that $V(C_k) \subseteq \mathcal{J}_{conn}^k(P)$. Note that if $J_a \in \mathcal{J}_{conn}^s(P)$ and $J_b \in \mathcal{J}_{conn}^t(P)$ for some $s \neq t$, the first assertion of Lemma 11 tells us that J_a and J_b are not adjacent in G_P. Thus, by the connectivity of C_k in G_P, all members of $V(C_k)$ must belong to $\mathcal{J}_{conn}^k(P)$ since we already have $V(D_k) \subseteq \mathcal{J}_{conn}^k(P)$. And then, we get that $V(D_k) \subseteq V(C_k) \subseteq \mathcal{J}_{conn}^k(P)$. That is to say, for any $J' \in V(C_k)$, the graph $\chi_{J'}$ is a subgraph of D_k. Therefore, $J' \subseteq \bigcup_{\Lambda_i^P \in V(D_k)} \Lambda_i^P$ for any $J' \in V(C_k)$. This leads to the following equality:

$$J_k^{max} = \bigcup_{J' \in V(C_k)} J' = \bigcup_{\Lambda_i^P \in V(D_k)} \Lambda_i^P. \tag{24}$$

For the given J, we assume that χ_J is a subgraph of the connected component D_r of H_P for some $1 \leqslant r \leqslant \ell$, and then D_r is a subgraph of C_r. Thus in view of (24), when $J \neq J_r^{max}$, it follows that $J \neq \bigcup_{\Lambda_i^P \in V(D_r)} \Lambda_i^P$. By the second assertion of Lemma 11, in the graph G_P we see that J is adjacent to some vertex of D_r, therefore, J is also a vertex of C_r. □

We are almost ready for the proof of Theorem 2. Note that the definition of $\mathrm{Des}(M)$ ($M \in \mathscr{M}(G_P)$) is indirect, which uses the map Ψ from $\mathscr{M}(G_P)$ to $\mathscr{F}(P)$. In order to make the proof of Theorem 2 more clear, we shall give another characterization of $\mathrm{Des}(M)$ which only uses the information of M. Before doing this, we shall introduce one more notation. Given $J_a, J_b \in M$, we say that $J_a \prec_M J_b$ if $J_a \subset J_b$ and there exists no $J \in M$ such that $J_a \subset J \subset J_b$. Our new characterization of $\mathrm{Des}(M)$ is as follows.

Lemma 13. *Given $M \in \mathscr{M}(G_P)$, then $i \in \mathrm{Des}(M)$ if and only if there exists $j < i$ such that $J_a \prec_M J_b$, where $J_a, J_b \in M$ are connected order ideals uniquely determined by i, j respectively as in Lemma 7.*

Proof. By definition, $i \in \mathrm{Des}(M) = \mathrm{Des}(F_M)$ if and only if the parent of i, say j, is greater than i with respect to the natural order on integers. Recall that if j is the parent of i, then $i <_{F_M} j$ and there exists no k such that $i <_{F_M} k <_{F_M} i$. It follows from Lemma 7 that there exist two connected order ideals J_a, J_b in M satisfying $J_a \setminus \mu(M, J_a) = \{i\}, J_b \setminus \mu(M, J_b) = \{j\}$. By the construction of F_M, we have $J_a \subset J_b$ but there exists no $J \in M$ such that $J_a \subset J \subset J_b$, namely $J_a \prec_M J_b$. □

As shown above, the relation $\prec_M$ plays an important role for the new characterization of $\mathrm{Des}(M)$. To prove Theorem 2, we also need the following lemma, which is evident by definition. Recall that the set $U_{max}(M, J)$ is defined by (12).

Lemma 14. *Given $J_a, J_b \in M$, if $J_a \prec_M J_b$ then $J_a \in U_{max}(M, J_b)$.*

Now we are in the position to prove Theorem 2. From now on we shall assume that P is naturally labeled.

Proof of Theorem 2. There are two cases to consider.

(1). The connected component C_r has only one vertex, say J_r. Thus M_r can only be the unique one maximum independent set $\{J_r\}$ of C_r. By Lemma 7, we have $J_r \setminus \mu(M^1, J_r) = \{i\}$ for some $i \in \{1, 2, \ldots, n\}$. In this case, we first prove that

$$\mathrm{Des}(M_r, M^1) = \mathrm{Des}(M_r, M^2) = \varnothing. \tag{25}$$

Otherwise, suppose that $\mathrm{Des}(M_r, M^1) = \{i\}$. By the definition of $\mathrm{Des}(M_r, M^1)$, we have $i \in \mathrm{Des}(M^1)$. By Lemma 13, there exist $j < i$ and $J \in M^1$ such that $J \setminus \mu(M^1, J) = \{j\}$ and $J_r \prec_{M^1} J$.

We proceed to show that it is impossible to have such a pair (i, j). Let us consider the order relation between i and j in the poset P. It cannot be $j <_P i$, since $i \in J_r \subset J$ and Lemma 6 tells us that j is a maximal element of J. Then it might be $i <_P j$, or i and j are incomparable in P. Since P is naturally labeled and $j < i$, it can not be $i <_P j$. Suppose that i and j are incomparable in P. Since $J_r \setminus \mu(M^1, J_r) = \{i\}$, it follows from Lemma 6 that i is a maximal element of J_r. We proceed to prove that i is also a maximal elements of J. To see this, it is enough to show that there exists no $k \in J$ satisfying $i <_P k$. Note that

$$J = \{j\} \cup \mu(M^1, J) = \{j\} \cup \left(\bigcup_{J' \in U(M^1, J)} J' \right) = \{j\} \cup \left(\bigcup_{J' \in U_{max}(M^1, J)} J' \right).$$

By Lemma 14, the relation $J_r \prec_{M^1} J$ implies that $J_r \in U_{max}(M^1, J)$. Then there are three cases to consider:

(i) If $k = j$, then i and k are incomparable in P;

(ii) If $k \in J_r$, in this case we have $k \leqslant_P i$, or i and k are incomparable in P, because i is a maximal element of J_r;

(iii) If $k \in J'$ for some $J' \in U_{max}(M^1, J)$ but $J' \neq J_r$, we obtain that i and k are incomparable in P, since by Lemma 4 we have $J' \cap J_r = \varnothing$, which implies that for any $u \in J_r$, $v \in J'$, u and v are incomparable in P.

Hence there exists no $k \in J$ such that $i <_P k$, i.e., i is a maximal element of J. It follows that $\{i, j\} \subseteq gs(J)$ and then the graphs χ_{J_r} and χ_J have a common vertex Λ_i^P. Then by Lemma 9, the graphs χ_{J_r} and χ_J belong to the same connected component C_s of G_P. Hence C_s has at least two vertices Λ_i^P and Λ_j^P. By Lemma 12 and the hypothesis that J_r is an isolated vertex of G_P, we obtain $J_r = \bigcup_{J' \in V(C_s)} J'$ and $J \subseteq \bigcup_{J' \in V(C_s)} J'$. This contradicts with the assumption that $J_r \prec_{M^1} J$. Hence i and j cannot be incomparable in P, a contradiction.

Since such a pair (i, j) can not exist, it follows that $\mathrm{Des}(M_r, M^1) = \varnothing$. By using a similar argument, one can also prove that $\mathrm{Des}(M_r, M^2) = \varnothing$. Moreover, by the definition of $\overline{\mathrm{Des}}(M_r, M)$, it is clear that

$$\overline{\mathrm{Des}}(M_r, M^1) = \overline{\mathrm{Des}}(M_r, M^2) = \varnothing.$$

(2). C_r has at least two vertices. In this case, $M_r \subset V(C_r)$. By Lemma 12, we see that $J_r^{max} = \bigcup_{J' \in V(C_r)} J'$ is an isolated vertex of G_P. Hence $J_r^{max} \in M$ holds for any maximum independent set of G_P, and in particular $J_r^{max} \in M^1$ as well as $J_r^{max} \in M^2$.

We first prove that for any $J \in M_r$ or $J = J_r^{max}$,

$$J \setminus \mu(M^1, J) = J \setminus \mu(M^2, J). \tag{26}$$

To see this, we partition the set $U(M^2, J)$ into two subsets B_1 and B_2, where

$$\begin{aligned} B_1 &= \{J_1 \in U(M^2, J) \mid J_1 \in V(C_r)\}, \\ B_2 &= \{J_2 \in U(M^2, J) \mid J_2 \notin V(C_r)\}. \end{aligned}$$

Assume $J \setminus \mu(M^1, J) = \{j\}$. We claim that $j \notin J_2$ for any $J_2 \in B_2$. Otherwise, suppose to the contrary that there exists some $J_2 \in B_2$ such that $j \in J_2$. It follows from Lemma 6 that $j \in gs(J)$. On the other hand, since $J_2 \subset J$, we obtain that $j \in gs(J_2)$. Hence the graph χ_J and χ_{J_2} have a common vertex Λ_j^P. Then by Lemma 9 the graphs χ_J and χ_{J_2} belong to the same connected component of G_P. We proceed to show that χ_{J_2} is a subgraph of C_r. To see this, there are two cases to consider.

(i) Suppose that $J \in M_r \subset V(C_r)$ (then $J \neq J_r^{max}$), namely, J is a vertex of the connected component C_r. It follows from the second assertion of Lemma 12 that χ_J and J are contained in the same connected component C_r of G_P. Hence both χ_J and χ_{J_2} are subgraphs of C_r.

(ii) Suppose that $J = J_r^{max} = \bigcup_{J' \in V(C_r)} J'$. Let $i \in gs(J)$ be a maximal element of J, then there exists some $J' \in V(C_r)$ such that $i \in J'$. It follows that i is also a maximal element of J', namely, $i \in gs(J')$. Hence the graphs χ_J and $\chi_{J'}$ have at least one common vertex Λ_i^P, and then χ_J and $\chi_{J'}$ belong to the same connected component of G_P. The second assertion of Lemma 12 tells us that for any $J' \in V(C_r)$, $\chi_{J'}$ and J' are contained in the same connected component C_r of G_P. Hence $\chi_J, \chi_{J'}$ and χ_{J_2} are all subgraphs of C_r.

On the other hand, because $J_2 \subset J$, we have $J_2 \neq J_r^{max}$. Then by the second assertion of Lemma 12 we get $J_2 \in V(C_r)$, leading to a contradiction. Hence the claim, that $j \notin J_2$ for any $J_2 \in B_2$, is true.

Recall that $M^1 \cap V(C_r) = M^2 \cap V(C_r) = M_r$. It is routine to verify that

$$U(M^1, J) \cap M_r = U(M^2, J) \cap M_r = B_1,$$

Combining (13) and the above identity, we get that

$$j \in J \setminus \mu(M^1, J) \subseteq J \setminus \bigcup_{J_1 \in B_1} J_1.$$

As we have shown that $j \notin J_2$ for any $J_2 \in B_2$, so again by (13) there holds

$$j \in J \setminus \bigcup_{J' \in (B_1 \cup B_2)} J' = J \setminus \bigcup_{J' \in U(M^2, J)} J' = J \setminus \mu(M^2, J).$$

Thus, by Lemma 6, the set $J \setminus \mu(M^2, J)$ contains exactly one element, which can only be j. Therefore, we have

$$\{j\} = J \setminus \mu(M^2, J) = J \setminus \mu(M^1, J).$$

We proceed to show that $\mathrm{Des}(M_r, M^1) \subseteq \mathrm{Des}(M_r, M^2)$. Let $i \in \mathrm{Des}(M_r, M^1)$, and by the definition of $\mathrm{Des}(M_r, M^1)$ and Lemma 6 there exists $J_a \in M_r$ such that $J_a \setminus \mu(M^1, J_a) = \{i\}$. By Lemma 13, there exist $j < i$ and $J_b \in M^1$ such that $J_b \setminus \mu(M^1, J_b) = \{j\}$ and $J_a \prec_{M^1} J_b$. We claim that $J_b \in V(C_r)$ or $J_b = J_r^{max}$. Suppose otherwise that J_b is not a vertex of C_r and $J_b \neq J_r^{max}$. Since $J_a \in V(C_r)$ and $J_a \subset J_b$, it follows from Lemma 10 that $J' \subset J_b$ for any $J' \in V(C_r)$. Hence $J_r^{max} \subset J_b$. Thus we obtain $J_a \subset J_r^{max} \subset J_b$. Recall that $J_r^{max} \in M^1$, the relation $J_a \subset J_r^{max} \subset J_b$ contradicts the assumption that $J_a \prec_{M^1} J_b$. Recall also that we have shown $J_r^{max} \in M^2$. If $J_b = J_r^{max}$ then $J_b \in M^2$. If $J_b \in V(C_r)$, then $J_b \in M_r = M^2 \cap V(C_r)$, and hence also $J_b \in M^2$. We further show that $J_a \prec_{M^2} J_b$. Otherwise, suppose there exists some $J_c \in M^2$ such that $J_a \subset J_c \subset J_b$. By the hypothesis that $J_a \prec_{M^1} J_b$ and $M^1 \cap V(C_r) = M^2 \cap V(C_r) = M_r$, it follows that $J_c \notin M_r \subset V(C_r)$. Then by Lemma 10, for any $J' \in V(C_r)$, there is $J' \subset J_c$. Hence $J_b \subseteq \bigcup_{J' \in V(C_r)} \subset J_c$, leading to a contradiction. Thus, for any $i \in \mathrm{Des}(M_r, M^1)$, by (26) there exist $J_a, J_b \in M^2$ such that $J_a \setminus \mu(M^2, J_a) = \{i\}$, $J_b \setminus \mu(M^2, J_b) = \{j\}$, $J_a \prec_{M^2} J_b$ and $i > j$. This means $i \in \mathrm{Des}(M_r, M^2)$ for any $i \in \mathrm{Des}(M_r, M^1)$. Hence $\mathrm{Des}(M_r, M^1) \subseteq \mathrm{Des}(M_r, M^2)$.

It can be proved in a similar way that $\mathrm{Des}(M_r, M^2) \subseteq \mathrm{Des}(M_r, M^1)$. So we get $\mathrm{Des}(M_r, M^1) = \mathrm{Des}(M_r, M^2)$. Combining this and (26), we further obtain $\overline{\mathrm{Des}}(M_r, M^1) = \overline{\mathrm{Des}}(M_r, M^2)$, as desired. □

We proceed to prove Theorem 3.

Proof of Theorem 3. Given a maximum independent set M of G_P, let

$$\overline{\mathrm{Des}}(M) = \big\{J \in M \mid J \setminus \mu(M, J) = \{i\} \text{ for some } i \in \mathrm{Des}(M)\big\}.$$

Recall that $\mathscr{M}(C_r)$ is the set of maximum independent sets of C_r for each $1 \leqslant r \leqslant h$, respectively. It is clear that M admits the following natural decomposition:

$$M = M_1 \uplus M_2 \uplus \cdots \uplus M_h, \text{ where } M_r \in \mathscr{M}(C_r).$$

It follows from Theorem 2 that both $\mathrm{Des}(M_r)$ and $\overline{\mathrm{Des}}(M_r)$ are well-defined, and hence

$$\mathrm{Des}(M) = \mathrm{Des}(M_1) \uplus \mathrm{Des}(M_2) \uplus \cdots \uplus \mathrm{Des}(M_h), \tag{27}$$
$$\overline{\mathrm{Des}}(M) = \overline{\mathrm{Des}}(M_1) \uplus \overline{\mathrm{Des}}(M_2) \uplus \cdots \uplus \overline{\mathrm{Des}}(M_h). \tag{28}$$

Thus, by (6), Theorem 1 and Lemma 7, we get that

$$F_P(\mathbf{x}) = \sum_{M \in \mathscr{M}(G_P)} \frac{\prod_{J \in \overline{\mathrm{Des}}(M)} \prod_{k \in J} x_k}{\prod_{J \in M}(1 - \prod_{\ell \in J} x_\ell)}.$$

By (28), we then have

$$\begin{aligned} F_P(\mathbf{x}) &= \sum_{M_1 \in \mathscr{M}(C_1)} \sum_{M_2 \in \mathscr{M}(C_2)} \cdots \sum_{M_h \in \mathscr{M}(C_h)} \frac{\prod_{r=1}^h \prod_{J \in \overline{\mathrm{Des}}(M_r)} \prod_{k \in J} x_k}{\prod_{r=1}^h \prod_{J \in M_r}(1 - \prod_{\ell \in J} x_\ell)} \\ &= \prod_{r=1}^h \sum_{M_r \in \mathscr{M}(C_r)} \frac{\prod_{J \in \overline{\mathrm{Des}}(M_r)} \prod_{k \in J} x_k}{\prod_{J \in M_r}(1 - \prod_{\ell \in J} x_\ell)}. \end{aligned}$$

□

We would like to point out that Theorem 3 enables us to give an alternative proof to Féray and Reiner's formula (4). To this end, let P be a naturally labeled forest with duplications as defined by Féray and Reiner [4], namely, for any connected order ideal J_a of P, there exists at most one other connected order ideal J_b such that J_a and J_b intersect nontrivially. Assume that G_P has h connected components $C_1, C_2, \ldots, C_h$. Then each C_r has at most two vertices, and hence each connected component of H_P has also at most two vertices.

We claim that when a connected component C of G_P has two vertices, say J_a and J_b, then both J_a and J_b are principal order ideals of P. Otherwise, suppose that J_a is not a principal order ideal of P. Then the graph χ_{J_a} has more than one vertices. Recall that χ_{J_a} is a subgraph of H_P. By Lemma 9 and the fact that each connected component of the graph H_P has at most two vertices, the graph χ_{J_a} is a connected component of H_P.

It then follows from (24) and the first assertion of Lemma 12 that J_a is an isolated vertex of G_P, a contradiction. Similarly, J_b is also a principal order ideal of P.

Therefore, we may assume that for $1 \leqslant r \leqslant d$ the component C_r has two vertices (both of them are principal order ideals of P), say $\Lambda^P_{i_r}$ and $\Lambda^P_{j_r}$, and for $d < r \leqslant h$ the component C_r has only one vertex. Thus, for $1 \leqslant r \leqslant d$, there are two choices for M_r, namely, $M_r = \{\Lambda^P_{i_r}\}$ or $M_r = \{\Lambda^P_{j_r}\}$. We assume that $i_r > j_r$. Then

$$\overline{\mathrm{Des}}(\{\Lambda^P_{i_r}\}) = \Lambda^P_{i_r}, \quad \overline{\mathrm{Des}}(\{\Lambda^P_{j_r}\}) = \varnothing.$$

For $d < r \leqslant h$, let J_r be the only vertex of C_r, and then $\overline{\mathrm{Des}}(\{J_r\}) = \varnothing$. By Theorem 3, we obtain that

$$\begin{aligned} F_P(\mathbf{x}) &= \prod_{1\leqslant r\leqslant d} \left[\frac{\mathbf{x}^{\Lambda^P_{i_r}}}{\left(1-\mathbf{x}^{\Lambda^P_{i_r}}\right)} + \frac{1}{\left(1-\mathbf{x}^{\Lambda^P_{j_r}}\right)} \right] \prod_{d<r\leqslant h} \frac{1}{(1-\mathbf{x}^{J_r})} \\ &= \prod_{1\leqslant r\leqslant d} \left[\frac{1-\mathbf{x}^{\Lambda^P_{i_r}}\mathbf{x}^{\Lambda^P_{j_r}}}{\left(1-\mathbf{x}^{\Lambda^P_{i_r}}\right)\left(1-\mathbf{x}^{\Lambda^P_{j_r}}\right)} \right] \prod_{d<r\leqslant h} \frac{1}{(1-\mathbf{x}^{J_r})}, \end{aligned}$$

where $\mathbf{x}^A = \prod_{i\in A} x_i$ for a subset $A \subseteq \{1, 2, \ldots, n\}$. It is straightforward to verify that the above formula is equivalent to (4).

4 Counting linear extensions

In this section, we take an example to show that formula (11) can be used to derive the generating function of major index of linear extensions of P, as well as to count the number $|\mathcal{L}(P)|$ of linear extensions of P.

The generating function $F_P(q)$ of major index of linear extensions of P is denoted by $F_P(q) = \sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)}$, where $\mathrm{maj}(w) = \sum_{i\in\mathrm{Des}(w)} i$ is called the major index of w. By letting $x_1 = \cdots = x_n = q$ respectively in (1) and (11), we are led to the following identity

$$F_P(q) = [n]!_q \prod_{r=1}^{h} \sum_{M_r\in\mathscr{M}(C_r)} \frac{q^{\sum_{J\in\overline{\mathrm{Des}}(M_r)}|J|}}{\prod_{J\in M_r}[|J|]_q}, \tag{29}$$

where $[i]_q = 1 - q^i$ for any i and $[m]!_q = \prod_{i=1}^{m}[i]_q$.

Moreover, when q tends to 1 on both sides of (29), we arrive at the following formula for the number of linear extensions of P:

$$|\mathcal{L}(P)| = n! \prod_{r=1}^{h} \sum_{M_r\in\mathscr{M}(C_r)} \frac{1}{\prod_{J\in M_r}|J|}. \tag{30}$$

Note that the number of linear extensions of P is independent of the labelling of P. Thus formula (30) is also valid in the cases when P is not naturally labeled.

We would like to mention that calculating the number of linear extensions for general posets has been proved to be a $\sharp P$-hard problem by Brightwell and Winkler [3]. However, in the case when P is a poset such that each connected component C_r of G_P has small size of vertex set, we shall illustrate that formula (30) provides an efficient way to count the number of linear extensions of P. For example, take the naturally labeled poset P in Figure 4. From the graph of G_P as illustrated in Figure 5, we obtain that

1. For the connected component C_1, there are 6 choices for M_1:

M_1	$\{\Lambda_4^P, \Lambda_{4,5}^P\}$	$\{\Lambda_4^P, \Lambda_{4,6}^P\}$	$\{\Lambda_5^P, \Lambda_{4,5}^P\}$	$\{\Lambda_5^P, \Lambda_{5,6}^P\}$
$\mathrm{Des}(M_1)$	$\varnothing$	$\{6\}$	$\{5\}$	$\{6\}$
$\overline{\mathrm{Des}}(M_1)$	$\varnothing$	$\{\Lambda_{4,6}^P\}$	$\{\Lambda_5^P\}$	$\{\Lambda_{5,6}^P\}$

M_1	$\{\Lambda_6^P, \Lambda_{4,6}^P\}$	$\{\Lambda_6^P, \Lambda_{5,6}^P\}$
$\mathrm{Des}(M_1)$	$\{6\}$	$\{5{,}6\}$
$\overline{\mathrm{Des}}(M_1)$	$\{\Lambda_6^P\}$	$\{\Lambda_6^P, \Lambda_{5,6}^P\}$

2. For the connected component C_2, there are 5 choices for M_2:

M_2	$\{\Lambda_{10}^P, \Lambda_{15}^P, \Lambda_{13,15}^P\}$	$\{\Lambda_{10}^P, \Lambda_{10,13}^P, \Lambda_{14}^P\}$	$\{\Lambda_{10}^P, \Lambda_{10,13}^P, \Lambda_{13,15}^P\}$
$\mathrm{Des}(M_2)$	$\{15\}$	$\varnothing$	$\{15\}$
$\overline{\mathrm{Des}}(M_2)$	$\{\Lambda_{15}^P\}$	$\varnothing$	$\{\Lambda_{13,15}^P\}$

M_2	$\{\Lambda_{13}^P, \Lambda_{10,13}^P, \Lambda_{14}^P\}$	$\{\Lambda_{13}^P, \Lambda_{10,13}^P, \Lambda_{13,15}^P\}$
$\mathrm{Des}(M_2)$	$\{13\}$	$\{13, 15\}$
$\overline{\mathrm{Des}}(M_2)$	$\{\Lambda_{13}^P\}$	$\{\Lambda_{13}^P, \Lambda_{13,15}^P\}$

3. For the connected component C_3, there are 3 choices for M_3:

M_3	$\{\Lambda_{11}^P, \Lambda_{11,9}^P\}$	$\{\Lambda_9^P, \Lambda_{11,9}^P\}$	$\{\Lambda_9^P, \Lambda_{12}^P\}$
$\mathrm{Des}(M_3)$	$\{11\}$	$\varnothing$	$\{12\}$
$\overline{\mathrm{Des}}(M_3)$	$\{\Lambda_{11}^P\}$	$\varnothing$	$\{\Lambda_{12}^P\}$

4. For the connected component C_4, there are 2 choices for M_4:

M_4	$\{\Lambda_{16}^P\}$	$\{\Lambda_{17}^P\}$
$\mathrm{Des}(M_4)$	$\varnothing$	$\{17\}$
$\overline{\mathrm{Des}}(M_4)$	$\varnothing$	$\{\Lambda_{17}^P\}$

5. For connected components which have only one vertex, each of them has only one choice for each M_r, and $\mathrm{Des}(M_r) = \varnothing$ as well as $\overline{\mathrm{Des}}(M_r) = \varnothing$.

Therefore, invoking formula (29), we see that $F_P(q) = \sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)}$ equals

$$[17]!_q \left[\frac{1}{[6]_q}\left(\frac{1+2q^3+2q^5+q^8}{[3]_q[5]_q}\right)\right]\left[\frac{1}{[15]_q}\left(\frac{q^{13}+1+q^{14}}{[7]_q[13]_q[14]_q} + \frac{q^{12}+q^{26}}{[12]_q[13]_q[14]_q}\right)\right]$$
$$\times\left[\frac{1}{[5]_q}\left(\frac{q^3}{[3]_q[4]_q} + \frac{1}{[2]_q[4]_q} + \frac{q^3}{[2]_q[3]_q}\right)\right]\left[\frac{1}{[17]_q}\frac{(1+q^{16})}{[16]_q}\right] \times 1^5.$$

Letting $q \to 1$ in the above formula, we arrive at

$$\begin{aligned} |\mathcal{L}(P)| &= 17! \times \left(\frac{1}{6} \times \frac{6}{3 \times 5}\right) \times \left[\frac{1}{15} \times \left(\frac{3}{7 \times 13 \times 14} + \frac{2}{13 \times 12 \times 14}\right)\right] \\ &\quad \times \left[\frac{1}{5} \times \left(\frac{1}{3 \times 4} + \frac{1}{3 \times 2} + \frac{1}{4 \times 2}\right)\right] \times \left(\frac{1}{17} \times \frac{2}{16}\right) \times 1^5 \\ &= 2851200. \end{aligned}$$

This coincides with the result by listing all linear extensions by using Sage [10].

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