Covering a Graph with Cycles of Length at least 4

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Abstract

Let G be a graph of order $n \geqslant 4k$, where k is a positive integer. Suppose that the minimum degree of G is at least $\lceil n/2 \rceil$. We show that G contains k vertex-disjoint cycles covering all the vertices of G such that $k-1$ of them are quadrilaterals.

Keywords: cycles; disjoint cycles; cycle coverings

1 Introduction

Let G be a graph. A set of subgraphs of G is said to be *independent* if no two of them have any common vertex in G. Corrádi and Hajnal [3] investigated the maximum number of independent cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k independent cycles. In particular, when the order of G is exactly $3k$, then G contains k independent triangles. A cycle of length 4 is called a *quadrilateral*. Erdős and Faudree [6] conjectured that if G is a graph of order $4k$ with minimum degree at least $2k$, then G contains k independent quadrilaterals. Alon and Yuster [1] proved that for any $\epsilon > 0$, there exists k_0 such that if G is a graph of order $4k$ and has minimum degree at least $(2+\epsilon)k$ with $k \geqslant k_0$, then G contains k independent quadrilaterals. We proved this conjecture in [11], that is

Theorem *A* [11] *If G is a graph of order $4k$ and the minimum degree of G is at least $2k$, then G contains k independent quadrilaterals.*

In [9], we proved the following theorem.

Theorem *B* [9] *Let G be a graph of order n with $4k+1 \leqslant n \leqslant 4k+4$, where k is a positive integer. Suppose that the minimum degree of G is at least $2k+1$. Then G contains k independent quadrilaterals.*

In [4], El-Zahar conjectured that if G is a graph of order $n = n_1 + n_2 + \cdots + n_k$, where each n_i is an integer at least 3, such that $\delta(G) \geqslant \lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k independent cycles of lengths $n_1, n_2, \ldots, n_k$, respectively. Clearly, this conjecture generalizes the above conjecture by Erdős and Faudree. In [8], we confirmed the El-Zahar's conjecture for the case $n_1 = \cdots = n_{k-1} = 3$ and $n_k \geqslant 3$. In this paper, we will prove the following theorem:

Theorem *C* *Let G be a graph of order $n \geqslant 4k$, where k is a positive integer. Suppose that the minimum degree of G is at least $\lceil n/2 \rceil$. Then G contains k independent cycles covering all the vertices of G such that $k-1$ of them are quadrilaterals.*

The minimum degree condition in the theorem is sharp. To see this, we just need to observe $K_{(n-1)/2,(n+1)/2}$ when n is odd and $K_{(n-2)/2,(n+2)/2}$ when n is even.

We discuss only finite simple graphs and use standard terminology and notation from [2] except as indicated. Let G be a graph. For a vertex $u \in V(G)$ and a subgraph H of G or a subset H of $V(G)$, $N(u, H)$ is the set of neighbors of u contained in H. We let $d(u, H) = |N(u, H)|$. Thus $d(u, G)$ is the degree of u in G. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. For a subset X of $V(G)$, we use $G - X$ to denote $G[V(G) - X]$. If $u \in V(G)$, we also write $G - \{u\}$ as $G - u$.

If $C = x_1x_2 \ldots x_mx_1$ is a cycle, then the subscripts of x_i's will be taken modulo by m in $\{1, 2, \ldots, m\}$. A chord of a cycle C in G is an edge of $G - E(C)$ that joins two vertices of C. We use $\tau(C)$ to denote the number of chords of C in G.

2 Lemmas

In the following, $G = (V, E)$ is a graph of order $n \geqslant 3$.

Lemma 2.1. *Let $P = x_1 \ldots x_k$ be a path and u a vertex in G such that $u \notin V(P)$ and $d(u, P) + d(x_k, P) \geqslant k$. Then either G has a path P' from x_1 to u such that $V(P') = V(P) \cup \{u\}$, or $k \geqslant 2$, $x_1u \in E$ and $d(x_k, P) + d(u, P) = k$.*

Proof. Let $I = \{x_{i+1} | x_ix_k \in E, 1 \leqslant i \leqslant k\}$. Clearly, $x_1 \notin I$. If $N(u, P) \cap I \neq \varnothing$, say $x_{i+1} \in N(u, P) \cap I$, then $x_1 \ldots x_ix_kx_{k-1} \ldots x_{i+1}u$ is the required path from x_1 to u. If $N(u, P) \cap I = \varnothing$, then $N(u, P) \cup I = V(P)$ since $d(x_k, P) + d(u, P) \geqslant k$ and $|I| = d(x_k, P)$, and then the lemma follows. □

Lemma 2.2. *Let Q be a quadrilateral and let x and y be two distinct vertices of G not on Q. Suppose $d(x, Q) + d(y, Q) \geqslant 5$, then $G[V(Q) \cup \{x, y\}]$ contains a quadrilateral Q' and an edge e such that Q' and e are independent and e is incident with exactly one of x and y.*

Proof. The lemma is clearly true if $d(x, Q) = 4$ or $d(y, Q) = 4$. So we may assume w.l.o.g. that $d(x, Q) = 3$ and $d(y, Q) \geqslant 2$. Label $Q = a_1a_2a_3a_4a_1$ such that $N(x, Q) = \{a_1, a_2, a_3\}$. Then we see that the lemma is true if either $a_2y \in E$ or $a_4y \in E$. If $a_2y \notin E$ and $a_4y \notin E$, then $Q' = a_1a_4a_3ya_1$ and $e = a_2x$ satisfy the requirement. □

Lemma 2.3. *Let Q be a quadrilateral and let x and y be two distinct vertices of G not on Q. Suppose that $d(x, Q) + d(y, Q) \geqslant 5$ and $G[V(Q) \cup \{x, y\}]$ is not hamiltonian. Then*

$G[V(Q) \cup \{x,y\}]$ contains a quadrilateral Q' with $\tau(Q') \geqslant \tau(Q)$ and an edge e such that Q' and e are independent and e is incident with exactly one of x and y.

Proof. Let $Q = a_1a_2a_3a_4a_1$. We may assume that $d(x,Q) \geqslant d(y,Q)$ and $\{a_1,a_2,a_3\} \subseteq N(x,Q)$. Clearly, the lemma is true if $ya_4 \in E$ or $d(x,Q) = 4$. Hence we may assume that $ya_4 \notin E$ and $d(x,Q) = 3$. Thus $d(y,Q) \geqslant 2$. As $G[V(Q) \cup \{x,y\}]$ is not hamiltonian, we see that $\{a_1,a_2\} \nsubseteq N(y)$ and $\{a_2,a_3\} \nsubseteq N(y)$. It follows that $N(y,Q) = \{a_1,a_3\}$, and therefore $a_2a_4 \notin E$ for otherwise $G[V(Q) \cup \{x,y\}]$ is hamiltonian. Let $Q' = ya_1a_4a_3y$ and $P' = xa_2$. Clearly, $\tau(Q') = \tau(Q)$, and so the lemma holds. □

Lemma 2.4. *Suppose that $n \geqslant 5$ and $d(x,G) + d(y,G) \geqslant n$ for every two nonadjacent vertices x and y of G. Then for each $x \in V(G)$, G has a quadrilateral Q such that $G - V(Q)$ has a hamiltonian path starting at x unless that $n \leqslant 6$, and in addition, if $n = 5$ then $d(u,G) + d(v,G) = 5$ for some two nonadjacent vertices u and v of G, and if $n = 6$ then G has an edge uv such that G has a hamiltonian path from u to v, $G - u - v$ has a quadrilateral and $d(u,G) + d(v,G) = 6$.*

Proof. For the proof, we suppose that the lemma fails. Let x_0 be a vertex of G such that G does not have a quadrilateral Q such that $G - V(Q)$ has a hamiltonian path starting at x_0.

First, suppose that $G - x_0$ does not have a quadrilateral. Let x and y be two arbitrary nonadjacent vertices of $G - x_0$. Then $|N(x, G - x_0) \cap N(y, G - x_0)| \leqslant 1$. As $d(x,G) + d(y,G) \geqslant n$, we see that $N(x,G) \cup N(y,G) = V(G) - \{x,y\}$, $x_0 \in N(x,G) \cap N(y,G)$ and $|N(x, G - x_0) \cap N(y, G - x_0)| = 1$. Say $N(x,G) \cap N(y,G) = \{x_0, z\}$. Assume w.l.o.g. $d(x) \geqslant d(y)$. Suppose $d(x, G - x_0) \geqslant 4$. Let $\{x_1,x_2\} \subseteq N(x, G - x_0 - z)$ with $x_1 \neq x_2$. Then either $x_1z \notin E$ or $x_2z \notin E$ for otherwise $G - x_0$ has a quadrilateral. Say $x_1z \notin E$. For the same reason, $x_1y \notin E$ and $x_2y \notin E$. Similarly, we must have $N(x_1,G) \cup N(y,G) = V(G) - \{x_1,y\}$ and $|N(x_1,G) \cap N(y,G)| = 2$. In particular, we also have that $x_1x_2 \in E$. Let $y_1 \in N(y, G - x_0 - z)$ be such that $x_1y_1 \in E$. Clearly, $x_2z \notin E$ and $x_2y_1 \notin E$ for otherwise $G - x_0$ has a quadrilateral. Similarly, we have that $|N(x_2,G) \cap N(y,G)| = 2$ and $N(x_2,G) \cup N(y,G) = V(G) - \{x_2,y\}$. Let $y_2 \in N(y,G)$ be such that $x_2y_2 \in E$. Similarly, we can show $y_1y_2 \in E$, and thus $x_1x_2y_2y_1x_1$ is a quadrilateral in $G - x_0$, a contradiction. Therefore we must have $d(x,G) = 3$. Thus $n \leqslant 6$. If $n = 5$, we have that $d(x,G) + d(y,G) = 5$ and we are done. Hence we assume $n = 6$. Thus $d(x) = d(y) = 3$. Let $V(G) - \{x_0,x,y,z\} = \{x_1,y_1\}$ be such that $\{xx_1, yy_1\} \subseteq E$. As $xy_1 \notin E$ and $yx_1 \notin E$, we can show, as before, that $\{x_0x_1, x_0y_1\} \subseteq E$. If $x_1y_1 \notin E$, then $z \in N(x_1) \cap N(y_1)$ as $d(x_1,G) + d(y_1,G) \geqslant 6$, and consequently, the second statement of the lemma holds with $\{u,v\} = \{y,y_1\}$. Thus we assume that $x_1y_1 \in E$. Then $zx_1 \notin E$ and $zy_1 \notin E$ for otherwise $G - x_0$ has a quadrilateral. Then $x_0z \in E$ as $d(x_1,G) + d(z,G) \geqslant 6$. Again, we see that the second statement of the lemma holds with $\{u,v\} = \{x_1,y_1\}$.

Next, suppose that $G - x_0$ has a quadrilateral. We now choose a quadrilateral Q from $G - x_0$ such that

The length of a longest path starting at x_0 in $G - V(Q)$ is maximum. (1)

Let P be a longest path starting at x_0 in $G - V(Q)$. Subject to (1), we choose Q and P such that

$$\tau(Q) \text{ is maximum.} \tag{2}$$

Let $P = x_0x_1 \dots x_t$ and $Q = a_1a_2a_3a_4a_1$. We need to show that $t = n - 5$. On the contrary, suppose $t < n-5$. Let $D = G - V(P \cup Q)$ and $r = |V(D)|$. Then $t = n-5-r$. Let $y_0 \in V(D)$. By Lemma 2.1, we have

$$d(y_0, P) + d(x_t, P) \leqslant t + 1. \tag{3}$$

Therefore

$$d(y_0, Q) + d(x_t, Q) \geqslant n - (t+1) - (r-1) = 5. \tag{4}$$

We claim the following:

Claim A. For each $i \in \{1, 2\}$, $\{a_i, a_{i+2}\} \not\subseteq N(y_0, Q)$.

Proof of Claim A. On the contrary, say w.l.o.g. $\{a_1, a_3\} \subseteq N(y_0, Q)$. By (1), we see that $\{a_2, a_4\} \cap N(x_t, Q) = \varnothing$. Hence $d(y_0, Q) \geqslant 3$ by (4). Say $a_2y_0 \in E$. As $\tau(y_0a_1a_2a_3y_0) \leqslant \tau(Q)$ by (2), we must have $a_2a_4 \in E$. Thus $G[\{a_1, a_2, a_3, a_4, y_0\} - \{a_i\}]$ contains a quadrilateral for each $i \in \{1, 2, 3, 4\}$, and therefore $d(x_t, Q) = 0$ by (1), contradicting with (4). Hence the claim holds.

We now divide the proof into the following two cases.

Case 1. $d(y_0, Q) = 2$.

In this case, $d(x_t, Q) \geqslant 3$. By Claim A, we may assume w.l.o.g. $N(y_0, Q) = \{a_1, a_2\}$. We may also assume w.l.o.g. $\{a_2, a_4\} \subseteq N(x_t, Q)$ as $d(x_t, Q) \geqslant 3$. Then $a_1a_3 \notin E$ for otherwise $y_0a_1a_3a_2y_0$ is a quadrilateral and $P + x_ta_4$ is longer than P in G. As $y_0a_3 \notin E$, $d(y_0, G) + d(a_3, G) \geqslant n$ and so $|N(y_0, G) \cap N(a_3, G)| \geqslant 2$. Then it is easy to see that $t \geqslant 1$. Set $Q_1 = x_ta_2a_3a_4x_t$. Then we see that $x_{t-1}y_0 \notin E$ and $x_{t-1}a_1 \notin E$ by (1). For the same reason, $d(y_0, P - x_t) + d(x_{t-1}, P - x_t) \leqslant t$ and $N(y_0, D) \cap N(x_{t-1}, D) = \varnothing$. It follows that $d(y_0, P \cup D) + d(x_{t-1}, P \cup D) \leqslant n - 5$, and therefore $d(y_0, Q) + d(x_{t-1}, Q) \geqslant 5$. Therefore $N(x_{t-1}, Q) = \{a_2, a_3, a_4\}$. Furthermore, we see that $d(x_{t-1}, P - x_t) + d(y_0, P - x_t) = t$. By Lemma 2.1, $t - 1 \geqslant 1$. Let $Q_2 = x_{t-1}x_ta_3a_4x_{t-1}$. Then we see that $\{y_0, a_1, a_2\} \cap N(x_{t-2}, G) = \varnothing$ and $N(y_0, D) \cap N(x_{t-2}, D) = \varnothing$. This implies that $d(y_0, P - x_t - x_{t-1}) + d(x_{t-2}, P - x_t - x_{t-1}) \geqslant n - 5 - (r-1) = t + 1$. By Lemma 2.1, $G[V(P) \cup \{y_0\} - \{x_t, x_{t-1}\}]$ has a hamiltonian path P' from x_0 to y_0. Therefore $P'y_0a_1a_2$ is longer than P and independent of Q_2, contradicting (1).

Case 2. $d(y_0, Q) = 1$.

We have that $d(x_t, Q) = 4$. Say $y_0a_1 \in E$. As $|N(y_0, G) \cap N(a_3, G)| \geqslant 2$, we see that $t \geqslant 1$. By (1), $a_1x_{t-1} \notin E$ and $y_0x_{t-1} \notin E$. We also have, by (1), that $N(y_0, D) \cap N(x_{t-1}, D) = \varnothing$. It follows that $d(y_0, P - x_t) + d(x_{t-1}, P - x_t) \geqslant n - 5 - (r-1) = t + 1$. By Lemma 2.1, $G[V(P) \cup \{y_0\} - \{x_t\}]$ has a hamiltonian path P'' from x_0 to y_0. Then $P''y_0a_1$ is longer than P and independent of $x_ta_2a_3a_4x_t$, contradicting (1). This proves the lemma. □

Lemma 2.5. *Suppose that $d(x,G)+d(y,G) \geqslant n$ for every two nonadjacent vertices x and y of G. Then for any two distinct vertices u and v, G has a hamiltonian path from u to v unless either $\{u,v\}$ is a vertex-cut of G or G has an independent set X with $|X| \geqslant n/2$ and $\{u,v\} \subseteq V(G)-X$.*

Proof. For the proof, we suppose that there exist two distinct vertices u and v such that G does not have a hamiltonian path from u to v and $\{u,v\}$ is not a vertex-cut of G. Then we shall prove that G has an independent set X with $|X| \geqslant n/2$ and $\{u,v\} \subseteq V(G)-X$.

Let P be a longest path of G starting at one of u and v but not passing through the other. Let $\{u,v\} = \{x_0,x_1\}$ and $P = x_1x_2\ldots x_t$. Set $D = G - V(P)$ and $r = |V(D)|$. If $r = 1$, then $x_0x_t \notin E$ and so $d(x_0,P)+d(x_t,P) \geqslant n$. By Lemma 2.1, G has a hamiltonian path from x_1 to x_0, a contradiction. Hence $r \geqslant 2$. As $\{x_0,x_1\}$ is not a vertex-cut of G, we let s be the smallest integer in $\{2,3,\ldots,t-1\}$ such that $d(x_s, D-x_0) \geqslant 1$. Let $y_0 \in V(D)-\{x_0\}$ such that $y_0x_s \in E$. For each $y \in V(D)-\{x_0\}$, we must have that $d(y,P)+d(x_t,P) \leqslant t$ by Lemma 2.1, and therefore $d(y,D)+d(x_t,D) \geqslant r$. Thus $d(x_t,D) > 0$. It follows that $N(x_t,D) = \{x_0\}$ and D is a complete subgraph of G. Furthermore, $d(y,P)+d(x_t,P) = t$ for each $y \in V(D)-\{x_0\}$. Let $L_1 = x_1x_2\ldots x_s$ and $L_2 = x_{s+1}x_{s+2}\ldots x_t$. Set $I = \{x_{i-1}|x_iy_0 \in E, s+1 \leqslant i \leqslant t-1\}$. Clearly, $y_0x_{s+1} \notin E$ and $N(x_t,L_2) \cap I = \varnothing$ for otherwise $G[V(P)\cup\{y_0\}]$ has a hamiltonian path starting at x_1. This implies that $d(y_0,L_2)+d(x_t,L_2) \leqslant |V(L_2)|-1 = t-s-1$, and thus $d(y_0,L_1)+d(x_t,L_1) \geqslant s+1$. As $x_0y_0x_sx_{s+1}x_{s+2}\ldots x_t$ is a path in G and by the maximality of P, we must have $s \geqslant 3$. Clearly, $x_{s-1}x_t \notin E$ for otherwise $x_1x_2\ldots x_{s-1}x_tx_{t-1}\ldots x_sy_0$ is a longer path than P in G. Therefore $N(x_t,L_1) = V(L_1)-\{x_{s-1}\}$ and $y_0x_1 \in E$. If $s \geqslant 4$, then $x_0y_0x_sx_{s+1}\ldots x_tx_2x_3\ldots x_{s-1}$ is a longer path than P in G, a contradiction. Hence $s = 3$. If $r \geqslant 3$ then $x_0y'y_0x_3x_4\ldots x_t$ is a longer path than P in G with $y' \in V(D)-\{x_0,y_0\}$, a contradiction. Hence $r = 2$. Let $P' = x_0y_0x_3x_4\ldots x_t$. Then P' is a path in G starting at x_0 without passing through x_1. Furthermore, P' and P have the same length. Therefore we may assume w.l.o.g. that $d(y_0,G) \geqslant n/2$ as $d(y_0,G)+d(x_2,G) \geqslant n$. Let $X = \{x_{i+1}|x_iy_0 \in E, 1 \leqslant i \leqslant t-1\} \cup \{y_0\}$. We see that X is an independent set of G for otherwise $G[V(P)\cup\{y_0\}]$ has a hamiltonian path starting at x_1. Clearly, $|X| \geqslant n/2$ and $\{x_0,x_1\} \subseteq V(G)-X$. This proves the lemma. □

Lemma 2.6. [7] *If $P = x_1x_2\ldots x_m$ is a path of G with $m \geqslant 3$ such that $d(x_1,P)+d(x_m,P) \geqslant m$, then G has a cycle C such that $V(C) = V(P)$. Moreover, if $d(x,G)+d(y,G) \geqslant n$ for any two nonadjacent vertices x and y of G, then G is hamiltonian.*

Lemma 2.7. *Let t be a positive integer and let G be a graph of order $n \geqslant 4t$. Suppose that $d(x) \geqslant \lceil n/2 \rceil$ for each $x \in V(G)$. Then G has t independent quadrilaterals $Q_1,Q_2,\ldots,Q_t$ such that $G - V(\cup_{i=1}^t Q_i)$ has a hamiltonian path.*

Proof. Let $r = n-4t$. We use induction on r to prove the lemma. When $r \in \{0,1,4\}$, the lemma is true by Theorem A and Theorem B. Suppose $r = 2$. Then G has t independent quadrilaterals $Q_1,\ldots,Q_t$ by Theorem B. If the two vertices of $G-V(\cup_{i=1}^t Q_i)$, say x and y, are not adjacent, then we would have that $d(x,\cup_{i=1}^t Q_i)+d(y,\cup_{i=1}^t Q_i) \geqslant 4t+2$, and therefore $d(x, Q_i) + d(y, Q_i) \geqslant 5$ for some $i \in \{1, 2, \ldots, t\}$. By Lemma 2.2, the lemma holds. Next, suppose $r = 3$. Using the above proof, we see that G has t independent quadrilaterals $Q_1, \ldots, Q_t$ such that $G - V(\cup_{i=1}^t Q_i)$ has at least one edge. Subject to this, we let $\sum_{i=1}^t \tau(Q_i)$ be as large as possible. Let $V(G) - V(\cup_{i=1}^t Q_i) = \{x_1, x_2, x_3\}$ be such that $x_1x_2 \in E$. If $x_3x_1 \in E$ or $x_3x_2 \in E$, we have nothing to prove. Hence we assume that $x_3x_1 \notin E$ and $x_3x_2 \notin E$. Then we see that there exists Q_i, say $Q_i = Q_1$, such that $d(x_1, Q_1) + d(x_3, Q_1) \geqslant 5$. By Lemma 2.2, $G[V(Q_1) \cup \{x_1, x_3\}]$ contains a quadrilateral Q_1' and an edge e such that Q_1' and e are indepedent and e is incident with exactly one of x_1 and x_3. If e is incident with x_1, then e and x_1x_2 together contains a path of order 3 and we are done. Therefore we may assume that e is incident with x_3. Say $y_1 \in V(Q_1)$ with $y_1x_3 \in E$. Let $Q_1 = a_1a_2a_3a_4a_1$. Suppose that $d(x_3, Q_1) \geqslant 3$. Say $N(x_3, Q_1) \supseteq \{a_1, a_2, a_3\}$. Then $\tau(x_3a_1a_2a_3x_3) \geqslant \tau(Q_1)$ with equality only if $a_2a_4 \in E$. By our choice of $Q_i (1 \leqslant i \leqslant t)$, we must have that $a_2a_4 \in E$. Thus for each $i \in \{1, 2, 3, 4\}$, $G[\{a_1, a_2, a_3, a_4, x_3\} - \{a_i\}]$ contains a quadrilateral. As $d(x_1, Q_1) \geqslant 1$, we see that the lemma holds. Hence we may assume that $d(x_3, Q_1) \leqslant 2$. Thus $d(x_1, Q_1) \geqslant 3$. Suppose that we also have that $d(x_2, Q_1) + d(x_3, Q_1) \geqslant 5$. Then $d(x_2, Q_1) \geqslant 3$. This implies that there exists $\{i, j\} \subseteq \{1, 2, 3, 4\}$ with $i \neq j$ such that $x_1a_ia_{i+1}x_2x_1$ and $x_1a_ja_{j+1}x_2x_1$ are two quadrilaterals in G. Thus the lemma holds if x_3 is adjacent to some vertex of $\{a_{i+2}, a_{i+3}, a_{j+2}, a_{j+3}\}$. Hence we may assume that x_3 is not adjacent to any vertex of this set, which implies that $d(x_3, Q_1) \leqslant 1$. It follows that $d(x_1, Q_1) = d(x_2, Q_1) = 4$ and $d(x_3, Q_1) = 1$, and clearly, the lemma holds in this situation, too. To finish the proof, we finally assume that $d(x_2, Q_1) + d(x_3, Q_1) \leqslant 4$. Then $d(x_2, \cup_{i=2}^t Q_i) + d(x_3, \cup_{i=2}^t Q_i) \geqslant 4t + 2 - 5 = 4(t-1) + 1$. This implies that there exists Q_i with $i \geqslant 2$, say $i = 2$, such that $d(x_2, Q_2) + d(x_3, Q_2) \geqslant 5$. As above, with Q_2 and x_2 in place of Q_1 and x_1, we may assume that Q_2 has a vertex y_2 such that $G[V(Q_2) \cup \{x_2\} - \{y_2\}]$ contains a quadrilateral Q_2' and $x_3y_2 \in E$. Since $y_1x_3y_2$ is a path in G, we see the lemma holds. Therefore the lemma holds if $r \leqslant 4$. We now assume that the lemma is true if the value of $(n - 4t)$ is less than r with $r \geqslant 5$. Say $n - 4t = r$. Then $n - 4(t+1) = r - 4$. By the induction hypothesis, G has $t+1$ independent quadrilaterals $Q_1, \ldots, Q_{t+1}$ such that $G - V(\cup_{i=1}^{t+1} Q_i)$ has a path P of order $r - 4$. Let $P = x_1x_2 \ldots x_{r-4}$. Then we may assume that for each $j \in \{1, 2, \ldots, t+1\}$, $d(x_1, Q_j) = d(x_{r-4}, Q_j) = 0$ holds for otherwise $G[V(Q_j \cup P)]$ has a hamiltonian path and we are done. Thus $d(x_1, P) + d(x_{r-4}, P) \geqslant n$, and by Lemma 2.6, $G[V(P)]$ is hamiltonian. As G is connected, there exists Q_j such that $\sum_{i=1}^{r-4} d(x_i, Q_j) > 0$ and therefore $G[V(Q_j \cup P)]$ has a hamiltonian path. Thus the lemma is true for $n - 4t = r$. This proves the lemma. □

Lemma 2.8. [5] *Let $C = x_1x_2 \ldots x_mx_1$ be a cycle of G. Let $x_i, x_j \in V(C)$ with $i \neq j$. If $d(x_i, C) + d(x_j, C) \geqslant m+1$, then G has a path P from x_{i+1} to x_{j+1} such that $V(P) = V(C)$.*

Lemma 2.9. *Suppose that G has a hamiltonian path and that $d(x, G) + d(y, G) \geqslant n + s$ for any two endvertices x and y of a hamiltonian path of G, where s is a fixed nonnegative integer. Then for any two distinct vertices u and v of G, $d(u, G) + d(v, G) \geqslant n + s$ holds.*

Proof. By Lemma 2.6, G is hamiltonian. Let $C = x_1x_2 \ldots x_nx_1$ be a hamiltonian cycle. Suppose, for a contradiction, that $d(x_i, G) + d(x_j, G) \leqslant n + s - 1$ for some $1 \leqslant i <$

$j \leqslant n$. By the hypothesis, $d(x_{i-1}, G) + d(x_i, G) \geqslant n + s$ and $d(x_{j-1}, G) + d(x_j, G) \geqslant n + s$. Then we see that $d(x_{i-1}, G) + d(x_{j-1}, G) \geqslant n + s + 1$. By Lemma 2.8, G has a hamiltonian path from x_i to x_j, and by the hypothesis again, $d(x_i, G) + d(x_j, G) \geqslant n + s$, a contradiction. □

3 Proof of Theorem *C*

Let k be a positive integer and G a graph of order $n \geqslant 4k$. Assume $\delta(G) \geqslant \lceil n/2 \rceil$. Suppose, for a contradiction, that G does not contain k independent cycles covering all the vertices of G such that $k-1$ of them are quadrilaterals. By Theorem A, $n > 4k$.

Let $t = n - 4(k-1)$. By Lemma 2.7, G has k independent quadrilaterals $Q_1, \ldots, Q_k$ such that $G - V(\cup_{i=1}^k Q_i)$ has a hamiltonian path P. If $t - 4 = 1$, then we readily see that $d(u, Q_i) \geqslant 3$ where $V(P) = \{u\}$ for some $i \in \{1, \ldots, k\}$ because $\delta(G) \geqslant \lceil n/2 \rceil$ and so $G[V(Q_i \cup P)]$ is hamiltonian, a contradiction. Hence we have $t - 4 \geqslant 2$. For convenience, let $r = t - 4$. As $G[V(Q_i \cup P)]$ is not hamiltonian, for any two endvertices u and v of a hamiltonian path of $G - V(\cup_{i=1}^k Q_i)$, we have

$$d(u, Q_i) + d(v, Q_i) \leqslant 4 \text{ for all } i \in \{1, \ldots, k\} \tag{5}$$

and therefore

$$d(u, G - V(\cup_{i=1}^k Q_i)) + d(v, G - V(\cup_{i=1}^k Q_i)) \geqslant r + \sigma \tag{6}$$

where $\sigma = 1$ if r is odd and otherwise $\sigma = 0$. By Lemma 2.6, $G - V(\cup_{i=1}^k Q_i)$ is hamiltonian if $r \geqslant 3$. Let $H = \cup_{i=1}^k Q_i$ and $D = G - V(H)$. By (6) and Lemma 2.9, we have

$$d(x, D) + d(y, D) \geqslant r + \sigma \text{ for all } \{x, y\} \subseteq V(D) \text{ with } x \neq y. \tag{7}$$

We now divide the proof into the following two cases.

Case 1. $r \geqslant 5$.

In this case, By (7), D is hamiltonian. We choose a hamiltonian path P of D as follows. If for each $x \in V(D)$, D has a quadrilateral Q such that $D - V(Q)$ has a hamiltonian path starting at x, let P be a hamiltonian path with an endvertex u such that $d(u, Q_i) \geqslant 1$ for some $i \in \{1, \ldots, k\}$. Such a path exists because G is connected. We may assume $e(u, Q_1) \geqslant 1$ in this case. Otherwise by (7) and Lemma 2.4, we see that $\sigma = 0$ and D has order 6. Furthermore, D has an edge uv such that D has a hamiltonian path P from u to v, $D - u - v$ has a quadrilateral and $d(u, D) + d(v, D) = 6$. Then equality holds in (5) and (6) with respect to $\{u, v\}$, and therefore $d(u, Q_i) + d(v, Q_i) \geqslant 1$ for some $i \in \{1, \ldots, k\}$. In this case, we may assume $d(u, Q_1) \geqslant 1$. In the former case, let Q_1' be a quadrilateral of D such that $D - V(Q_1')$ has a hamiltonian path starting at u. Subject to this, we further choose Q_1' such that $D - V(Q_1')$ does not contain a vertex-cut of cardinality 2 if there exists such a choice. In the latter, let Q_1' be a quadrilateral of $D - u - v$ and $P = uv$. Set $D' = G[V(D \cup Q_1) - V(Q_1')]$. As $d(u, Q_1) \geqslant 1$, D' has a hamiltonian path.

Replacing Q_1' and D' in the above proof of (5), (6) and (7), we see that D' is hamiltonian, too. Let L be a hamiltonian cycle of D'. Then the number of edges of L in between Q_1 and $D' - V(Q_1)$ must be even. This allows us to see that there exist two independent edges x_1y_1 and x_2y_2 between Q_1 and $D' - V(Q_1)$ with $\{y_1, y_2\} \subseteq V(Q_1)$ such that $G[V(Q_1)]$ has a hamiltonian path from y_1 to y_2. If D has a hamiltonian path from x_1 to x_2, then $G[V(D \cup Q_1)]$ is hamiltonian and we are done. By Lemma 2.5, we see that either $\{x_1, x_2\}$ is a vertex-cut of D or D has an independent set X with $|X| \geqslant r/2$ and $\{x_1, x_2\} \subseteq V(D) - X$. Let us first assume that $\{x_1, x_2\}$ is a vertex-cut of D. By (7), we see that $D - x_1 - x_2$ has exactly two components, say D_1 and D_2 such that $D_1 \cong D_2 \cong K_{(r-2)/2}$ and $d(x_1, D_1 \cup D_2) = d(x_2, D_1 \cup D_2) = r - 2$. As $D_1 \cup D_2 \supseteq Q'_1$, we see that $r \geqslant 10$. Let Q''_1 be a quadrilateral in $D_1 + x_1$ with $x_1 \in V(Q''_1)$. Let $z_1 \in V(D_1) - V(Q''_1)$ and $z_2 \in V(D_2)$. Clearly, $d(z_1, D) + d(z_2, D) = r$ and $D - V(Q''_1)$ has a hamiltonian path from z_1 to z_2. Then equality holds in (5) and (6) with respect to $\{z_1, z_2\}$. It follows that $d(z_j, Q_i) \geqslant 1$ for some $j \in \{1, 2\}$ and $i \in \{1, \ldots, k\}$. But $D - V(Q''_1)$ does not contain a vertex-cut of D with cardinality 2. This contradicts the choice of Q'_1. Therefore D has an independent set X with $|X| \geqslant r/2$ and $\{x_1, x_2\} \subseteq V(D) - X$. By (7), we see that $|X| = r/2$ and D contains a complete bipartite subgraph with $(X, V(D) - X)$ as its bipartition. As mentioned in the beginning of this paragraph, D' is hamiltonian and so we readily see that $d(x, Q_1) > 0$ for some $x \in X$. It follows that $G[V(D \cup Q_1)]$ is hamiltonian, a contradiction.

Case 2. $2 \leqslant r \leqslant 4$.

In this case, we choose the k independent quadrilaterals $Q_1, \ldots, Q_k$ and the path P of order r such that

$$\sum_{i=1}^{k} \tau(Q_i) \text{ is maximum.} \tag{8}$$

By (6), D is hamiltonian if $r \geqslant 3$. We break into the following three cases.

Case 2.1. $r = 3$.

Then D is a triangle, say $D = x_1x_2x_3x_1$. We have that $\sum_{i=1}^{3} d(x_i, H) \geqslant 3\lceil n/2 \rceil - 6 = 3(2k + 2) - 6 = 6k$. This implies that there exists Q_i in H, say $Q_i = Q_1$, such that $\sum_{i=1}^{3} d(x_i, Q_1) \geqslant 6$. This further implies that there exist two independent edges e_1 and e_2 between D and Q_1. Say $Q_1 = a_1a_2a_3a_4a_1$. As $G[V(D \cup Q_1)]$ is not hamiltonian, we may assume that $e_1 = a_1x_1$ and $e_2 = a_3x_2$, and then we see that $d(a_2, D) = d(a_4, D) = 0$ and $a_2a_4 \notin E$. Consequently, $d(a_1, D) = d(a_3, D) = 3$. Let Q'_1 be a quadrilateral of $D + a_1$ and $P' = a_2a_3a_4$. Clearly, $\tau(Q'_1) > \tau(Q_1)$, contradicting (8).

Case 2.2. $r = 4$.

Since G is hamiltonian, we have nothing to prove when $k = 1$. Therefore $k \geqslant 2$. Let $Q_0 = x_1x_2x_3x_4x_1$ be a quadrilateral of D. Clearly, $\sum_{i=1}^{4} d(x_i, H) \geqslant 4\lceil n/2 \rceil - 12 = 8(k + 1) - 12 = 6k + 2k - 4$. This implies that there exists Q_i in H, say $Q_i = Q_1$, such that $\sum_{i=1}^{4} d(x_i, Q_1) \geqslant 6$. This further implies that there exists two independent edges, say u_1w_1 and u_2w_2 with $\{u_1, u_2\} \subseteq V(Q_0)$ and $\{w_1, w_2\} \subseteq V(Q_1)$, between Q_0 and Q_1 such that either $u_1u_2 \in E(Q_0)$ or $w_1w_2 \in E(Q_1)$. We may assume w.l.o.g. that $w_1w_2 \in E(Q_1)$. Therefore $u_1u_2 \notin E(Q_0)$ as $G[V(Q_0 \cup Q_1)]$ is not hamiltonian. Say w.l.o.g.

$\{u_1, u_2\} = \{x_1, x_3\}$. Then for the same reason, we see that $d(x_2, Q_1) = d(x_4, Q_1) = 0$ and $x_2x_4 \notin E$. Thus $d(x_2, H - V(Q_1)) + d(x_4, H - V(Q_1)) \geqslant 2\lceil n/2 \rceil - 4 = 4(k-1) + 4$. This implies that there exists Q_i in $H - V(Q_1)$, say $Q_i = Q_2$, such that $d(x_2, Q_2) + d(x_4, Q_2) \geqslant 5$. As $G[V(Q_0 \cup Q_2)]$ is not hamiltonian, we see that $x_1x_3 \notin E$. Then equality holds in (5) and (6) with respect to $\{x_j, x_{j+1}\}$ for each $j \in \{1, 2, 3, 4\}$ and $i \in \{1, 2, \dots, k\}$, that is, $d(x_j, Q_i) + d(x_{j+1}, Q_i) = 4$ for each $j \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \dots, k\}$. Thus $d(x_1, Q_1) = d(x_3, Q_1) = 4$. As $d(x_2, Q_2) + d(x_4, Q_2) \geqslant 5$, it is also easy to see that $d(x_1, Q_2) = d(x_3, Q_2) = 0$ for otherwise $G[V(Q_0 \cup Q_2)]$ is hamiltonian. Thus $d(x_2, Q_2) = d(x_4, Q_2) = 4$ and $d(x_1, Q_1) = d(x_3, Q_1) = 4$. Let y be an arbitrary vertex of Q_1 and z an arbitrary vertex of Q_2. Clearly, $Q_1 - y + x_3$ and $Q_2 - z + x_4$ are hamiltonian and yx_1x_2z is a path in G. Similar to the proof of (6), we see that $G[\{y, x_1, x_2, z\}]$ must be hamiltonian. Consequently, $yz \in E$. This argument implies that $d(w, Q_2) = 4$ for all $w \in V(Q_1)$. It follows that $G[V(Q_0 \cup Q_1 \cup Q_2) - \{y, x_1, x_2, z\}]$ is hamiltonian and we are done.

Case 2.3. $r = 2$.

Let $D = x_1x_2$. As $\delta(G) \geqslant 2k+1$ and by (5) and (6), we see that $d(x_1, Q_i) + d(x_2, Q_i) = 4$ for all $i \in \{1, 2, \dots, k\}$. We claim that for each $i \in \{1, 2, \dots, k\}$, either $d(x_1, Q_i) = 0$ or $d(x_2, Q_i) = 0$. If this is not true, say $d(x_1, Q_1) > 0$ and $d(x_2, Q_1) > 0$. Let $Q_1 = a_1a_2a_3a_4a_1$ with $x_1a_1 \in E$. As $G[V(D \cup Q_1)]$ is not hamiltonian, we see that $N(x_1, Q_1) = N(x_2, Q_1) = \{a_1, a_3\}$ and $a_2a_4 \notin E$. Let $Q'_1 = x_1a_1x_2a_3x_1$. Clearly, $\tau(Q'_1) = \tau(Q_1) + 1$. We also have that $d(a_2, H - V(Q_1)) + d(a_4, H - V(Q_1)) \geqslant 2(2k+1) - 4 = 4(k-1) + 2$. This implies that there exists Q_i in $H - V(Q_1)$, say $Q_i = Q_2$, such that $d(a_2, Q_2) + d(a_4, Q_2) \geqslant 5$. As $G[V(Q_2) \cup \{a_2, a_4\}]$ is not hamiltonian and by Lemma 2.3, $G[V(Q_2) \cup \{a_2, a_4\}]$ has a quadrilateral Q'_2 and a path P' of order 2 such that $\tau(Q'_2) \geqslant \tau(Q_2)$ and $V(Q'_2) \cap V(P') = \varnothing$. Replacing Q_1 and Q_2 by Q'_1 and Q'_2, we see that (8) is violated. Therefore our claim holds.

Next, we claim that $G[N(x_1, G - x_2)]$ is a complete subgraph of G and $G[N(x_2, G - x_1)]$ is a complete subgraph of G. For the proof, let u be an arbitrary vertex of $N(x_1, G - x_2)$. We shall show that u is adjacent to every vertex of $N(x_1, G - x_2 - u)$. Let $Q_1 = a_1a_2a_3a_4a_1$. Say w.l.o.g. $u = a_1$. Let $G_1 = G[V(D \cup Q_1)]$ and $H_1 = H - V(Q_1)$. Then $d(x_2, H_1) + d(a_1, H_1) \geqslant 2(2k+1) - 5 = 4(k-1) + 1$. This implies that there exists Q_i in H_1, say $Q_i = Q_2$, such that $d(x_2, Q_2) + d(a_1, Q_2) \geqslant 5$. Thus we must have $d(x_2, Q_2) = 4$ and $d(x_1, Q_2) = 0$. If $d(a_1, Q_2) \geqslant 2$, then $G[V(Q_2) \cup \{x_2, a_1\}]$ is hamiltonian. As $Q_1 - a_1 + x_1$ is hamiltonian, the theorem holds, a contradiction. Hence $d(a_1, Q_2) = 1$. Let $Q_2 = b_1b_2b_3b_4b_1$ with $a_1b_1 \in E$. Similarly, we must have that $d(b_1, Q_1) = 1$. Replacing Q_1 and Q_2 by $Q'_1 = x_1a_2a_3a_4x_1$ and $Q'_2 = x_2b_2b_3b_4x_2$, we see, by (8), that $\{a_1a_3, b_1b_3\} \subseteq E$. Note that this argument implies that $G[V(Q_i)]$ is a complete graph of order 4 for all $i \in \{1, \dots, k\}$. With respect to the choice of $\{a_1b_1, Q'_1, Q'_2, Q_3, \dots, Q_k\}$, we can also show that for each $i \in \{3, \dots, k\}$, either $d(a_1, Q_i) = 4$ or $d(b_1, Q_i) = 4$. If there exists Q_i in $\{Q_3, \dots, Q_k\}$ such that $d(x_1, Q_i) = d(b_1, Q_i) = 4$, then we would see that each of $Q_i + x_1 + b_1$ and $Q_2 - b_1 + x_2$ is hamiltonian and we are done. Hence we must have $d(a_1, Q_i) = 4$ for each $i \in \{3, \dots, k\}$ with $d(x_1, Q_i) = 4$. Therefore $G[N(x_1, G - x_2)]$ is a complete subgraph of G. Similarly, $G[N(x_2, G - x_1)]$ is a complete subgraph of G. The above argument also implies that $d(w, G[N(x_i, G - x_j)]) = 1$ for each $w \in N(x_j, G - x_i)$ with $\{i, j\} = \{1, 2\}$, that is, there are $2k$ independent edges between $N(x_1, G - x_2)$ and $N(x_2, G - x_1)$. It is easy to see that G contains k required independent cycles in this case. This completes the proof of the theorem.

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