Rook placements and Jordan forms of upper-triangular nilpotent matrices

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Abstract

The set of n by n upper-triangular nilpotent matrices with entries in a finite field $\mathbb{F}_q$ has Jordan canonical forms indexed by partitions $\lambda \vdash n$. We present a combinatorial formula for computing the number $F_\lambda(q)$ of matrices of Jordan type λ as a weighted sum over standard Young tableaux. We construct a bijection between paths in a modified version of Young's lattice and non-attacking rook placements, which leads to a refinement of the formula for $F_\lambda(q)$.

Keywords: nilpotent matrices, finite fields, Jordan form, rook placements, Young tableaux, set partitions.

1 Introduction

In the beautiful paper *Variations on the Triangular Theme* [7], Kirillov studied various structures on the set of triangular matrices. Let $G = G_n(\mathbb{F}_q)$ denote the group of n by n invertible upper-triangular matrices over the field $\mathbb{F}_q$ having q elements, and let $\mathfrak{g} = \mathfrak{g}_n(\mathbb{F}_q) = \mathrm{Lie}(G_n(\mathbb{F}_q))$ denote the corresponding Lie algebra of n by n upper-triangular nilpotent matrices over $\mathbb{F}_q$. The problem of determining the set $\mathcal{O}_n(\mathbb{F}_q)$ of adjoint G-orbits in $\mathfrak{g}$ remains challenging, and a more tractable task is to study a decomposition of $\mathcal{O}_n(\mathbb{F}_q)$ via the Jordan canonical form. Let $\lambda \vdash n$ be a partition of n with r positive parts

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$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, and let

$$J_\lambda = J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_r}, \qquad \text{where} \qquad J_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{i \times i}$$

is the i by i elementary Jordan matrix with all eigenvalues equal to zero. If $X \in \mathfrak{g}_n(\mathbb{F}_q)$ is similar to J_λ under $GL_n(\mathbb{F}_q)$, then X is said to have *Jordan type* λ. Each conjugacy class contains a unique Jordan matrix J_λ, so these classes are indexed by the partitions of n. Evidently, the Jordan type of X depends only on its adjoint G-orbit.

Let $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q) \subseteq \mathfrak{g}_n(\mathbb{F}_q)$ be the set of upper-triangular nilpotent matrices of fixed Jordan type λ, and let

$$F_\lambda(q) = |\, \mathfrak{g}_{n,\lambda}(\mathbb{F}_q)\, | \,. \tag{1}$$

Springer showed that $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ is an algebraic manifold with f^λ irreducible components, where f^λ is the number of standard Young tableaux of shape λ, and each of which has dimension $\binom{n}{2} - n_\lambda$, where n_λ is an integer defined in Equation 10. These quantities appear in the study of $F_\lambda(q)$.

In Section 2, we show that the numbers $F_\lambda(q)$ satisfy a simple recurrence equation, and that they are polynomials in q with integer coefficients. As a consequence of the recurrence equation in Theorem 8, it follows that the coefficient of the highest degree term in $F_\lambda(q)$ is f^λ, and $\deg F_\lambda(q) = \binom{n}{2} - n_\lambda$. Equation (9) is a combinatorial formula for $F_\lambda(q)$ as a sum over standard Young tableaux of shape λ that can be derived from the recurrence equation.

The cases $F_{(1^n)}(q) = 1$ and $F_{(n)}(q) = (q-1)^{n-1} q^{\binom{n-1}{2}}$ are readily computed, since the matrix in $\mathfrak{g}_n(\mathbb{F}_q)$ of Jordan type (1^n) is the matrix of zero rank, and the matrices in $\mathfrak{g}_n(\mathbb{F}_q)$ of Jordan type (n) are the matrices of rank equal to $n-1$. Section 2 concludes with explicit formulas for $F_\lambda(q)$ in several other special cases of λ, including hook shapes, two-rowed partitions and two-columned partitions.

In Section 3, we explore a connection of $F_\lambda(q)$ with rook placements. In their study of a formula of Frobenius, Garsia and Remmel [4] introduced the *q-rook polynomial*

$$R_{B,k}(q) = \sum_{c \in \mathcal{C}(B,k)} q^{\mathrm{inv}(c)},$$

which is a sum over the set $\mathcal{C}(B,k)$ of non-attacking placements of k rooks on a Ferrers board B, and $\mathrm{inv}(c)$, defined in Equation (13), is the number of inversions of c. In the case when $B = B_n$ is the staircase-shaped board, Garsia and Remmel showed that $R_{B_n,k}(q) = S_{n,n-k}(q)$ is a *q-Stirling number of the second kind.* These numbers are defined by the recurrence equation

$$S_{n,k}(q) = q^{k-1} S_{n-1,k-1}(q) + [k]_q S_{n-1,k}(q) \quad \text{for} \quad 0 \leqslant k \leqslant n,$$

with initial conditions $S_{0,0}(q) = 1$, and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.

It was shown by Solomon [12] that non-attacking placements of k rooks on rectangular $m \times n$ boards are naturally associated to m by n matrices with rank k over $\mathbb{F}_q$. By identifying a Ferrers board B inside an n by n grid with the entries of an n by n matrix, Haglund [5] generalized Solomon's result to the case of non-attacking placements of k rooks on Ferrers boards, and obtained a formula for the number of n by n matrices with rank k whose support is contained in the Ferrers board region. A special case of Haglund's formula shows that the number of n by n nilpotent upper-triangular matrices of rank k is

$$P_{B_n,k}(q) = (q-1)^k q^{\binom{n}{2}-k} R_{B_n,k}(q^{-1}). \tag{2}$$

Now, a matrix in $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ has rank $n - \ell(\lambda)$, where $\ell(\lambda)$ is the number of parts of λ, so the number of matrices in $\mathfrak{g}_n(\mathbb{F}_q)$ with rank k is

$$P_{B_n,k}(q) = \sum_{\lambda \vdash n:\ \ell(\lambda)=n-k} F_\lambda(q). \tag{3}$$

Given Equations 2 and 3, it would be natural to ask whether it is possible to partition the placements $\mathcal{C}(B_n, k)$ into disjoint subsets so that the sum over each subset of placements gives $F_\lambda(q)$. A central goal of this paper is to study the connection between upper-triangular nilpotent matrices over $\mathbb{F}_q$ and non-attacking rook placements on the staircase-shaped board B_n. Theorem 28 shows that there is a weight-preserving bijection Φ between rook placements on B_n and paths in a graph $\mathcal{Z}$ (see Figure 5), which is a multi-edged version of Young's lattice. As a result, we obtain Corollary 30, which gives a formula for $F_\lambda(q)$ as a sum over certain rook placements that can be viewed as a generalization of Haglund's formula in Equation (2).

There is a classically known bijection between rook placements in $\mathcal{C}(B_n, k)$ and set partitions of $[n]$ with $n - k$ parts, so it is logical to next study the connection between $F_\lambda(q)$ and set partitions. We do this in Section 4. Theorem 34 describes the construction of a new (weight-preserving) bijection Ψ between rook placements and set partitions. These bijections allow us to refine Equation (9) to a sum over set partitions (or rook placements). We also discuss the significance of the polynomials $F_C(q)$ indexed by rook placements in a special case.

This paper is the full version of the extended abstract [15].

2 Formulas for $F_\lambda(q)$

The recurrence equation for $F_\lambda(q)$ in Theorem 8 can be found in [1, Division Theorem], where Borodin considers the matrices as particles of a certain mass and studies the asymptotic behaviour of the formula. A preliminary version of the idea first appeared in [7]. In this section, we give an elementary proof of the formula, and investigate some of the combinatorial properties of $F_\lambda(q)$.

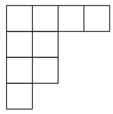
2.1 The recurrence equation for $F_\lambda(q)$

A *partition* λ of a nonnegative integer n, denoted by $\lambda \vdash n$, is a non-increasing sequence of nonnegative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ with $|\lambda| = \sum_{i=1}^{n} \lambda_i = n$. If λ has r positive parts, write $\ell(\lambda) = r$. A partition λ can be represented by its Ferrers diagram in the English notation, which is an array of λ_i boxes in the ith row, with the boxes justified upwards and to the left. Let λ'_j denote the size of the jth column of λ.

Young's lattice $\mathcal{Y}$ is the lattice of partitions ordered by the inclusion of their Ferrers diagrams; that is, $\mu \leqslant \lambda$ if and only if $\mu_i \leqslant \lambda_i$ for every i. In particular, μ is covered by λ in the Hasse diagram of $\mathcal{Y}$ and we write $\mu \lessdot \lambda$ if the Ferrers diagram of λ can be obtained by adding a box to the Ferrers diagram of μ. See Figure 1.

Example 1. The partition

$$\lambda = (4,2,2,1) \vdash 9 \qquad \text{has diagram}$$



and columns $\lambda'_1 = 4, \lambda'_2 = 3, \lambda'_3 = 1, \lambda'_4 = 1$.

Lemma 2. *Let $\lambda \vdash n$ be a partition whose Ferrers diagram has r rows and c columns. The Jordan matrix J_λ satisfies*

$$\operatorname{rank}\left(J_\lambda^k\right) = \begin{cases} \lambda'_{k+1} + \cdots + \lambda'_c, & \text{if } 0 \leqslant k < c, \\ 0, & \text{if } k \geqslant c. \end{cases}$$

Proof. The i by i elementary Jordan matrix J_i has $\operatorname{rank}\left(J_i^k\right) = i - k$ if $0 \leqslant k \leqslant i$, and its rank is zero otherwise, so the Jordan matrix $J_\lambda = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ has

$$\operatorname{rank}\left(J_\lambda^k\right) = \sum_{i=1}^{r} \operatorname{rank}\left(J_{\lambda_i}^k\right) = \sum_{i:\lambda_i \geqslant k} \operatorname{rank}\left(J_{\lambda_i}^k\right) = \sum_{j=k+1}^{c} \lambda'_j,$$

for $0 \leqslant k < c$, which is the number of boxes in the last $c - k$ columns of λ. □

Remark 3. Matrices which are similar have the same rank, so if $X \in \mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$, then $\operatorname{rank}\left(X^k\right) = \operatorname{rank}\left(J_\lambda^k\right)$ for all $k \geqslant 0$. Conversely, let $\lambda, \nu \vdash n$. It follows from Lemma 2 that $\operatorname{rank}\left(J_\lambda^k\right) = \operatorname{rank}\left(J_\nu^k\right)$ for all $k \geqslant 0$ if and only if $\lambda = \nu$. Thus if $X \in \mathfrak{g}_n(\mathbb{F}_q)$ is a matrix such that $\operatorname{rank}\left(X^k\right) = \operatorname{rank}\left(J_\lambda^k\right)$ for all $k \geqslant 0$, then X is similar to J_λ.

Example 4. If a matrix $X \in \mathfrak{g}_n(\mathbb{F}_q)$ has Jordan type $\lambda = (4,2,2,1)$, then $\operatorname{rank}(X) = 5$, $\operatorname{rank}(X^2) = 2$, $\operatorname{rank}(X^3) = 1$, and $\operatorname{rank}(X^4) = 0$.

If $X \in \mathfrak{g}_n(\mathbb{F}_q)$ is a matrix of the form

$$X = \begin{bmatrix} J_\mu & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix},$$

where $\mu \vdash n-1$, and $\mathbf{v} = [v_1, \ldots, v_{n-1}]^T \in \mathbb{F}_q^{n-1}$, then the first order leading principal submatrix of X^k is J_μ^k, and for $1 \leqslant k \leqslant n$, we define column vectors $\mathbf{v}^k = [v_1^k, \ldots, v_{n-1}^k]^T \in \mathbb{F}_q^{n-1}$ by

$$X^k = \begin{bmatrix} J_\mu^k & \mathbf{v}^k \\ \mathbf{0} & 0 \end{bmatrix}.$$

For $i \geqslant 1$, let $\alpha_i = \mu_1 + \cdots + \mu_i$ be the sum of the first i parts of μ. The (i,j)th entry of J_μ^k is nonzero if and only if $j = i + k$, and $i, i+k \leqslant \alpha_b$ for all $b \geqslant 1$. It follows from this that

$$v_i^k = \begin{cases} v_{i+k-1}, & \text{if } i, i+k-1 \leqslant \alpha_b \text{ for all } b \geqslant 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

There is a simple way to visualize the vectors $\mathbf{v}^k$, which we illustrate with an example.

Example 5. Let $\mu = (4,2,1,1)$, so that $\alpha_1 = 4, \alpha_2 = 6, \alpha_3 = 7$, and $\alpha_4 = 8$. Let

$$X = \left[\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & & & & & v_1 \\ 0 & 0 & 1 & 0 & & & & & v_2 \\ 0 & 0 & 0 & 1 & & & & & v_3 \\ 0 & 0 & 0 & 0 & & & & & v_4 \\ \hline & & & & 0 & 1 & & & v_5 \\ & & & & 0 & 0 & & & v_6 \\ \hline & & & & & & 0 & & v_7 \\ \hline & & & & & & & 0 & v_8 \\ \hline & & & & & & & & 0 \end{array}\right] \quad \text{so that} \quad X^2 = \left[\begin{array}{ccccccccc} 0 & 0 & 1 & 0 & & & & & v_2 \\ 0 & 0 & 0 & 1 & & & & & v_3 \\ 0 & 0 & 0 & 0 & & & & & v_4 \\ 0 & 0 & 0 & 0 & & & & & 0 \\ \hline & & & & 0 & 0 & & & v_6 \\ & & & & 0 & 0 & & & 0 \\ \hline & & & & & & 0 & & 0 \\ \hline & & & & & & & 0 & 0 \\ \hline & & & & & & & & 0 \end{array}\right].$$

We may visualize the vectors $\mathbf{v}$ and $\mathbf{v}^2$ as fillings of the Ferrers diagram for μ:

$$\mathbf{v} = \begin{array}{|c|c|c|c|} \hline v_4 & v_3 & v_2 & v_1 \\ \hline v_6 & v_5 \\ \cline{1-2} v_7 \\ \cline{1-1} v_8 \\ \cline{1-1} \end{array} \quad \text{and} \quad \mathbf{v}^2 = \begin{array}{|c|c|c|c|} \hline 0 & v_4 & v_3 & v_2 \\ \hline 0 & v_6 \\ \cline{1-2} 0 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array}.$$

This way, a basis of $\ker X^k$ is the set of vectors filling the first k columns of the diagram.

Lemma 6. *If $X \in \mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ and its first order leading principal submatrix $Y \in \mathfrak{g}_{n-1,\mu}(\mathbb{F}_q)$, then $\lambda \gtrdot \mu$.*

Proof. We first consider the case $Y = J_\mu$. If μ has s parts, let $\alpha_i = \mu_1 + \cdots + \mu_i$ for $1 \leqslant i \leqslant s$. Then

$$\operatorname{rank}\left(X^k\right) - \operatorname{rank}\left(J_\mu^k\right) = \begin{cases} 0, & \text{if } v_{\alpha_i} = 0 \text{ for all } i \text{ such that } \mu_i \geqslant k, \\ 1, & \text{otherwise.} \end{cases} \tag{5}$$

Let $c \leqslant n$ be the smallest positive integer for which $\operatorname{rank}(X^c) - \operatorname{rank}(J_\mu^c) = 0$. Then Equation (5) implies that

$$\operatorname{rank}(X^k) - \operatorname{rank}(J_\mu^k) = \begin{cases} 0, & \text{if } k \geqslant c, \\ 1, & \text{if } k < c. \end{cases}$$

Together with Lemma 2, we deduce that

$$\lambda'_k - \mu'_k = \left(\operatorname{rank}(X^{k-1}) - \operatorname{rank}(X^k)\right) - \left(\operatorname{rank}(J_\mu^{k-1}) - \operatorname{rank}(J_\mu^k)\right) = \begin{cases} 1, & \text{if } k = c, \\ 0, & \text{if } k \neq c. \end{cases}$$

Therefore, $\lambda \gtrdot \mu$ in the case $Y = J_\mu$.

In the general case where Y is any matrix of Jordan type μ, then $\operatorname{rank}(Y^k) = \operatorname{rank}(J_\mu^k)$ for all $k \geqslant 0$, so the argument is the same. □

Let λ be the partition whose diagram is obtained by adding a box to the ith row and jth column of the diagram of the partition μ. Define the coefficient

$$c_{\mu,\lambda}(q) = \begin{cases} q^{|\mu| - \mu'_j}, & \text{if } j = 1, \\ q^{|\mu| - \mu'_{j-1}} \left(q^{\mu'_{j-1} - \mu'_j} - 1\right), & \text{if } j \geqslant 2. \end{cases} \tag{6}$$

Note that in the case $j \geqslant 2$, we have $\mu'_{j-1} - \mu'_j \geqslant 1$.

Lemma 7. *Let Y be an upper-triangular nilpotent matrix of Jordan type $\mu \vdash n-1$. If $\mu \lessdot \lambda$, then there are $c_{\mu,\lambda}(q)$ upper-triangular nilpotent matrices X of Jordan type λ whose first order leading principal submatrix is Y.*

Proof. By similarity, it suffices to consider the case $Y = J_\mu = J_{\mu_1} \oplus \cdots \oplus J_{\mu_m}$, where $\ell(\mu) = m$. Suppose X is a matrix of the form

$$X = \begin{bmatrix} J_\mu & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

of Jordan type λ such that λ is obtained by adding a box to μ in the ith row and jth column.

First consider the case $j \geqslant 2$. Following the proof of Lemma 6, we know that j is the unique integer where $\operatorname{rank}(X^{j-1}) = \operatorname{rank}(J_\mu^{j-1}) + 1$, and $\operatorname{rank}(X^j) = \operatorname{rank}(J_\mu^j)$. In order to satisfy the first condition, the entries in the vector $\mathbf{v}^{j-1}$ corresponding to the boxes in the $(j-1)$th column and rows $\geqslant i$ must not simultaneously be zero (refer to Equation (4) and Example 5), while in order to satisfy the second condition, the entries in the vector $\mathbf{v}^j$ corresponding to the boxes in the jth column of μ must all be zero. The remaining $n - 1 - \mu'_{j-1}$ entries of the vector $\mathbf{v}$ are free to be any element in $\mathbb{F}_q$, so there are

$$q^{n-1-\mu'_{j-1}} \left(q^{\mu'_{j-1} - \mu'_j} - 1\right)$$

possible matrices X whose leading principal submatrix is J_μ.

The case $j = 1$ is simpler. The necessary and sufficient condition that X and J_μ must satisfy is that $\operatorname{rank}(X^k) = \operatorname{rank}(J_\mu^k)$ for all $k \geqslant 1$, so the entries in the vector $\mathbf{v}$ corresponding to the boxes in the first column of the diagram for $\mathbf{v}^1$ must all be zero, while the remaining $n-1-\mu'_1$ entries are free to be any element in $\mathbb{F}_q$, so there are $q^{n-1-\mu'_1}$ matrices X whose leading principal submatrix is J_μ in this case. □

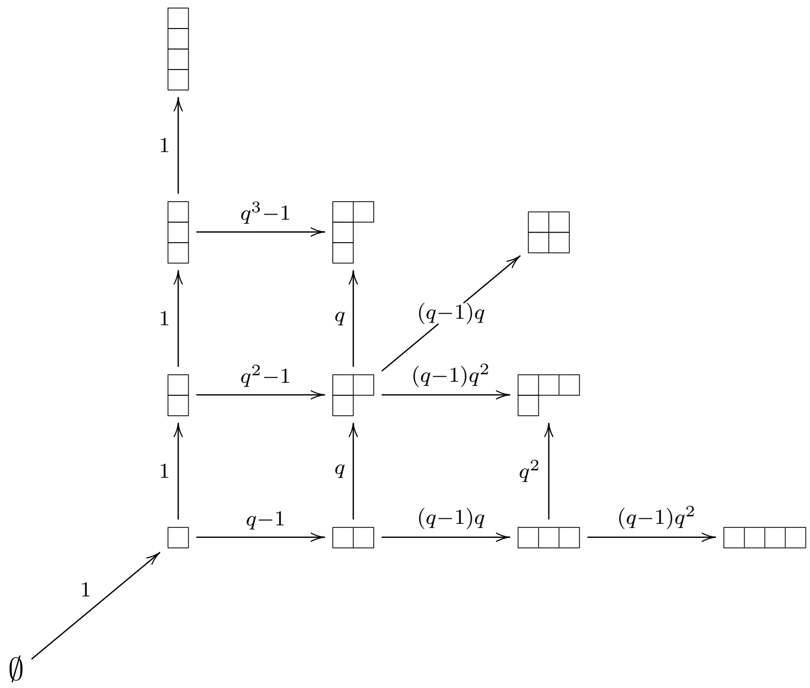


Figure 1: Young's lattice with edge weights $c_{\mu,\lambda}(q)$, up to $n = 4$.

The following recurrence equation for $F_\lambda(q)$ was first obtained by Borodin [1, Division Theorem]. Here, we provide an elementary proof before investigating some of the combinatorial properties of $F_\lambda(q)$.

Theorem 8. *The number of n by n upper-triangular nilpotent matrices over $\mathbb{F}_q$ of Jordan type $\lambda \vdash n$ is*

$$F_\lambda(q) = \sum_{\mu\colon \mu \lessdot \lambda} c_{\mu,\lambda}(q) F_\mu(q),$$

with $F_\emptyset(q) = 1$.

Proof. Proceed by induction on n. For $n = 1$, the zero matrix is the only upper-triangular nilpotent matrix, and it has Jordan type (1), agreeing with the formula $c_{\emptyset,(1)}(q) = 1$.

Suppose $\lambda \vdash n$. By Lemma 6, any matrix of Jordan type λ has a leading principal submatrix of type $\mu \vdash n-1$ for some $\mu \lessdot \lambda$. Furthermore, by Lemma 7, for each matrix $Y \in \mathfrak{g}_{n-1,\mu}(\mathbb{F}_q)$, there are $c_{\mu\lambda}(q)$ matrices $X \in \mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ having Y as its leading principal submatrix. Summing over all $\mu \lessdot \lambda$ gives the desired formula. □

Remark 9 (Formulation in terms of standard Young tableaux)*.* The formula for $F_\lambda(q)$ in Theorem 8 can be re-phrased as a sum over the set $\mathcal{P}_\mathcal{Y}(\lambda)$ of paths in Young's lattice $\mathcal{Y}$ from the empty partition $\emptyset$ to λ. If $\mu \lessdot \lambda$ in $\mathcal{Y}$, we assign the weight $c_{\mu,\lambda}(q)$ to the corresponding edge in $\mathcal{Y}$. Figure 1 shows Young's lattice with weighted edges for partitions with up to four boxes. Let $P = (\emptyset = \pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)} = \lambda)$ denote a path in $\mathcal{Y}$ from $\emptyset$ to

λ, where $\pi^{(i)}$ is a partition of i. To simplify notation, let $\epsilon_i(q) = c_{\pi^{(i-1)},\pi^{(i)}}(q)$. Theorem 8 is equivalently re-phrased as

$$F_\lambda(q) = \sum_{P \in P_{\mathcal{Y}}(\lambda)} F_P(q), \tag{7}$$

where the weight of the path P is $F_P(q) = \prod_{i=1}^n \epsilon_i(q)$.

The set of paths $\mathcal{P}_{\mathcal{Y}}(\lambda)$ is in bijection with the set $\mathrm{SYT}(\lambda)$ of standard Young tableaux of shape λ, so we can also give an equation for $F_\lambda(q)$ as a sum over standard Young tableaux.

A *standard Young tableau* T of shape λ is a filling of the Ferrers diagram of $\lambda \vdash n$ with the integers $1, \ldots, n$ such that the integers increase weakly along each row and strictly along each column. For $1 \leqslant i \leqslant n$, Let $T^{(i)}$ denote the Young tableau of shape $\lambda^{(i)}$ consisting of the boxes containing $1, \ldots, i$, and define weights

$$T^{(i)}(q) = \begin{cases} q^{i-\ell(\lambda^{(i)})}, & \text{if the } i\text{th box is in the first column,} \\ q^{i-\lambda_j^{(i)\prime}} - q^{i-1-\lambda_{j-1}^{(i)\prime}}, & \text{if the } i\text{th box is in the } j\text{th column, } j \geqslant 2. \end{cases} \tag{8}$$

Then

$$F_\lambda(q) = \sum_{T \in \mathrm{SYT}(\lambda)} F_T(q), \tag{9}$$

where the weight of the standard Young tableau T is $F_T(q) = \prod_{i=1}^n T^{(i)}(q)$.

2.2 Properties of $F_\lambda(q)$

Several properties of $F_\lambda(q)$ follow readily from Theorem 8. For $\lambda \vdash n$, let

$$n_\lambda = \sum_{i \geqslant 1} (i-1)\lambda_i = \sum_{b \in \lambda} \mathrm{coleg}(b), \tag{10}$$

where if a box $b \in \lambda$ lies in the ith row of λ, then $\mathrm{coleg}(b) = i - 1$.

Corollary 10. *Let $\lambda \vdash n$. As a polynomial in q,*

$$\deg F_\lambda(q) = \binom{n}{2} - n_\lambda.$$

Moreover, the coefficient of the highest degree term in $F_\lambda(q)$ is f^λ, the number of standard Young tableaux of shape λ.

Proof. Suppose $P = (\emptyset = \pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)} = \lambda)$ is a path in $\mathcal{Y}$ such that $\pi^{(k)}$ is obtained by adding a box to the ith row and jth column of $\pi^{(k-1)}$. Then $\deg c_{\pi^{(k-1)},\pi^{(k)}}(q) = k - i$, and therefore

$$\deg F_P(q) = \sum_{k=1}^n \deg c_{\pi^{(k-1)},\pi^{(k)}}(q) = \sum_{k=1}^n k - \sum_{k \geqslant 1} k\lambda_k = \binom{n}{2} - n_\lambda.$$

In particular, every polynomial $F_P(q)$ arising from a path $P \in \mathcal{P}_{\mathcal{Y}}(\lambda)$ has the same degree, so $\deg F_\lambda(q) = \binom{n}{2} - n_\lambda$. Moreover, each $F_P(q)$ is monic, so the coefficient of the highest degree term in $F_\lambda(q)$ is the number of paths in $\mathcal{Y}$ from $\emptyset$ to λ, which is f^λ. □

Corollary 11. *Let $\lambda \vdash n$. The multiplicity of the factor $q-1$ in $F_\lambda(q)$ is $n-\ell(\lambda)$.*

Proof. The weight $c_{\pi^{(k-1)},\pi^{(k)}}(q)$ corresponding to the kth step in the path P contributes a single factor of $q-1$ to $F_P(q)$ if and only if the kth box added is not in the first column of λ. Therefore, the multiplicity of $q-1$ in $F_P(q)$ is $n-\ell(\lambda)$, and it follows that the multiplicity of $q-1$ in $F_\lambda(q)$ is $n-\ell(\lambda)$. □

Example 12. There are two partitions of 4 with two parts, namely $(3,1)$ and $(2,2)$.

There are three paths from $\emptyset$ to $(3,1)$ in $\mathcal{Y}$, giving

$$\begin{aligned} F_{(3,1)}(q) &= (q-1)\cdot(q-1)q\cdot q^2 + (q-1)\cdot q\cdot(q-1)q^2 + \cdot(q^2-1)\cdot(q-1)q^2 \\ &= (q-1)^2\left(3q^3+q^2\right), \end{aligned}$$

and there are two paths from $\emptyset$ to $(2,2)$ in $\mathcal{Y}$, giving

$$\begin{aligned} F_{(2,2)}(q) &= (q-1)\cdot q\cdot(q-1)q + (q^2-1)\cdot(q-1)q \\ &= (q-1)^2(2q^2+q). \end{aligned}$$

Summing these gives a shift of the q-Stirling polynomial $(q-1)^2q^4S_{4,2}(q^{-1}) = (q-1)^2(3q^3+3q^2+q)$.

2.3 Explicit formulas

In this section, we derive non-recursive formulas for some special cases of λ. Previously, we have noted the simple cases $F_{(1^n)} = 1$ and $F_{(n)} = q^{\binom{n-1}{2}}(q-1)^{n-1}$.

Proposition 13 (Hook shapes)**.** *Let $n > k \geqslant 2$, and let $\lambda = (n-k+1, 1^{k-1})$ be a hook-shaped partition of n with $\ell(\lambda) = k$ parts. Then*

$$F_\lambda(q) = (q-1)^{n-k}\sum_{i=0}^{k-1}\binom{n-i-1}{k-i-1}q^{\alpha-i}, \qquad \textit{where } \alpha = \binom{n-1}{2} - \binom{k-1}{2}.$$

Proof. We make use of Equation (7). We enumerate paths from $\emptyset$ to λ according to the first time a box is added to the second column, so for $0 \leqslant r \leqslant k-1$, let S_r be the set of paths in the sublattice $[\emptyset, \lambda]$ which contains the edge $((1,1^r),(2,1^r))$. Such paths are formed by the concatenation of the unique path between $\emptyset$ and $\nu = (2,1^r)$, which has weight $q^{r+1}-1$, with any path in the sublattice $[\nu,\lambda]$. The sublattice $[\nu,\lambda]$ is the Cartesian product of a $(n-k)$-chain and a $(k-r-1)$-chain, so it forms a rectangular grid, and therefore $|\mathrm{S}_r| = \binom{n-r-1}{k-r-1}$. Notice that in any sublattice of the form

$$\begin{array}{ccc} (a,1^{b+1}) & \xrightarrow{(q-1)q^{a+b}} & (a+1,1^{b+1}) \\ \uparrow{\scriptstyle q^{a-1}} & & \uparrow{\scriptstyle q^{a}} \\ (a,1^b) & \xrightarrow{(q-1)q^{a+b-1}} & (a+1,1^b) \end{array}$$

the product of the edge weights is $(q-1)q^{2a+b-1}$ no matter which path is taken from $(a,1^b)$ to $(a+1,1^{b+1})$, so it follows that every path from ν to λ has the same weight. By considering the path $(\nu, 21^{r+1}, \ldots, 21^{k-1}, 31^{k-1}, \ldots, \lambda)$, this weight is easily seen to be $(q-1)^{n-k-1}q^{\alpha-r}$, for $\alpha = \binom{n-1}{2} - \binom{k-1}{2}$. Altogether,

$$F_\lambda(q) = (q-1)^{n-k} \sum_{r=0}^{k-1} \binom{n-r-1}{k-r-1} \left(q^\alpha + q^{\alpha-1} + \cdots + q^{\alpha-r}\right).$$

For $0 \leqslant i \leqslant k-1$, the coefficient of $q^{\alpha-i}$ in $F_\lambda(q)/(q-1)^{n-k}$ is

$$\sum_{r=i}^{k-1} \binom{n-r-1}{k-r-1} = \sum_{u=0}^{k-i-1} \binom{n-k+u}{u} = \binom{n-i-1}{k-i-1},$$

since $\sum_{u=0}^{M} \binom{N+u}{u} = \binom{N+M+1}{M}$. Therefore,

$$F_\lambda(q) = (q-1)^{n-k} \sum_{i=0}^{k-1} \binom{n-i-1}{k-i-1} q^{\alpha-i},$$

as claimed. □

We next consider the case when λ is a partition with two parts. For $n \geqslant k \geqslant 1$, let

$$C_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1}, \tag{11}$$

and let $C_{n,0} = 1$ for all $n \geqslant 0$. These generalized Catalan numbers $C_{n,k}$ (see OEIS [10, A009766]) enumerate lattice paths from $(0,0)$ to (n,k), using the steps $(1,0)$ and $(0,1)$, which do not rise above the line $y = x$. In the remainder of this section, we shall refer to these as *Dyck paths*.

The generalized Catalan numbers satisfy the simple recursive formula $C_{n,k} = C_{n,k-1} + C_{n-1,k}$. Also, these are the usual Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n} = C_{n,n} = C_{n,n-1}$ when $k = n$ or $n-1$. These facts will be used in the computations which follow.

Proposition 14 (Partitions with two parts)**.** *If $\lambda = (r,s) \vdash n$ such that $r > s \geqslant 1$, then*

$$F_{(r,s)}(q) = (q-1)^{r+s-2} q^{\binom{r+s-1}{2}-2s+1} \sum_{i=0}^{s} C_{r+s-i,i}\, q^i.$$

If $r = s$, then

$$F_{(r,r)}(q) = (q-1)^{2r-2} q^{\binom{2r-2}{2}} \sum_{i=0}^{r-1} C_{2r-1-i,i}\, q^i.$$

Proof. Proceed by induction on $r+s$. The base cases are $F_{(r)}(q) = (q-1)^{r-1}q^{\binom{r-1}{2}}$ for $r \geqslant 1$ and $F_{(1,1)}(q) = 1$.

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	2				
3	1	3	5	5			
4	1	4	9	14	14		
5	1	5	14	28	42	42	
6	1	6	20	48	90	132	132

Figure 2: The Catalan triangle $C_{n,k}$.

We first handle the case $s = 1$ separately. For $r \geqslant 2$,

$$\begin{aligned} F_{(r,1)}(q) &= q^{r-1}F_{(r)}(q) + (q-1)q^{r-1}F_{(r-1,1)}(q) \\ &= q^{r-1} \cdot (q-1)^{r-1}q^{\binom{r-1}{2}} + (q-1)q^{r-1} \cdot (q-1)^{r-2}q^{\binom{r-1}{2}-1}\left((r-1)q+1\right) \\ &= (q-1)^{r-1}q^{\binom{r}{2}-1}\left(rq+1\right). \end{aligned}$$

Next, consider the case $s = r$. For $r \geqslant 2$,

$$\begin{aligned} F_{(r,r)}(q) &= (q-1)q^{2r-3}F_{(r,r-1)}(q) \\ &= (q-1)q^{2r-3} \cdot (q-1)^{2r-3}q^{\binom{2r-2}{2}-2(r-1)+1}\sum_{i=0}^{r-1} C_{2r-1-i,i}q^i \\ &= (q-1)^{2r-2}q^{\binom{2r-2}{2}}\sum_{i=0}^{r-1} C_{2r-1-i,i}q^i. \end{aligned}$$

The case $s = r-1$ is obtained as follows. For $r \geqslant 3$,

$$\begin{aligned} F_{(r,r-1)}(q) &= (q-1)q^{2r-4}F_{(r,r-2)}(q) + (q^2-1)q^{2r-4}F_{(r-1,r-1)}(q) \\ &= (q-1)^{2r-3}q^{2r-4}q^{\binom{2r-4}{2}}\left(q\sum_{i=0}^{r-2} C_{2r-2-i,i}q^i + (q+1)\sum_{i=0}^{r-2} C_{2r-3-i,i}q^i\right) \\ &= (q-1)^{2r-3}q^{\binom{2r-3}{2}}\Bigg(\left(C_{r,r-2} + C_{r-1,r-2}\right)q^{r-1} \\ &\qquad + \sum_{i=1}^{r-2}\left(C_{2r-1-i,i-1} + C_{2r-2-i,i-1} + C_{2r-3-i,i}\right)q^i + C_{2r-3,0}q^0\Bigg). \end{aligned}$$

Since $C_{n,n-1} = C_{n,n}$, then $C_{r,r-2} + C_{r-1,r-2} = C_{r,r-1}$. Similarly, we obtain $C_{2r-1-i,i-1} + C_{2r-2-i,i-1} + C_{2r-3-i,i} = C_{2r-1-i,i}$ by applying the recurrence equation for the generalized Catalan numbers. Lastly, $C_{n,0} = 1$ for all $n \geqslant 0$, thus

$$F_{(r,r-1)}(q) = (q-1)^{2r-3}q^{\binom{2r-3}{2}}\sum_{i=0}^{r-1} C_{2r-1-i,i}q^i,$$

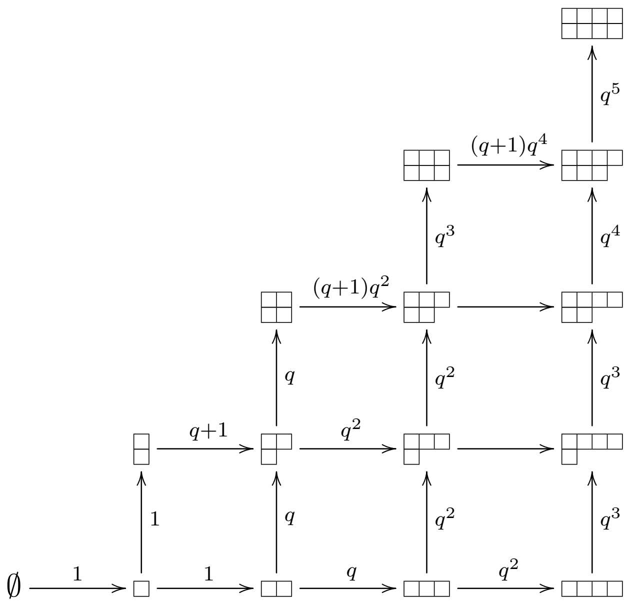


Figure 3: Factors of $q-1$ are omitted from the edge weights in this sublattice of partitions with at most two rows.

which agrees with the formula for the case $s = r - 1$.

The last case to consider is the general case $r - s \geqslant 2$ where $s \geqslant 2$.

$$\begin{aligned}
F_{(r,s)} &= (q-1)q^{r+s-3}F_{(r,s-1)} + (q-1)q^{r+s-2}F_{(r-1,s)} \\
&= (q-1)^{r+s-2}q^{r+s-3}q^{\binom{r+s-2}{2}-2s+1}\left(q^2\sum_{i=0}^{s-1} C_{r+s-1-i,i}q^i + q\sum_{i=0}^{s} C_{r-1+s-i,i}q^i\right) \\
&= (q-1)^{r+s-2}q^{\binom{r+s-1}{2}-2s+1}\left(C_{r+s,0}q^0 + \sum_{i=1}^{s}\left(C_{r+s-i,i-1} + C_{r-1+s-i,i}\right)q^i\right) \\
&= (q-1)^{r+s-2}q^{\binom{r+s-1}{2}-2s+1}\sum_{i=0}^{s} C_{r+s-i,i}q^i.
\end{aligned}$$

□

The next equation is a formula for $F_{(r,r)}(q)$ with a different factorization.

Proposition 15 (Two equal parts). *Let $\lambda = (r,r) \vdash n$, and $r \geqslant 1$. Then*

$$F_{(r,r)}(q) = (q-1)^{2r-2}\sum_{i=0}^{r-1} C_{r-1,r-1-i}\ q^{2(r-1)^2-i}(q+1)^i.$$

Proof. The set of paths in the sublattice $[\emptyset, \lambda]$ are in bijection with the set of lattice paths from $(0,0)$ to (r,r). In any sublattice of the form

$$\begin{CD} (a,b+1) @>{(q-1)q^{a+b}}>> (a+1,b+1) \\ @A{(q-1)q^{a+b-2}}AA @AA{(q-1)q^{a+b-1}}A \\ (a,b) @>{(q-1)q^{a+b-1}}>> (a+1,b) \end{CD}$$

where $b \leqslant a-2$, the product of the edge weights is $(q-1)^2 q^{2a+2b-2}$ no matter which path is taken from (a,b) to $(a+1,b+1)$. As for sublattices of the form

$$\begin{CD} (a,a) @>{(q^2-1)q^{2a-2}}>> (a+1,a) \\ @A{(q-1)q^{2a-3}}AA @AA{(q-1)q^{2a-2}}A \\ (a,a-1) @>{(q-1)q^{2a-2}}>> (a+1,a-1) \end{CD}$$

the product of the edge weights is $(q-1)^2 q^{4a-4}$ via the lower horizontal edge, versus $(q-1)^2 q^{4a-5}(q+1)$ via the upper horizontal edge. It follows that if a path P from $\emptyset$ to λ contains i partitions of the form (a,a), then it has the weight

$$F_p(q) = (q-1)^{2r-2} q^{2(r-1)^2-i}(q+1)^i.$$

Dyck paths may be enumerated according to the points at which they touch the diagonal line $y = x$, and the set of touch points are indexed by compositions $\alpha = (\alpha_1, \ldots, \alpha_{i+1}) \vDash r$ where $\alpha_j \geqslant 1$. The number of Dyck paths from $(0,0)$ to (r,r) which touch the diagonal exactly i times, not including the initial and the end points, is

$$\sum_{\substack{\alpha \vDash r \\ \ell(\alpha)=i+1}} \prod_{j=1}^{i+1} C_{\alpha_j - 1}.$$

On the other hand, the number $C_{r-1,r-1-i}$ of Dyck paths from $(0,0)$ to $(r-1, r-1-i)$ satisfies the same recurrence equation

$$C_{r-1,r-1-i} = \sum_{\substack{\beta \vDash r-1-i \\ \ell(\beta)=i+1}} \prod_{j=1}^{i+1} C_{\beta_j},$$

but the sum is over the set of weak compositions so that $\beta_j \geqslant 0$. Under the appropriate shift in indices, it follows that the number of Dyck paths from $(0,0)$ to (r,r) which touch the diagonal exactly i times is $C_{r-1,r-1-i}$. The result follows from this. □

Corollary 16. *For $k \geqslant m \geqslant 0$,*

$$\sum_{j=m}^{k} \binom{j}{m} C_{k,j} = C_{2k+1-m,m}.$$

Proof. The two formulas for $F_{(k+1,k+1)}(q)$ yields the identity

$$\sum_{i=0}^{k} C_{k,k-i}\, q^{2k^2-i}(q+1)^i = q^{\binom{2k}{2}} \sum_{i=0}^{k} C_{2k-i,i} q^i.$$

Extracting the coefficient of q^{2k^2-m} in the above expressions yields the result. $\square$

Remark 17. The formula for $F_{(r,r)}(q)$ provided in Proposition 15 can be viewed as a sum over Dyck paths, where each Dyck path π contributes a term of the form $q^{s_1(\pi)}(q+1)^{s_2(\pi)}$ for some statistics s_1 and s_2 on the Dyck paths. This particular factorization for $F_{(r,r)}(q)$ is related to the work of Cai and Readdy on the q-Stirling numbers of the second kind, since the polynomials $F_\lambda(q)$ can be viewed as a refinement of $S_{n,k}(q)$, as explained in Section 3.

Cai and Readdy obtained a formula [2, Theorem 3.2] for $\widetilde{S}_{n,k}(q)$ (they use a different recursive formula to define the q-Stirling numbers, and the two are related by $S_{n,k}(q) = q^{\binom{k}{2}}\widetilde{S}_{n,k}(q)$) as a sum over allowable restricted-growth words, where each allowable word w gives rise to a term of the form $q^{a(w)}(q+1)^{b(w)}$ for some statistics $a(w)$ and $b(w)$. They also showed that this enumerative result has an interesting extension to the study of the Stirling poset of the second kind, providing a decomposition of that poset into Boolean sublattices.

For example, if we define polynomials $G_\lambda(q)$ by letting $G_\lambda(q) = F_\lambda(q)/(q-1)^{n-\ell(\lambda)}$ (see Equation (12)), then $G_{(3,1)}(q) + G_{(2,2)}(q) = q^3\widetilde{S}_{4,2}(q^{-1})$. The formula of Cai and Readdy yields $q^3 S_{4,2}(q^{-1}) = q(q+1)^2 + q^2(q+1) + q^3$, while our factorization yields $G_{(3,1)}(q) + G_{(2,2)}(q) = (q^3 + q^3 + q^2(q+1)) + (q^2 + q(q+1))$. So, the result of Proposition 15 gives a different expression for $S_{n,n-2}(q)$ as a sum with terms of the form $q^{s_1(\pi)}(q+1)^{s_2(\pi)}$, and it may be interesting to further investigate such factorizations of $F_\lambda(q)$.

Example 18. The first few $F_{(k,k)}$ are

$$\begin{aligned}
F_{(1,1)} &= 1\\
F_{(2,2)} &= (q-1)^2\left(q^2 + q(q+1)\right)\\
&= (q-1)^2\left(2q^2+q\right)\\
F_{(3,3)} &= (q-1)^4\left(2q^8 + 2q^7(q+1) + q^6(q+1)^2\right)\\
&= (q-1)^4\left(5q^8 + 4q^7 + q^6\right)\\
F_{(4,4)} &= (q-1)^6\left(5q^{18} + 5q^{17}(q+1) + 3q^{16}(q+1)^2 + q^{15}(q+1)^3\right)\\
&= (q-1)^6\left(14q^{18} + 14q^{17} + 6q^{16} + q^{15}\right)\\
F_{(5,5)} &= (q-1)^8\left(14q^{32} + 14q^{31}(q+1) + 9q^{30}(q+1)^2 + 4q^{29}(q+1)^3 + q^{28}(q+1)^4\right)\\
&= (q-1)^8\left(42q^{32} + 48q^{31} + 27q^{30} + 8q^{29} + q^{28}\right).
\end{aligned}$$

We end this section with one more closed formula for $F_\lambda(q)$ where λ is a rectangular shape with two columns. Let $\mathcal{D}(n,k)$ denote the set of Dyck paths from $(0,0)$ to (n,k). The *coarea* of a Dyck path π is the number of whole unit squares lying between the path

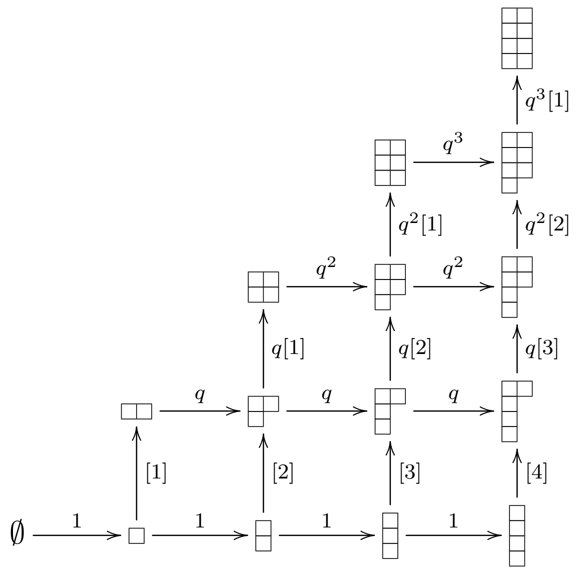
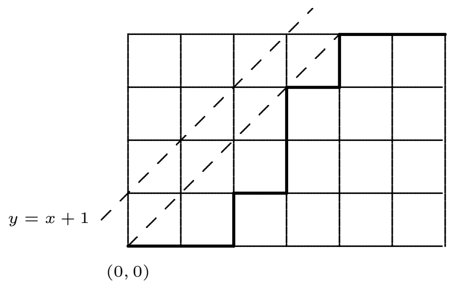


Figure 4: Factors of $q-1$ are omitted from the edge weights in this sublattice of partitions with at most two columns.

and the x-axis. For $i = 1, \ldots, n$, let $\rho_i(\pi)$ be one plus the number of unit squares lying between the path and the line $y = x + 1$ in the ith row. For example, the following Dyck path π has $\text{coarea}(\pi) = 12$, and $(\rho_1(\pi), \rho_2(\pi), \rho_3(\pi), \rho_4(\pi)) = (2, 2, 1, 1)$.



For $n \geqslant 1$, let $[n]_q = 1 + q + \cdots + q^{n-1}$.

Proposition 19. *(Partitions with two columns) Let* $\lambda = (2^r, 1^s) \vdash n$ *such that* $r, s \geqslant 0$. *Then*

$$F_{(2^r,1^s)}(q) = (q-1)^r q^{\binom{r}{2}} \sum_{\pi \in \mathcal{D}(r+s,r)} q^{\text{coarea}(\pi)} \prod_{i=1}^{r} [\rho_i(\pi)]_q .$$

Proof. By Corollary 11, we know the multiplicity of the factor $q - 1$ in $F_{(2^r,1^s)}(q)$ is $n - \ell(\lambda) = \lambda'_2 = r$, so we focus on computing $F_{(2^r,1^s)}(q)/(q-1)^r$. The paths in Young's lattice from $\emptyset$ to $(2^r, 1^s)$ are in bijection with the Dyck paths $\mathcal{D}(r + s, r)$, so we identify these paths; adding a box in the first column of a partition corresponds to a $(1, 0)$ step in the Dyck path, and adding a box in the second column of a partition corresponds to a $(0, 1)$ step in the Dyck path. As seen in Figure 4, a vertical step (i, j) to $(i, j + 1)$ has weight $q^j[i]_q$, while a horizontal step (i,j) to $(i+1,j)$ has weight q^j. Thus the product of the edge weights of the r vertical steps of a given Dyck path π is $q^{\binom{r}{2}}\prod_{i=1}^{r}[\rho_i(\pi)]_q$, while the product of the edge weights of the $r+s$ horizontal steps of a given Dyck path is $q^{\mathrm{coarea}(\pi)}$. The result follows. □

Example 20. The first few $F_{(2^n)}$ are

$$\begin{aligned}
F_{(2)} &= (q-1)(q+1) \\
F_{(2^2)} &= (q-1)^2 q\,(q+(q+1)) \\
F_{(2^3)} &= (q-1)^3 q^3\left(q^3+2q^2(q+1)+q(q+1)^2+(q^2+q+1)(q+1)\right) \\
F_{(2^4)} &= (q-1)^4 q^6\left(q^6+3q^5[2]+3q^4[2]^2+q^3[2]^3+2q^3[3]!+2q^2[2][3]!+q[3][3]!+[4]!\right).
\end{aligned}$$

Remark 21. Kirillov and Melnkov [8] considered the number $A_n(q)$ of n by n upper-triangular matrices over $\mathbb{F}_q$ satisfying $X^2=0$. In their first characterization of these polynomials, they considered the number $A_n^r(q)$ of matrices of a given rank r, so that $A_n(q)=\sum_{r\geqslant 0}A_n^r(q)$, and observed that $A_n^r(q)$ satisfies the recurrence equation

$$A_n^r(q) = q^r A_{n-1}^r(q) + \left(q^{n-r}-q^r\right)A_n^r(q), \qquad A_n^0(q)=1.$$

We may think of $A_n(q)$ as the sum of $F_\lambda(q)$ over $\lambda \vdash n$ with at most two columns, so Theorem 8 is a generalization of this recurrence equation.

It was also conjectured in [8] that the same sequence of polynomials arise in a number of different ways. Ekhad and Zeilberger [3] proved that one of the conjectured alternate definitions of $A_n(q)$, namely

$$C_n(q) = \sum_s c_{n+1,s}\, q^{\frac{n^2}{4}+\frac{1-s^2}{12}},$$

is a sum over all $s\in[-n-1,n+1]$ which satisfy $s\equiv n+1 \bmod 2$ and $s\equiv(-1)^n \bmod 3$, and $c_{n+1,s}$ are entries in the signed Catalan triangle, is indeed the same as $A_n(q)$. It would be interesting to see what other combinatorics may arise from considering the sum of $F_\lambda(q)$ over $\lambda\vdash n$ with at most k columns for a fixed k.

3 Jordan canonical forms and q-rook placements

In light of Corollary 11, we define polynomials $G_\lambda(q)\in\mathbb{Z}[q]$ by

$$F_\lambda(q) = (q-1)^{n-\ell(\lambda)} G_\lambda(q). \tag{12}$$

In fact, we can deduce from Corollary 11 that $G_\lambda(q)\in\mathbb{N}[q]$. In this section, we explore the connection between the nonnegative coefficients of $G_\lambda(q)$ and rook placements.

3.1 Background on rook polynomials

A *board* B is a subset of an n by n grid of squares. In this paper, we follow Haglund [5] and Solomon [12], and index the squares using the convention for the entries of a matrix. A *Ferrers board* is a board B where if a square $s \in B$, then every square lying north and/or east of s is also in B. Our Ferrers boards have squares justified upwards and to the right. Let B_n denote the staircase-shaped board with n columns of sizes $0, 1, \ldots, n-1$. Let $\mathrm{area}(B)$ be the number of squares in B, so that in particular, $\mathrm{area}(B_n) = \binom{n}{2}$.

A placement of k rooks on a board B is *non-attacking* if there is at most one rook in each row and each column of B. Let $\mathcal{C}(B, k)$ denote the set of non-attacking placements of k rooks on B. All rook placements considered in this article are non-attacking, so from this point forward, we drop the qualifier. For a placement $C \in \mathcal{C}(B, k)$, let $\mathrm{ne}(C)$ be the number of squares in B lying directly north or directly east of a rook. The *inversion* of the placement is the number

$$\mathrm{inv}(C) = \mathrm{area}(B) - k - \mathrm{ne}(C). \tag{13}$$

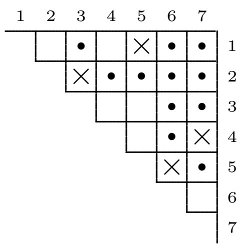
As noted in [4], the statistic $\mathrm{inv}(C)$ is a generalization of the number of inversions of a permutation, since permutations can be identified with rook placements on a square-shaped board.

For $i = 1, \ldots, n$, the weight of the ith column C_i of C is

$$C_i(q) = (q-1)^{\#\text{rooks in } C_i} q^{\mathrm{ne}(C_i)}, \tag{14}$$

and the *weight* of C is defined by $F_C(q) = \prod_{i=1}^{n} C_i(q)$. Alternatively, if $C \in \mathcal{C}(B, k)$, then $F_C(q) = (q-1)^k q^{\mathrm{ne}(C)}$.

Example 22. We use $\times$ to mark a rook and use $\bullet$ to mark squares lying directly north or directly east of a rook (these squares shall be referred to as the north-east squares of the placement). The following illustration is a placement of four rooks on the staircase-shaped board B_7.



This rook placement has $\mathrm{ne}(C) = 11$, $\mathrm{inv}(C) = 6$, and weight $F_C(q) = (q-1)^4 q^{11}$.

For $k \geqslant 0$, the *q-rook polynomial* of a Ferrers board B is defined by Garsia and Remmel [4, I.4] as

$$R_{B,k}(q) = \sum_{C \in \mathcal{C}(B,k)} q^{\mathrm{inv}(C)}. \tag{15}$$

The following result explains the role of rook polynomials in the enumeration of matrices of given rank. The *support* of a matrix X is $\{(i,j) \mid x_{ij} \neq 0\}$. Given a Ferrers board B with n columns, we may identify the squares in B with the entries in an n by n matrix.

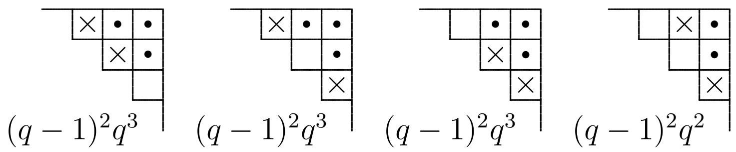
Theorem 23 (Haglund)**.** *If B is a Ferrers board, then the number $P_{B,k}(q)$ of n by n matrices of rank k with support contained in B is*

$$P_{B,k}(q) = (q-1)^k q^{\mathrm{area}(B)-k} R_{B,k}(q^{-1}).$$

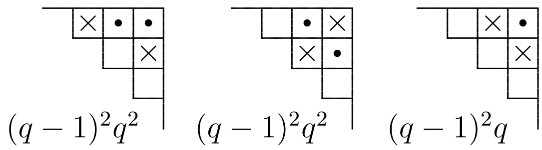
Looking ahead, it will be convenient to consider Theorem 23 in the following equivalent form:

$$P_{B,k}(q) = \sum_{C \in \mathcal{C}(B,k)} (q-1)^k q^{\mathrm{ne}(C)} = \sum_{C \in \mathcal{C}(B,k)} F_C(q). \tag{16}$$

Example 24. We list the seven rook placements on B_4 with two rooks, along with their weights.



(17)



(18)

Thus $P_{B_4,2}(q) = (q-1)^2(3q^3 + 3q^2 + q)$.

3.2 Rook placements and Jordan forms

The purpose of this section is to generalize Haglund's formula (16) to a formula for $F_\lambda(q)$ (Corollary 30) as a sum over a set of rook placements. We achieve this by defining a multigraph $\mathcal{Z}$ that is related to $\mathcal{Y}$, and show that paths in $\mathcal{Z}$ are equivalent to rook placements.

The multigraph $\mathcal{Z}$ is constructed from $\mathcal{Y}$ by replacing each edge of $\mathcal{Y}$ by one or more edges as follows. If there is an edge from μ to λ in $\mathcal{Y}$ of weight $q^{|\mu|-\mu'_{j-1}}\left(q^{\mu'_{j-1}-\mu'_j} - 1\right)$, then this edge is replaced by $\mu'_{j-1} - \mu'_j$ edges from μ to λ with weights

$$(q-1)q^{|\mu|-\mu'_j-1}, \ldots, (q-1)q^{|\mu|-\mu'_{j-1}} \tag{19}$$

in $\mathcal{Z}$. All other edges remain as before. See Figure 5.

Let $\mathcal{P}_{\mathcal{Z}}(\lambda)$ denote the set of paths in the graph $\mathcal{Z}$ from the empty partition $\emptyset$ to λ. For a path $P = (\emptyset = \pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)} = \lambda)$ in $\mathcal{P}_{\mathcal{Z}}(\lambda)$, let $\epsilon_i(q)$ denote the weight of the ith edge, for $i = 1, \ldots, n$. Naturally, we define the weight of the path by $F_P(q) = \prod_{i=1}^n \epsilon_i(q)$, so that

$$F_\lambda(q) = \sum_{P \in \mathcal{P}_{\mathcal{Z}}(\lambda)} F_P(q). \tag{20}$$

Lemma 25. *Let $\mu \vdash n-1$ be a partition with $\ell(\mu) = \ell$ parts. Then there are $\ell + 1$ edges leaving μ in the graph $\mathcal{Z}$, with weights*

$$(q-1)q^{|\mu|-1}, (q-1)q^{|\mu|-2}, \ldots, (q-1)q^{|\mu|-\ell}, \text{ and } q^{|\mu|-\ell}.$$

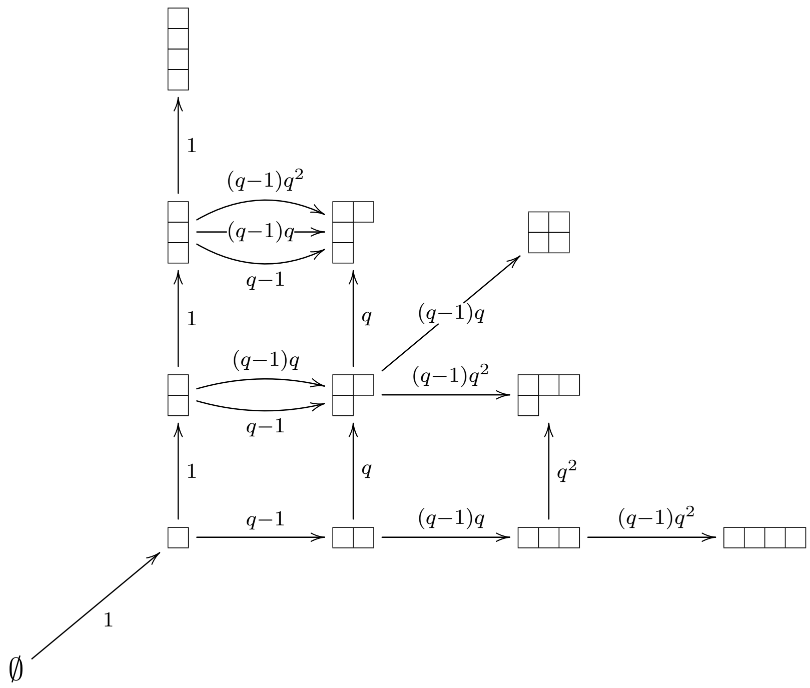


Figure 5: The multigraph $\mathcal{Z}$, up to $n = 4$.

Proof. If a partition $\lambda \vdash n$ is obtained by adding a box to the first column of μ, then there is a unique edge from μ to λ in $\mathcal{Z}$ with weight $q^{|\mu|-\ell}$. Otherwise, if we consider the set of all partitions which can be obtained from μ by adding a box anywhere except in the first column, then there are a total of

$$\sum_{j \geqslant 2} \left(\mu'_{j-1} - \mu'_j\right) = \ell$$

edges from μ to some partition of n. Moreover, by Equation (19), these ℓ weights are $(q-1)q^{|\mu|-i}$ for $i = 1, \ldots, \ell$. □

A sequence of nonnegative integers is $\mathcal{P}_{\mathcal{Z}}$-admissible if it is the degree sequence of a path $P = (\emptyset, \pi^{(1)}, \ldots, \pi^{(n)})$ in $\mathcal{Z}$. That is, $(d_1, \ldots, d_n) = (\deg \epsilon_1(q), \ldots, \deg \epsilon_n(q))$.

Corollary 26. *A $\mathcal{P}_{\mathcal{Z}}$-admissible sequence determines a unique path in $\mathcal{Z}$.*

Proof. Induct on n. When $n = 1$, the only path is the from $\emptyset$ to (1), and it has degree sequence (0).

Given a $\mathcal{P}_{\mathcal{Z}}$-admissible sequence $(d_1, \ldots, d_n)$, the subsequence $(d_1, \ldots, d_{n-1})$ determines a unique path $P' = (\emptyset, \pi^{(1)}, \ldots, \pi^{(n-1)})$. Suppose $\mu = \pi^{(n-1)}$ has ℓ parts. Then $|\mu| - \ell + 1 \leqslant d_n \leqslant |\mu|$, and by Lemma 25, there is a unique edge leaving μ with degree d_n. □

3.3 The construction of Φ

Let $\mathcal{P}_\mathcal{Z}(n, n-k)$ denote the set of paths in $\mathcal{Z}$ from $\emptyset$ to a partition of n with $n-k$ parts. In this section, we define a weight-preserving bijection $\Phi : \mathcal{C}(B_n, k) \to \mathcal{P}_\mathcal{Z}(n, n-k)$.

Proposition 27. *Let $n \geqslant 1$ and $k = 0, \dots, n-1$. Let $C \in \mathcal{C}(B_n, k)$ be a rook placement with columns $C_1, \dots, C_n$. There exists a unique path $P \in \mathcal{P}_\mathcal{Z}(n, n-k)$ with edge weights $(\epsilon_1(q), \dots, \epsilon_n(q)) = (C_1(q), \dots, C_n(q))$.*

Proof. Proceed by induction on $n+k$. When $n = 1$ and $k = 0$, there is a unique rook placement on the empty board B_1 with no rooks having weight one, corresponding to the unique path $P = (\emptyset, (1))$ in $\mathcal{Z}$ with the same weight.

Assume the result holds for all rook placements in $\mathcal{C}(B_{n-1}, k)$ and $\mathcal{C}(B_{n-1}, k-1)$. Given a rook placement $C \in \mathcal{C}(B_n, k)$, let C' be the sub-placement consisting of the first $n-1$ columns of C. By induction, the sequence $(C_1(q), \dots, C_{n-1}(q))$ determines a unique path $(\emptyset, \pi^{(1)}, \dots, \pi^{(n-1)})$ in $\mathcal{Z}$ such that $\epsilon_i(q) = C_i(q)$ for $i = 1, \dots, n-1$.

There are now two cases two consider. The first case is if $C' \in \mathcal{C}(B_{n-1}, k)$, so that $\ell(\pi^{(n-1)}) = n-k-1$. There are k rooks in C', so the nth column of C does not contain any rooks, and $C_n(q) = q^k$. By Lemma 25, there exists a unique edge in the graph $\mathcal{Z}$ originating at $\pi^{(n-1)}$ with weight q^k. Thus C corresponds to the path $P = (\emptyset, \pi^{(1)}, \dots, \pi^{(n-1)}, \pi^{(n)})$ where $\pi^{(n)}$ is obtained from $\pi^{(n-1)}$ by adding a box to the first column, and $\epsilon_n(q) = q^k$. Moreover, $\ell(\pi^{(n)}) = n-k$.

The second case is if $C' \in \mathcal{C}(B_{n-1}, k-1)$, so that $\ell(\pi^{(n-1)}) = n-k$. There must be $k-1$ 'northeast' squares in the nth column of C, and there are $n-k$ remaining squares in that column where a rook may be placed. Label these available squares $a_0, a_1, \dots, a_{n-k-1}$ from the top to the bottom. Observe that $C_n(q) = (q-1)q^{k-1+i}$ if a rook is placed in the square a_i, for $0 \leqslant i \leqslant n-k-1$. Again by Lemma 25, there exists $n-k$ edges in the graph $\mathcal{Z}$ originating at $\pi^{(n-1)}$ with the weights $(q-1)q^h$ for $k-1 \leqslant h \leqslant n-2$. Thus if the kth rook of C is placed in the square a_i, then C corresponds to the path $P = (\emptyset, \pi^{(1)}, \dots, \pi^{(n-1)}, \pi^{(n)})$ with $\epsilon_n(q) = (q-1)q^{k-1+i}$, and $\ell(\pi^{(n)}) = n-k$. □

Given a rook placement $C \in \mathcal{C}(B_n, k)$, let $\Phi(C)$ be the path in $\mathcal{P}_\mathcal{Z}(n, n-k)$ with edge weights $(\epsilon_1(q), \dots, \epsilon_n(q)) = (C_1(q), \dots, C_n(q))$.

Theorem 28. *The map $\Phi : \mathcal{C}(B_n, k) \to \mathcal{P}_\mathcal{Z}(n, n-k)$ is a weight-preserving bijection.*

Proof. Proposition 27 shows that the map Φ is an injective weight-preserving map, since each column of the rook placement determines each edge of the path $\Phi(C)$:

$$F_C(q) = \prod_{i=1}^{n} C_i(q) = \prod_{i=1}^{n} \epsilon_i(q) = F_{\Phi(C)}(q).$$

In fact, the proof of the Proposition also shows that Φ is surjective because the number of possible ways to add a column to an existing rook placement is equal to the number of possible ways to extend a path in $\mathcal{Z}$ by one edge. Therefore, Φ is a weight-preserving bijection. □

A sequence of nonnegative integers is $\mathcal{C}$-*admissible* if it is the degree sequence of a rook placement. That is, $(d_1, \ldots, d_n) = (\deg C_1(q)), \ldots, \deg C_n(q))$ for a $C \in \mathcal{C}(B_n, k)$. The next Corollary follows easily from Theorem 28.

Corollary 29. *A* $\mathcal{C}$*-admissible sequence determines a unique rook placement.* □

It follows from Theorem 28 that we may associate a partition type to each rook placement on B_n. The *partition type* of a rook placement C is the partition at the endpoint of the path $\Phi(C)$ in $\mathcal{Z}$. Let $\mathcal{C}(\lambda) = \Phi^{-1}(P_{\mathcal{Z}}(\lambda))$ denote the set of rook placements of partition type λ.

Corollary 30. *Let* $\lambda \vdash n$ *be a partition with* $\ell(\lambda) = n - k$ *parts. Then*

$$F_\lambda(q) = \sum_{C \in \mathcal{C}(\lambda)} F_C(q) = (q-1)^{n-\ell(\lambda)} \sum_{C \in \mathcal{C}(\lambda)} q^{\mathrm{ne}(C)}.$$

Proof. The result follows from Equation 20 and the bijection Φ. □

Remark 31. The polynomial $G_\lambda(q) \in \mathbb{N}[q]$ defined in Equation (12) is simply a sum over the rook placements of type λ involving the north-east statistic.

4 A connection with set partitions

The results of the previous section naturally leads to a decomposition of $F_T(q)$, indexed by some tableau T, into a sum of polynomials indexed by set partitions, which we explain below.

A *set partition* is a set $S = \{s_1, \ldots, s_k\}$ of nonempty disjoint subsets of $[n]$ such that $\bigcup_{i=1}^k s_i = [n]$. The s_i's are the *blocks* of σ. Let $\ell(S)$ denote the number of blocks of S, and let $\mathcal{S}(n, n-k)$ denote the set of set partitions of $[n]$ with $n-k$ blocks. We adopt the convention of listing the blocks in order so that

$$|s_1| \geqslant |s_2| \geqslant \cdots \geqslant |s_k|, \text{ and } \min s_i < \min s_{i+1} \text{ if } |s_i| = |s_{i+1}|. \tag{21}$$

This allows us to represent a set partition with a diagram similar to that of a standard Young tableau; the ith row of the diagram consists of the elements in the block s_i listed in increasing order, but there are no restrictions on the entries in each column of the diagram. A set partition $S = (s_1, \ldots, s_m)$ has *partition type* λ if $\lambda = (|s_1|, \ldots, |s_m|)$.

For $i = 1, \ldots, n$, let $S^{(i)}$ denote the sub-diagram of S consisting of the boxes containing $1, \ldots, i$, with rows ordered according to the convention set forth in Equation (21). If the box containing i is not in the first column of the diagram, let u be the least element in the same row as i in $S^{(i)}$, and suppose u is in the rth row of $S^{(i-1)}$ for some $1 \leqslant r \leqslant \ell(S^{(i-1)})$. The weight arising from the ith box is

$$S^{(i)}(q) = \begin{cases} q^{i-1-\ell(S^{(i-1)})}, & \text{if the } i\text{th box is in the first column,} \\ (q-1)q^{i-1-r}, & \text{if the } i\text{th box is in the } j\text{th column, } j \geqslant 2. \end{cases} \tag{22}$$

We define the *weight* of S as $F_S(q) = \prod_{i=1}^{n} S^{(i)}(q)$.

A sequence of nonnegative integers is $\mathcal{S}$*-admissible* if it is the degree sequence of a set partition. That is, $(d_1, \ldots, d_n) = (\deg S^{(1)}(q)), \ldots, \deg S^{(n)}(q))$ for a $S \in \mathcal{S}(n)$.

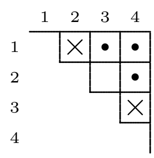
Lemma 32. *An $\mathcal{S}$-admissible sequence determines a unique set partition.*

Proof. Induct on n. When $n = 1$, the only set partition is $\{\{1\}\}$, and its degree sequence is (0).

Given an $\mathcal{S}$-admissible sequence $(d_1, \ldots, d_n)$, the subsequence $(d_1, \ldots, d_{n-1})$ determines a unique set partition $S^{(n-1)} = (S_1^{(n-1)}, \ldots, S_m^{(n-1)})$. By Equation (22), $n - 1 - m \leqslant d_n \leqslant n - 1$, and each of the $m + 1$ choices for d_n determines the block of $S^{(n-1)}$ into which n should be inserted. □

We have already constructed a weight-preserving bijection Φ between rook placements and paths in $\mathcal{Z}$. We now construct a weight-preserving bijection Ψ between rook placements and set partitions, effectively showing that paths in $\mathcal{Z}$ are equivalent to set partitions, so that $F_Z(q) = F_C(q) = F_S(q)$ if $Z \longleftrightarrow C \longleftrightarrow S$ for $Z \in \mathcal{P}_{\mathcal{Z}}(n, n-k)$, $C \in \mathcal{C}(B_n, k)$, and $S \in \mathcal{S}(n, n-k)$.

Remark 33. There is a classically known bijection (see [14]) between the set of rook placements on the staircase board B_n with k rooks and the set of set partitions of $[n] = \{1, \ldots, n\}$ with $n - k$ blocks: the placement C corresponds to the set partition where the integers i and j are in the same block if and only if there is a rook in the square $(i, j) \in C$. This bijection is different from the one described in Theorem 34. For example, the classical bijection associates the rook placement



to the set partition $(\{1, 2\}, \{3, 4\})$ and so has partition type $(2, 2)$, but as we shall see below, this placement is associated to the set partition $(\{1, 2, 4\}, \{3\})$ under the bijection in Theorem 34 and has partition type $(3, 1)$.

4.1 The construction of Ψ

Let $C \in \mathcal{C}(B_n, k)$ be a rook placement. The main idea is that the degree of $C_i(q)$ arising from the ith column of C determines the block of the set partition in which we place i. In the construction of the set partition $\Psi(C)$, we will create a sequence of intermediate set partitions $S^{(i)}$ of $[i]$ for $i = 1, \ldots, n$.

The initial case is always $\deg(C_1(q)) = \deg(1) = 0$, so $S^{(1)} = \{\{1\}\}$. Assume that $S^{(i-1)} = \{S_1^{(i-1)}, \ldots, S_m^{(i-1)}\}$ is the set partition which corresponds to the first $i-1$ columns of C, so that $m = \ell(S^{(i-1)})$. Observe that there are $m + 1$ possible blocks in which to insert i to obtain $S^{(i)}$. By Corollary 26, we know that

$$i - 1 - \ell(S^{(i-1)}) \leqslant \deg(C_i(q)) \leqslant i - 1,$$

so we construct $S^{(i)}$ by placing i in the jth block of $S^{(i-1)}$, where $j = i - \deg(C_i(q))$, and then rearranging the blocks to fit the convention in Equation (21) if necessary.

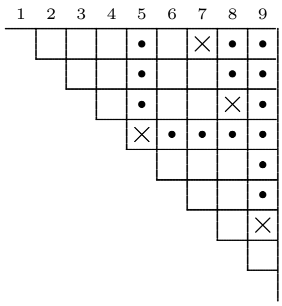
Theorem 34. *The map* $\Psi : \mathcal{C}(n,k) \to \mathcal{S}(n, n-k)$ *is a weight-preserving bijection.*

Proof. Let $S = \Psi(C)$. The map Ψ is weight-preserving, as $C_i(q) = S^{(i)}(q)$ by construction, for each $i = 1, \dots, n$. Now, since the degrees $\deg C_i(q) = \deg S^{(i)}(q)$, and by Corollary 29 and Lemma 32 the sequences of degrees completely determine C and S respectively, then Ψ is injective. Finally, we note that $|\mathcal{C}(n,k)| = |\mathcal{S}(n,n-k)|$, so Ψ is a bijection. □

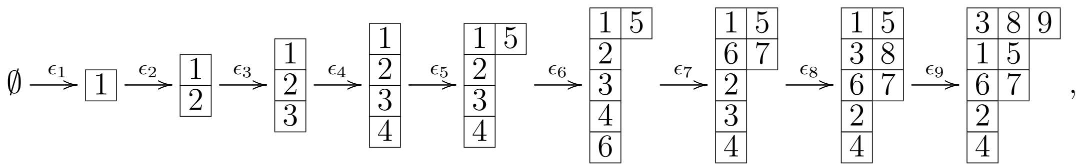
Corollary 35. *Let* $\mathcal{S}(\lambda)$ *denote the set of all set partitions of partition type* λ. *Then*

$$F_\lambda(q) = \sum_{S \in \mathcal{S}(\lambda)} F_S(q). \qquad \square$$

Example 36. Let C be the rook placement



The associated sequence of set partition diagrams associated to C is



so the set partition associated to the rook placement C is

$$S = \Psi(C) = (\{3,8,9\}, \{1,5\}, \{6,7\}, \{2\}, \{4\}).$$

Remark 37. An intriguing question is to ask for a geometric interpretation of the polynomials $F_C(q)$, indexed by rook placements (or set partitions or paths in $\mathcal{Z}$).

The problem of determining the number of adjoint $G_n(\mathbb{F}_q)$ orbits on $\mathfrak{g}_n(\mathbb{F}_q)$ remains open. In the case $q = 2$, this number has been computed for $n \leqslant 16$ by Pak and Soffer [11, Appendix B]. Let $\mathcal{O}_n(k)$ denote the orbits of rank k matrices. When $k = 1$, it turns out that the polynomials $F_C(q)$ indexed by rook placements with exactly one rook gives the sizes of the $\binom{n}{2}$ orbits in $\mathcal{O}_n(1)$. For $2 \leqslant i < j \leqslant n$, each orbit contains a unique matrix E_{ij} whose ijth entry is 1, and is zero everywhere else. The orbit containing E_{ij} is associated to the rook placement $C(i,j)$ with a single rook in the ijth square, and the size of the associated orbit is $F_{C(i,j)}(q) = (q-1)q^{n-1-(j-i)}$.

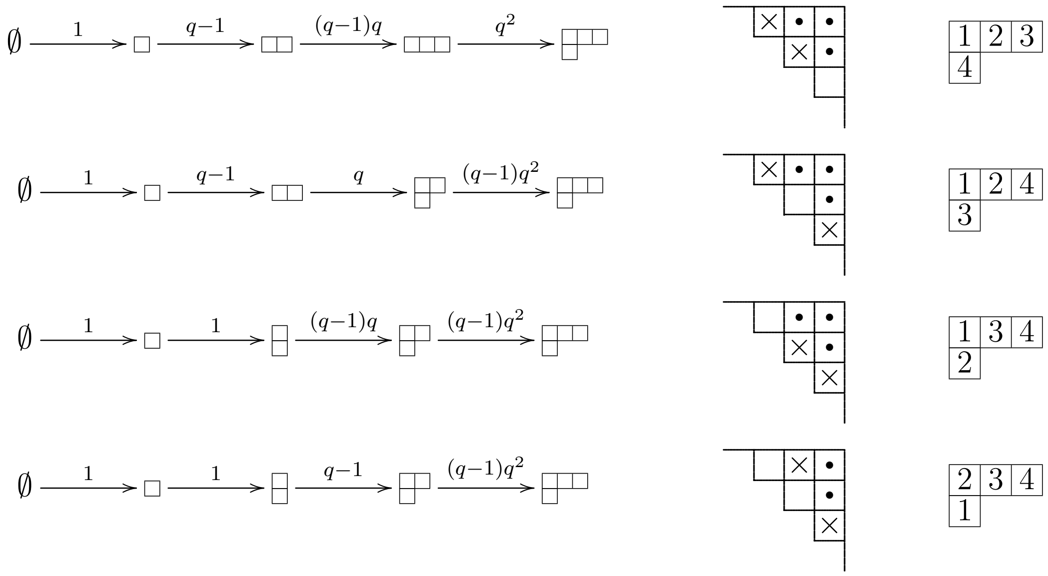


Figure 6: Paths, rook placements, and set partitions related to the computation of $F_{(3,1)}(q) = (q-1)^2(3q^3 + q^2)$.

In particular, the formula in Proposition 13 applied to the partition $\lambda = (2, 1^{n-2})$ gives the generating function

$$F_{(2,1^{n-2})}(q) = (q-1)\left((n-1)q^{n-2} + (n-2)q^{n-3} + \cdots + 3q^2 + 2q + 1\right)$$

for rank one orbits of $G_n(\mathbb{F}_q)$ on $\mathfrak{g}_n(\mathbb{F}_q)$.

Remark 38. To close, we mention a related problem which may provide a geometric interpretation of $F_C(q)$ for every rook placement C. Let N be an $n \times n$ nilpotent matrix with entries in an algebraically closed field k containing $\mathbb{F}_q$, and suppose N has Jordan type $\lambda \vdash n$. A complete flag $f = (f_1, \ldots, f_n)$ is a sequence of subspaces in k^n such that $f_1 \subset \cdots \subset f_n$ and $\dim f_i = i$ for all i. A flag is N-*stable* if $N(f_i) \subseteq f_i$ for all i. Spaltenstein [13] showed that the variety X_λ of N-stable flags is a disjoint union of f^λ smooth irreducible subvarieties X_T indexed by the standard Young tableaux of shape λ. Moreover, the closures $\overline{X}_T$ are the irreducible components of X_λ, each of which has dimension n_λ. The number of $\mathbb{F}_q$-rational points in X_λ is given by Green's polynomials $Q^\lambda_{(1^n)}(q)$ [9, III.7]. Evidently,

$$\left(\prod_{i \geqslant 1} [m_i(\lambda)]_q!\right)^{-1} Q^\lambda_{(1^n)}(q) = \left((q-1)^{n-\ell(\lambda)} q^m\right)^{-1} F_\lambda(q),$$

with $m = \min_{C \in \mathcal{C}(\lambda)} \mathrm{ne}(C)$. Based on some computations for small values of n, we expect that $F_C(q)$ plays a role in counting points in certain intersections of the irreducible components $\overline{X}_T$.

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